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Pointwise Convergence of the Fractional Schrödinger Equation in $\mathbb{R}^2$

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Abstract. We investigate the pointwise convergence of the solution to the fractional Schrödinger equation in $\mathbb{R}^2$. By establishing $H^s(\mathbb{R}^2) - L^3(\mathbb{R}^2)$ estimates for the associated maximal operator provided that $s > 1/3$, we improve the previous result obtained by Miao, Yang, and Zheng [19]. Our estimates extend the refined Strichartz estimates obtained by Du, Guth, and Li [10] to a general class of elliptic functions.

1. Introduction

For $\alpha > 1$, we consider the fractional Schrödinger equation

$$i\partial_t u + (-\Delta)^{\alpha/2} = 0, \quad u(x,0) = f(x) \tag{1.1}$$

for $f \in H^s(\mathbb{R}^2)$. Here, H^s is the L^2 Sobolev space of order s. Formally, the solution of (1.1) can be written as

$$U_\alpha f(x,t) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{i(x\cdot\xi + t|\xi|^\alpha)} \widehat{f}(\xi)\, d\xi.$$

In this study, we investigate the order of s for which

$$\lim_{t\to 0} U_\alpha f(x,t) = f(x) \quad \text{a.e. } x \tag{1.2}$$

holds whenever $f \in H^s(\mathbb{R}^2)$.

The problem of determining the optimal regularity s for which (1.2) holds for the Schrödinger equation was initially studied by Carleson [6]. When $d = 1$, he proved the convergence of (1.2) with $\alpha = 2$ for $s \geq 1/4$, whereas it generally fails for $s < 1/4$ in any dimension, as shown by Dahlberg and Kenig [9].

In higher dimensions, Sjölin [21] and Vega [24] independently showed that (1.2) with $\alpha = 2$ holds for $s > 1/2$. This result was improved to $s > 1/2 - 1/(4d)$ by Lee [16] for $d = 2$ and by Bourgain [2] for $d \geq 3$. Subsequently, Bourgain [3] showed that $s \geq d/(2d+2)$ is

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necessary for the almost everywhere convergence. The sufficiency part of the convergence was shown by Du, Guth, and Li [10] when $d = 2$ and by Du and Zhang [12] when $d \geq 3$ for a sharp range except for the endpoint (see [1, 5, 8, 11, 17, 18, 20, 23] for previous work).

For the fractional Schrödinger operator ($\alpha > 1$), Sjölin [21] proved that (1.2) holds if and only if $s \geq 1/4$ when $d = 1$. He also obtained some positive results in higher dimensions: (1.2) is valid for $s \geq 1/2$ when $d = 2$ and for $s > 1/2$ when $d \geq 3$. Subsequently, this result was improved by Miao, Yang, and Zheng [19] to $s > 3/8$ when $d = 2$ and $s > s_0$ for some $s_0 < 1/2$ when $d \geq 3$. We extend the result for $d = 2$.

Theorem 1.1. *Let $\alpha > 1$. Then,* (1.2) *holds for $f \in H^s(\mathbb{R}^2)$ whenever $s > 1/3$.*

The result in Theorem 1.1 extends to the solution of the linear dispersive equation

$$iu_t - \Phi(D)u = 0, \quad u(x, 0) = f(x).$$

Here, $\Phi(D)$ is a multiplier operator defined on $\mathbb{R}^2$, where Φ is a smooth function except for the origin and satisfies the following property: for $\alpha > 1$, there is a constant $C \geq 1$ such that $|\nabla\Phi(\xi)| \geq C^{-1}|\xi|^{\alpha-1}$ and $|\partial_\xi^\gamma \Phi(\xi)| \leq C|\xi|^{\alpha-|\gamma|}$ for any multi-indices γ. See Remark 3.11.

We denote by $B^d(x, r)$ a ball of radius r centered at x in $\mathbb{R}^d$. Theorem 1.1 follows from the maximal estimate.

Theorem 1.2. *Let $\alpha > 1$. Then, for $s > 1/3$, there exists a constant $C > 0$ such that*

$$\|U_\alpha f\|_{L_x^3 L_t^\infty(B^2(0,1)\times[0,1])} \leq C\|f\|_{H^s(\mathbb{R}^2)}.$$

The proof of Theorem 1.2 is motivated by the argument used in [10] and proceeds by using polynomial partitioning to decompose $U_\alpha f$ into cells as well as transversal and tangential parts of a wall. The first two parts are easy to handle by induction, whereas the tangential term is much more complicated. To treat the tangential part, we need to prove refined Strichartz estimate for U_α. We prove the estimate by using the decoupling inequality for elliptic parabola and induction on scales via rescaling. In contrast to the Schrödinger operator, $U_\alpha f$ ($\alpha \neq 2$) does not preserve the form after parabolic rescaling. To circumvent this issue, we consider a class of general elliptic functions as in [14]. Thus, we obtain the refined Strichartz estimates for a general class of operators (see Proposition 3.6).

Structure of the paper. The remainder of this paper is organized as follows. By applying polynomial partitioning, we reduce the problem to a problem of proving bilinear tangential estimate (Theorem 2.6). Section 3 establishes the linear refined Strichartz estimates (Proposition 3.6) and bilinear refined Strichartz estimates (Proposition 3.10). Accordingly, we prove Theorem 2.6.

Notation. Throughout the paper, $\mathcal{F}(f)$ denotes the Fourier transform of f. Further, $A \lesssim B$ denotes $A \le CB$ for some constant $C > 0$ and $\#\mathcal{D}$ denotes the cardinality of a set $\mathcal{D}$.

2. Proof of Theorem 1.2

Let $\alpha > 1$ and set $\mathbb{A}_r$ be the annulus given by

$$\mathbb{A}_r := \{\xi \in \mathbb{R}^2 : 2^{-1}r \le |\xi| \le 2r\}.$$

To prove Theorem 1.2, by the Littlewood–Paley decomposition and the triangle inequality, it suffices to show that for any $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$\Big\| \sup_{0<t\le 1} |U_\alpha f| \Big\|_{L^3(B^2(0,1))} \le C_\epsilon R^{1/3+\epsilon} \|f\|_2,$$

provided that $\widehat{f}$ is supported on $\mathbb{A}_R$ for $R \ge 1$. By a parabolic rescaling $\xi \to R\xi$ and $(x,t) \to (R^{-1}x, R^{-\alpha}t)$, the estimate is reduced to showing that

$$\Big\| \sup_{0<t\le R^\alpha} |U_\alpha f| \Big\|_{L^3(B^2(0,R))} \le C_\epsilon R^{\epsilon} \|f\|_2, \tag{2.1}$$

whenever $\widehat{f}$ is supported on $\mathbb{A}_1$. Now we reduce the matter to showing (2.1) in which the supremum is taken over a smaller interval $[0,R]$ instead of $[0,R^\alpha]$. More precisely, to prove (2.1) it suffices to show that for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$\Big\| \sup_{0<t\le R} |U_\alpha f| \Big\|_{L^3(B^2(0,R))} \le C_\epsilon R^{\epsilon} \|f\|_2$$

whenever $\widehat{f}$ is supported on $\mathbb{A}_1$. This reduction can be obtained by applying the time localization lemma in [19, Lemma 2.11] for the fractional Schrödinger operator (see also [16]). Alternatively, one may verify the lemma by using TT^* argument as in [7, Lemma 2.1]. After finite decomposition, we may assume that $\widehat{f}$ is supported on a ball $B^2(\xi_0, r) \subset \mathbb{A}_1$. Hence, Theorem 1.2 is a consequence of the following. For simplicity, let $B_R = B^2(0,R) \times [0,R]$.

Theorem 2.1. *Let $p \ge 3$ and $R \ge 1$. Then, for any $\epsilon > 0$, $q > \epsilon^{-4}$, and $r \le 1$ such that $B(\xi_0, r) \subset \mathbb{A}_1$, there exists a constant $C_\epsilon > 0$ such that*

$$\|U_\alpha f\|_{L^p_x L^q_t(B_R)} \le C_\epsilon r^{\epsilon^2} R^{\epsilon} \|f\|_2 \tag{2.2}$$

whenever $\widehat{f}$ is supported on $B(\xi_0, r)$.

Indeed, by the dominated convergence theorem, the estimate (2.2) implies that

$$\|U_\alpha f\|_{L^p_x L^\infty_t(B_R)} \le C_\epsilon R^\epsilon \|f\|_2$$

for any $p > 3$. By interpolating this with a trivial estimate $\|U_\alpha f\|_{L^2_x L^\infty_t(B_R)} \lesssim R^{1/2}\|f\|_2$, we have Theorem 1.2.

We begin by stating a wave packet decomposition of $U_\alpha f$ (see, for example, [10, 22]). For later use, we state the following for a more general operator $e^{it\Phi} f$ defined by

$$e^{it\Phi} f(x,t) = \int e^{i(x\cdot\xi + t\Phi(\xi))} \widehat{f}(\xi)\, d\xi$$

where Φ is smooth and the Hessian matrix of Φ is nondegenerate. Let ψ be a smooth function such that $\widehat{\psi}$ is supported on $B^2(0,3/2)$ and $\sum_{k\in\mathbb{Z}^2} |\widehat{\psi}(\,\cdot - k)|^2 = (2\pi)^{-2}$ on $\mathbb{R}^2$. For $\delta > 0$ and $(y,\nu) \in [R^{1/2}\mathbb{Z}^2 \cap B^2(0,R)] \times [R^{-1/2}\mathbb{Z}^2 \cap B^2(0,2)]$, we define a tube $T = T_{y,v}$ by

$$T = \big\{(x,t) \in \mathbb{R}^3 : |x - y + t\nabla\Phi(v)| \le R^{1/2+\delta}, 0 \le t \le R\big\}, \tag{2.3}$$

and denote the direction of tube by $D(T) = (-\nabla\Phi(v), 1)$ and the set of all tubes T by $\mathcal{T}$. We define $\psi_T = \psi_{T_{y,v}}$ by

$$\widehat{\psi_T}(\xi) = e^{-iy\cdot\xi} R^{1/2} \widehat{\psi}(R^{1/2}(\xi - v))$$

so that $\sum_{T\in\mathcal{T}} \psi_T(x)\overline{\mathcal{F}(\psi_T)(\xi)} = (2\pi)^{-2} e^{ix\cdot\xi}$ by the Poisson summation formula (see for example [13]).

Lemma 2.2. *Let Φ, T and ψ_T be as above. Suppose $\widehat{f}$ is supported on the ball $B^2(0,1)$. By setting $f_T = \langle f, \psi_T\rangle \psi_T$, we have*

$$f = \sum_{T\in\mathcal{T}} f_T$$

such that

$$\sum_{T\in\mathcal{T}} |\langle f, \psi_T\rangle|^2 \lesssim \|f\|_2^2$$

and for sufficiently large $N \ge 1$ and $(x,t) \in B^3(0,R)$,

$$|e^{it\Phi}\psi_T(x,t)| \lesssim R^{-1/2}\chi_T(x,t) + O(R^{-N})\|f\|_2.$$

Let $\epsilon > 0$ and $0 < r \le 1 \le R$. Suppose that the support of $\widehat{f}$ is contained in $B^2(\xi_0, r)$ for some $\xi_0 \in \mathbb{A}_1$. The proof proceeds by induction on the size of r and R. Note that (2.2) holds trivially for $R \sim 1$; hence, it suffices to consider $R \gg 1$. Furthermore, we only need to consider $r \ge R^{-1/2}$. In fact, if $r \le R^{-10}$, then (2.2) follows trivially since

$|U_\alpha f(x,t)| \le r\|f\|_2$ by Hölder's inequality. On the other hand, if $R^{-10} \le r \le R^{-1/2}$, then all the wave packets have the same direction. Therefore, we apply Hölder's inequality and obtain

$$\begin{aligned}\Big\|\sum_T U_\alpha f_T\Big\|_{L_x^pL_t^q(B_R)} &\lesssim R^{-1/2+1/q}\Big\|\sum_T \langle f,\psi_T\rangle\chi_T\Big\|_{L_x^pL_t^\infty(B_R)} \\ &\lesssim R^{-1/2+1/q}\sum_T |\langle f,\psi_T\rangle|\Big\|\sup_T \chi_T\Big\|_{L_x^pL_t^\infty(B_R)} \\ &\lesssim R^{-1/2+1/q}R^{(3/2+\delta)/p}\|f\|_2.\end{aligned} \tag{2.4}$$

For the last inequality, we use (2.3). Thus, (2.2) follows for $p \ge 3$ and sufficiently large $q > \epsilon^{-4}$ with small $\delta = \delta(\epsilon)$. Hereafter, we only consider $r \ge R^{-1/2}$. Now, we may assume that (2.2) holds if the radius of balls in physical space is less than $R/2$ or the radius of balls in physical space is less than R and that of balls in frequency space is less than $r/2$. Then, it suffices to show (2.2) for $R \gg 1$ and $r \ge R^{-1/2}$.

Now, we reduce the matter to showing the *bilinear tangential estimate* (Theorem 2.6) by a standard argument using polynomial partitioning. Let us denote by $Z(P)$ the zero set of a polynomial P. We say that P is a nonsingular polynomial if $\nabla P(z) \ne 0$ for all $z \in Z(P)$. Throughout this paper, we may assume that the polynomial P is a product of nonsingular polynomials by the density argument (see [14]). We recall the polynomial partitioning in [10].

Theorem 2.3. *Let $g \in L_x^1L_t^s(\mathbb{R}^{d+1})$ be a nonzero function, $1 \le s < \infty$, and $D > 0$. Then, there exists a nonzero polynomial P defined on $\mathbb{R}^{d+1}$ of degree $\le D$, which is a product of distinct nonsingular polynomials, and there exists a collection of disjoint open sets $\{O_i\}_{i\in\mathcal{I}}$ such that $\#\mathcal{I} \sim D^{d+1}$ and*

$$(\mathbb{R}^d \times \mathbb{R}) \setminus Z(P) = \bigcup_{i\in\mathcal{I}} O_i.$$

Moreover, there exists a constant C_1 independent of i such that

$$\|g\|_{L_x^1L_t^s(\mathbb{R}^{d+1})} \le C_1D^{d+1}\|\chi_{O_i}g\|_{L_x^1L_t^s(\mathbb{R}^{d+1})}$$

for each $i \in \mathcal{I}$.

By taking $s = q/p$, $D = R^{\epsilon^4}$, and $g = \chi_{B_R}|U_\alpha f|^p$, and applying Theorem 2.3, we have

$$\|U_\alpha f\|^p_{L_x^pL_t^q(B_R)} \le C_1D^3\|\chi_{O_i}U_\alpha f\|^p_{L_x^pL_t^q(B_R)}. \tag{2.5}$$

We denote by W a wall that is an $R^{1/2+\delta}$-neighborhood of $Z(P)$ and a cell $\widetilde{O}_i = O_i \setminus W$. It is clear that

$$\|U_\alpha f\|^p_{L_x^pL_t^q(B_R)} = \sum_{i\in\mathcal{I}}\|\chi_{\widetilde{O}_i}U_\alpha f\|^p_{L_x^pL_t^q(B_R)} + \|\chi_W U_\alpha f\|^p_{L_x^pL_t^q(B_R)}.$$

If the terms $\sum_{i\in\mathcal{I}}\|\chi_{\widetilde{O}_i}U_\alpha f\|^p_{L^p_xL^q_t(B_R)}$ dominate the other in the above-mentioned equation, the estimate is easy to handle. Hence, we consider this case first.

Cellular part. Let us consider a subcollection $\widetilde{\mathcal{I}}$ of an index set $\mathcal{I}$ such that

$$\widetilde{\mathcal{I}}=\big\{i\in\mathcal{I}:\|U_\alpha f\|^p_{L^p_xL^q_t(B_R)}\le 2C_1D^3\|\chi_{\widetilde{O}_i}U_\alpha f\|^p_{L^p_xL^q_t(B_R)}\big\}$$

where the constant C_1 is given by (2.5). To treat the case in which the cellular part dominates the walls, we may assume that $\widetilde{\mathcal{I}}=\mathcal{I}$. For each $i\in\mathcal{I}$, we set

$$f_i=\sum_{T:\,T\cap\widetilde{O}_i\neq\emptyset}f_T.$$

By Lemma 2.2, if $(x,t)\in\widetilde{O}_i$, then $|U_\alpha f(x,t)|\le|U_\alpha f_i(x,t)|+O(R^{-N})\|f\|_2$ for sufficiently large N. Since each tube T intersects at most $(D+1)$ cells O_i, we have

$$\sum_{i\in\mathcal{I}}\|f_i\|_2^2=\sum_{\substack{i\in\mathcal{I},T\in\mathcal{T}:\\ T\cap\widetilde{O}_i\neq\emptyset}}\|f_T\|_2^2\lesssim D\|f\|_2^2.$$

Since $\#\mathcal{I}\sim D^3$, by pigeonholing, there exists an index $i_\circ\in\mathcal{I}$ such that $\|f_{i_\circ}\|_2^2\lesssim D^{-2}\|f\|_2^2$. We cover B_R by $\{B'_{R/2}\}$, which are translations of $B_{R/2}$, and obtain

$$\|U_\alpha f\|^p_{L^p_xL^q_t(B_R)}\le 2C_1D^3\sum_{B'_{R/2}}\Big\|\sum_{T:\,T\cap\widetilde{O}_{i_\circ}\neq\emptyset}U_\alpha f_T\Big\|^p_{L^p_xL^q_t(B'_{R/2})}+O(R^{-N})\|f\|_2^p.$$

By applying the induction hypothesis, it follows that

$$\begin{aligned}\|U_\alpha f\|^p_{L^p_xL^q_t(B_R)}&\le CD^3\big[C_\epsilon r^{\epsilon^2}(R/2)^\epsilon\|f_{i_\circ}\|_2\big]^p+O(R^{-N})\|f\|_2^p\\ &\le 2C2^{-p\epsilon}D^3\big[C_\epsilon r^{\epsilon^2}R^\epsilon D^{-1}\|f\|_2\big]^p.\end{aligned}$$

Since $p>3$ and $D=R^{\epsilon^4}$, we see that $2C2^{-p\epsilon}D^{3-p}\le 2^{-p}$ for sufficiently large R. Therefore, we get (2.2) when the cell part dominates.

Now, we consider the opposite case in which the wall part dominates. To this end, we present some definitions. We denote by $T_z(Z(P))$ the tangent plane of $Z(P)$ at a fixed point z. Let us partition B_R into balls B_j of radius $R^{1-\delta}$. We say that a tube T is *tangent* to the wall W in B_j if T intersects B_j and W and satisfies

$$\text{Angle}\,\big(D(T),T_z(Z(P))\big)\le R^{-1/2+2\delta}$$

for any nonsingular point $z\in Z(P)\cap 10T\cap 2B_j$. Otherwise, we say that T is *transversal* to the wall W in B_j. Let $\mathcal{T}_{j,\mathrm{tang}}$ be the collection of all tubes $T\in\mathcal{T}$ such that T is tangent to the wall in B_j and let $\mathcal{T}_{j,\mathrm{trans}}$ be the collection of tubes such that T is transversal. We also set

$$f_{j,\mathrm{tang}} = \sum_{T \in \mathcal{T}_{j,\mathrm{tang}}} f_T \quad \text{and} \quad f_{j,\mathrm{trans}} = \sum_{T \in \mathcal{T}_{j,\mathrm{trans}}} f_T.$$

For a given $\delta' > 0$, we say that a tube T is $R^{-1/2+\delta'}$-tangent to Z if it satisfies

$$T \subset N_{R^{1/2+\delta'}} Z \cap B_R, \quad \operatorname{Angle}\big(D(T), T_z Z(P)\big) \le R^{-1/2+\delta'} \tag{2.6}$$

for all nonsingular points $z \in N_{2R^{1/2+\delta'}}(T) \cap 2B_R \cap Z$. The collection of tubes that are $R^{-1/2+\delta'}$-tangent to Z is denoted by $T_Z(R^{-1/2+\delta'})$. We say that f is concentrated on wave packets from $T_Z(R^{-1/2+\delta'})$ if

$$\sum_{T \notin T_Z(R^{-1/2+\delta'})} \|f_T\|_2 = O(R^{-N})\|f\|_2$$

holds for sufficiently large $N > 0$.

Wall part. Now we consider the case $\mathcal{I} \ne \widetilde{\mathcal{I}}$; hence, we can choose $i_\circ \in \mathcal{I} \setminus \widetilde{\mathcal{I}}$. From (2.5),

$$\|U_\alpha f\|^p_{L^p_x L^q_t(B_R)} \le C_1 D^3 \|\chi_{\widetilde{O}_{i_\circ}} U_\alpha f\|^p_{L^p_x L^q_t(B_R)} + C_1 D^3 \|\chi_{O_{i_\circ} \cap W} U_\alpha f\|^p_{L^p_x L^q_t(B_R)}.$$

Since $i_\circ \in \mathcal{I} \setminus \widetilde{\mathcal{I}}$, we have $C_1 D^3 \|\chi_{\widetilde{O}_{i_\circ}} U_\alpha f\|^p_{L^p_x L^q_t(B_R)} \le 2^{-1} \|U_\alpha f\|^p_{L^p_x L^q_t(B_R)}$. Therefore,

$$\|U_\alpha f\|^p_{L^p_x L^q_t(B_R)} \le 2C_1 D^3 \|\chi_W U_\alpha f\|^p_{L^p_x L^q_t(B_R)}.$$

Thus, it suffices to consider the wave packets concentrated on the wall.

Recalling that $\operatorname{supp} \widehat{f} \subset B^2(\xi_0, r)$, for $1 \ll K \ll R^\epsilon$, we cover $B^2(\xi_0, r)$ by boundedly overlapping collection of balls ω of radius $K^{-1}r$ and let $f = \sum_\omega f_\omega$, where $\widehat{f_\omega}$ is supported on ω. For each fixed B_j, we set

$$f_{\omega,j,\mathrm{tang}} = (f_\omega)_{j,\mathrm{tang}}, \quad f_{\omega,j,\mathrm{trans}} = (f_\omega)_{j,\mathrm{trans}}.$$

We define a bilinear tangential operator by

$$\operatorname{Bil}(U_\alpha f_{j,\mathrm{tang}})(x,t) = \sum_{\operatorname{dist}(\omega_1,\omega_2) \ge K^{-1} r} |U_\alpha f_{\omega_1,j,\mathrm{tang}}(x,t)|^{1/2} |U_\alpha f_{\omega_2,j,\mathrm{tang}}(x,t)|^{1/2}.$$

We set

$$B = \Big\{(x,t) \in B_R : K^{\epsilon^3} \max_\omega |U_\alpha f_\omega(x,t)| \le |U_\alpha f(x,t)|\Big\} \tag{2.7}$$

and for $(x,t) \in W \cap B$,

$$\Omega = \big\{\omega : |U_\alpha f_{\omega,j,\mathrm{tang}}(x,t)| \le K^{-4} |U_\alpha f(x,t)|\big\}.$$

By fixing B_j and $(x,t) \in B_j \cap W \cap B$, we first consider the case in which all balls ω in Ω^{c} are adjacent. Hence, $\#\Omega^{\mathsf{c}} \lesssim 1$, and it follows that $\sum_{\omega\in\Omega^{\mathsf{c}}} |U_\alpha f_\omega(x,t)| \le \frac{1}{2}|U_\alpha f(x,t)|$ by (2.7). Thus, we have $\frac{1}{2}|U_\alpha f(x,t)| \le \big|\sum_{\omega\in\Omega} U_\alpha f_\omega(x,t)\big|$. Then,

$$\frac{1}{2}|U_\alpha f(x,t)| \le \Big|\sum_{\omega\in\Omega} U_\alpha f_{\omega,j,\mathrm{trans}}(x,t)\Big| + \Big|\sum_{\omega\in\Omega} U_\alpha f_{\omega,j,\mathrm{tang}}(x,t)\Big| + O(R^{-N})\|f\|_2.$$

Since the total number of ω is $\le 10K^2$, we get

$$\frac{1}{2}|U_\alpha f(x,t)| \lesssim \Big|\sum_{\omega\in\Omega} U_\alpha f_{\omega,j,\mathrm{trans}}(x,t)\Big| + K^{-2}|U_\alpha f(x,t)| + O(R^{-N})\|f\|_2.$$

Otherwise, for $(x,t) \in B_j \cap W \cap B$, there are $\omega_1, \omega_2 \in \Omega^{\mathsf{c}}$ such that $\mathrm{dist}(\omega_1,\omega_2) \gtrsim K^{-1}r$. Then, by the definition of Ω, $|U_\alpha f(x,t)| \le K^4 \,\mathrm{Bil}(U_\alpha f_{j,\mathrm{tang}})(x,t)$. Therefore, we have the following.

Lemma 2.4. *For each point $(x,t) \in W \cap B_R$, there exists a collection Ω of balls ω of radius $K^{-1}r$ such that*

$$\begin{aligned}|\chi_W U_\alpha f(x,t)|^p \lesssim |\chi_{W\cap B^{\mathsf{c}}} U_\alpha f(x,t)|^p &+ \sum_j \Big|\sum_{\omega\in\Omega} \chi_{W\cap B_j} U_\alpha f_{\omega,j,\mathrm{trans}}(x,t)\Big|^p \\ &+ \sum_j K^{4p} |\chi_{W\cap B_j}\,\mathrm{Bil}(U_\alpha f_{j,\mathrm{tang}})(x,t)|^p + O(R^{-N})\|f\|_2^p.\end{aligned}$$

By Lemma 2.4, we have

$$\begin{aligned}\|U_\alpha f\|^p_{L^p_x L^q_t(W\cap B_R)} \lesssim \|U_\alpha f\|^p_{L^p_x L^q_t(W\cap B^{\mathsf{c}})} &+ \sum_j \Big\|\sum_{\omega\in\Omega} U_\alpha f_{\omega,j,\mathrm{trans}}\Big\|^p_{L^p_x L^q_t(W\cap B_j)} \\ &+ \sum_j \|K^4\,\mathrm{Bil}(U_\alpha f_{j,\mathrm{tang}})\|^p_{L^p_x L^q_t(W\cap B_j)} + O(R^{-N})\|f\|_2^p.\end{aligned} \tag{2.8}$$

From (2.7), the first term on the right-hand side of (2.8) is bounded by

$$\|U_\alpha f\|^p_{L^p_x L^q_t(W\cap B^{\mathsf{c}})} \le K^{\epsilon^3 p} \sum_\omega \|U_\alpha f_\omega\|^p_{L^p_x L^q_t(B_R)}.$$

By applying the induction hypothesis (2.2) to the right-hand side, we have

$$\|U_\alpha f\|^p_{L^p_x L^q_t(W\cap B^{\mathsf{c}})} \le K^{\epsilon^3 p} \sum_\omega \big[C_\epsilon (K^{-1}r)^{\epsilon^2} R^\epsilon \|f_\omega\|_2\big]^p \le 10K^{(\epsilon^3-\epsilon^2)p}\big[C_\epsilon r^{\epsilon^2} R^\epsilon \|f\|_2\big]^p.$$

Since $K \ll R^\epsilon$ and $R \gg 1$, we have $10K^{\epsilon^3-\epsilon^2} \le 1/6$. This completes the induction step for the first term on the right-hand side of (2.8). In the remainder of this section, we treat the second and third terms on the right-hand side of (2.8).

Transversal case. Now, we deal with the transversal term in (2.8). In [14], it was shown that for each tube $T \in \mathcal{T}$,

$$\#\{j : T \in \mathcal{T}_{j,\mathrm{trans}}\} \le R^{O(\epsilon^4)}. \tag{2.9}$$

Since $\#\Omega \le 2^{10K^2}$, the second term on the right-hand side of (2.8) is controlled by

$$\sum_j \left\| \max_{\Omega} \left| \sum_{\omega\in\Omega} U_\alpha f_{\omega,j,\mathrm{trans}} \right| \right\|^p_{L^p_x L^q_t(W\cap B_j)} \le \sum_j 2^{10K^2} \left\| \sum_{\omega\in\Omega} U_\alpha f_{\omega,j,\mathrm{trans}} \right\|^p_{L^p_x L^q_t(B_j)}. \tag{2.10}$$

By applying the induction hypothesis to a ball B_j of radius $R^{1-\delta}$, the right-hand side of (2.10) is bounded by $\sum_j 2^{10K^2}\big[C_\epsilon r^{\epsilon^2} R^{(1-\delta)\epsilon}\|f_{j,trans}\|_2\big]^p$. Therefore, by applying (2.9), we get

$$\sum_j \left\| \max_{\Omega} \left| \sum_{\omega\in\Omega} U_\alpha f_{\omega,j,\mathrm{trans}} \right| \right\|^p_{L^p_x L^q_t(W\cap B_j)} \le R^{O(\epsilon^4)} 2^{10K^2} R^{-\delta\epsilon p}\big[C_\epsilon r^{\epsilon^2} R^\epsilon \|f\|_2\big]^p.$$

Since $K \ll R^\epsilon$, we take $\delta = \epsilon^2$ and obtain $2^{10K^2} R^{O(\epsilon^4)-\epsilon^3 p} \le 1/6$ for sufficiently large $R > 0$. Therefore, the induction closes for the transversal term.

Bilinear tangential case. To estimate the third term on the right-hand side of (2.8), it remains to prove the following bilinear maximal estimates.

Theorem 2.5. *For any $\epsilon > 0$ and $p > 3$, there exists $C_\epsilon > 0$ such that*

$$\left(\int_{B_R} \sup_{t\colon (x,t)\in W\cap B_j} |\operatorname{Bil}(U_\alpha f_{j,\mathrm{tang}})(x,t)|^p \, dx \right)^{1/p} \le C_\epsilon R^{\epsilon/2} \|f\|_2.$$

Indeed, assuming Theorem 2.5, by Hölder's inequality, it follows that for $q > \epsilon^{-4}$,

$$\begin{aligned} \sum_j K^{4p} \|\operatorname{Bil}(U_\alpha f_{j,\mathrm{tang}})\|^p_{L^p_x L^q_t(W\cap B_j)} &\le \sum_j K^{4p} R^{\epsilon^4 p} \|\operatorname{Bil}(U_\alpha f_{j,\mathrm{tang}})\|^p_{L^p_x L^\infty_t(W\cap B_j)} \\ &\le R^{3\delta} K^{4p} R^{\epsilon^4 p} \big[C_\epsilon R^{\epsilon/2}\|f\|_2\big]^p \end{aligned}$$

since the number of j is $\lesssim R^{3\delta}$. Because $\delta = \epsilon^2$, $K \ll R^\epsilon$ and $r \ge R^{-1/2}$, we obtain $K^4 R^{\epsilon^4+\epsilon/2+3\delta/p} \le \frac{1}{6} R^\epsilon r^{\epsilon^2}$. This completes the proof of Theorem 2.1.

To prove Theorem 2.5, we show the following maximal estimate.

Theorem 2.6. *Let $0 < r \le 1$, $\xi_0 \in \mathbb{A}_1$, and $K = K(\epsilon)$ be a sufficiently large constant. Suppose that the supports of $\widehat{f}$ and $\widehat{g}$ are contained in $B^2(\xi_0, r)$ and separated by $K^{-1}r$. If f, g are concentrated on the wave packets from $T_Z(R^{-1/2+\delta'})$ with $\delta' \le 100\delta$, then there exist constants c and C such that*

$$\big\| |U_\alpha f|^{1/2} |U_\alpha g|^{1/2} \big\|_{L^3_x L^\infty_t(B_R)} \le C R^{c\delta'} \|f\|_2^{1/2} \|g\|_2^{1/2}. \tag{2.11}$$

In Theorem 2.5, we are concerned with the function $f_{j,\mathrm{tang}}$ defined on a smaller ball B_j of radius $R_1 = R^{1-\delta}$. We can easily see that the wave packets of $f_{j,\mathrm{tang}}$ on the ball B_j are concentrated in $T_Z(R_1^{-1/2+\delta'})$ for some $\delta' \le 100\delta$ (see [15]), and we omit the details. Since $R_1^{\delta'} \le R^{c\delta}$, we obtain the desired bound in Theorem 2.5 from the estimate (2.11).

3. Proof of Theorem 2.6

In this section, we prove Theorem 2.6 by considering linear and bilinear refined Strichartz estimates (Propositions 3.6 and 3.10, respectively), which are variants of the estimates presented in [10] for a class of elliptic functions. We begin by defining a class of elliptic phase functions.

3.1. Class of elliptic functions

We consider the class of phase functions that are small perturbations of $\phi_0(\xi) = |\xi|^2/2$.

Definition 3.1. Let $0 < \epsilon_0 \ll 1$, $\rho > 0$, and $n \ge 10^3$ be a positive integer. We define a class of normalized phase functions by

$$\mathcal{P}(\epsilon_0, n) = \{\phi \in \mathcal{C}^n(B^2(0,2)) : \|\phi - \phi_0\|_{\mathcal{C}^n(B^2(0,2))} \le \epsilon_0\}.$$

Let $\phi \in \mathcal{P}(\epsilon_0, n)$, $\xi_0 \in \mathbb{A}_1$, and $\mathrm{H}\phi(\xi_0)$ be the Hessian matrix of ϕ at $\xi = \xi_0$. Then, $\mathrm{H}\phi$ is positive definite on $B^2(0,2)$ and $\mathrm{H}\phi(\xi_0) = \mathrm{T}^{-1}\mathrm{D}\mathrm{T}$, where D is a diagonal matrix with eigenvalues $\lambda_1 > 0$ and $\lambda_2 > 0$, and T is a symmetric matrix. If we set $\mathrm{H}_{\xi_0} := \sqrt{\mathrm{H}\phi(\xi_0)}$, then $\mathrm{H}_{\xi_0} = \mathrm{T}^{-1}\mathrm{D}^{1/2}\mathrm{T}$ with $\mathrm{D}^{1/2} = (\sqrt{\lambda_1}e_1, \sqrt{\lambda_2}e_2)$. We denote the normalization of ϕ by

$$\phi^\rho_{\xi_0}(\xi) = \rho^{-2}\big(\phi(\rho\mathrm{H}_{\xi_0}^{-1}\xi + \xi_0) - \phi(\xi_0) - \rho\nabla\phi(\xi_0)\cdot\mathrm{H}_{\xi_0}^{-1}\xi\big). \tag{3.1}$$

Then, we observe the following, which plays an important role in the induction argument. We denote by $\operatorname{int}\mathbb{A}_r$ the interior of $\mathbb{A}_r$.

Lemma 3.2. *Let $\epsilon_0 > 0$ and $\xi_0 \in \operatorname{int}\mathbb{A}_1$. Suppose that $\phi \in \mathcal{C}^n(\mathbb{A}_1)$ and the Hessian matrix of ϕ is positive definite. Then, there exists a constant $\rho_0 > 0$ such that $\phi^\rho_{\xi_0} \in \mathcal{P}(\epsilon_0, n)$ whenever $\rho \le \rho_0$. Moreover, if $\phi \in \mathcal{P}(\epsilon_0, n)$, then for sufficiently small $\epsilon_0 > 0$, there exists a constant ρ_1 such that $\phi^\rho_{\xi_0} \in \mathcal{P}(\epsilon_0, n)$ whenever $\rho \le \rho_1$.*

Proof. By (3.1) and Taylor's expansion, we may write

$$\phi^\rho_{\xi_0}(\xi) = \frac{|\xi|^2}{2} + \mathcal{E}(\xi, \xi_0, \rho)$$

where $\|\mathcal{E}(\,\cdot, \xi_0, \rho)\|_{\mathcal{C}^n(\mathbb{A}_1)} = O(\rho|\mathrm{H}_{\xi_0}^{-1}\xi|^3)$. Thus, we can take ρ_0 such that $\|\phi^\rho_{\xi_0} - \phi_0\|_{\mathcal{C}^n(\mathbb{A}_1)} \le C\rho \le \epsilon_0$ holds for any $\rho \le \rho_0$. Similarly, if $\phi \in \mathcal{P}(\epsilon_0, n)$ and $\xi_0 \in B^2(0,1)$, then we can take $\rho_1 > 0$ such that $\phi^\rho_{\xi_0} \in \mathcal{P}(\epsilon_0, n)$ whenever $\rho \le \rho_1$. $\square$

Suppose that $\phi \in \mathcal{C}^n(\mathbb{A}_1)$ and the Hessian matrix of ϕ is positive definite. For a given small $\epsilon_0 > 0$, by partitioning $B^2(0,1)$ into smaller balls of radius ρ_0 and applying Lemma 3.2, $\phi^\rho_{\xi_0} \in \mathcal{P}(\epsilon_0, n)$ for any $\rho \le \rho_0$. Therefore, hereafter, we may fix $n \ge 10^3$ and simply denote $\mathcal{P}(\epsilon_0, n)$ by $\mathcal{P}(\epsilon_0)$. To prove Theorem 2.6, it suffices to consider $\phi \in \mathcal{P}(\epsilon_0)$.

3.2. Parabolic rescaling

We define a linear map $\mathcal{A}^\rho_{\xi_0} : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$(\mathcal{A}^\rho_{\xi_0})^{-1}(x,t) = (\rho^{-1}\mathrm{H}^t_{\xi_0}x - \rho^{-2}t\nabla\phi(\xi_0), \rho^{-2}t). \tag{3.2}$$

Lemma 3.3. *Let $\epsilon_0 > 0$ be sufficiently small, $\xi_0 \in \operatorname{int}\mathbb{A}_1$, and $\phi \in \mathcal{P}(\epsilon_0)$. Suppose that $\widehat{f}$ is supported on a ball $B^2(\xi_0, \rho)$ for $\rho \le \rho_1$, where ρ_1 is given in Lemma 3.2. Then, there exist $\widetilde{f}$, $\widetilde{T}$, and a constant $C = C(\xi_0, \rho)$ such that*

$$\|e^{it\phi}f\|_{L^q(T)} = C\rho^{1-4/q}\|e^{it\phi^\rho_{\xi_0}}\widetilde{f}\|_{L^q(\widetilde{T})}, \tag{3.3}$$

where $\mathcal{F}(\widetilde{f})$ is supported on $B^2(0,1)$ such that $\|\widetilde{f}\|_2 = \|f\|_2$, and

$$\widetilde{T} = \{(x,t) : (\mathcal{A}^\rho_{\xi_0})^{-1}(x,t) \in T\}.$$

Remark 3.4. Let $\phi \in \mathcal{P}(\epsilon_0)$. Suppose that T is a tube of dimensions $\rho^{-1}M \times \rho^{-1}M \times \rho^{-2}M$ centered at the origin with its long axis parallel to $(-\nabla\phi(\xi_0), 1)$, i.e.,

$$T = \{(x,t) : |x + t\nabla\phi(\xi_0)| \le \rho^{-1}M, |t| \le \rho^{-2}M\}.$$

Then, $\widetilde{T} = \{(x,t) : |\rho^{-1}\mathrm{H}_{\xi_0}x| \le \rho^{-1}M, |\rho^{-2}t| \le \rho^{-2}M\}$ and $\widetilde{T}$ is contained a cube of side length CM for some $C = C(\epsilon_0)$.

Proof of Lemma 3.3. Let $|\mathrm{H}|$ denote the determinant of a matrix H. By change of variables $\xi \to \rho\mathrm{H}^{-1}_{\xi_0}\xi + \xi_0$, we have

$$\left|e^{it\phi}f(x)\right| = \rho|\mathrm{H}^{-1}_{\xi_0}|^{1/2}\left|\int e^{i(x,t)\cdot(\rho\mathrm{H}^{-1}_{\xi_0}\xi, \phi(\rho\mathrm{H}^{-1}_{\xi_0}\xi+\xi_0))}\rho|\mathrm{H}^{-1}_{\xi_0}|^{1/2}\widehat{f}(\rho\mathrm{H}^{-1}_{\xi_0}\xi + \xi_0)\,d\xi\right|.$$

We define $\widetilde{f}$ by

$$\mathcal{F}\widetilde{f}(\xi) = \rho|\mathrm{H}^{-1}_{\xi_0}|^{1/2}\widehat{f}(\rho\mathrm{H}^{-1}_{\xi_0}\xi + \xi_0). \tag{3.4}$$

Then, $\widetilde{f}$ has a Fourier support on $B^2(0,1)$ and $\|\widetilde{f}\|_2 = \|f\|_2$. By the definition of $\phi^\rho_{\xi_0}$ (see (3.1)), we note that

$$\big(\rho^{-1}\mathrm{H}^t_{\xi_0}x - \rho^{-2}t\nabla\phi(\xi_0)\big)\cdot\rho\mathrm{H}^{-1}_{\xi_0}\xi + \rho^{-2}t\big(\phi(\rho\mathrm{H}^{-1}_{\xi_0}\xi + \xi_0) - \phi(\xi_0)\big) = x\cdot\xi + t\phi^\rho_{\xi_0}(\xi).$$

Thus, by change of variables $x \to \rho^{-1}\mathrm{H}^t_{\xi_0}x - t\nabla\phi(\xi_0)$ and $t \to \rho^{-2}t$ (i.e., $(x,t) \to (\mathcal{A}^\rho_{\xi_0})^{-1}(x,t)$), the desired bound (3.3) follows by taking $C = |\mathrm{H}_{\xi_0}|^{1/q-1/2}$. $\square$

Now, we observe that the condition (2.6) is preserved after parabolic rescaling.

Lemma 3.5. *Let $0 < \rho \le \rho_0$ and $\xi_0 \in \operatorname{int} \mathbb{A}_1$. Suppose that f is concentrated on the wave packets from $T_Z(R^{-1/2+\delta'})$ associated with ϕ for some $Z = Z(P)$. If $\widetilde{f}$ is given by (3.4), then $\widetilde{f}$ is concentrated on the wave packets from $T_{Z'}(\rho^{-1}R^{-1/2+\delta'})$ associated with $\phi^{\rho}_{\xi_0}$ and $Z' = Z'(\widetilde{P})$, where $\widetilde{P} = P \circ (\mathcal{A}^{\rho}_{\xi_0})^{-1}$.*

Proof. It suffices to show that

$$|(-\nabla_{\xi'}\phi^{\rho}_{\xi_0}(\xi'), 1) \cdot \nabla_{x',t'}\widetilde{P}|/|\nabla_{x',t'}\widetilde{P}| \lesssim \rho^{-1}R^{-1/2+\delta'}, \tag{3.5}$$

where $\xi' = \rho^{-1}\mathrm{H}_{\xi_0}(\xi - \xi_0)$ and $(x', t') = \mathcal{A}^{\rho}_{\xi_0}(x, t)$. Since $\widetilde{P}(x', t') = P(\rho^{-1}\mathrm{H}^{t}_{\xi_0}x' - \rho^{-2}t'\nabla\phi(\xi_0), \rho^{-2}t')$, we have

$$\nabla_{x'}\widetilde{P} = \rho^{-1}\mathrm{H}_{\xi_0}\nabla_x P \quad \text{and} \quad \partial_{t'}\widetilde{P} = \rho^{-2}(\partial_t P - \nabla_x P \cdot \nabla\phi(\xi_0)). \tag{3.6}$$

Combining this with $\nabla_{\xi'}\phi^{\rho}_{\xi_0}(\xi') = \rho^{-1}\mathrm{H}^{-t}_{\xi_0}(\nabla\phi(\xi) - \nabla\phi(\xi_0))$, we get

$$(-\nabla_{\xi'}\phi^{\rho}_{\xi_0}(\xi'), 1) \cdot (\nabla_{x'}\widetilde{P}, \partial_{t'}\widetilde{P}) = \rho^{-2}(-\nabla\phi(\xi), 1) \cdot (\nabla_x P, \partial_t P).$$

On the other hand, from (3.6), we can easily deduce that $|\nabla_{x',t'}\widetilde{P}| \gtrsim \rho^{-1}|\nabla_{x,t}P|$ by considering the cases $|\partial_t P| \ge 2|\nabla_x P \cdot \nabla\phi(\xi_0)|$ and $|\partial_t P| \le 2|\nabla_x P \cdot \nabla\phi(\xi_0)|$ separately. By the assumption $|(-\nabla\phi, 1) \cdot \nabla_{x,t}P|/|\nabla_{x,t}P| \lesssim R^{-1/2+\delta'}$, the desired estimate (3.5) follows. □

3.3. Linear refined Strichartz estimates

Before proving Theorem 2.6, we consider the linear and bilinear refined Strichartz estimates. We first prove the linear refined Strichartz estimates (Proposition 3.6) and then prove the bilinear estimates (Proposition 3.10) by using Proposition 3.6.

Proposition 3.6. *Let $\phi \in \mathcal{P}(\epsilon_0)$. Suppose that f is concentrated on the wave packets from $T_Z(R^{-1/2+\delta'})$ and $\widehat{f}$ is supported on $B^2(0, 1)$. Let $Q_1, Q_2, \dots$ be lattice cubes of side length $R^{1/2}$ in $B^3(0, R)$. Suppose that M cubes Q_j are contained in $B^2(0, R) \times [t_0, t_0 + R^{1/2}]$ for each $t_0 \in R^{1/2}\mathbb{Z} \cap [0, R]$, and for each Q_j,*

$$\|e^{it\phi}f\|_{L^6(Q_j)} \ \text{ is essentially constant.} \tag{3.7}$$

Then, for any $\epsilon > 0$, there exist constants $C_\epsilon, C \ge 1$ such that

$$\|e^{it\phi}f\|_{L^6(\bigcup_j Q_j)} \le C_\epsilon R^{-1/6+\epsilon+C\delta'} M^{-1/3}\|f\|_2. \tag{3.8}$$

We start by recalling the l^2-decoupling inequality for an elliptic paraboloid, which was obtained by Bourgain and Demeter [4].

Theorem 3.7. *Suppose that $\widehat{g}$ is supported in a σ-neighborhood of an elliptic paraboloid S in $\mathbb{R}^2$. Let τ be rectangles of dimensions $\sigma^{1/2} \times \sigma$, which cover a σ-neighborhood of S. If $\widehat{g}_\tau = \widehat{g}\chi_\tau$, then for $\epsilon > 0$ and $2 \le p \le 6$, we have*

$$\|g\|_{L^p(\mathbb{R}^2)} \le C_\epsilon \sigma^{-\epsilon} \left(\sum_\tau \|g_\tau\|^2_{L^p(\mathbb{R}^2)} \right)^{1/2}.$$

Let us consider the wave packet decomposition

$$f = \sum_T f_T \tag{3.9}$$

such that the Fourier support of f_T is contained in a ball of radius $R^{-1/4}$ and f_T is essentially supported on a ball of radius $R^{3/4}$. Then, $e^{it\phi} f_T$ restricted to the ball $B^3(0,R)$ is essentially supported on a tube T of dimensions $R^{3/4} \times R^{3/4} \times R$. Since f is concentrated on the wave packets from $T_Z(R^{-1/2+\delta'})$ for some Z, we can apply Theorem 3.7 and obtain the following.

Proposition 3.8. *Let $\phi \in \mathcal{P}(\epsilon_0)$ for sufficiently small $\epsilon_0 > 0$. Let $f = \sum_T f_T$ be as stated above. Suppose that f is concentrated on the wave packets from $T_Z(R^{-1/2+\delta'})$ for some $Z = Z(P)$ and suppose that Q is a cube of side length $R^{1/2}$ contained in the $2R^{1/2+\delta'}$-neighborhood of Z. Then,*

$$\|e^{it\phi} f\|_{L^6(Q)} \le C_\epsilon R^\epsilon \left(\sum_T \|e^{it\phi} f_T\|^2_{L^6(\omega_Q)} \right)^{1/2} + O(R^{-N})\|f\|_2. \tag{3.10}$$

Here, $\omega_Q(z) = (1 + R^{-1/2}|z - c_Q|)^{-100}$, where c_Q is the center of the cube Q.

Proof. Let $\psi \in \mathcal{S}(\mathbb{R}^3)$ such that $\psi = 1$ on $B^3(0,1)$ and $\widehat{\psi}$ is supported on $B^3(0,1)$. By letting $\psi_Q(z) = \psi(CR^{-1/2}(z - c_Q))$ for some constant $C > 0$ such that $\psi = 1$ on Q, we get

$$\|e^{it\phi} f\|_{L^6(Q)} \le \|\psi_Q e^{it\phi} f\|_{L^6(\mathbb{R}^3)}.$$

Since f is concentrated on the wave packets from $T_Z(R^{-1/2+\delta'})$, it suffices to consider the wave packets that are contained in the $R^{1/2+\delta'}$-neighborhood of a plane $W = T_z Z(P)$. We claim that

$$\|\psi_Q e^{it\phi} f\|_{L^6(W)} \le C_\epsilon R^{\epsilon/2} \left(\sum_T \|\psi_Q e^{it\phi} f_T\|^2_{L^6(W)} \right)^{1/2} + O(R^{-N})\|f\|_2. \tag{3.11}$$

By assuming (3.11), we prove (3.10). By integrating along the $W^\perp$ axis and using Minkowski's inequality and Fubini's theorem, we obtain

$$\|e^{it\phi} f\|_{L^6(Q)} \le C_\epsilon R^{\epsilon/2} \left(\sum_T \|\psi_Q e^{it\phi} f_T\|^2_{L^6(\mathbb{R}^3)} \right)^{1/2} + O(R^{-N})\|f\|_2.$$

Here, we use the fact that the number of tubes T intersecting Q is $\lesssim R^{\epsilon/100}$. Since ψ_Q decays rapidly outside Q, we get the desired result (3.10).

Now, we prove (3.11). It suffices to show that the restriction of $\mathcal{F}(\psi_Q e^{it\phi} f)$ to W is contained in an $R^{-1/4}$-neighborhood of an elliptic paraboloid. Since $\mathcal{F}(\psi_Q)$ is supported on $B^3(0, R^{-1/2})$, it suffices to consider the restriction of $\mathcal{F}(e^{it\phi} f)$ to W. Let $\mathbf{n}$ be the unit normal vector of W. Since $\widehat{f}$ is supported on $B^2(0,1)$, we have $|\mathbf{n} \cdot e_3| < 1/2$. By rotation and dilation, we may assume that $\mathbf{n} = \mathbf{n}'/|\mathbf{n}'|$, where $\mathbf{n}' = (0, 1, n_3)$ for some $|n_3| \lesssim 1$. Since $\partial^2_{\xi_2} \phi \neq 0$ on $B^2(0,1)$, by the implicit function theorem, there is a function $g \in \mathcal{C}^1_0((-1,1))$ such that

$$(-\nabla\phi(\xi_1, g(\xi_1)), 1) \cdot \mathbf{n} = 0, \tag{3.12}$$

and equivalently, $n_3 = \partial_{\xi_2}\phi(\xi_1, g(\xi_1))$.

Note that $|(-\nabla\phi(\xi), 1) \cdot \mathbf{n}| = | - \partial_{\xi_2}\phi + n_3| \lesssim R^{-1/4}$ on the support of $\operatorname{supp} \widehat{f}$. Since $\partial^2_{\xi_2}\phi \neq 0$, by the mean value theorem, we have

$$|\xi_2 - g(\xi_1)| \lesssim R^{-1/4}. \tag{3.13}$$

Therefore, we may write

$$(\xi, \phi(\xi)) = (\xi_1, 0, \widetilde{\phi}(\xi_1)) + \xi_2 \mathbf{n}' + \mathcal{E}(\xi) e_3,$$

where $\widetilde{\phi}(\xi_1) = \phi(\xi_1, g(\xi_1)) - n_3 g(\xi_1)$ and $\mathcal{E}(\xi) = \phi(\xi) - \phi(\xi_1, g(\xi_1)) - n_3(\xi_2 - g(\xi_1))$. By (3.13) and the mean value theorem, it is easy to see that $\mathcal{E}(\xi) = O(R^{-1/2})$. Moreover, by (3.12), $\partial_{\xi_1}\widetilde{\phi}(\xi_1) = (\partial_{\xi_1}\phi)(\xi_1, g(\xi_1))$; hence, $\partial^2_{\xi_1}\widetilde{\phi}(\xi_1) = (\partial^2_{\xi_1}\phi)(\xi_1, g(\xi_1)) + O(\epsilon_0)$ whenever $\phi \in \mathcal{P}(\epsilon_0)$ for sufficiently small $\epsilon_0 > 0$. Thus, the curve parameterized by $\xi_1 \to \widetilde{\phi}(\xi_1)$ is contained in an $R^{-1/2}$-neighborhood of the elliptic paraboloid. By Theorem 3.7, we obtain the desired bound (3.11). □

Using Proposition 3.8, we can prove Proposition 3.6 by induction.

Proof of Proposition 3.6. Let us set $\mathcal{A}_0 := \mathcal{A}^{R^{-1/4}}_{\xi_0}$ for simplicity. Recalling (3.9), for each tube T, we choose a cube Q_T of side length $R^{1/2}$ that is contained in $\mathcal{A}_0(T)$. We decompose Q_T into horizontal strips S' of height $R^{1/4}$ in Q_T such that $Q_T = \cup S'$. Furthermore, each strip S' is decomposed into cubes Q' of side length $R^{1/4}$. We set $T' := \mathcal{A}_0^{-1}(Q')$, which is a tube of size $\sim R^{1/2} \times R^{1/2} \times R^{3/4}$ such that the union of all T' covers T (see Figure 3.1).

By dyadic pigeonholing on the size of $\|f_T\|_2$, we may assume that $\|f_T\|_2$ is essentially constant for each tube T. In fact, the number of h is only $C \log R$ such that $\|f_T\|_2 \sim h$ since it is negligible if $\|f_T\|_2 \leq R^{-N}\|f\|_2$. Therefore, we choose one of h such that $\|f_T\|_2$

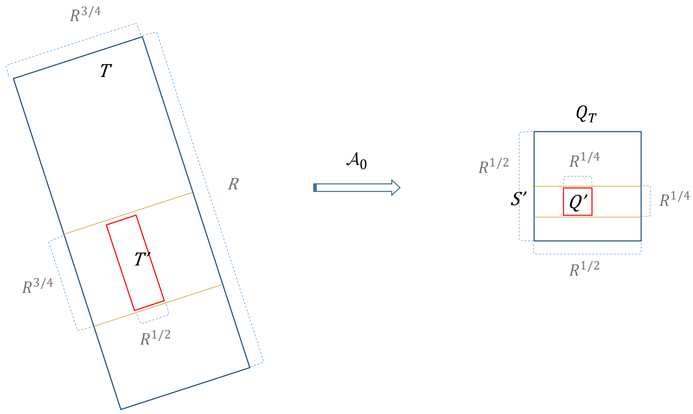

Figure 3.1

is essentially constant for a fraction $\sim 1/(\log R)$ of tubes T. For a fixed such tube T, we again perform dyadic pigeonholing on the size of $\|e^{it\phi} f_T\|_{L^6(T')}$ such that

$$\|e^{it\phi} f_T\|_{L^6(T')} \text{ is essentially constant,} \tag{3.14}$$

which is greater than $R^{-N}\|f\|_2$ since this part can be absorbed in the error term in (3.10) if the constant is less than $R^{-N}\|f\|_2$. We sort T' further according to the number of T' arranged along the short axis of T. Precisely, we may assume that

$$\#\{T' : T' \subset \mathcal{A}_0^{-1}(S')\} \sim M'$$

for a dyadic number M'. For simplicity, by abuse of notation, we denote such tubes by T'. Then, it follows that

$$\|e^{it\phi} f\|_{L^6(Q_j)} \lesssim (\log R)^3 \Big\| \sum_T \sum_{T'} \chi_{\cup T'} e^{it\phi} f_T \Big\|_{L^6(Q_j)} \tag{3.15}$$

for a fraction $\sim 1/(\log R)^3$ of all cubes Q_j in $\bigcup_j Q_j$. Finally, we sort the cubes Q_j further such that Q_j satisfies (3.15) and each Q_j is contained in $\sim \mu$ tubes T such that $Q_j \subset T' \subset T$. Let $\mathcal{Q}$ denote the set of such cubes Q_j; then, we see that $\#\mathcal{Q} \gtrsim (\log R)^{-4} M$ by dyadic pigeonholing. By (3.7), we have $\|e^{it\phi} f\|^6_{L^6(\bigcup_j Q_j)} \lesssim (\log R)^4 \|e^{it\phi} f\|^6_{L^6(\bigcup_{Q_j \in \mathcal{Q}} Q_j)}$. Hence, it suffices to consider cubes $Q_j \in \mathcal{Q}$.

By (3.15), (3.10), and Hölder's inequality, we get

$$\|e^{it\phi} f\|_{L^6(Q_j)} \le C_\varepsilon (\log R)^4 R^\varepsilon \mu^{1/3} \left(\sum_T \|\chi_{\cup T'} e^{it\phi} f_T\|^6_{L^6(\omega_{Q_j})} \right)^{1/6} + O(R^{-N})\|f\|_2. \tag{3.16}$$

Since $T' = \mathcal{A}_0^{-1}(Q')$ for some cube Q' of side length $R^{1/4}$, by (3.3), we have

$$\sum_{Q_j} \|\chi_{\cup T'} e^{it\phi} f_T\|^6_{L^6(\omega_{Q_j})} \lesssim R^{-1/2} \|e^{it\widetilde{\phi}} \widetilde{f}_T\|^6_{L^6(\cup Q')}$$

where $\mathcal{F}(\widetilde{f_T})$ is given by (3.4) for f_T in place of f. Then, $\mathcal{F}(\widetilde{f_T})$ is supported on $B^2(0,1)$ and $\|\widetilde{f_T}\|_2 = \|f_T\|_2$. Combining this with (3.16), we have

$$(3.17)\qquad \|e^{it\phi}f\|_{L^6(\bigcup_j Q_j)} \le C_\varepsilon R^{-1/12+2\varepsilon}\mu^{1/3}\left(\sum_T \|e^{it\widetilde{\phi}}\widetilde{f_T}\|^6_{L^6(\cup Q')}\right)^{1/6} + O(R^{-N})\|f\|_2.$$

By the choice of (3.14) combined with (3.3) for $f = f_T$, we observe that $\|e^{it\widetilde{\phi}}\widetilde{f_T}\|_{L^6(Q')}$ is essentially constant for each Q'. Since $\phi \in \mathcal{P}(\epsilon_0)$, we have $\widetilde{\phi} \in \mathcal{P}(\epsilon_0)$ by Lemma 3.2 provided that $R^{-1/2} \le \rho_1$. Thus, by applying the induction hypothesis to the right-hand side of (3.17) with $R^{1/2}$ and M' instead of R and M, respectively, we have

$$(3.18)\qquad \|e^{it\phi}f\|_{L^6(\bigcup_j Q_j)} \le C_\varepsilon R^{-\frac{1}{6}+2\epsilon+\frac{1}{2}C\delta'}\mu^{1/3}(M')^{-1/3}\left(\sum_T \|f_T\|_2^6\right)^{1/6}.$$

Since $\|f_T\|_2$ is essentially constant, it is easy to see that $\big(\sum_T \|f_T\|_2^6\big)^{1/6} \lesssim (\#T)^{-1/3}\|f\|_2$. We claim that

$$(3.19)\qquad M\mu \lesssim (\log R)^4 M'\#T.$$

Once we have (3.19), (3.18) is, in turn, bounded by

$$\|e^{it\phi}f\|_{L^6(\bigcup_j Q_j)} \le C_\varepsilon R^{-\frac{1}{6}+3\varepsilon+\frac{1}{2}C\delta'}M^{-1/3}\|f\|_2.$$

Taking $\varepsilon = \epsilon/C$ for sufficiently large $C > 0$ gives the desired bound (3.8).

Now, we prove (3.19). Let $S^{(t_0)}$ be the strip given by

$$S^{(t_0)} = \mathbb{R}^2 \times [t_0, t_0 + R^{1/2}]$$

for some $t_0 \in [0,R] \cap R^{1/2}\mathbb{Z}$. It suffices to consider cubes Q_j arranged along the strip $S^{(t_0)}$. By the choice of Q_j, T, and T', we note that

$$(3.20)\qquad (\log R)^{-4}M\mu \lesssim \#\{(Q_j,T) : Q_j \subset T' \cap S^{(t_0)} \text{ for some } T' \subset T\}.$$

Note that the angle between the long axis of T' and the vectors contained in the x-plane $\mathbb{R}^2$ is away from 0. Hence, each tube T' contains at most a finite number of cubes Q_j in $S^{(t_0)}$. Similarly, since $T' \subset \mathcal{A}_0^{-1}(S')$ for some S', the number of S' such that $S^{(t_0)} \cap \mathcal{A}_0^{-1}(S') \neq \emptyset$ is at most C. Hence, the number of Q_j is smaller than C times that of T' such that $T' \subset \mathcal{A}_0^{-1}(S')$. Therefore, $\#Q_j \le CM'$. Thus, the right-hand side of (3.20) is bounded by $M'\#T$, which gives the desired bound (3.19). $\square$

3.4. Bilinear refined Strichartz estimates

Now, we establish the bilinear refined Strichartz estimates using Proposition 3.6.

Proposition 3.9. *Let $\phi \in \mathcal{P}(\epsilon_0)$, $R^{-1/2} \le r \le 1$, and K be a sufficiently large constant. Suppose that the Fourier supports of f, g are contained in $B^2(\xi_0, r)$ and separated by $K^{-1}r$, and f, g are concentrated on the wave packets from $T_Z(R^{-1/2+\delta'})$ for $Z = Z(P)$ with $\delta' \le 100\delta$. Let $Q_1, \ldots, Q_M$ be cubes of side length $R^{1/2}$ contained in B_R and*

$$\big\||e^{it\phi}fe^{it\phi}g|^{1/2}\big\|_{L^6(Q_j)} \text{ is essentially constant} \tag{3.21}$$

for each Q_j. Then, for any $\epsilon > 0$, there exist $C_\epsilon, C \ge 1$ such that

$$\big\||e^{it\phi}fe^{it\phi}g|^{1/2}\big\|_{L^6(\bigcup_{j=1}^M Q_j)} \le C_\epsilon r^{-1/6} M^{-1/6} R^{-1/6+\epsilon+C\delta'} \|f\|_2^{1/2}\|g\|_2^{1/2}. \tag{3.22}$$

Before proving Proposition 3.9, we first consider the special case $r = 1$ of Proposition 3.9 as follows, which is a direct consequence of Proposition 3.6.

Proposition 3.10. *Let $\phi \in \mathcal{P}(\epsilon_0)$. Suppose that the Fourier supports of f, g are contained in $B^2(0, 1)$ and separated by K^{-1}, and f and g are concentrated on the wave packets from $T_Z(R^{-1/2+\delta'})$ for $\delta' \le 100\delta$. Suppose that $Q_1, \ldots, Q_M$ are cubes of side length $R^{1/2}$, and $\|e^{it\phi}f\|_{L^6(Q_j)}$ and $\|e^{it\phi}g\|_{L^6(Q_j)}$ are essentially constant for each Q_j. Then, for any $\epsilon > 0$, there exist $C_\epsilon, C \ge 1$ such that*

$$\big\||e^{it\phi}fe^{it\phi}g|^{1/2}\big\|_{L^6(\bigcup_j Q_j)} \le C_\epsilon R^{-1/6+\epsilon+C\delta'} M^{-1/6}\|f\|_2^{1/2}\|g\|_2^{1/2}. \tag{3.23}$$

Proof. Let us consider the wave packet decompositions $f = \sum_T f_T$ and $g = \sum_{\widetilde{T}} g_{\widetilde{T}}$ (see (3.9)). By repeating the pigeonholing arguments as in the proof of Proposition 3.6, we can choose T_f, T_f', M_f', μ_f and T_g, T_g', M_g', μ_g, respectively, in place of T, T', M, μ. Then, by pigeonholing, there are $C(\log R)^{-8}M$ cubes Q_j such that (3.18) holds for f and g simultaneously. We claim that

$$M\mu_f\mu_g \lesssim M_f'M_g'\#T_f\#T_g. \tag{3.24}$$

First, by assuming (3.24), we prove (3.23). By Hölder's inequality,

$$\big\||e^{it\phi}fe^{it\phi}g|^{1/2}\big\|_{L^6(\bigcup_j Q_j)} \le \|e^{it\phi}f\|_{L^6(\bigcup_j Q_j)}^{1/2}\|e^{it\phi}g\|_{L^6(\bigcup_j Q_j)}^{1/2}.$$

We apply (3.18) to the right-hand side and get

$$\big\||e^{it\phi}fe^{it\phi}g|^{1/2}\big\|_{L^6(\bigcup_j Q_j)} \le C_\epsilon R^{-\frac{1}{6}+\epsilon+\frac{1}{2}C\delta'}(\mu_f\mu_g)^{1/6}(\#T_f\#T_gM_f'M_g')^{-1/6}\|f\|_2^{1/2}\|g\|_2^{1/2}.$$

Substituting (3.24), we obtain the desired bound (3.23).

Now, we prove (3.24). For each fixed T_f and T_g, we first consider the intersection of $T_f' \subset T_f$ and $T_g' \subset T_g$. Since the angle between T_f' and T_g' is about $1/K$, the intersection of T_f' and T_g' is essentially a cube of side length $C_K R^{1/2}$. Therefore, if cubes Q_j are contained in $(\cup T_f') \cap (\cup T_g')$, then the number of such Q_j is at most $CM_f' M_g'$.

Recall that μ_f is the number of T_f containing Q_j such that $Q_j \subset T_f'$ and μ_g similarly. Considering all pairs $\{T_f, T_g, Q_j\}$ such that $Q_j \subset T_f' \cap T_g'$ for some $T_f' \subset T_f$ and $T_g' \subset T_g$, we obtain (3.24). □

Now, we prove Proposition 3.9. If $r \sim 1$, Proposition 3.9 is a direct consequence of Proposition 3.10. Hence, we are only concerned with $R^{-1/2} \le r \ll 1$. In this case, Proposition 3.9 follows from Proposition 3.10 via rescaling in Lemma 3.3.

Proof of Proposition 3.9. Let $\phi \in \mathcal{P}(\epsilon_0)$. By (3.2), $\mathcal{A}^r_{\xi_0}(B_R)$ is a tube of dimensions $CrR \times CrR \times r^2 R$ for a constant $C \ge 1$. Let us set

$$P_j = \mathcal{A}^r_{\xi_0}(Q_j).$$

Since Q_j is a cube of side length $R^{1/2}$, P_j is a tube of dimensions $CrR^{1/2} \times CrR^{1/2} \times r^2 R^{1/2}$ contained in a larger tube $\mathcal{A}^r_{\xi_0}(B_R)$.

We define $\widetilde{f}$ and $\widetilde{g}$ by (3.4) for f and g, respectively, with $r = \rho$. Further, let $\widetilde{\phi} = \phi^r_{\xi_0}$. By Lemma 3.2, we see that $\widetilde{\phi} \in \mathcal{P}(\epsilon_0)$ by taking sufficiently small $r \le \rho_1$. Moreover, by Lemma 3.3, we have

$$\big\| |e^{it\phi} f e^{it\phi} g|^{1/2} \big\|_{L^6(\bigcup_{j=1}^M Q_j)} \lesssim r^{1/3} \big\| |e^{it\widetilde{\phi}} \widetilde{f} e^{it\widetilde{\phi}} \widetilde{g}|^{1/2} \big\|_{L^6(\bigcup_j P_j)}$$

such that $\|\widetilde{f}\|_2 = \|f\|_2$, $\|\widetilde{g}\|_2 = \|g\|_2$ and the Fourier supports of $\widetilde{f}$ and $\widetilde{g}$ are contained in $B^2(0,1)$ and separated by K^{-1}. Moreover, by Lemma 3.5, $\widetilde{f}$ and $\widetilde{g}$ are $r^{-1}R^{-1/2+\delta'}$-concentrated in the wave packets from $T_Z(r^{-1}R^{-1/2+\delta'})$ associated with $\widetilde{\phi}$ for $Z = Z(\widetilde{P})$ for some polynomial $\widetilde{P} = P \circ (\mathcal{A}^r_{\xi_0})^{-1}$. Thus, to prove (3.22), it suffices to show that

$$\big\| |e^{it\widetilde{\phi}} \widetilde{f} e^{it\widetilde{\phi}} \widetilde{g}|^{1/2} \big\|_{L^6(\bigcup_j P_j)} \le C_\epsilon r^{-1/2} M^{-1/6} R^{-1/6+\epsilon+C\delta'} \|\widetilde{f}\|_2^{1/2} \|\widetilde{g}\|_2^{1/2}. \tag{3.25}$$

Now, we make couple of reductions by pigeonholing. For this purpose, let us set

$$r_1 = r^2 R \quad \text{and} \quad r_2 = rR^{1/2}. \tag{3.26}$$

Recall that $\mathcal{A}^r_{\xi_0}(B_R)$ is a tube of dimensions $Cr^{-1}r_1 \times Cr^{-1}r_1 \times r_1$. Now, we decompose $\mathcal{A}^r_{\xi_0}(B_R)$ into cubes Q_{r_1} of side length r_1. For each fixed cube Q_{r_1}, we consider cubes Q'_{r_2} of side length r_2 contained in Q_{r_1} such that $\{Q'_{r_2}\}$ covers $(\bigcup_j P_j) \cap Q_{r_1}$. By dyadic pigeonholing on the size of $\|e^{it\widetilde{\phi}} \widetilde{f}\|_{L^2(Q'_{r_2})}$ and $\|e^{it\widetilde{\phi}} \widetilde{g}\|_{L^2(Q'_{r_2})}$, we may sort Q'_{r_2}, which satisfy

$$\|e^{it\widetilde{\phi}} \widetilde{f}\|_{L^2(Q'_{r_2})}, \quad \|e^{it\widetilde{\phi}} \widetilde{g}\|_{L^2(Q'_{r_2})} \quad \text{are essentially constant.} \tag{3.27}$$

Let $\mathcal{Q}'_{r_2}$ be the collection of such cubes Q'_{r_2}. We also choose a subcollection of $\{Q_{r_1}\}$ that satisfy

$$\big\| |e^{it\widetilde{\phi}}\widetilde{f}e^{it\widetilde{\phi}}\widetilde{g}|^{1/2}\big\|_{L^6(\bigcup_{Q'_{r_2}\in\mathcal{Q}'_{r_2}} Q'_{r_2})} \text{ is essentially constant} \tag{3.28}$$

for $Q'_{r_2}\in\mathcal{Q}'_{r_2}$ such that $Q'_{r_2}\subset Q_{r_1}$, and also satisfy

$$\|e^{it\widetilde{\phi}}\widetilde{f}\|_{L^2(R^{2a}Q_{r_1})},\quad \|e^{it\widetilde{\phi}}\widetilde{g}\|_{L^2(R^{2a}Q_{r_1})} \text{ are essentially constant.} \tag{3.29}$$

Here, $R^{2a}Q_{r_1}$ is a cube that has the same center as Q_{r_1} and the side length is $R^{2a}r_1$ for a sufficiently small constant $a=a(\epsilon)$, which will be determined later. We denote such a collection of Q_{r_1} by $\mathcal{Q}_{r_1}$. We also denote by $\mathcal{P}=\mathcal{P}(r_1,r_2)$ the collection of tubes P_j that intersect with $Q'_{r_2}\subset Q_{r_1}$ for $Q'_{r_2}\in\mathcal{Q}'_{r_2}$ and $Q_{r_1}\in\mathcal{Q}_{r_1}$. By dyadic pigeonholing, we have

$$\#\mathcal{P}\gtrsim(\log R)^{-5}M. \tag{3.30}$$

We put

$$M_1=\#\{Q_{r_1}\in\mathcal{Q}_{r_1}\},\quad M_2=\#\{Q'_{r_2}\in\mathcal{Q}'_{r_2}\}.$$

Since the number of $P_j\in\mathcal{P}$ contained in a cube Q'_{r_2} is at most r^{-1}, it is easy to see that

$$(\log R)^{-5}M\lesssim r^{-1}M_1M_2. \tag{3.31}$$

Thus, from (3.21) and (3.30), we get

$$\big\| |e^{it\widetilde{\phi}}\widetilde{f}e^{it\widetilde{\phi}}\widetilde{g}|^{1/2}\big\|^6_{L^6(\bigcup_j P_j)}\lesssim(\log R)^5\big\| |e^{it\widetilde{\phi}}\widetilde{f}e^{it\widetilde{\phi}}\widetilde{g}|^{1/2}\big\|^6_{L^6(\bigcup_{P_j\in\mathcal{P}} P_j)}. \tag{3.32}$$

By (3.28), for any $Q_{r_1}\in\mathcal{Q}_{r_1}$ and $\cup Q'_{r_2}\subset Q_{r_1}$, we have

$$\big\| |e^{it\widetilde{\phi}}\widetilde{f}e^{it\widetilde{\phi}}\widetilde{g}|^{1/2}\big\|_{L^6(\bigcup_{P_j\in\mathcal{P}} P_j)}\lesssim M_1^{1/6}\big\| |e^{it\widetilde{\phi}}\widetilde{f}e^{it\widetilde{\phi}}\widetilde{g}|^{1/2}\big\|_{L^6(\cup Q'_{r_2})}. \tag{3.33}$$

The right-hand side of (3.33) can be estimated by applying the induction hypothesis. By Lemma 2.2, we have $\widetilde{f}=\sum_T\widetilde{f}_T$ and $\widetilde{g}=\sum_{T'}\widetilde{g}_{T'}$. Here, the Fourier transforms of $\widetilde{f}_T$ and $\widetilde{g}_{T'}$ are supported on balls of radius $r_1^{-1/2}$, and $e^{it\widetilde{\phi}}\widetilde{f}_T$ and $e^{it\widetilde{\phi}}\widetilde{g}_T$ restricted to $B^3(0,r_1)$ are essentially supported on tubes T, T' of dimensions $\sim r_1^{1/2}\times r_1^{1/2}\times r_1$.

Note that it suffices to consider the wave packets that intersect with Q_{r_1}. Indeed, let us fix t_0 such that $(x_0,t_0)\in Q_{r_1}$ for some x_0 and denote the slice of Q_{r_1} by

$$Q_{r_1}^{(t_0)}=Q_{r_1}\cap\{t=t_0\}.$$

Since $\cup Q'_{r_2}\subset Q_{r_1}$, to estimate the right-hand side of (3.33), it suffices to consider the wave packets T, T' intersecting with $R^aQ_{r_1}^{(t_0)}$, which is an R^a-neighborhood of $Q_{r_1}^{(t_0)}$. Therefore,

$$\big\| |e^{it\widetilde{\phi}}\widetilde{f}e^{it\widetilde{\phi}}\widetilde{g}|^{1/2}\big\|_{L^6(\cup Q'_{r_2})}\lesssim\Bigg\|\Bigg|\sum_{\substack{T\cap R^aQ_{r_1}^{(t_0)}\neq\emptyset,\\ T'\cap R^aQ_{r_1}^{(t_0)}\neq\emptyset}} e^{it\widetilde{\phi}}\widetilde{f}_Te^{it\widetilde{\phi}}\widetilde{g}_{T'}\Bigg|^{1/2}\Bigg\|_{L^6(\cup Q'_{r_2})}+\mathcal{E}_N$$

where $\mathcal{E}_N = O(R^{-N})\|f\|_2^{1/2}\|g\|_2^{1/2}$ for sufficiently large N. Under the assumption (3.27), we apply (3.23) for $R = r_1$ to the above inequality and using Plancherel's theorem, we obtain

$$\big\||e^{it\widetilde{\phi}}\widetilde{f}e^{it\widetilde{\phi}}\widetilde{g}|^{1/2}\big\|_{L^6(\cup Q'_{r_2})} \le C_\varepsilon r_1^{-1/6+\varepsilon+C\delta'}M_2^{-1/6}\Big\|\sum_T e^{it_0\widetilde{\phi}}\widetilde{f}_T\Big\|_{L_x^2}^{1/2}\Big\|\sum_{T'} e^{it_0\widetilde{\phi}}\widetilde{g}_{T'}\Big\|_{L_x^2}^{1/2} + \mathcal{E}_N$$

where the summation is taken over T and T' satisfying $T\cap R^aQ_{r_1}^{(t_0)} \neq \emptyset$ and $T'\cap R^aQ_{r_1}^{(t_0)} \neq \emptyset$.

Note that the length $Cr_1^{1/2}$ of the short axis of a tube T is smaller than the side length $R^a r_1$ of the cube $R^aQ_{r_1}$. Hence, the intersection of the tube T and $Q_{r_1}^{(t_0)}$ is contained in $R^{2a}Q_{r_1}\cap\{t = t_0\}$ for sufficiently large $R > 0$. Thus, we have

$$\begin{aligned}
&\big\||e^{it\widetilde{\phi}}\widetilde{f}e^{it\widetilde{\phi}}\widetilde{g}|^{1/2}\big\|_{L^6(\cup Q'_{r_2})} \\
&\le C_\varepsilon r_1^{-1/6+\varepsilon+C\delta'}M_2^{-1/6}\|e^{it_0\widetilde{\phi}}\widetilde{f}\|_{L_x^2(R^{2a}Q_{r_1}^{(t_0)})}^{1/2}\|e^{it_0\widetilde{\phi}}\widetilde{g}\|_{L_x^2(R^{2a}Q_{r_1}^{(t_0)})}^{1/2} + \mathcal{E}_N.
\end{aligned} \tag{3.34}$$

By taking the average with respect to t, we have

$$\|e^{it_0\widetilde{\phi}}\widetilde{f}\|_{L_x^2(R^{2a}Q_{r_1}^{(t_0)})} \le (r^2R^{1+2a})^{-1/2}\|e^{it\widetilde{\phi}}\widetilde{f}\|_{L^2(R^{2a}Q_{r_1})}. \tag{3.35}$$

Since $\cup R^{2a}Q_{r_1} \subset \mathcal{A}_{\xi_0}^r(B_{R^{1+3a}})$, by (3.29) and Lemma 3.3, we get

$$\|e^{it\widetilde{\phi}}\widetilde{f}\|_{L^2(R^{2a}Q_{r_1})} \lesssim M_1^{-1/2}\|e^{it\widetilde{\phi}}\widetilde{f}\|_{L^2(\mathrm{A}_{\xi_0}^r(B_{R^{1+3a}}))} \lesssim M_1^{-1/2}r\|e^{it\phi}f\|_{L^2(B_{R^{1+3a}})}. \tag{3.36}$$

By combining (3.35) and (3.36) with the trivial estimate $\|e^{it\phi}f\|_{L^2(B_{R^{1+3a}})} \lesssim R^{(1+3a)/2}\|f\|_2$ via Plancherel's theorem, we have

$$\big\|e^{it_0\widetilde{\phi}}\widetilde{f}\big\|_{L_{x,t}^2(R^{2a}Q_{r_1}^{(t_0)})} \lesssim M_1^{-1/2}R^{a/2}\|f\|_2.$$

Since $r_1 = r^2R$ (see (3.26)), disregarding the error term $\mathcal{E}_N$, (3.34) is bounded by

$$\big\||e^{it\widetilde{\phi}}\widetilde{f}e^{it\widetilde{\phi}}\widetilde{g}|^{1/2}\big\|_{L^6(\cup Q'_{r_2})} \le C_\varepsilon(r^2R)^{-1/6+\varepsilon+C\delta'}M_2^{-1/6}M_1^{-1/2}R^{a/2}\|f\|_2^{1/2}\|g\|_2^{1/2}.$$

By combining this with (3.32) and (3.33), and taking sufficiently small $a = a(\varepsilon)$ for $R^{-1/2} \le r$, we obtain

$$\big\||e^{it\widetilde{\phi}}\widetilde{f}e^{it\widetilde{\phi}}\widetilde{g}|^{1/2}\big\|_{L^6(\bigcup_j P_j)} \le C_\varepsilon R^{-1/6+3\varepsilon+C\delta'}r^{-1/3}M_1^{-1/3}M_2^{-1/6}\|f\|_2^{1/2}\|g\|_2^{1/2}.$$

Therefore, using (3.31), the above is bounded by $C_\varepsilon R^{-1/6+4\varepsilon+C\delta'}r^{-1/2}M^{-1/6}M_1^{-1/6}\|f\|_2^{1/2}$ $\|g\|_2^{1/2}$. Since $M_1 \ge 1$, by taking $\varepsilon \le \epsilon/C_1$ for sufficiently large $C_1 \ge 1$, we obtain (3.25). This completes the proof of (3.22). $\square$

We conclude this section by proving Theorem 2.6.

3.5. Proof of Theorem 2.6

Let $\epsilon > 0$. Assume that $\widehat{f}$, $\widehat{g}$ are supported on $B^2(\xi_0, r)$ for some $0 < r < 1$ and $\xi_0 \in \operatorname{int} \mathbb{A}_1$. From (2.4), we may assume that $r \geq R^{-1/2}$. For a dyadic number λ, we define a set X_λ by

$$X_\lambda = \left\{ x \in B^2(0, R) : \sup_{0<t<R} |U_\alpha f(x) U_\alpha g(x)| \sim \lambda \right\}.$$

It suffices to consider $R^{-N} \leq \lambda \leq R^c$.

We cover $N_{R^{1/2+\delta'}}(Z)$ with cubes Q_j, $j = 1, \dots, M$, of side length $R^{1/2}$. Since $\mathcal{F}(U_\alpha f)$ is supported on a ball of radius r, we consider smaller cubes $Q' = Q'(Q_j)$ of side length $r^{-1} \leq R^{1/2}$ such that $Q' \subset Q_j$. We define S_λ by the union of Q' contained in B_R such that

$$\sup_{(x,t)\in Q'} |U_\alpha f(x) U_\alpha g(x)| \gtrsim \lambda$$

and all projections of Q' onto the x-plane $\mathbb{R}^2$ are boundedly overlapping. Clearly, we have

$$|X_\lambda| \lesssim r |S_\lambda|. \tag{3.37}$$

For each Q_j, the number of Q' contained in Q_j is at most $C(R^{1/2}/r^{-1})^2 = CRr^2$ since the projections of Q' onto x-space are finitely overlapping. Hence, by pigeonholing, we also have

$$|S_\lambda| \lesssim (\log R) M R r^{-1}. \tag{3.38}$$

Now, we prove Theorem 2.6. By construction, it suffices to show that

$$\lambda^{1/2} |X_\lambda|^{1/3} \leq C R^{c\delta'} \|f\|_2^{1/2} \|g\|_2^{1/2}. \tag{3.39}$$

After finite decomposition, we may assume that $\widehat{f}$ and $\widehat{g}$ are supported on a ball $B^2(\eta_0, \epsilon_1 r)$ for some small ϵ_1 and $\eta_0 \in \operatorname{int} \mathbb{A}_1$ since f and g have Fourier supports separated by $K^{-1} r$ such that $K^{-1} \ll \epsilon_1$. By Lemmas 3.2 and 3.5 with $\rho = \epsilon_1 > 0$ and $\xi_0 = \eta_0$, we have $f_{\epsilon_1} = \widetilde{f}$, $g_{\epsilon_1} = \widetilde{g}$ (see (3.4)), and $\phi_{\epsilon_1} = \phi_{\eta_0}^{\epsilon_1}$ for $\phi(\xi) = |\xi|^\alpha$ such that $\phi_{\epsilon_1} \in \mathcal{P}(\epsilon_0)$ and f_{ϵ_1} and g_{ϵ_1} have Fourier supports on a ball $B^2(0, r)$ separated by $\epsilon_1^{-1} K^{-1} r$ and concentrated on the wave packets from $T_Z(\epsilon_1^{-1} R^{-1/2+\delta'})$.

By pigeonholing on the size of $\| |e^{it\phi_{\epsilon_1}} f_{\epsilon_1} e^{it\phi_{\epsilon_1}} g_{\epsilon_1}|^{1/2} \|_{L^6(Q_j)}$, we sort the fraction $\sim 1/(\log R)$ of cubes Q_j such that

$$\left\| |e^{it\phi_{\epsilon_1}} f_{\epsilon_1} e^{it\phi_{\epsilon_1}} g_{\epsilon_1}|^{1/2} \right\|_{L^6(Q_j)} \text{ is essentially constant.} \tag{3.40}$$

We denote by $\mathcal{Q}$ the set of such cubes Q_j. Thus, to prove (3.39), it suffices to show that

$$\lambda^3 |S_\lambda| \lesssim R^{c_1 \delta'} \left\| |e^{it\phi_{\epsilon_1}} f_{\epsilon_1} e^{it\phi_{\epsilon_1}} g_{\epsilon_1}|^{1/2} \right\|^6_{L^6(\bigcup_{Q_j \in \mathcal{Q}} Q_j)} \tag{3.41}$$

for some constant c_1. Indeed, by (3.37), we have $|X_\lambda|^{1/3} \lesssim r^{1/3}|S_\lambda|^{1/6}|S_\lambda|^{1/6}$ and $r^{1/3}|S_\lambda|^{1/6} \le (\log R)^{1/6}(rMR)^{1/6}$ by (3.38). Hence, we have

$$\lambda^{1/2}|X_\lambda|^{1/3} \le (\log R)^{1/6}(rMR)^{1/6}(\lambda^3|S_\lambda|)^{1/6}.$$

Combining this with (3.41), we obtain

$$\lambda^{1/2}|X_\lambda|^{1/3} \lesssim r^{1/6}M^{1/6}R^{1/6+c\delta'}\big\||e^{it\phi_{\epsilon_1}}f_{\epsilon_1}e^{it\phi_{\epsilon_1}}g_{\epsilon_1}|^{1/2}\big\|_{L^6(\bigcup_{Q_j\in\mathcal{Q}}Q_j)}.$$

By applying Proposition 3.9 to the right-hand side, we get the desired bound (3.39) and hence (2.11).

It remains to show (3.41). Let $Q' \subset Q_j$ and let $\psi_{Q'} \in \mathcal{C}_0^\infty(\mathbb{R}^3)$ such that $\mathcal{F}(\psi_{Q'})$ is supported on $B^3(0,2r)$ and $\psi_{Q'}$ decreases rapidly outside Q'. Since $\widehat{f}$, $\widehat{g}$ are supported on $B^2(\xi_0,r)$, we see that

$$\mathcal{F}(U_\alpha f U_\alpha g)(\xi,\xi_3) = \mathcal{F}(U_\alpha f U_\alpha g)(\xi,\xi_3)\mathcal{F}(\psi_{Q'})(\xi,\xi_3).$$

Since $\psi_{Q'}$ decreases rapidly outside a ball of radius r^{-1}, $\int U_\alpha f U_\alpha g((x,t)-(y,s))\psi_{Q'}(y,s)\, dyds$ is negligible when $|(y,s)| \ge R^a r^{-1}$ for sufficiently small a. Hence, we have

$$\sup_{(x,t)\in Q'}|U_\alpha f(x,t)U_\alpha g(x,t)| \lesssim r^3\int_{R^aQ'}|U_\alpha f(x,t)U_\alpha g(x,t)|\,dxdt + E_N$$

where $E_N = O(R^{-N}\|f\|_2\|g\|_2)$ for sufficiently large N. By applying Lemma 3.3 with $\rho = \epsilon_1$ and $\xi_0 = \eta_0$, we get

$$|Q'|\sup_{(x,t)\in Q'}|U_\alpha f(x,t)U_\alpha g(x,t)| \lesssim \int_{\widetilde{A}_{Q'}}|e^{it\phi_{\epsilon_1}}f_{\epsilon_1}(x)e^{it\phi_{\epsilon_1}}g_{\epsilon_1}(x)|\,dxdt + E_N \tag{3.42}$$

where $\widetilde{A}_{Q'} = \mathcal{A}_{\eta_0}^{\epsilon_1}(R^aQ')$. By applying Hölder's inequality, we have

$$|Q'|^{1/3}\sup_{(x,t)\in Q'}|U_\alpha f(x,t)U_\alpha g(x,t)| \lesssim R^{ca}\left(\int_{\widetilde{A}_{Q'}}|e^{it\phi_{\epsilon_1}}f_{\epsilon_1}(x)e^{it\phi_{\epsilon_1}}g_{\epsilon_1}(x)|^3\,dxdt\right)^{1/3} + E_N$$

for some constant $c > 1$. If a is sufficiently small, then $\widetilde{A}_{Q'}$ is contained in a cube of side length $R^{1/2}$; hence, we may assume that $\widetilde{A}_{Q'} \subset Q_j$. Taking the third power and summing over all Q' such that $\widetilde{A}_{Q'} \subset \bigcup_j Q_j$ and using (3.40), we have (3.41) with a minor error term E_N. This completes the proof.

Remark 3.11. Without difficulty, we can see that the result of Theorem 1.1 extends to the solution of the general dispersive equation. Let $\alpha > 1$ and Φ be a smooth function such that $|\nabla\Phi(\xi)| \ge C^{-1}|\xi|^{\alpha-1}$ and $|\partial_\xi^\gamma\Phi(\xi)| \le C|\xi|^{\alpha-|\gamma|}$ for any γ for some constant $C > 0$. The reduction to the bilinear estimate in Theorem 2.6 (see Section 2) via polynomial partitioning is nearly identical. Since Φ is positive definite, by rescaling and Lemma 3.2, we only need to consider $e^{it\phi}$ for $\phi \in \mathcal{P}(\epsilon_0)$ (see (3.42)). Then, by following the proof of Theorem 2.6, we can obtain (2.11) for $e^{it\Phi}$ in place of $U_\alpha f$.

Remark 3.12. A modification of the proof of Theorem 1.1 combined with the argument in [12] provides (1.2) for $d \geq 3$ whenever $s > d/(2d+2)$.

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