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Moments of $S(t,f)$ Associated with Holomorphic Hecke Cusp Forms

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Abstract. Let $S(t,f) := \pi^{-1} \arg L(1/2+it,f)$, where f is a holomorphic Hecke cusp form for $\mathrm{SL}_2(\mathbb{Z})$ of weight k. We establish an asymptotic formula for the moments of $S(t,f)$.

1. Introduction

It is well known that the function $S(t) = \frac{1}{\pi} \arg \zeta(1/2+it)$ is related to the number of nontrivial zeros ρ of the Riemann zeta-function $\zeta(s)$ with $0 < \mathrm{Im}(\rho) \leq t$. There are many research papers that have been devoted to studying the behavior of $S(t)$ (see, for example, [1,2,11,13,14,18]). In particular, Selberg [15,17] showed that for $n \in \mathbb{N}$,

$$\frac{1}{T} \int_T^{2T} S(t)^{2n} \,\mathrm{d}t = \frac{(2n)!}{n!(2\pi)^{2n}} (\log\log T)^n + O\big((\log\log T)^{n-1/2}\big).$$

Using the same method one can derive a similar result for $S(t,\chi) = \pi^{-1} \arg L(1/2+it,\chi)$, where χ is a primitive Dirichlet character of modulus q. In the subsequent paper [16] Selberg proved an analog result in the conductor aspect. More precisely, for a prime q, he showed

$$\frac{1}{q} \sideset{}{^*}\sum_{\chi \ (\mathrm{mod}\ q)} S(t,\chi)^{2n} = \frac{(2n)!}{n!(2\pi)^{2n}} (\log\log q)^n + O_{t,n}\big((\log\log q)^{n-1/2}\big), \tag{1.1}$$

where the summation runs over the primitive characters $\chi \pmod q$.

In [6], Hejhal and Luo considered a GL_2 analog of (1.1). They proved an asymptotic formula for the spectral moments of $S(t,f_j) = \pi^{-1} \arg L(1/2+it,f_j)$ assuming the Generalized Riemann Hypothesis (GRH), i.e., $S(t,f_j)^n$ is averaged over the Hecke–Maass cusp forms f_j for $\mathrm{SL}_2(\mathbb{Z})$ with a smooth test function, for each fixed positive t. Recently, a GL_3 analogous result was obtained in [12] assuming the GRH. In this paper, we establish a GL_2 analogous result of Hejhal and Luo in the weight aspect *without assuming the GRH.*

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To state our main result, let H_k denote the set of holomorphic Hecke cusp forms f of weight k for $\mathrm{SL}_2(\mathbb{Z})$, where $f(z)$ has the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z}$$

with $\lambda_f(1) = 1$. For $f \in H_k$, the L-function associated to f is given by

$$L(s, f) := \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s}, \quad \mathrm{Re}(s) > 1$$

and this has Euler product

$$\begin{aligned} L(s, f) &= \prod_p \left(1 - \lambda_f(p) p^{-s} + p^{-2s}\right)^{-1} \\ &= \prod_p \left(1 - \alpha_f(p) p^{-s}\right)^{-1} \left(1 - \beta_f(p) p^{-s}\right)^{-1}. \end{aligned}$$

The Ramanujan–Petersson conjecture (proved by Deligne [4]) asserts that

$$|\alpha_f(p)| = |\beta_f(p)| = 1, \quad \text{and thus} \quad |\lambda_f(p)| \leq 2. \tag{1.2}$$

The complete L-function

$$\Lambda(s, f) := \pi^{-s} \Gamma\left(\frac{s + (k-1)/2}{2}\right) \Gamma\left(\frac{s + (k+1)/2}{2}\right) L(s, f)$$

admits an entire continuation to $s \in \mathbb{C}$ and satisfies the functional equation

$$\Lambda(s, f) = i^k \Lambda(1 - s, f).$$

Next we define the analog of $S(t)$ for f by

$$S(t, f) := \frac{1}{\pi} \arg L(1/2 + it, f),$$

where the argument $\arg L(1/2 + it, f)$ is obtained by continuous variation along the line $\mathrm{Im}(s) = t$ from $\sigma = +\infty$ to $\sigma = 1/2$. Our main result is the following theorem.

Theorem 1.1. *Let $t > 0$ and $n \in \mathbb{N}$ be given. For sufficiently large even integer k we have*

$$\frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{S(t, f)^n}{L(1, \mathrm{sym}^2 f)} = C_n (\log\log k)^{n/2} + O_{t,n}\big((\log\log k)^{(n-1)/2}\big),$$

where

$$C_n = \begin{cases} \frac{n!}{(n/2)!(2\pi)^n} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Remark 1.2. Theorem 1.1 and Corollary 2.2 indicate that the values of $|S(t, f)|$ on average have order of magnitude $\sqrt{\log\log k}$.

2. Preliminaries

The following proposition is the well-known Petersson trace formula, which can be found in [8].

Proposition 2.1 (Petersson trace formula)**.**

$$\sum_{f\in H_k}\omega_f^{-1}\lambda_f(m)\lambda_f(n)=\delta_{m,n}+2\pi i^{-k}\sum_{c=1}^{\infty}\frac{S(m,n;c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

where $\omega_f=\frac{(4\pi)^{k-1}}{\Gamma(k-1)}\|f\|^2$, $\delta_{m,n}$ *equals* 1 *if* $m=n$ *and* 0 *otherwise,* $S(m,n;c)$ *is the Kloosterman sum, and* $J_{k-1}(x)$ *is the J-Bessel function.*

From the integral representation (see [5, 8.411 10])

$$J_\nu(x)=\frac{1}{\Gamma(\nu+1/2)\Gamma(1/2)}\left(\frac{x}{2}\right)^\nu\int_{-1}^{1}e^{ixt}(1-t^2)^{\nu-1/2}\,\mathrm{d}t$$

and the Stirling's formula, we deduce that

$$J_{k-1}(x)\ll\left(\frac{ex}{2k}\right)^{k-1}. \tag{2.1}$$

Using the bound (2.1) and the relation (see [10])

$$\omega_f=\frac{k-1}{2\pi^2}L(1,\operatorname{sym}^2 f),$$

one can deduce the following corollary.

Corollary 2.2. *For any* $m,n\geq 1$ *with* $8\pi\sqrt{mn}\leq k$ *we have*

$$\frac{2\pi^2}{k-1}\sum_{f\in H_k}\frac{\lambda_f(n)\lambda_f(m)}{L(1,\operatorname{sym}^2 f)}=\delta_{m,n}+O(k^{-A})$$

for any $A>0$.

We will need the following zero-density estimate which was established by Hough [7]. In fact, [7, Theorem 1.1] states a result without the weight $1/L(1,\operatorname{sym}^2 f)$. However, it is easy to derive a weighted version as Proposition 2.3 below by using [7, Proposition 5.1].

Proposition 2.3. *Let*

$$N_f(\sigma,T):=\#\{\rho=\beta+i\gamma\mid L(\rho,f)=0,\sigma<\beta,|\gamma|<T\}.$$

Let $\frac{1}{2}+\frac{2}{\log k}<\sigma<1$. *For some sufficiently small* $\delta_1,\theta_1>0$ *we have uniformly in* $\frac{10}{\log k}<T<k^{\delta_1}$,

$$\sum_{f\in H_k}\frac{N_f(\sigma,T)}{L(1,\operatorname{sym}^2 f)}\ll Tk^{1-\theta_1(\sigma-1/2)}\log k.$$

3. Proof of the main theorem

We will follow the framework of Selberg [16,17] and Hejhal–Luo [6]. The argument here is more complicated than in [6] since we did not assume the GRH.

For a positive parameter x (to be determined later), let

$$M(t,f) := \frac{1}{\pi}\operatorname{Im}\sum_{p\le x^3}\frac{C_f(p)}{p^{1/2+it}} \quad \text{and} \quad R(t,f) := S(t,f) - M(t,f),$$

where $C_f(p) := \alpha_f(p) + \beta_f(p) = \lambda_f(p)$.

Proposition 3.1. *Let $t > 0$ be given. For even $k \in \mathbb{N}$ sufficiently large and $x = k^{\delta/3}$ with sufficiently small $\delta > 0$, we have*

$$\frac{2\pi^2}{k-1}\sum_{f\in H_k}\frac{M(t,f)^n}{L(1,\mathrm{sym}^2 f)} = C_n(\log\log k)^{n/2} + O_{t,n}\big((\log\log k)^{n/2-1}\big), \tag{3.1}$$

$$\frac{2\pi^2}{k-1}\sum_{f\in H_k}\frac{|R(t,f)|^{2n}}{L(1,\mathrm{sym}^2 f)} = O_{t,n}(1). \tag{3.2}$$

The proof of Proposition 3.1 shall be given in Section 5. Now we deduce the main theorem from this result.

Proof of Theorem 1.1. By the binomial theorem, we have

$$S(t,f)^n = M(t,f)^n + O_n\left(\sum_{\ell=1}^{n}|M(t,f)|^{n-\ell}|R(t,f)|^{\ell}\right). \tag{3.3}$$

For $1 \le \ell < n$, we apply the generalized Hölder's inequality with exponents

$$\mathrm{p} = 2, \quad \mathrm{q} = \frac{2n}{n-\ell} \quad \text{and} \quad \mathrm{r} = \frac{2n}{\ell}$$

and Proposition 3.1 to deduce that

$$\begin{aligned}
&\frac{2\pi^2}{k-1}\sum_{f\in H_k}\frac{1}{L(1,\mathrm{sym}^2 f)}|M(t,f)|^{n-\ell}|R(t,f)|^{\ell}\\
&\ll \left(\frac{2\pi^2}{k-1}\sum_{f\in H_k}\frac{1}{L(1,\mathrm{sym}^2 f)}\right)^{1/\mathrm{p}}\left(\frac{2\pi^2}{k-1}\sum_{f\in H_k}\frac{1}{L(1,\mathrm{sym}^2 f)}|M(t,f)|^{2n}\right)^{1/\mathrm{q}}\\
&\quad\times\left(\frac{2\pi^2}{k-1}\sum_{f\in H_k}\frac{1}{L(1,\mathrm{sym}^2 f)}|R(t,f)|^{2n}\right)^{1/\mathrm{r}}\\
&\ll_{t,n} (\log\log k)^{(n-\ell)/2} \ll_{t,n} (\log\log k)^{(n-1)/2}.
\end{aligned}$$

The assertion follows from (3.3), Proposition 3.1 and this bound. □

4. An approximation of $S(t,f)$

In this section we will prove several technical lemmas and derive an approximation of $S(t,f)$.

We denote by $\rho=\beta+i\gamma$ a typical zero of $L(s,f)$ inside the critical strip, i.e., $0<\beta<1$. For $\mathrm{Re}(s)>1$, we have

$$-\frac{L'}{L}(s,f)=\sum_{n=1}^{\infty}\frac{\Lambda(n)C_f(n)}{n^s}, \tag{4.1}$$

where $\Lambda(n)$ denotes the von Mangoldt function, and

$$C_f(n)=\begin{cases}\alpha_f(p)^m+\beta_f(p)^m & \text{if } n=p^m \text{ for a prime } p,\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 4.1. *Let $x>1$. For $s\neq\rho$, and $s\neq -2m-\frac{k\pm1}{2}$ $(m=0,1,2,\dots)$, we have the following identity*

$$\frac{L'}{L}(s,f)=-\sum_{n\le x^3}\frac{C_f(n)\Lambda_x(n)}{n^s}-\frac{1}{\log^2 x}\sum_{\rho}\frac{x^{\rho-s}(1-x^{\rho-s})^2}{(\rho-s)^3}$$
$$-\frac{1}{\log^2 x}\sum_{m=0}^{\infty}\frac{x^{-2m-\frac{k\pm1}{2}-s}(1-x^{-2m-\frac{k\pm1}{2}-s})^2}{(-2m-\frac{k\pm1}{2}-s)^3},$$

where

$$\Lambda_x(n)=\begin{cases}\Lambda(n) & \textit{if } n\le x,\\ \Lambda(n)\frac{\log^2(x^3/n)-2\log^2(x^2/n)}{2\log^2 x} & \textit{if } x\le n\le x^2,\\ \Lambda(n)\frac{\log^2(x^3/n)}{2\log^2 x} & \textit{if } x^2\le n\le x^3,\\ 0 & \textit{if } n\ge x^3.\end{cases}$$

Proof. First we recall the discontinuous integral

$$\frac{1}{2\pi i}\int_{(\alpha)}\frac{y^s}{s^3}\,\mathrm{d}s=\begin{cases}\frac{\log^2 y}{2} & \text{if } y\ge1,\\ 0 & \text{if } 0<y\le 1\end{cases} \tag{4.2}$$

for $\alpha>0$. It follows from (4.1) and (4.2) that

$$-\log^2 x\sum_{n=1}^{\infty}\frac{C_f(n)\Lambda_x(n)}{n^s}=\frac{1}{2\pi i}\int_{(\alpha)}\frac{x^u(1-x^u)^2}{u^3}\frac{L'}{L}(s+u,f)\,\mathrm{d}u,$$

where $\alpha=\max\{2,1+\mathrm{Re}(s)\}$. By moving the line of integration all way to the left, we pick up the residues at $u=0$, $u=\rho-s$ and $u=-2m-\frac{k\pm1}{2}-s$ $(m=0,1,2,\dots)$ and deduce that

$$\begin{aligned}&\frac{1}{2\pi i}\int_{(2)}\frac{x^u(1-x^u)^2}{u^3}\frac{L'}{L}(s+u,f)\,\mathrm{d}u\\&=\frac{L'}{L}(s,f)\log^2 x+\sum_\rho\frac{x^{\rho-s}(1-x^{\rho-s})^2}{(\rho-s)^3}+\sum_{m=0}^{\infty}\frac{x^{-2m-\frac{k\pm1}{2}-s}(1-x^{-2m-\frac{k\pm1}{2}-s})^2}{(-2m-\frac{k\pm1}{2}-s)^3}.\end{aligned}$$

Thus the lemma follows immediately. □

Lemma 4.2. *For $s=\sigma+it$, $s'=\sigma'+it'$ such that $1/2\le\sigma,\sigma'\le 10$, $s\neq\rho$, and $s'\neq\rho$, we have*

$$\mathrm{Im}\left(\frac{L'}{L}(s,f)-\frac{L'}{L}(s',f)\right)=\mathrm{Im}\sum_\rho\left(\frac{1}{s-\rho}-\frac{1}{s'-\rho}\right)+O(1),$$

and

$$\mathrm{Re}\,\frac{L'}{L}(s,f)=\sum_\rho\frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2}+O(\log(|t|+k)). \tag{4.3}$$

Proof. By Hadamard's factorization of the entire function $\Lambda(s,f)$ we have

$$\frac{L'}{L}(s,f)=b_f+\sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+(k-1)/2}{2}\right)-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+(k+1)/2}{2}\right)+\log\pi$$

for some $b_f\in\mathbb{C}$ with $\mathrm{Re}(b_f)=-\mathrm{Re}\sum_\rho\frac{1}{\rho}$ (see [9, Propsoition 5.7]). Now Lemma 4.2 follow from

$$\frac{\Gamma'}{\Gamma}(s)=\log s+O\left(\frac{1}{|s|}\right).$$ □

Let $x\ge 4$. We define

$$\sigma_x=\sigma_{x,f}=\sigma_{x,f,t}:=\frac{1}{2}+2\max_\rho\left\{\left|\beta-\frac{1}{2}\right|,\frac{5}{\log x}\right\}, \tag{4.4}$$

where $\rho=\beta+i\gamma$ runs through the zeros of $L(s,f)$ for which

$$|t-\gamma|\le\frac{x^{3|\beta-1/2|}}{\log x}. \tag{4.5}$$

We shall display the dependence of σ_x as needed.

Lemma 4.3. *Let $x\ge 4$. For $\sigma\ge\sigma_x$, we have*

$$\begin{aligned}\frac{L'}{L}(\sigma+it,f)=-\sum_{n\le x^3}\frac{C_f(n)\Lambda_x(n)}{n^{\sigma+it}}&+O\left(x^{1/4-\sigma/2}\left|\sum_{n\le x^3}\frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x+it}}\right|\right)\\&+O\big(x^{1/4-\sigma/2}\log(|t|+k)\big),\end{aligned}$$

and

$$\sum_\rho\frac{\sigma_x-1/2}{(\sigma_x-\beta)^2+(t-\gamma)^2}=O\left(\sum_{n\le x^3}\frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x+it}}\right)+O(\log(|t|+k)).$$

Proof. By (4.3), we have

$$\operatorname{Re}\frac{L'}{L}(\sigma_x+it,f)=\sum_\rho\frac{\sigma_x-\beta}{(\sigma_x-\beta)^2+(t-\gamma)^2}+O(\log(|t|+k)). \tag{4.6}$$

On the other hand, if $\beta+i\gamma$ is a zero of $L(s,f)$, then $(1-\beta)+i\gamma$ is also a zero of $L(s,f)$. Thus we have

$$\begin{aligned}
&\sum_\rho\frac{\sigma_x-\beta}{(\sigma_x-\beta)^2+(t-\gamma)^2}\\
&=\frac{1}{2}\sum_\rho\left(\frac{\sigma_x-\beta}{(\sigma_x-\beta)^2+(t-\gamma)^2}+\frac{\sigma_x-(1-\beta)}{(\sigma_x-1-\beta)^2+(t-\gamma)^2}\right)\\
&=\left(\sigma_x-\frac{1}{2}\right)\sum_\rho\frac{(\sigma_x-1/2)^2-(\beta-1/2)^2+(t-\gamma)^2}{\big((\sigma_x-\beta)^2+(t-\gamma)^2\big)\big((\sigma_x-1+\beta)^2+(t-\gamma)^2\big)}.
\end{aligned}\tag{4.7}$$

Case (i): If $|\beta-1/2|\le\frac{\sigma_x-1/2}{2}$, then

$$\begin{aligned}
(\sigma_x-1/2)^2-(\beta-1/2)^2&\ge\frac{1}{2}\big((\sigma_x-1/2)^2+(\beta-1/2)^2\big)\\
&=\frac{1}{4}\big((\sigma_x-\beta)^2+(\sigma_x-1+\beta)^2\big).
\end{aligned}$$

Thus

$$(\sigma_x-1/2)^2-(\beta-1/2)^2+(t-\gamma)^2\ge\frac{1}{4}\big((\sigma_x-1+\beta)^2+(t-\gamma)^2\big).$$

Case (ii): If $|\beta-1/2|>\frac{\sigma_x-1/2}{2}$, then by (4.4) and (4.5) we have

$$|t-\gamma|>\frac{x^{3|\beta-1/2|}}{\log x}>3|\beta-1/2|.$$

Thus

$$\begin{aligned}
&(\sigma_x-1/2)^2-(\beta-1/2)^2+(t-\gamma)^2\\
&=\big((\sigma_x-1/2)^2+(\beta-1/2)^2\big)+(t-\gamma)^2-2(\beta-1/2)^2\\
&\ge\frac{1}{2}\big((\sigma_x-\beta)^2+(\sigma_x-1+\beta)^2\big)+\frac{7}{9}(t-\gamma)^2\\
&\ge\frac{1}{4}\big((\sigma_x-1+\beta)^2+(t-\gamma)^2\big).
\end{aligned}$$

From Cases (i), (ii) and (4.7), we have

$$\sum_\rho\frac{\sigma_x-\beta}{(\sigma_x-\beta)^2+(t-\gamma)^2}\ge\frac{1}{4}(\sigma_x-1/2)\sum_\rho\frac{1}{(\sigma_x-\beta)^2+(t-\gamma)^2}.$$

Using this bound and (4.6), we obtain

$$\sum_\rho\frac{1}{(\sigma_x-\beta)^2+(t-\gamma)^2}\le\frac{4}{\sigma_x-1/2}\left|\frac{L'}{L}(\sigma_x+it,f)\right|+O\left(\frac{\log(|t|+k)}{\sigma_x-1/2}\right). \tag{4.8}$$

On the other hand, we have (by Lemma 4.1)

$$\frac{L'}{L}(\sigma+it,f) = -\sum_{n\le x^3}\frac{\Lambda_x(n)C_f(n)}{n^{\sigma+it}} + \frac{w(x,\sigma,t)}{\log^2 x}\sum_{\rho}\frac{x^{\beta-\sigma}(1+x^{\beta-\sigma})^2}{\big((\sigma-\beta)^2+(t-\gamma)^2\big)^{3/2}} + O\left(\frac{x^{-\sigma}}{\log^2 x}\right) \tag{4.9}$$

with $|w(x,\sigma,t)|\le 1$.

Next we claim that

$$\frac{x^{\beta-\sigma}(1+x^{\beta-\sigma})^2}{\big((\sigma-\beta)^2+(t-\gamma)^2\big)^{3/2}} \le 2\log x\frac{x^{1/4-\sigma/2}}{(\beta-\sigma_x)^2+(t-\gamma)^2}. \tag{4.10}$$

If $\beta\le\frac{\sigma_x+1/2}{2}$, then

$$\begin{aligned}\frac{x^{\beta-\sigma}(1+x^{\beta-\sigma})^2}{((\sigma-\beta)^2+(t-\gamma)^2)^{3/2}} &\le \frac{4x^{1/4-\sigma/2}}{(\sigma_x-\beta)((\sigma_x-\beta)^2+(t-\gamma)^2)}\\ &\le \frac{8}{\sigma_x-1/2}\frac{x^{1/4-\sigma/2}}{(\sigma_x-\beta)^2+(t-\gamma)^2}\\ &\le \frac{4}{5}\log x\frac{x^{1/4-\sigma/2}}{(\beta-\sigma_x)^2+(t-\gamma)^2}.\end{aligned}$$

If $\beta>\frac{\sigma_x+1/2}{2}$, then by the definition of σ_x in (4.4) and (4.5),

$$|t-\gamma| > \frac{x^{3|\beta-1/2|}}{\log x} > 3|\beta-1/2| \ge 3|\beta-\sigma_x|.$$

Thus $(t-\gamma)^2>\frac{8}{9}\big((\beta-\sigma_x)^2+(t-\gamma)^2\big)$. Hence

$$\begin{aligned}\frac{x^{\beta-\sigma}(1+x^{\beta-\sigma})^2}{((\sigma-\beta)^2+(t-\gamma)^2)^{3/2}} &\le \frac{x^{\beta-\sigma}(1+x^{\beta-1/2})^2}{|t-\gamma|(t-\gamma)^2} \le \frac{\log x}{x^{3|\beta-1/2|}}\cdot\frac{9}{8}\frac{x^{\beta-\sigma}(1+x^{\beta-1/2})^2}{(\beta-\sigma_x)^2+(t-\gamma)^2}\\ &= \frac{9}{8}(\log x)(1+x^{-(\beta-1/2)})^2\frac{x^{1/2-\sigma}}{(\beta-\sigma_x)^2+(t-\gamma)^2}\\ &< \frac{9}{8}(1+e^{-5})^2(\log x)\frac{x^{1/2-\sigma}}{(\beta-\sigma_x)^2+(t-\gamma)^2}\\ &< 2(\log x)\frac{x^{1/2-\sigma}}{(\beta-\sigma_x)^2+(t-\gamma)^2}.\end{aligned}$$

So in both cases, we have (4.10). Using (4.8) and (4.10), we get

$$\begin{aligned}&\sum_{\rho}\frac{x^{\beta-\sigma}(1+x^{\beta-\sigma})^2}{((\sigma-\beta)^2+(t-\gamma)^2)^{3/2}}\\ &\le 8\log x\frac{x^{1/4-\sigma/2}}{\sigma_x-1/2}\left|\frac{L'}{L}(\sigma_x+it,f)\right| + O\left(\frac{(\log x)x^{1/4-\sigma/2}\log(|t|+k)}{\sigma_x-1/2}\right)\\ &\le \frac{4}{5}(\log x)^2x^{1/4-\sigma/2}\left|\frac{L'}{L}(\sigma_x+it,f)\right| + O\big((\log x)^2x^{1/4-\sigma/2}\log(|t|+k)\big).\end{aligned}$$

Inserting this bound into (4.9), we have

$$\begin{aligned}\frac{L'}{L}(\sigma+it,f) &= -\sum_{n\le x^3}\frac{C_f(n)\Lambda_x(n)}{n^{\sigma+it}} + \frac{4}{5}w'(x,\sigma,t)x^{1/4-\sigma/2}\frac{L'}{L}(\sigma_x+it,f) \\ &\quad + O\big(x^{1/4-\sigma/2}\log(|t|+k)\big)\end{aligned} \tag{4.11}$$

with $|w'(x,\sigma,t)|\le 1$. By taking $\sigma=\sigma_x$,

$$\begin{aligned}\left(1-\frac{4}{5}w'(x,\sigma_x,t)x^{1/4-\sigma_x/2}\right)\frac{L'}{L}(\sigma_x+it,f) &= O\left(\left|\sum_{n\le x^3}\frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x+it}}\right|\right) \\ &\quad + O(x^{1/4-\sigma/2}\log(|t|+k)).\end{aligned}$$

Since $\left|1-\frac{4}{5}w'(x,\sigma_x,t)\right|\ge 1-\frac{4}{5}=\frac{1}{5}$,

$$\frac{L'}{L}(\sigma_x+it,f) = O\left(\left|\sum_{n\le x^3}\frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x+it}}\right|\right) + O\big(x^{1/4-\sigma/2}\log(|t|+k)\big). \tag{4.12}$$

The results follow from substituting (4.12) into (4.11) and (4.8). □

The following theorem provides an approximation of $S(t,f)$.

Theorem 4.4. *For $t\ne 0$, and $x\ge 4$, we have*

$$\begin{aligned}S(t,f) &= \frac{1}{\pi}\operatorname{Im}\sum_{n\le x^3}\frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x+it}\log n} + O\left((\sigma_x-1/2)\left|\sum_{n\le x^3}\frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x+it}}\right|\right) \\ &\quad + O\big((\sigma_x-1/2)\log(|t|+k)\big),\end{aligned}$$

where σ_x is defined in (4.4).

Proof. We begin with

$$\begin{aligned}\pi S(t,f) &= -\int_{1/2}^{\infty}\operatorname{Im}\frac{L'}{L}(\sigma+it,f)\,\mathrm{d}\sigma \\ &= -\int_{\sigma_x}^{\infty}\operatorname{Im}\frac{L'}{L}(\sigma+it,f)\,\mathrm{d}\sigma - (\sigma_x-1/2)\operatorname{Im}\frac{L'}{L}(\sigma_x+it,f) \\ &\quad + \int_{1/2}^{\sigma_x}\operatorname{Im}\left(\frac{L'}{L}(\sigma_x+it,f)-\frac{L'}{L}(\sigma+it,f)\right)\mathrm{d}\sigma \\ &=: J_1+J_2+J_3.\end{aligned}$$

By Lemma 4.3, we have

$$
\begin{aligned}
J_1 &= -\int_{\sigma_x}^{\infty} \operatorname{Im} \frac{L'}{L}(\sigma+it,f)\,\mathrm{d}\sigma \\
&= \int_{\sigma_x}^{\infty} \operatorname{Im} \sum_{n\le x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma+it}}\,\mathrm{d}\sigma + O\left(\left|\sum_{n\le x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x+it}}\right| \int_{\sigma_x}^{\infty} x^{1/4-\sigma/2}\,\mathrm{d}\sigma\right) \\
&\quad + O\left(\log(|t|+k)\int_{\sigma_x}^{\infty} x^{1/4-\sigma/2}\,\mathrm{d}\sigma\right) \\
&= \operatorname{Im} \sum_{n\le x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x+it}\log n} + O\left(\frac{1}{\log x}\left|\sum_{n\le x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x+it}}\right|\right) + O\left(\frac{\log(|t|+k)}{\log x}\right).
\end{aligned}
$$

Taking $\sigma=\sigma_x$ in Lemma 4.3 we see

$$
\begin{aligned}
|J_2| &\le (\sigma_x-1/2)\left|\frac{L'}{L}(\sigma_x+it,f)\right| \\
&\ll (\sigma_x-1/2)\left|\sum_{n\le x^2} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x+it}}\right| + (\sigma_x-1/2)\log(|t|+k).
\end{aligned}
$$

To bound J_3, we first apply Lemma 4.2 to see that

$$
\begin{aligned}
&\operatorname{Im}\left(\frac{L'}{L}(\sigma_x+it,f) - \frac{L'}{L}(\sigma+it,f)\right) \\
&= \sum_{\rho} \frac{(\sigma_x-\sigma)(\sigma+\sigma_x-2\beta)(t-\gamma)}{\big\{(\sigma_x-\beta)^2+(t-\gamma)^2\big\}\big\{(\sigma-\beta)^2+(t-\gamma)^2\big\}} + O(1).
\end{aligned}
$$

Hence

$$
\begin{aligned}
|J_3| &\le \sum_{\rho} \int_{1/2}^{\sigma_x} \frac{(\sigma_x-\sigma)|\sigma+\sigma_x-2\beta||t-\gamma|}{\big\{(\sigma_x-\beta)^2+(t-\gamma)^2\big\}\big\{(\sigma-\beta)^2+(t-\gamma)^2\big\}}\,\mathrm{d}\sigma + O(\sigma_x-1/2) \\
&\le \sum_{\rho} \frac{\sigma_x-1/2}{(\sigma_x-\beta)^2+(t-\gamma)^2} \int_{1/2}^{\sigma_x} \frac{|\sigma+\sigma_x-2\beta||t-\gamma|}{(\sigma-\beta)^2+(t-\gamma)^2}\,\mathrm{d}\sigma + O(\sigma_x-1/2).
\end{aligned}
$$

Case (i): If $|\beta-1/2| \le \frac{1}{2}(\sigma_x-1/2)$, then for $1/2\le\sigma\le\sigma_x$,

$$
\begin{aligned}
|\sigma+\sigma_x-2\beta| &= \big|(\sigma-1/2)+(\sigma_x-1/2)-2(\beta-1/2)\big| \\
&\le |\sigma-1/2| + |\sigma_x-1/2| + 2|\beta-1/2| \le 3(\sigma_x-1/2).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\int_{1/2}^{\sigma_x} \frac{|\sigma+\sigma_x-2\beta||t-\gamma|}{(\sigma-\beta)^2+(t-\gamma)^2}\,\mathrm{d}\sigma &\le 3(\sigma_x-1/2)\int_{-\infty}^{\infty} \frac{|t-\gamma|}{(\sigma-\beta)^2+(t-\gamma)^2}\,\mathrm{d}\sigma \\
&< 10(\sigma_x-1/2).
\end{aligned}
$$

Case (ii): If $|\beta - 1/2| > \frac{1}{2}(\sigma_x - 1/2)$, then by (4.4) and (4.5),

$$|t - \gamma| > \frac{x^{3|\beta-1/2|}}{\log x} > 3|\beta - 1/2|$$

and for $1/2 \le \sigma \le \sigma_x$,

$$|\sigma + \sigma_x - 2\beta| \le (\sigma - 1/2) + (\sigma_x - 1/2) + 2|\beta - 1/2| \le 6|\beta - 1/2|.$$

Thus

$$\begin{aligned}\int_{1/2}^{\sigma_x} \frac{|\sigma + \sigma_x - 2\beta||t - \gamma|}{(\sigma - \beta)^2 + (t - \gamma)^2}\, d\sigma &< \int_{1/2}^{\sigma_x} \frac{|\sigma + \sigma_x - 2\beta|}{|t - \gamma|}\, d\sigma \\ &< \int_{1/2}^{\sigma_x} \frac{6|\beta - 1/2|}{3|\beta - 1/2|}\, d\sigma = 2(\sigma_x - 1/2).\end{aligned}$$

It follows from Cases (i) and (ii) that

$$\begin{aligned}|J_3| &\le 10(\sigma_x - 1/2) \sum_{\rho} \frac{\sigma_x - 1/2}{(\sigma_x - \beta)^2 + (t - \gamma)^2} + O(\sigma_x - 1/2) \\ &= O\left((\sigma_x - 1/2) \left| \sum_{n \le x^3} \frac{\Lambda_x(n) C_f(n)}{n^{\sigma_x + it}} \right| \right) + O((\sigma_x - 1/2) \log(|t| + k)),\end{aligned}$$

where the last step is obtained by Lemma 4.3. The theorem follows from these bounds for J_1, J_2 and J_3. □

We end this section by giving a consequence of Theorem 4.4, which is an analog to those obtained by Littlewood [11], Selberg [16], and Hejhal–Luo [6].

Theorem 4.5. *Assuming the GRH, we have*

$$S(t, f) \ll \frac{\log(|t| + k)}{\log\log(|t| + k)}.$$

Proof. We have $\sigma_x = \frac{1}{2} + \frac{10}{\log x}$ by virtue of the GRH. By (1.2),

$$|C_f(p^m)\Lambda_x(p^m)| \le 2 \log p.$$

Thus

$$\left| \operatorname{Im} \sum_{n \le x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x + it} \log n} \right| \ll \sum_{p \le x^3} p^{-1/2} \ll \frac{x^{3/2}}{\log x}$$

and

$$(\sigma_x - 1/2) \left| \sum_{n \le x^3} \frac{C_f(n)\Lambda_x(n)}{n^{\sigma_x + it}} \right| \ll \frac{1}{\log x} \sum_{p \le x^3} p^{-1/2} \log p \ll \frac{x^{3/2}}{\log x}.$$

The theorem follows by taking $x = \{\log(|t| + k)\}^{2/3}$ in Theorem 4.4 and above estimates. □

5. Proof of Proposition 3.1

5.1. The main term (3.1)

First recall that $C_f(p) = \alpha(p) + \beta(p) = \lambda_f(p)$ and

$$M(t, f) = \frac{1}{\pi} \operatorname{Im} \sum_{p \le x^3} \frac{C_f(p)}{p^{1/2+it}} = \frac{-i}{2\pi} \left(\sum_{p \le x^3} \frac{\lambda_f(p)}{p^{1/2+it}} - \sum_{p \le x^3} \frac{\lambda_f(p)}{p^{1/2-it}} \right).$$

Set $x = k^{\delta/3}$ for a suitably small $\delta > 0$ (to be specified later). Thus we have

$$M(t, f)^n = \frac{(-i)^n}{(2\pi)^n} \left(\sum_{p \le k^\delta} \frac{\lambda_f(p)}{p^{1/2+it}} - \sum_{p \le k^\delta} \frac{\lambda_f(p)}{p^{1/2-it}} \right)^n. \tag{5.1}$$

A general term in the expansion of (5.1) has the form

$$\frac{\lambda_f(p_1)^{m(p_1)+n(p_1)}}{p_1^{m(p_1)(1/2+it)}(-1)^{n(p_1)} p^{n(p_1)(1/2-it)}} \times \cdots \times \frac{\lambda_f(p_r)^{m(p_r)+n(p_r)}}{p_r^{m(p_r)(1/2+it)}(-1)^{n(p_r)} p_r^{n(p_r)(1/2-it)}}, \tag{5.2}$$

where $p_1 < p_2 < \cdots < p_r < k^\delta$, $m(p_j) + n(p_j) \ge 1$, and $\sum_{j=1}^r (m(p_j) + n(p_j)) = n$.

To study the general term (5.2), we first recall the following Hecke relation

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right). \tag{5.3}$$

Now we discuss the contribution from the general term (5.2) in the following cases.

Case (I)*: In the general term* (5.2)*,* $m(p_{j_0}) \not\equiv n(p_{j_0}) \pmod 2$ *for some* j_0. By the Heceke relation (5.3) and Corollary 2.2, we have the contribution from these terms are negligible by taking $0 < \delta < 2/n$.

Thus for any $A > 0$ and odd $n \ge 1$, we have

$$\frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{M(t, f)^n}{L(1, \operatorname{sym}^2 f)} \ll k^{-A}.$$

Case (II)*: In the general term* (5.2)*,* $m(p_j) \equiv n(p_j) \pmod 2$ *for all* j*, and* $m(p_{j_0}) + n(p_{j_0}) \ge 4$ *for some* j_0. In this case, n is even and $n > 2r$. By the bound (1.2) and $\sum_{p \le x} p^{-1} \ll \log\log x$, we deduce that the contribution from these terms to (3.1) is at most $O\big((\log\log k)^{n/2-1}\big)$.

It remains to discuss the following: In the general term (5.2), $m(p_j) \equiv n(p_j) \pmod 2$ and $m(p_j) + n(p_j) = 2$ for all j. Clearly n must be even, say $n = 2m$ and so $r = m$.

Case (III)*:* $m(p_{j_0}) = 2$ *or* $n(p_{j_0}) = 2$ *for some* j_0. We deduce that the contribution from these terms to (3.1) is at most $O\big((\log\log k)^{n/2-1}\big)$ by using (1.2) and the convergence of $\sum_p p^{-1-it}$ for $t \neq 0$.

Case (IV)*: $m(p_j)=n(p_j)=1$ for all j.* The contribution from these terms to (3.1) is

$$\begin{aligned}&\frac{2\pi^2}{k-1}\sum_{f\in H_k}\frac{1}{L(1,\mathrm{sym}^2 f)}\binom{2m}{m}(m!)(m!)\frac{(-1)^m(-i)^{2m}}{(2\pi)^{2m}}\\&\times\sum_{p_1<\cdots<p_m<k^\delta}\frac{\lambda_f(p_1)^2\cdots\lambda_f(p_m)^2}{p_1\cdots p_m}.\end{aligned}\tag{5.4}$$

By the Hecke relations, we have $\lambda_f(p_1)^2\cdots\lambda_f(p_m)^2=\lambda_f(p_1\cdots p_m)\lambda_f(p_1\cdots p_m)$.

Applying Corollary 2.2, we see that (5.4) equals

$$\begin{aligned}&\frac{(2m)!}{(2\pi)^{2m}}\sum_{p_1<\cdots<p_m<k^\delta}\frac{1}{p_1\cdots p_m}\cdot\big(1+O(k^{-A})\big)\\&=\frac{(2m)!}{m!(2\pi)^{2m}}\sum_{\substack{p_1,\ldots,p_m<k^\delta\\ p_j\ \text{distinct}}}\frac{1}{p_1\cdots p_m}\cdot\big(1+O(k^{-A})\big)\\&=C_{2m}(\log\log k+O(1))^m\cdot\big(1+O(k^{-A})\big)\\&=C_{2m}(\log\log k)^m+O_m\big((\log\log k)^{m-1}\big).\end{aligned}$$

Now the asymptotic formula (3.1) follows from Cases (I)–(IV).

5.2. The remainder term (3.2)

Lemma 5.1. *Let $t>0$ be given. For even $k\in\mathbb{N}$ sufficiently large and $x=k^{\delta/3}$ with $0<\delta<3\theta_1/(8n+3)$, we have*

$$\sum_{f\in H_k}\frac{(\sigma_{x,f}-1/2)^{4n}x^{4n(\sigma_{x,f}-1/2)}}{L(1,\mathrm{sym}^2 f)}\ll_{t,n,\delta}\frac{k}{(\log k)^{4n}},$$

where $\sigma_{x,f}$ is defined in (4.4) *and θ_1 is as in Proposition* 2.3.

Proof. By the definition of $\sigma_{x,f}$,

$$\begin{aligned}&(\sigma_{x,f}-1/2)^{4n}x^{4n(\sigma_{x,f}-1/2)}\\&\le\left(\frac{10}{\log x}\right)^{4n}x^{40n/\log x}+2^{4n+1}\sum_{\substack{\beta>\frac12+\frac{5}{\log x}\\ |t-\gamma|\le\frac{x^{3(\beta-1/2)}}{\log x}}}(\beta-1/2)^{4n}x^{8n(\beta-1/2)}.\end{aligned}$$

On the other hand,

$$\sum_{\substack{\beta>\frac12+\frac{5}{\log x}\\ |t-\gamma|\le\frac{x^{3(\beta-1/2)}}{\log x}}}(\beta-1/2)^{4n}x^{8n(\beta-1/2)}$$

$$
\begin{aligned}
&\leq \sum_{j=5}^{\frac{1}{2}\lfloor \log x\rfloor} \left(\frac{j+1}{\log x}\right)^{4n} x^{8n\frac{j+1}{\log x}} \sum_{\substack{\frac{1}{2}+\frac{j}{\log x}<\beta\leq\frac{1}{2}+\frac{j+1}{\log x}\\ |t-\gamma|\leq \frac{x^{3(\beta-1/2)}}{\log x}}} 1\\
&\leq \frac{1}{(\log x)^{4n}} \sum_{j=5}^{\frac{1}{2}\lfloor \log x\rfloor} (j+1)^{4n} e^{8n(j+1)} N_f\left(\frac{1}{2}+\frac{j}{\log x}, |t|+\frac{e^{3(j+1)}}{\log x}\right).
\end{aligned}
$$

By Proposition 2.3,

$$
\begin{aligned}
&\sum_{f\in H_k} \frac{1}{L(1,\mathrm{sym}^2 f)} \sum_{\substack{\beta>\frac{1}{2}+\frac{5}{\log x}\\ |t-\gamma|\leq \frac{x^{3(\beta-1/2)}}{\log x}}} (\beta-1/2)^{4n} x^{8n(\beta-1/2)}\\
&\ll_t \frac{1}{(\log x)^{4n}} \sum_{j=5}^{\infty} (j+1)^{4n} e^{8n(j+1)} \frac{e^{3(j+1)}}{\log x} k^{1-\theta_1\frac{j}{\log x}} \log k\\
&\ll_t \frac{k\log k}{(\log x)^{4n+1}} \sum_{j=5}^{\infty} (j+1)^{4n} e^{8n(j+1)+3(j+1)-\theta_1\frac{\log k}{\log x} j}\\
&\ll_{t,n,\delta} \frac{k}{(\log k)^{4n}} \sum_{j=5}^{\infty} (j+1)^{4n} e^{(8n+3-\frac{3\theta_1}{\delta})j}\\
&\ll_{t,n,\delta} \frac{k}{(\log k)^{4n}}
\end{aligned}
$$

provided that $0<\delta<3\theta_1/(8n+3)$. In addition,

$$
\sum_{f\in H_k} \frac{1}{L(1,\mathrm{sym}^2 f)} \left(\frac{10}{\log x}\right)^{4n} x^{40n/\log x} \ll \frac{k}{(\log x)^{4n}} \ll \frac{k}{(\log k)^{4n}}.
$$

The lemma follows from above estimates. □

Lemma 5.2.

$$
\begin{aligned}
R(t,f) = {}& O\left(\left|\mathrm{Im} \sum_{p\leq x^3} \frac{C_f(p)(\Lambda(p)-\Lambda_x(p))}{p^{1/2+it}\log p}\right|\right) + O\left(\left|\mathrm{Im} \sum_{p\leq x^{3/2}} \frac{C_f(p^2)\Lambda_x(p^2)}{p^{1+2it}\log p}\right|\right)\\
&+ O\left((\sigma_x-1/2)x^{\sigma_x-1/2}\int_{1/2}^{\infty} x^{1/2-\sigma}\left|\sum_{p\leq x^3} \frac{C_f(p)\Lambda_x(p)\log(xp)}{p^{\sigma+it}}\right| \mathrm{d}\sigma\right)\\
&+ O\big((\sigma_x-1/2)\log(|t|+k)\big) + O(1).
\end{aligned}
$$

Proof. By Theorem 4.4

$$
R(t,f) = S(t,f) - M(t,f)
$$

$$
\begin{aligned}
&= \frac{1}{\pi}\operatorname{Im}\sum_{p\le x^3}\frac{C_f(p)(\Lambda_x(p)p^{1/2-\sigma_x}-\Lambda(p))}{p^{1/2+it}\log p}+\frac{1}{\pi}\operatorname{Im}\sum_{m=2}^{\infty}\sum_{p^m\le x^3}\frac{C_f(p^m)\Lambda_x(p^m)}{p^{m(\sigma_x+it)}\log p^m}\\
&\quad+O\left((\sigma_x-1/2)\left|\sum_{m=1}^{\infty}\sum_{p^m\le x^3}\frac{C_f(p^m)\Lambda_x(p^m)}{p^{m(\sigma_x+it)}}\right|\right)+O\big((\sigma_x-1/2)\log(|t|+k)\big).
\end{aligned}
$$

Using the bound (1.2), we deduce that

$$
\sum_{m=3}^{\infty}\sum_{p^m\le x^3}\frac{C_f(p^m)\Lambda_x(p^m)}{p^{m(\sigma_x+it)}}=O(1)\quad\text{and}\quad\sum_{m=3}^{\infty}\sum_{p^m\le x^3}\frac{C_f(p^m)\Lambda_x(p^m)}{p^{m(\sigma_x+it)}\log p}=O(1).
$$

Note that

$$
\begin{aligned}
(\sigma_x-1/2)\left|\sum_{p\le x^{3/2}}\frac{C_f(p^2)\Lambda_x(p^2)}{p^{2(\sigma_x+it)}}\right| &\ll(\sigma_x-1/2)\sum_{p\le x^{3/2}}\frac{\log p}{p}\\
&\ll(\sigma_x-1/2)\log x\ll(\sigma_x-1/2)\log k.
\end{aligned}
$$

Thus

$$
\begin{aligned}
R(t,f)&=\frac{1}{\pi}\operatorname{Im}\sum_{p\le x^3}\frac{C_f(p)(\Lambda_x(p)-\Lambda(p))}{p^{1/2+it}\log p}-\frac{1}{\pi}\operatorname{Im}\sum_{p\le x^3}\frac{C_f(p)\Lambda_x(p)(1-p^{1/2-\sigma_x})}{p^{1/2+it}\log p}\\
&\quad+\frac{1}{\pi}\operatorname{Im}\sum_{p\le x^{3/2}}\frac{C_f(p^2)\Lambda_x(p^2)}{p^{1+it}\log p}+\frac{1}{\pi}\operatorname{Im}\sum_{p\le x^{3/2}}\frac{C_f(p^2)\Lambda_x(p^2)(p^{1-2\sigma_x}-1)}{p^{1+it}\log p}\\
&\quad+O\left((\sigma_x-1/2)\left|\sum_{p\le x^3}\frac{C_f(p)\Lambda_x(p)}{p^{\sigma_x+it}}\right|\right)+O\big((\sigma_x-1/2)\log(|t|+k)\big)+O(1).
\end{aligned}
$$

Note that for $1/2\le a\le\sigma_x$,

$$
\begin{aligned}
\left|\sum_{p\le x^3}\frac{C_f(p)\Lambda_x(p)}{p^{a+it}}\right|&=x^{a-1/2}\left|\int_a^\infty x^{1/2-\sigma}\sum_{p\le x^3}\frac{C_f(p)\Lambda_x(p)\log(xp)}{p^{\sigma+it}}\,\mathrm{d}\sigma\right|\\
&\le x^{\sigma_x-1/2}\int_{1/2}^\infty x^{1/2-\sigma}\left|\sum_{p\le x^3}\frac{C_f(p)\Lambda_x(p)\log(xp)}{p^{\sigma+it}}\right|\mathrm{d}\sigma.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\left|\sum_{p\le x^3}\frac{C_f(p)\Lambda_x(p)(1-p^{1/2-\sigma_x})}{p^{1/2+it}\log p}\right|\\
&=\left|\int_{1/2}^{\sigma_x}\sum_{p\le x^3}\frac{C_f(p)\Lambda_x(p)}{p^{a+it}}\,\mathrm{d}a\right|
\end{aligned}
$$

$$\leq (\sigma_x - 1/2)x^{\sigma_x - 1/2} \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p)\log(xp)}{p^{\sigma+it}} \right| \mathrm{d}\sigma$$

and

$$(\sigma_x - 1/2) \left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p)}{p^{\sigma_x+it}} \right| \leq (\sigma_x - 1/2)x^{\sigma_x - 1/2} \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p)\log(xp)}{p^{\sigma+it}} \right| \mathrm{d}\sigma.$$

Moreover,

$$\begin{aligned} \left| \sum_{p \leq x^{3/2}} \frac{C_f(p^2)\Lambda_x(p^2)(p^{1-2\sigma_x} - 1)}{p^{1+it}\log p} \right| &\ll \sum_{p \leq x^{3/2}} \frac{1}{p}(1 - p^{1-2\sigma_x}) \ll \sum_{p \leq x^{3/2}} (\sigma_x - 1/2)\frac{\log p}{p} \\ &\ll (\sigma_x - 1/2)\log k. \end{aligned}$$

The results follows from above estimates. □

Now we are ready to prove (3.2). Recall that $\sigma_x = \sigma_{x,f}$ depending on f. By Lemma 5.2,

(5.5)

$$\begin{aligned} &\sum_{f \in H_k} \frac{|R(t,f)|^{2n}}{L(1, \mathrm{sym}^2 f)} \\ &\ll \sum_{f \in H_k} \frac{1}{L(1, \mathrm{sym}^2 f)} \left| \mathrm{Im} \sum_{p \leq x^3} \frac{C_f(p)(\Lambda(p) - \Lambda_x(p))}{p^{1/2+it}\log p} \right|^{2n} \\ &\quad + \sum_{f \in H_k} \frac{1}{L(1, \mathrm{sym}^2 f)} \left| \mathrm{Im} \sum_{p \leq x^{3/2}} \frac{C_f(p^2)\Lambda_x(p^2)}{p^{1+2it}\log p} \right|^{2n} \\ &\quad + \sum_{f \in H_k} \frac{(\sigma_{x,f} - 1/2)^{2n} x^{2n(\sigma_{x,f}-1/2)}}{L(1, \mathrm{sym}^2 f)} \left(\int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p \leq x^3} \frac{C_f(p)\Lambda_x(p)\log(xp)}{p^{\sigma+it}} \right| \mathrm{d}\sigma \right)^{2n} \\ &\quad + \sum_{f \in H_k} \frac{1}{L(1, \mathrm{sym}^2 f)} (\sigma_{x,f} - 1/2)^{2n} (\log(|t| + k))^{2n}. \end{aligned}$$

Since

$$|\Lambda(p) - \Lambda_x(p)| = O\left(\frac{\log^3 p}{\log^2 x}\right) \quad \text{and} \quad C_f(p^2) = \lambda_f(p^2) - 1,$$

the first two terms is of $O(k)$ by the same argument as in Section 5.1. The last term is of $O(k)$ by Lemma 5.1.

For the third term, it follows from Cauchy's inequality that

$$\sum_{f\in H_k}\frac{(\sigma_{x,f}-1/2)^{2n}x^{2n(\sigma_{x,f}-1/2)}}{L(1,\mathrm{sym}^2 f)}\left(\int_{1/2}^{\infty}x^{1/2-\sigma}\left|\sum_{p\le x^3}\frac{C_f(p)\Lambda_x(p)\log(xp)}{p^{\sigma+it}}\right|\mathrm{d}\sigma\right)^{2n}$$
$$\le\left(\sum_{f\in H_k}\frac{(\sigma_{x,f}-1/2)^{4n}x^{4n(\sigma_{x,f}-1/2)}}{L(1,\mathrm{sym}^2 f)}\right)^{1/2}$$
$$\times\left(\sum_{f\in H_k}\frac{1}{L(1,\mathrm{sym}^2 f)}\left(\int_{1/2}^{\infty}x^{1/2-\sigma}\left|\sum_{p\le x^3}\frac{C_f(p)\Lambda_x(p)\log(xp)}{p^{\sigma+it}}\right|\mathrm{d}\sigma\right)^{4n}\right)^{1/2}.$$

By Hölder's inequality with the exponents $4n/(4n-1)$ and $4n$,

$$\left(\int_{1/2}^{\infty}x^{1/2-\sigma}\left|\sum_{p\le x^3}\frac{C_f(p)\Lambda_x(p)\log(xp)}{p^{\sigma+it}}\right|\mathrm{d}\sigma\right)^{4n}$$
$$\le\left(\int_{1/2}^{\infty}x^{1/2-\sigma}\,\mathrm{d}\sigma\right)^{4n-1}\left(\int_{1/2}^{\infty}x^{1/2-\sigma}\left|\sum_{p\le x^3}\frac{C_f(p)\Lambda_x(p)\log(xp)}{p^{\sigma+it}}\right|^{4n}\mathrm{d}\sigma\right)$$
$$=\frac{1}{(\log x)^{4n-1}}\int_{1/2}^{\infty}x^{1/2-\sigma}\left|\sum_{p\le x^3}\frac{C_f(p)\Lambda_x(p)\log(xp)}{p^{\sigma+it}}\right|^{4n}\mathrm{d}\sigma.$$

Using the same argument as in Section 5.1, we have

$$\sum_{f\in H_k}\frac{1}{L(1,\mathrm{sym}^2 f)}\left|\sum_{p\le x^3}\frac{C_f(p)\Lambda_x(p)\log(xp)}{p^{\sigma+it}}\right|^{4n}\ll\left(\sum_{p\le x^3}\frac{|\Lambda_x(p)\log(xp)p^{1/2-\sigma}|^2}{p}\right)^{2n}$$
$$\ll\left(\sum_{p\le x^3}\frac{(\log p)^2}{p}(\log x)^2\right)^{2n}\ll(\log x)^{8n},$$

where the last inequality is obtained by $\sum_{p\le y}\frac{(\log p)^2}{p}\ll(\log y)^2$. Thus

$$\sum_{f\in H_k}\frac{1}{L(1,\mathrm{sym}^2 f)}\left(\int_{1/2}^{\infty}x^{1/2-\sigma}\left|\sum_{p\le x^3}\frac{C_f(p)\Lambda_x(p)\log(xp)}{p^{\sigma+it}}\right|\mathrm{d}\sigma\right)^{4n}$$
$$\ll\frac{k}{(\log x)^{4n-1}}\int_{1/2}^{\infty}x^{1/2-\sigma}(\log x)^{8n}\,\mathrm{d}\sigma\ll k(\log x)^{4n}.$$

Using this bound and Lemma 5.1, we have the third term in (5.5) is of $O(k)$.

Finally (3.2) follows from the above discussion.

6. A weighted central limit theorem

As pointed out in Remark 1.2, for a weight k Hecke cusp forms f, $S(t,f)$ has order of magnitude $\sqrt{\log\log k}$. Thus it makes sense to consider the following probability measure μ_k on $\mathbb{R}$, defined by

$$\mu_k(E) := \left(\sum_{f\in H_k} \frac{1}{L(1,\mathrm{sym}^2 f)}\mathbf{1}_E\left(\frac{S(t,f)}{\sqrt{\log\log k}}\right)\right) \Big/ \left(\sum_{f\in H_k}\frac{1}{L(1,\mathrm{sym}^2 f)}\right)$$

where $\mathbf{1}_E$ is the characteristic function on a Borel measurable set E in $\mathbb{R}$. As a consequence of Theorem 1.1 we obtain the following weighted central limit theorem, which should be compared with that in [6].

Theorem 6.1. *As $k\to\infty$, the probability measure μ_k converges to the Gaussian distribution of mean 0 and variance $(2\pi^2)^{-1}$; that is,*

$$\lim_{k\to\infty}\mu_k([a,b]) = \int_a^b \sqrt{\pi}\exp(-\pi^2\xi^2)\,\mathrm{d}\xi$$

for any $a<b$.

Proof. The nth moment of μ_k is

$$\int_{\mathbb{R}} \xi^n\,\mathrm{d}\mu_k(\xi) = \left(\sum_{f\in H_k}\frac{1}{L(1,\mathrm{sym}^2 f)}\left(\frac{S(t,f)}{\sqrt{\log\log k}}\right)^n\right)\Big/\left(\sum_{f\in H_k}\frac{1}{L(1,\mathrm{sym}^2 f)}\right).$$

By Theorem 1.1 and Corollary 2.2, we deduce that for all n,

$$\lim_{k\to\infty}\int_{\mathbb{R}}\xi^n\,\mathrm{d}\mu_k(\xi) = C_n = \int_{\mathbb{R}}\xi^n\sqrt{\pi}\exp(-\pi^2\xi^2)\,\mathrm{d}\xi.$$

Now the result follows from the theory of moments in probability theory (see, for example, [3, Theorem 30.2]). □

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