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Global Existence and Blow-up of Solutions for a System of Fractional Wave Equations

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Abstract. We investigate the Cauchy problem for a 2×2-system of weakly coupled semi-linear fractional wave equations with polynomial nonlinearities posed in $\mathbb{R}^+ \times \mathbb{R}^N$. Under appropriate conditions on the exponents and the fractional orders of the time derivatives, it is shown that there exists a threshold value of the dimension N, for which, small data-global solutions as well as finite time blowing-up solutions exist. Furthermore, we investigate the L^∞-decay estimates of global solutions.

1. Introduction

We consider the following Cauchy problem

$$\begin{cases} {}^C D^{\gamma_1}_{0|t} u - \Delta u = f(v(t,\cdot)), & t > 0,\ x \in \mathbb{R}^n, \\ {}^C D^{\gamma_2}_{0|t} v - \Delta v = g(u(t,\cdot)), & t > 0,\ x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

subject to the initial conditions

$$\begin{cases} u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

where $1 < \gamma_1, \gamma_2 < 2$, ${}^C D^{\alpha}_{0|t} u$ denotes the Caputo derivative, defined for a function u of class C^2, as (see, e.g., [31])

$$({}^C D^{\alpha}_{0|t} u)(t) := \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{u_{tt}(s,\cdot)}{(t-s)^{\alpha-1}}\, ds, \quad 1 < \alpha < 2,$$

Δ is the Laplacian, $f(v) = \pm |v|^{p-1} v$ or $\pm |v|^p$, $g(u) = \pm |u|^{q-1} u$ or $\pm |u|^q$, $p, q \geq 1$, and u_0, v_0, u_1, v_1 are given initial data.

Observe that system (1.1) interpolates reaction-diffusion system ($\gamma_1 = \gamma_2 = 1$) and hyperbolic system ($\gamma_1 = \gamma_2 = 2$).

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Before we present our results and comment on them, let us dwell on some related existing results.

Escobedo and Herrero [13] studied the global existence and blowing-up solutions of the system

$$\begin{cases} u_t - \Delta u = v^p, & t > 0,\ x \in \mathbb{R}^N, \\ v_t - \Delta v = u^q, & t > 0,\ x \in \mathbb{R}^N. \end{cases} \tag{1.3}$$

In particular, for

$$pq > 1, \quad \frac{N}{2} \leq \frac{\max\{p,q\}+1}{pq-1},$$

they have shown that every nontrivial solution of (1.3) blows-up in a finite time $T^* = T^*(u,v)$, and

$$\limsup_{t\to T^*} \|u(t)\|_\infty = \limsup_{t\to T^*} \|v(t)\|_\infty = +\infty.$$

Some related results concerning global existence or blowing-up solutions can be found in [14,27–30], etc. In particular, see the review papers [4,11] and the authoritative paper [26].

Blowing-up solutions and global solutions for time-fractional differential systems have been studied, for example, in [1–3, 12, 16–18, 20, 22, 34].

Concerning the system of wave equations

$$\begin{cases} u_{tt} - \Delta u = |v|^p, & 0 < t < T,\ x \in \mathbb{R}^N, \\ v_{tt} - \Delta v = |u|^q, & 0 < t < T,\ x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

subject to initial data

$$\begin{cases} u(0,x) = f(x), \ \ u_t(0,x) = g(x), & x \in \mathbb{R}^N, \\ v(0,x) = h(x), \ \ v_t(0,x) = k(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.5}$$

where $f,g,h,k \in C_0^\infty(R^N)$, we may mention the works [8–10]. For $N = 3$ in [8], the following optimal results were obtained:

▷ If $p,q > 1$ and

$$\max\left\{\frac{p+2+q^{-1}}{pq-1}, \frac{q+2+p^{-1}}{pq-1}\right\} > 1,$$

then the classical solution to (1.4)–(1.5) blows-up in a finite time.

▷ If $p,q > 1$ and

$$\max\left\{\frac{p+2+q^{-1}}{pq-1}, \frac{q+2+p^{-1}}{pq-1}\right\} < 1,$$

then there exists a global classical solution to (1.4)–(1.5) for sufficiently "small" initial data.

Our interest in (1.1) stems from the fact that it interpolates different situations; for example, reaction-diffusion systems with fractional derivatives can model chemical reactions taking place in porous media. In this case, fractional (nonlocal) terms with order in $(0,1)$ account for the anomalous diffusion [23, 25]. Experimental results show that several complex systems have a non-local dynamics.

On the other hand, equations/systems of fractional differential equations with order in $(1,2)$ have been studied in [7, 24, 33], etc. Examples include mechanical, acoustical, biological phenomena, marine sediments, etc. [19, 32].

In the present paper, we consider the problem (1.1)–(1.2) and present conditions, relating the space dimension N with the parameters γ_1, γ_2, p, and q, for which the solution of (1.1)–(1.2) exists globally in time and satisfies L^∞-decay estimates. We also investigate blowing-up in finite time solutions with initial data having positive average. Our study of global existence employs the mild formulation of the solution via Mittag–Leffler's function, while we use the test function approach due to Mitidieri and Pohozaev [26] for the case of blowing-up solutions. The test function approach has been used by several authors, (for instance, see [5, 6, 15, 22, 27, 34]). To the best of our knowledge, there do not exist global existence and large time behavior results for the time-fractional diffusion system with two different fractional powers. Thus our results are new and contribute significantly to the existing literature on the topic.

The rest of this paper is organized as follows. In next section, we present some preliminary lemmas, basic facts and useful tools such as time fractional derivative, L^p-L^q-estimates of the fundamental solution of the linear time fractional wave equation. Section 3 contains the main results of the paper. Finally, Section 4 is devoted to the proof of small data global existence and blow-up in finite time of the solution of problem (1.1)–(1.2).

In the sequel, C will be a positive constant which may have different values from line to line. The space $L^p(\mathbb{R}^N)$ $(1 \le p < \infty)$ will be equipped with the norm:

$$\|u\|^p_{L^p(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |u(t,x)|^p \, dx.$$

2. Preliminaries

The Riemann–Liouville fractional integral of order $0 < \alpha < 1$ of $f(t) \in L^1(0,T)$ is defined as

$$(J^\alpha_{0|t} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) \, d\tau,$$

where Γ stands for the usual Euler gamma function.

The left-sided Riemann–Liouville derivative $D^\alpha_{0|t} f$ (see [31]), for $f \in C^{m-1}(0,T)$, of order α is defined as follows:

$$(D^{\alpha}_{0|t}f)(t) = \frac{d^m}{dt^m}(J^{m-\alpha}_{0|t}f)(t), \quad t > 0,\ m-1 < \alpha < m,\ m \in \mathbb{N}.$$

The Caputo fractional derivative of a function $f \in C^m(0,T)$ is defined as

$$({}^{C}D^{\alpha}_{0|t}f)(t) = J^{m-\alpha}_{0|t}f^{(m)}(t), \quad t > 0,\ m-1 < \alpha < m,\ m \in \mathbb{N}.$$

For $0 < \alpha < 1$ and f of class C^1, we have

$$(D^{\alpha}_{0|t}f)(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(0)}{t^{\alpha}} + \int_0^t \frac{f'(\sigma)}{(t-\sigma)^{\alpha}}\,d\sigma\right],$$

and

$$(D^{\alpha}_{t|T}f)(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(T)}{(T-t)^{\alpha}} - \int_t^T \frac{f'(\sigma)}{(\sigma-t)^{\alpha}}\,d\sigma\right].$$

The Caputo derivative is related to the Riemann–Liouville derivative for $f \in AC[0,T]$ (the space of absolutely continuous functions defined on $[0,T]$) by

$$({}^{C}D^{\alpha}_{0|t}f)(t) = D^{\alpha}_{0|t}(f(t) - f(0)).$$

Assume that $0 < \alpha < 1$, $f \in C^1([a,b])$ and $g \in C(a,b)$. Then the formula of integration by parts is

$$\int_a^b f(t)(D^{\alpha}_{0|t}g)(t)\,dt = \int_a^b g(t)({}^{C}D^{\alpha}_{t|T}f)(t)\,dt + f(a)(I^{1-\alpha}_{a|t}g)(t)\Big|_{t=a}^{t=b}.$$

The Mittag–Leffler function is defined (see [31]) by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha,\beta \in \mathbb{C},\ \Re(\alpha) > 0,\ z \in \mathbb{C};$$

its Riemann–Liouville fractional integral satisfies

$$J^{1-\alpha}_{0|t}(t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})) = E_{\alpha,1}(\lambda t^{\alpha}) \quad \text{for } \lambda \in \mathbb{C},\ 0 < \alpha < 1.$$

For later use, let

$$\varphi(t) = \left(1 - \frac{t}{T}\right)_+^l, \quad l \geq 2;$$

then

$${}^{C}D^{\alpha}_{t|T}\varphi(t) = \frac{\Gamma(l+1)}{\Gamma(l+1-\alpha)}T^{-\alpha}\left(1 - \frac{t}{T}\right)_+^{l-\alpha}, \quad t \leq T,$$

(see, for example, [22]).

2.1. Linear estimates

In this section, we present fundamental estimates which will be used to prove Theorem 3.3.

For $1 < \alpha < 2$, we define the operators $\widetilde{E}_{\alpha,1}(t,x)$ and $\widetilde{E}_{\alpha,\alpha}(t,x)$ as follows:

$$\begin{aligned}
\widetilde{E}_{\alpha,1}(t,x) &= (2\pi)^{-N/2}\mathcal{F}^{-1}(E_{\alpha,1}(-4\pi^2 t^\alpha|\xi|^2))(x), \qquad x\in\mathbb{R}^N,\ t>0,\\
\widetilde{E}_{\alpha,2}(t,x) &= (2\pi)^{-N/2}\mathcal{F}^{-1}(E_{\alpha,2}(-4\pi^2 t^\alpha|\xi|^2))(x), \qquad x\in\mathbb{R}^N,\ t>0,\\
\widetilde{E}_{\alpha,\alpha}(t,x) &= (2\pi)^{-N/2}\mathcal{F}^{-1}(E_{\alpha,\alpha}(-|\xi|^2 t^\alpha))(x), \qquad x\in\mathbb{R}^N,\ t>0.
\end{aligned}$$

Consider the following linear inhomogeneous time fractional equation with initial data:

$$\text{(2.1)} \qquad \begin{cases} {}^C D^\alpha_{0|t} u - \Delta u = f(t,x), & t>0,\ x\in\mathbb{R}^N,\ 1<\alpha<2,\\ u(0,x)=u_0(x),\ \ u_t(0,x)=u_1(x), & x\in\mathbb{R}^N. \end{cases}$$

If $u_0 \in S(R^N)$ (the Schwartz space), $u_1 \in S(R^N)$ and $f \in L^1((0,+\infty),\mathcal{S}(\mathbb{R}^N))$, then by [18] (see also [1]) problem (2.1) admits a solution $u \in C^\alpha([0,+\infty);\mathcal{S}(\mathbb{R}^N))$, which satisfies

$$u(t,x) = \widetilde{E}_{\alpha,1}(t,x)u_0(x) + t\widetilde{E}_{\alpha,2}(t,x)u_1(x) + \int_0^t (t-s)^{\alpha-1}\widetilde{E}_{\alpha,\alpha}(t-s)f(s,x)\,ds,$$

where $\widetilde{E}_{\alpha,\beta}(t,\cdot)h$, $(\beta = 1,2,\alpha)$ is defined for $h \in S'(R^N)$ by

$$\widetilde{E}_{\alpha,\beta}(t,\cdot)h := \widetilde{E}_{\alpha,\beta}(t,\cdot) * h(x) = \int_{\mathbb{R}^N} \widetilde{E}_{\alpha,\beta}(t,x-y)h(y)\,dy.$$

The following lemmas contain the so called smoothing effect of the Mittag–Leffler operators family $\{\widetilde{E}_{\alpha,1}(t)\}_{t\ge0}$ and $\{\widetilde{E}_{\alpha,\alpha}(t)\}_{t\ge0}$ in Lebesgue spaces and play an important role in obtaining the first result of this paper; they appear in [18, Lemma 5.1] and [3, Lemma 5.1]. Their proofs are based on the Fourier multiplier theorem combined with a scaling argument (see [2, Lemma 3.1-(i)] or [3, Propositions 4.2 and 4.3]).

Lemma 2.1. [3, Lemma 5.1] *Let $1 < p_1 \le p_2 < \infty$, $1 < \alpha < 2$ and $\lambda = \frac{N}{p_1} - \frac{N}{p_2}$. Then there is a constant $C > 0$ such that*

$$\begin{aligned}
\big\|\widetilde{E}_{\alpha,1}(t)f\big\|_{L^{p_2}} &\le Ct^{-\frac{\alpha}{2}\lambda}\|f\|_{L^{p_1}} && \text{if } \lambda<2,\\
\big\|t\widetilde{E}_{\alpha,2}(t)f\big\|_{L^{p_2}} &\le Ct^{1-\frac{\alpha}{2}\lambda}\|f\|_{L^{p_1}} && \text{if } \frac{2}{\alpha}<\lambda<2,\\
\big\|t\widetilde{E}_{\alpha,2}(t)f\big\|_{L^{p_2}} &\le Ct^{-\frac{\alpha}{2}\lambda}\|f\|_{\dot{\mathcal{H}}_{p_1}^{-\frac{2}{\alpha}}} && \text{if } \frac{2}{\alpha}<\lambda<2,\\
\big\|\widetilde{E}_{\alpha,\alpha}(t)f\big\|_{L^{p_2}} &\le Ct^{-\frac{\alpha}{2}\lambda}\|f\|_{L^{p_1}} && \text{if } \left(2-\frac{2}{\alpha}\right)<\lambda<2
\end{aligned}$$

for all $f \in S'(R^N)$, where $\dot{H}_{p_1}^{-\frac{2}{\alpha}}$ is the homogeneous Sobolev spaces of negative order $-\frac{2}{\alpha}$.

Lemma 2.2. *The family of operators* $\{\widetilde{E}_{\alpha,1}(t)\}_{t>0}$, $\{\widetilde{E}_{\alpha,1}(t)\}_{t>0}$ *and* $\{\widetilde{E}_{\alpha,\alpha}(t)\}_{t>0}$ *enjoy the following* L^{p_1}-L^{p_1} *estimates property:*

(i) *If* $h \in L^{p_1}(\mathbb{R}^N)$ $(1 \le p_1 \le +\infty)$*, then* $\widetilde{E}_{\alpha,\beta}(t)h \in L^{p_1}(\mathbb{R}^N)$ *and*
$$\big\|\widetilde{E}_{\alpha,\beta}(t)h\big\|_{L^{p_1}(\mathbb{R}^N)} \le C\|h\|_{L^{p_1}(\mathbb{R}^N)}, \quad t>0 \quad \textit{for } \beta = 1,2,\alpha$$
for some positive constant $C > 0$.

(ii) *Let* $p_1 > N/2$. *If* $h \in L^{p_1}(\mathbb{R}^N)$*, then* $\widetilde{E}_{\alpha,\beta}(t)h \in L^{\infty}(\mathbb{R}^N)$ *and we have*
$$\big\|\widetilde{E}_{\alpha,\beta}(t)h\big\|_{L^{\infty}(\mathbb{R}^N)} \le Ct^{-\frac{\alpha}{2}\frac{N}{p_1}}\|h\|_{L^{p_1}(\mathbb{R}^N)}, \quad t>0 \quad \textit{for } \beta = 1,2,\alpha.$$

Proof. We use the following pointwise estimates that are shown in [21, Theorem 5.1]:
$$\big|\widetilde{E}_{\alpha,\alpha}(t,x)\big| \le |x|^{-N}\exp\Big\{-c(t^{-\alpha}|x|^2)^{\frac{1}{2-\alpha}}\Big\} \quad \text{if } R := |x|^2t^{-\alpha} \ge 1,$$
and if $R := |x|^2t^{-\alpha} < 1$, then we have
$$\big|\widetilde{E}_{\alpha,\alpha}(t,x)\big| \le \begin{cases} t^{-\frac{\alpha N}{2}}, & N<2,\\ t^{-\alpha}|x|^{-N+2}\big(1+|\ln(|x|^2t^{-\alpha})|\big), & N=2,\\ |x|^{-N+2}t^{-\alpha}, & N>2. \end{cases}$$
Concerning the operator $t\widetilde{E}_{\alpha,2}(t)$, we have the pointwise estimates
$$\big|t\widetilde{E}_{\alpha,2}(t)\big| \le C|x|^{-N}t\exp\Big\{-c(t^{-\alpha}|x|^2)^{\frac{1}{2-\alpha}}\Big\} \quad \text{if } R := |x|^2t^{-\alpha} \ge 1,$$
and if $R := |x|^2t^{-\alpha} < 1$, then
$$\big|t\widetilde{E}_{\alpha,2}(t,x)\big| \le \begin{cases} t^{1-\frac{\alpha N}{2}}, & N<2,\\ |x|^{-N+2}t^{1-\alpha}\big(1+|\ln(|x|^2t^{-\alpha})|\big), & N=2,\\ |x|^{-N+2}t^{1-\alpha}, & N>2. \end{cases}$$
Arguing as in Zacher et al. [20], $\widetilde{E}_{\alpha,1}(t,\cdot\,)$, $\widetilde{E}_{\alpha,2}(t,\cdot\,)$ and $\widetilde{E}_{\alpha,\alpha}(t,\cdot\,)$ are Lebesgue integrable.

In fact, we have
$$\int_{\mathbb{R}^N}\big|\widetilde{E}_{\alpha,\alpha}(t,x)\big|\,dx = \int_{\{R\ge 1\}}\big|\widetilde{E}_{\alpha,\alpha}(t,x)\big|\,dx + \int_{\{R<1\}}\big|\widetilde{E}_{\alpha,\alpha}(t,x)\big|\,dx.$$
Using the first pointwise estimate, we get
$$\begin{aligned}\int_{\{R\ge 1\}}\big|\widetilde{E}_{\alpha,\alpha}(t,x)\big|\,dx &\le \int_{\{R\ge 1\}}|x|^{-N}\exp\Big\{-c(t^{-\alpha}|x|^2)^{\frac{1}{2-\alpha}}\Big\}\,dx\\ &= \int_{t^{\frac{\alpha}{2}}}^{+\infty} r^{-N}\exp\Big\{-c(t^{-\alpha}r^2)^{\frac{1}{2-\alpha}}\Big\}\,r^{N-1}\,dr\\ &= \int_{t^{\frac{\alpha}{2}}}^{+\infty} r^{-1}\exp\Big\{-c(t^{-\alpha}r^2)^{\frac{1}{2-\alpha}}\Big\}\,dr, \quad \text{set } z = t^{-\frac{\alpha}{2}}r\\ &= \int_1^{+\infty} z^{-1}\exp\Big\{-c(z^2)^{\frac{1}{2-\alpha}}\Big\}\,dz \le C.\end{aligned}$$

On the other hand, if $N < 2$, we have

$$\int_{\{R\leq 1\}} \left|\widetilde{E}_{\alpha,\alpha}(t,x)\right| dx \leq \int_{\{R\leq 1\}} t^{-\frac{\alpha N}{2}}\, dx = t^{-\frac{\alpha N}{2}} \int_0^{t^{\frac{\alpha}{2}}} r^{N-1}\, dr = t^{-\frac{\alpha N}{2}} \frac{t^{\frac{\alpha N}{2}}}{N} = C.$$

For $N = 2$, we have

$$\begin{aligned}\int_{\{R\leq 1\}} \left|\widetilde{E}_{\alpha,\alpha}(t,x)\right| dx &\leq \int_{\{R\leq 1\}} |x|^{-N+2} t^{-\alpha}(1-\ln(|x|^2 t^{-\alpha}))\, dx\\ &= t^{-\alpha}\int_0^{t^{\frac{\alpha}{2}}} (1+|\ln(r^2t^{-\alpha})|) r\, dr = t^{-\alpha} t^{\frac{\alpha}{2}N} \int_0^1 (1-\ln(z^2)) z\, dz = C.\end{aligned}$$

When $N > 2$, we have

$$\begin{aligned}\int_{\{R\leq 1\}} \left|\widetilde{E}_{\alpha,\alpha}(t,x)\right| dx &\leq \int_{\{R\leq 1\}} |x|^{-N+2} t^{-\alpha}\, dx\\ &= t^{-\alpha}\int_0^{t^{\frac{\alpha}{2}}} r^{-N+2} r^{N-1}\, dr = t^{-\alpha}\int_0^{t^{\frac{\alpha}{2}}} r\, dr,\end{aligned}$$

so

$$\int_{\{R\leq 1\}} \left|\widetilde{E}_{\alpha,\alpha}(t,x)\right| dx \leq \frac{1}{2}.$$

The first result (i) follows from Young's convolution inequality, that is,

$$\begin{aligned}\left\|\widetilde{E}_{\alpha,\alpha}(t,\cdot)h\right\|_{L^{p_1}(\mathbb{R}^N)} &= \left\|\widetilde{E}_{\alpha,\alpha}(t,\cdot) * h(x)\right\|_{L^{p_1}(\mathbb{R}^N)}\\ &\leq \left\|\widetilde{E}_{\alpha,\alpha}(t,\cdot)\right\|_{L^1(\mathbb{R}^N)} \|h\|_{L^{p_1}(\mathbb{R}^N)} \leq C\|h\|_{L^{p_1}(\mathbb{R}^N)}.\end{aligned}$$

In a s similar manner, it can be shown that the operators $\widetilde{E}_{\alpha,1}(t,\cdot)$ and $\widetilde{E}_{\alpha,2}(t,\cdot)$ are bounded.

In order to show statement (ii), we need to prove that $\widetilde{E}_{\alpha,\alpha}(t,\cdot)$, belongs to $L^{p_2}(\mathbb{R}^N)$

$$\begin{aligned}\int_{\{R\geq 1\}} \left|\widetilde{E}_{\alpha,\alpha}(t,x)\right|^{p_2} dx &\leq \int_{\{R\geq 1\}} |x|^{-Np_2} \exp\left\{-c(t^{-\alpha}|x|^2)^{\frac{1}{2-\alpha}}\right\} dx\\ &= \int_{t^{\frac{\alpha}{2}}}^{+\infty} r^{-Np_2} \exp\left\{-c(t^{-\alpha}r^2)^{\frac{1}{2-\alpha}}\right\} r^{N-1}\, dr\\ &= \int_{t^{\frac{\alpha}{2}}}^{+\infty} r^{-Nr+N-1} \exp\left\{-c(t^{-\alpha}r^2)^{\frac{1}{2-\alpha}}\right\} dr, \quad \text{set } z = t^{-\frac{\alpha}{2}} r\\ &= t^{-\frac{\alpha}{2}N(p_2-1)} \int_1^{+\infty} z^{-Np_2+N-1} \exp\left\{-c(z^2)^{\frac{1}{2-\alpha}}\right\} dz\\ &\leq Ct^{-\frac{\alpha}{2}N(p_2-1)}.\end{aligned}$$

On the other hand, if $N = 1$, we have

$$\begin{aligned}\int_{\{R\leq 1\}} \left|\widetilde{E}_{\alpha,\alpha}(t,x)\right|^{p_2} dx &\leq \int_{\{R\leq 1\}} t^{-\frac{\alpha N}{2}p_2}\, dx = t^{-\frac{\alpha N}{2}p_2} \int_0^{t^{\frac{\alpha}{2}}} r^{N-1}\, dr\\ &= \frac{1}{N} t^{-\frac{\alpha N}{2}p_2 + \frac{\alpha}{2}} = Ct^{-\frac{\alpha}{2}N(p_2-1)}.\end{aligned}$$

For $N = 2$, we have

$$\begin{aligned}\int_{\{R\le 1\}} \left|\widetilde{E}_{\alpha,\alpha}(t,x)\right|^{p_2} dx &\le \int_{\{R\le 1\}} t^{-\alpha p_2}(1-\ln(|x|^2 t^{-\alpha}))^{p_2}\, dx \\ &= t^{-\alpha p_2}\int_0^{t^{\frac{\alpha}{2}}} (1+|\ln(r^2 t^{-\alpha})|)^{p_2} r^{N-1}\, dr \\ &= t^{-\alpha(p_2-1)}\int_0^1 (1-\ln(z^2))^{p_2} z\, dz = Ct^{-\alpha(p_2-1)}.\end{aligned}$$

When $N > 2$, we have

$$\begin{aligned}\int_{\{R\le 1\}} \left|\widetilde{E}_{\alpha,\alpha}(t,x)\right|^{p_2} dx &\le \int_{\{R\le 1\}} |x|^{-(N-2)p_2} t^{-\alpha p_2}\, dx \\ &= t^{-\alpha p_2}\int_0^{t^{\frac{\alpha}{2}}} r^{-(N-2)p_2} r^{N-1}\, dr = t^{-\alpha p_2}\int_0^{t^{\frac{\alpha}{2}}} r^{-(N-2)p_2+N-1}\, dr,\end{aligned}$$

provided $N > (N-2)p_2$. So

$$\int_{\{R\le 1\}} \left|\widetilde{E}_{\alpha,\alpha}(t,x)\right|^{p_2} dx \le Ct^{-\alpha p_2 - \frac{\alpha}{2}(N-2)p_2 + \frac{\alpha}{2}N} = Ct^{-\frac{\alpha}{2}N(p_2-1)}.$$

Hence $\left\|\widetilde{E}_{\alpha,\alpha}(t,\cdot)\right\|_{p_2} \le Ct^{-\frac{\alpha}{2}N\left(1-\frac{1}{p_2}\right)}$, for $p_2 < N/(N-2)$.

Now (ii) follows by Young's convolution inequality and the last estimate

$$\left\|\widetilde{E}_{\alpha,\alpha}(t,\cdot) * f\right\|_{L^\infty} \le \left\|\widetilde{E}_{\alpha,\alpha}(t,\cdot)\right\|_{L^{p_1'}} \|f\|_{L^{p_1}} \le Ct^{-\frac{\alpha}{2}\frac{N}{p_1}}\|f\|_{L^{p_1}} \quad \text{for } p_1 > \frac{N}{2},$$

where p_1' is the conjugate of p_1 ($1/p_1 + 1/p_1' = 1$). Arguing in a similar way, we obtain L^{p_1}-L^∞ estimates to the operators $\widetilde{E}_{\alpha,\beta}(t)$ for $\beta = 1, 2$. □

Lemma 2.3. *Let $l \ge 1$, and let the function $f(t,x)$ satisfy*

$$\|f(t,\cdot)\|_l \le C_1,\ \ 0 \le t \le 1, \quad \|f(t,\cdot)\|_l \le C_2 t^{-\alpha},\ \ t > 0$$

for some positive constants C_1, C_2 and α. Then

$$\|f(t,\cdot)\|_l \le \max\{C_1, C_2\}(1+t)^{-\beta} \quad \textit{for all } 0 < \beta \le \alpha \textit{ and } t \ge 0.$$

3. Main results

In this section, we state our main results. Let us begin with the definition of a mild solution of problem (1.1)–(1.2).

Definition 3.1. Let $u_0, v_0, u_1, v_1 \in X$, $(X := L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$, $1 < \gamma_1, \gamma_2 < 2$, $f, g \in L^1((0,T), \mathcal{S}(\mathbb{R}^N))$ and $T > 0$. We call $(u,v) \in C([0,T];\mathbb{X}) \times C([0,T];\mathbb{X})$ a mild solution of system (1.1)–(1.2) if (u,v) satisfies the following integrals

$$\begin{aligned} u(t,x) &= \widetilde{E}_{\gamma_1,1}(t,x)u_0(x) + t\widetilde{E}_{\gamma_1,2}(t,x)u_1(x) \\ &\quad + \int_0^t (t-\tau)^{\gamma_1-1}\widetilde{E}_{\gamma_1,\gamma_1}(t-\tau,x)f(v(\tau,x))\,d\tau, \end{aligned} \tag{3.1}$$

$$\begin{aligned} v(t,x) &= \widetilde{E}_{\gamma_2,1}(t,x)v_0(x) + t\widetilde{E}_{\gamma_2,2}(t,x)v_1(x) \\ &\quad + \int_0^t (t-\tau)^{\gamma_1-1}\widetilde{E}_{\gamma_2,\gamma_2}(t-\tau,x)g(u(\tau,x))\,d\tau. \end{aligned} \tag{3.2}$$

The existence and uniqueness of a local solution of (1.1) can be established by using the Banach fixed point theorem and Gronwall's inequality.

Proposition 3.2 (Local existence of a mild solution)**.** *Let* $u_0, v_0, u_1, v_1 \in X$*,* $1 < \gamma_1, \gamma_2 < 2$*,* $p, q \geq 1$ *such that* $pq > 1$*. Then there exist a maximal time* $T_{\max} > 0$ *and a unique mild solution to problem* (1.1)–(1.2)*, such that either*

(i) $T_{\max} = \infty$ *(the solution is global), or*

(ii) $T_{\max} < \infty$ *and* $\lim_{t\to T_{\max}} \left(\|u(t)\|_\infty + \|v(t)\|_\infty\right) = \infty$ *(the solution blows up in a finite time).*

Moreover, for any $s_1, s_2 \in (1, +\infty)$*,* $(u,v) \in C([0,T]; L^{s_1}(\mathbb{R}^N) \times L^{s_2}(\mathbb{R}^N))$*.*

Now, we are in a position to state the first main result of this section concerning global existence and large time behavior of solutions of (1.1)–(1.2).

Theorem 3.3 (Global existence of a mild solution)**.** *Let* $N \geq 2$*,* $q \geq p \geq 1$*,* $pq > 1$*,* $1 < \gamma_1 \leq \gamma_2 < 2$*. If*

$$\frac{N}{2} \geq \max\left\{\frac{1}{\gamma_1} + \frac{q+1}{pq-1}, \frac{1}{\gamma_1} + \frac{p\gamma_2 + \gamma_1}{\gamma_1(pq-1)}\right\}, \tag{3.3}$$

the initial data satisfy

$$\|u_0\|_{\mathbb{X}} + \|u_1\|_{\mathbb{X}} + \|v_0\|_{\mathbb{X}} + \|v_1\|_{\mathbb{X}} \leq \varepsilon_0$$

for some $\varepsilon_0 > 0$*, then problem* (1.1)–(1.2) *admits a global mild solution and that*

$$\begin{aligned} u &\in L^\infty([0,\infty), L^\infty(\mathbb{R}^N)) \cap L^\infty([0,\infty), L^{s_1}(\mathbb{R}^N)), \\ v &\in L^\infty([0,\infty), L^\infty(\mathbb{R}^N)) \cap L^\infty([0,\infty), L^{s_2}(\mathbb{R}^N)), \end{aligned}$$

where $s_1 > q$ *and* $s_2 > p$*.*

Furthermore, for any δ *satisfying* $1-\frac{1+q}{(p+1)q\gamma_2}<\delta<\min\left\{1,\frac{N(pq-1)}{2q(p+1)}\right\}$,

$$\|u(t)\|_{s_1}\le C(t+1)^{-\frac{(1-\delta)(\gamma_1+p\gamma_2)}{pq-1}},\quad t\ge 0,$$
$$\|v(t)\|_{s_2}\le C(t+1)^{-\frac{(1-\delta)(\gamma_2+q\gamma_1)}{pq-1}},\quad t\ge 0.$$

If, in addition,

$$\frac{pN}{2s_2}<1\quad \textit{and}\quad \frac{qN}{2s_1}<1,$$

or

$$N>2,\quad \frac{pN}{2s_2}<1\quad \textit{and}\quad \frac{qN}{2s_1}\ge 1,$$

or

$$N>2,\quad \frac{qN}{2s_1}\ge 1,\quad \frac{pN}{2s_2}\ge 1\quad \textit{and}\quad q\ge p>1\quad \textit{with}\ \sqrt{\frac{(p+1)q\gamma_1}{(q+1)p}}<\gamma_1\le\gamma_2<2,$$

then

$$u,v\in L^\infty([0,\infty),L^\infty(\mathbb{R}^N)),$$
$$\|u(t)\|_\infty\le C(t+1)^{-\widetilde{\sigma}},\quad \|v(t)\|_\infty\le C(t+1)^{-\widehat{\sigma}}\quad \textit{for all}\ t\ge 0$$

for some positive constants $\widetilde{\sigma}$ *and* $\widehat{\sigma}$.

Definition 3.4 (Weak solution)**.** Let $u_0,v_0\in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, $u_1,v_1\in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, $T>0$. We say that $(u,v)\in L^q((0,T),L^\infty_{\mathrm{loc}}(\mathbb{R}^N))\times L^p((0,T),L^\infty_{\mathrm{loc}}(\mathbb{R}^N))$ is a weak solution of (1.1)–(1.2) if for all nonnegative test functions $\varphi\in C^{1,2}_{t,x}([0,T]\times\mathbb{R}^N)$ with compact support, such that $\varphi(T,\cdot)=0$, the system of integral equalities

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^N}uD^{\gamma_1}_{t|T}\varphi(t,x)\,dxdt-\int_0^T\int_{\mathbb{R}^N}u\Delta\varphi(t,x)\,dxdt\\
&=\int_{\mathbb{R}^N}u_0(x)(D^{\gamma_1-1}_{t|T}\varphi)(0,\cdot)\,dx+\int_0^T\int_{\mathbb{R}^N}u_1D^{\gamma_1-1}_{t|T}\varphi(t,x)\,dxdt\\
&\quad+\int_0^T\int_{\mathbb{R}^N}f(v(\tau,x))\varphi(t,x)\,dxdt,\\
&\int_0^T\int_{\mathbb{R}^N}vD^{\gamma_2}_{t|T}\varphi(t,x)\,dxdt-\int_0^T\int_{\mathbb{R}^N}v\Delta\varphi(t,x)\,dxdt\\
&=\int_{\mathbb{R}^N}v_0(x)(D^{\gamma_2-1}_{t|T}\varphi)(0,\cdot)\,dx+\int_0^T\int_{\mathbb{R}^N}v_1D^{\gamma_2-1}_{t|T}\varphi(t,x)\,dxdt\\
&\quad+\int_0^T\int_{\mathbb{R}^N}g(u(\tau,x))\varphi(t,x)\,dxdt
\end{aligned}$$

holds.

Similar to the proof in [15], we can obtain the following lemma asserting that the mild solution is the weak solution.

Lemma 3.5. *Assume that* $(u_0, v_0), (u_1, v_1) \in S(\mathbb{R}^N) \times S(\mathbb{R}^N)$ *and let* $(u, v) \in C^{\gamma_1}([0, T], \mathcal{S}(\mathbb{R}^N)) \times C^{\gamma_2}([0, T], \mathcal{S}(\mathbb{R}^N))$ *be a mild solution of* (1.1)–(1.2). *Then* (u, v) *is also a weak solution of* (1.1)–(1.2).

Proof. As (u, v) is a mild solution, we have

$$u(t,x) = \widetilde{E}_{\gamma_1,1}(t,x)u_0(x) + t\widetilde{E}_{\gamma_1,2}(t,x)u_1(x) + \int_0^t (t-s)^{\gamma_1-1}\widetilde{E}_{\gamma_1,\gamma_1}(t-s)f(v(s,x))\,ds.$$

Differentiating with respect to t and noting that $1 < \gamma_1 < 2$, we get

$$\begin{aligned} u_t(t,x) - u_1(x) &= \partial_t\widetilde{E}_{\gamma_1,1}(t,x)u_0(x) + \partial_t(t\widetilde{E}_{\gamma_1,2}(t,x))u_1(x) - u_1(x) \\ &\quad + \int_0^t (t-s)^{\gamma_1-2}\widetilde{E}_{\gamma_1,\gamma_1-1}(t-s)f(v(s,x))\,ds, \end{aligned} \tag{3.4}$$

where we have used the following formula

$$\left(\frac{d}{dz}\right)^{(m)} \left[z^{\beta-1}E_{\alpha,\beta}(z^\alpha)\right] = z^{\beta-m-1}E_{\alpha,\beta-m}(z^\alpha), \quad \Re(\beta-m) > 0,\ m = 0, 1, \ldots.$$

Applying $J_{0|t}^{2-\gamma_1}$ to both sides of (3.4), we obtain

$$\begin{aligned} J_{0|t}^{2-\gamma_1}(u_t - u_1) &= J_{0|t}^{2-\gamma_1}(\partial_t\widetilde{E}_{\gamma_1,1}(t,x))u_0(x) + J_{0|t}^{2-\gamma_1}(\partial_t(t\widetilde{E}_{\gamma_1,2}(t,\cdot))u_1(x) - u_1(x)) \\ &\quad + J_{0|t}^{2-\gamma_1}\left(\int_0^t (t-s)^{\gamma_1-2}\widetilde{E}_{\gamma_1,\gamma_1-1}(t-s,\cdot)f(v(s,x))\right)ds. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & J_{0|t}^{2-\gamma_1}\left(\int_0^t (t-s)^{\gamma_1-2}E_{\gamma_1,\gamma_1-1}\big(-|\xi|^2(s-\tau)^{\gamma_1}\big)\widehat{f}(s,\xi)\,ds\right) \\ &= \frac{1}{\Gamma(2-\gamma_1)}\int_0^t (t-s)^{1-\gamma_1}\int_0^s (s-\tau)^{\gamma_1-2}E_{\gamma_1,\gamma_1-1}\big(-|\xi|^2(s-\tau)^{\gamma_1}\big)\widehat{f}(\tau,\xi)\,d\tau ds \\ &= \sum_{k=0}^{+\infty}\frac{(-1)^k|\xi|^{2k}}{\Gamma(2-\gamma_1)\Gamma(\gamma_1 k+\gamma_1-1)}\int_0^t (t-s)^{1-\gamma_1}\int_0^s (s-\tau)^{\gamma_1-2+\gamma_1 k}\widehat{f}(\tau,\xi)\,d\tau ds \\ &= \sum_{k=0}^{+\infty}\frac{(-1)^k|\xi|^{2k}}{\Gamma(2-\gamma_1)\Gamma(\gamma_1 k+\gamma_1-1)}\int_0^t\int_\tau^t (t-s)^{1-\gamma_1}(s-\tau)^{\gamma_1-2+\gamma_1 k}\,ds\widehat{f}(\tau,\xi)\,d\tau \\ &= \sum_{k=0}^{+\infty}\frac{(-1)^k|\xi|^{2k}}{\Gamma(2-\gamma_1)\Gamma(\gamma_1 k+\gamma_1-1)}\mathbf{B}(2-\gamma_1, \gamma_1 k+\gamma_1-1)\int_0^t (t-s)^{\gamma_1 k}\widehat{f}(s,\xi)\,ds \\ &= \int_0^t E_{\gamma_1,1}\big(-|\xi|^2(s-\tau)^{\gamma_1}\big)\widehat{f}(s,\xi)\,ds. \end{aligned} \tag{3.5}$$

Here B denotes to the beta function.

Applying the Fourier inverse transform to both sides of (3.5) yields

$$J_{0|t}^{2-\gamma_1}\left(\int_0^t (t-s)^{\gamma_1-2}\widetilde{E}_{\gamma_1,\gamma_1-1}(t-s,\cdot)f(v(s,x))\right)ds = \int_0^t \widetilde{E}_{\gamma_1,1}(t-s,\cdot)f(v(s,x))\,ds.$$

Then, for every test function $\varphi \in C_{x,t}^{2,1}(\mathbb{R}^N \times [0,T])$, $\operatorname{supp}\varphi \subset\subset R^N \times [0,T]$ and $\varphi(T,x) = 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} J_{0|t}^{2-\gamma_1}(u_t-u_1)\varphi\,dx = &\int_{\mathbb{R}^N} J_{0|t}^{2-\gamma_1}\big(\partial_t\widetilde{E}_{\gamma_1,1}(t,x)\big)u_0(x)\varphi\,dx\\ &+\int_{\mathbb{R}^N} J_{0|t}^{2-\gamma_1}\big(\partial_t(t\widetilde{E}_{\gamma_1,2}(t,\cdot))u_1(x)-u_1(x)\big)\varphi\,dx\\ &+\int_{\mathbb{R}^N}\int_0^t \widetilde{E}_{\gamma_1,1}(t-s)f(v(s,x))\,ds\,\varphi\,dx.\end{aligned}$$

Setting

$$I := \int_{\mathbb{R}^N} J_{0|t}^{2-\gamma_1}(u_t-u_1)\varphi\,dx,$$

we get

$$\begin{aligned}\frac{\partial}{\partial t}I &= \int_{\mathbb{R}^N}\frac{\partial}{\partial t}\left[J_{0|t}^{2-\gamma_1}(u_t-u_1)\varphi\right]dx\\ &= \int_{\mathbb{R}^N}\frac{\partial}{\partial t}\left[J_{0|t}^{2-\gamma_1}\big(\partial_t\widetilde{E}_{\gamma_1,1}(t,x)\big)u_0(x)\varphi\right]dx\\ &\quad+\int_{\mathbb{R}^N}\frac{\partial}{\partial t}\left[J_{0|t}^{2-\gamma_1}\big(\partial_t(t\widetilde{E}_{\gamma_1,2}(t,\cdot))u_1(x)-u_1(x)\big)\varphi\right]dx\\ &\quad+\int_{\mathbb{R}^N}\frac{\partial}{\partial t}\left(\int_0^t \widetilde{E}_{\gamma_1,1}(t-s)\,d\tau\,f(s,x)\,ds\,\varphi\right)dx.\end{aligned}$$

On the other hand, using the relations

$$\begin{aligned}D_{0|t}^{\gamma_1}\widetilde{E}_{\gamma_1,1}(t,\cdot)u_0(x) &= \Delta\widetilde{E}_{\gamma_1,1}(t,\cdot)u_0(x),\\ D_{0|t}^{\gamma_1}(t\widetilde{E}_{\gamma_1,2}(t,\cdot))u_1(x) &= \Delta(t\widetilde{E}_{\gamma_1,2}(t,\cdot))u_1(x),\end{aligned}$$

we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}\frac{\partial}{\partial t}\big[J_{0|t}^{2-\gamma_1}\big(\partial_t(t\widetilde{E}_{\gamma_1,2}(t,\cdot))u_1(x)-u_1(x)\big)\varphi\big]\,dx\\ &= \int_{\mathbb{R}^N} D_{0|t}^{\gamma_1}(t\widetilde{E}_{\gamma_1,2}(t,\cdot))u_1(x)\varphi(t,x)\,dx\\ &\quad+\int_{\mathbb{R}^N} J_{0|t}^{2-\gamma_1}\big(\partial_t(t\widetilde{E}_{\gamma_1,2}(t,\cdot))u_1(x)-u_1(x)\big)\varphi_t(t,x)\,dx\\ &= \int_{\mathbb{R}^N} t\widetilde{E}_{\gamma_1,2}(t,\cdot)u_1(x)\Delta\varphi(t,x)\,dx\\ &\quad+\int_{\mathbb{R}^N} J_{0|t}^{2-\gamma_1}\big(\partial_t(t\widetilde{E}_{\gamma_1,2}(t,\cdot))u_1(x)-u_1(x)\big)\varphi_t(t,x)\,dx,\end{aligned}$$

and

$$\int_{\mathbb{R}^N} \frac{\partial}{\partial t}\big[J_{0|t}^{2-\gamma_1}(\partial_t \widetilde{E}_{\gamma_1,1}(t,x))u_0(x)\varphi\big]\,dx = \int_{\mathbb{R}^N} \widetilde{E}_{\gamma_1,1}(t,x)u_0(x)\Delta\varphi(t,x)\,dx + \int_{\mathbb{R}^N} J_{0|t}^{2-\gamma_1}\big(\partial_t \widetilde{E}_{\gamma_1,1}(t,x)\big)u_0(x)\varphi_t(t,x)\,dx.$$

Using the Leibniz formula, we get

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^t \widetilde{E}_{\gamma_1,1}(t-s)f(v(s,x))\,ds &= \widetilde{E}_{\gamma_1,1}(0)f(v(t,x)) + \int_0^t \partial_t \widetilde{E}_{\gamma_1,1}(t-s)f(v(s,x))\,ds \\ &= f(v(t,x)) + \int_0^t \partial_t \widetilde{E}_{\gamma_1,1}(t-s)f(v(t,x))\,ds.\end{aligned}$$

So

$$\begin{aligned}\frac{\partial}{\partial t}I &= \int_{\mathbb{R}^N} \widetilde{E}_{\gamma_1,1}(t,x)u_0(x)\Delta\varphi\,dx + \int_{\mathbb{R}^N} t\widetilde{E}_{\gamma_1,2}(t,\cdot\,)u_1(x)\Delta\varphi\,dx \\ &\quad + \int_{\mathbb{R}^N} f(v(t,x))\varphi\,dx + \int_{\mathbb{R}^N}\int_0^t (t-s)^{\gamma_1-1}\widetilde{E}_{\gamma_1,\gamma_1}(t-s)f(v(s,x))\Delta\varphi\,dsdx \\ &\quad + \int_{\mathbb{R}^N} J_{0|t}^{2-\gamma_1}\big(\partial_t \widetilde{E}_{\gamma_1,1}(t,x)\big)u_0(x)\varphi_t\,dx \\ &\quad + \int_{\mathbb{R}^N} J_{0|t}^{2-\gamma_1}\big(\partial_t(t\widetilde{E}_{\gamma_1,2}(t,\cdot\,))u_1(x) - u_1(x)\big)\varphi_t\,dx \\ &\quad + \int_{\mathbb{R}^N}\int_0^t \widetilde{E}_{\gamma_1,1}(t-s)f(v(s,x))\,ds\,\varphi_t\,dx.\end{aligned}$$

Using the fact that u is a mild solution, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}I &= \int_{\mathbb{R}^N} u\Delta\varphi\,dx + \int_{\mathbb{R}^N} f(v(t,x))\varphi\,dx + \int_{\mathbb{R}^N} J_{0|t}^{2-\gamma_1}\big(\partial_t \widetilde{E}_{\gamma_1,1}(t,x)\big)u_0(x)\varphi_t\,dx \\ &\quad + \int_{\mathbb{R}^N} J_{0|t}^{2-\gamma_1}\big(\partial_t(t\widetilde{E}_{\gamma_1,2}(t,\cdot\,))u_1(x) - u_1(x)\big)\varphi_t\,dx \\ &\quad + \int_{\mathbb{R}^N}\int_0^t \widetilde{E}_{\gamma_1,1}(t-s)f(v(s,x))\,ds\,\varphi_t\,dx \\ &= \int_{\mathbb{R}^N} u\Delta\varphi\,dx + \int_{\mathbb{R}^N} f(v(t,x))\varphi\,dx + \int_{\mathbb{R}^N} J_{0|t}^{2-\gamma_1}(u_t - u_1)\varphi_t\,dx.\end{aligned} \tag{3.6}$$

On the other hand, we have

$$\frac{\partial}{\partial t}I = \int_{\mathbb{R}^N} \frac{\partial}{\partial t}\big[J_{0|t}^{2-\gamma_1}(u_t - u_1)\big]\varphi\,dx + \int_{\mathbb{R}^N} J_{0|t}^{2-\gamma_1}(u_t - u_1)\varphi_t\,dx. \tag{3.7}$$

Integrating both sides of (3.6) and (3.7) on $[0,T]$, and then identifying the terms, we get

$$\int_0^T\int_{\mathbb{R}^N} \frac{\partial}{\partial t}J_{0|t}^{2-\gamma_1}(u_t - u_1)\varphi\,dxdt = \int_0^T\int_{\mathbb{R}^N} u\Delta\varphi\,dxdt + \int_0^T\int_{\mathbb{R}^N} f(v(t,x))\varphi\,dxdt.$$

The formula of integration by parts allows to write

$$\int_0^T \int_{\mathbb{R}^N} (u_t - u_1) D_{t|T}^{\gamma_1} \varphi \, dxdt = \int_0^T \int_{\mathbb{R}^N} u \Delta \varphi \, dxdt + \int_0^T \int_{\mathbb{R}^N} f(v(t,x)) \varphi \, dxdt.$$

By an analogous calculation, we can show that

$$\int_0^T \int_{\mathbb{R}^N} (v_t - v_1) D_{t|T}^{\gamma_2} \varphi \, dxdt = \int_0^T \int_{\mathbb{R}^N} v \Delta \varphi \, dxdt + \int_0^T \int_{\mathbb{R}^N} g(u(t,x)) \varphi \, dxdt.$$

This completes the proof. □

Our next result concerns the blow-up of solutions of (1.1).

Theorem 3.6 (Blow-up of mild solution). *Let* $N \geq 1$, $p > 1$, $q > 1$, $u_0, v_0, u_1, v_1 \in L^p_{\text{loc}}(\mathbb{R}^N)$, $1 < \gamma_1, \gamma_2 < 2$, *be such that* $\int_{\mathbb{R}^N} u_0(x)\, dx \geq 0$, $\int_{\mathbb{R}^N} v_0(x)\, dx \geq 0$, $\int_{\mathbb{R}^N} u_1(x)\, dx > 0$ *and* $\int_{\mathbb{R}^N} v_1(x)\, dx > 0$. *If*

$$\frac{N}{2} < \min\left\{ \frac{1}{\gamma_1} + \frac{\gamma_2 p + \gamma_1}{\gamma_1(pq-1)}, \frac{1}{\gamma_1} + \frac{p+1}{pq-1} \right\},$$

or

$$\frac{N}{2} < \min\left\{ \frac{1-\gamma_2}{\gamma_1} + \frac{(p\gamma_2 + \gamma_1)q}{\gamma_1(pq-1)}, \frac{1-\gamma_2}{\gamma_1} + \frac{(p+1)q}{pq-1} \right\},$$

or

$$\frac{N}{2} < \min\left\{ \frac{(1-\gamma_1)}{\gamma_2} + \frac{(\gamma_1 q + \gamma_2)p}{\gamma_2(pq-1)}, \frac{1-\gamma_1}{\gamma_2} + \frac{(q+1)p}{pq-1} \right\},$$

or

$$\frac{N}{2} < \min\left\{ \frac{1}{\gamma_2} + \frac{q\gamma_1 + \gamma_2}{(pq-1)\gamma_2}, \frac{1}{\gamma_2} + \frac{q+1}{pq-1} \right\},$$

then the mild solution (u, v) *of* (1.1)–(1.2) *blows up in a finite time.*

4. Global existence and decay estimates

Proof of Theorem 3.3. The proof proceeds in three steps. Without loss of generality, we assume that $1 < \gamma_1 \leq \gamma_2 < 2$ and $q \geq p \geq 1$ such that $pq > 1$.

First step: Global existence for (u, v) in $L^{s_1}(\mathbb{R}^N) \times L^{s_2}(\mathbb{R}^N)$.

Since $pq > 1$, from (3.3) we have for $N \geq 2$ that

$$\frac{N}{2} \geq \max\left\{ \frac{1}{\gamma_1} + \frac{q+1}{pq-1}, \frac{1}{\gamma_1} + \frac{p\gamma_2 + \gamma_1}{\gamma_1(pq-1)} \right\}.$$

If $\max\left\{ \frac{1}{\gamma_1} + \frac{q+1}{pq-1}, \frac{1}{\gamma_1} + \frac{p\gamma_2+\gamma_1}{\gamma_1(pq-1)} \right\} = \frac{1}{\gamma_1} + \frac{q+1}{pq-1}$, then $\frac{N}{2} \geq \frac{1}{\gamma_1} + \frac{q+1}{pq-1}$, which gives

$$1 - \frac{pq-1}{q(p+1)\gamma_2} < 1 - \frac{pq-1}{2q(p+1)} < \frac{pq - 1 + q\gamma_1 + \gamma_1}{\gamma_1 q(p+1)} \leq \frac{N(pq-1)}{2q(p+1)}.$$

If $\max\left\{\frac{1}{2}+\frac{q+1}{pq-1},\frac{1}{\gamma_1}+\frac{p\gamma_2+\gamma_1}{\gamma_1(pq-1)}\right\}=\frac{1}{\gamma_1}+\frac{p\gamma_2+\gamma_1}{\gamma_1(pq-1)}$. That is $\frac{1}{\gamma_1}+\frac{q+1}{pq-1}\le\frac{1}{\gamma_1}+\frac{p\gamma_2+\gamma_1}{\gamma_1(pq-1)}$, in this case

$$\frac{N}{2}\ge\frac{1}{\gamma_1}+\frac{p\gamma_2+\gamma_1}{\gamma_1(pq-1)}\ge\frac{1}{\gamma_1}+\frac{q+1}{pq-1},$$

which gives again $\frac{N(pq-1)}{2q(p+1)}>1-\frac{pq-1}{q(p+1)\gamma_2}$, and since $1-\frac{pq-1}{q(p+1)\gamma_2}<1$, we can choose $\delta>0$ such that

$$1-\frac{pq-1}{q(p+1)\gamma_2}<\delta<\min\left\{1,\frac{N(pq-1)}{2q(p+1)}\right\}.$$

We set

$$\begin{aligned}
r_1&=\frac{N\gamma_1(pq-1)}{2[\gamma_1(1+\delta p)+\gamma_2 p(1-\delta)]},\quad r_2=\frac{N\gamma_2(pq-1)}{2[\gamma_2(1+\delta q)+\gamma_1 q(1-\delta)]},\\
\frac{1}{s_1}&=\frac{2\delta}{N}\frac{p+1}{pq-1},\quad \frac{1}{s_2}=\frac{2\delta}{N}\frac{q+1}{pq-1},\\
\sigma_1&=\frac{(1-\delta)(\gamma_1+\gamma_2 p)}{pq-1},\quad \sigma_2=\frac{(1-\delta)(\gamma_2+\gamma_1 q)}{pq-1}.
\end{aligned}\tag{4.1}$$

Clearly, we have

$$\begin{aligned}
\frac{1}{r_1}&=\frac{2}{N\gamma_1}\frac{(1-\delta)(\gamma_1+\gamma_2 p)}{pq-1}+\frac{2\delta}{N}\frac{(p+1)}{pq-1},\\
\frac{1}{r_2}&=\frac{2}{N\gamma_2}\frac{(1-\delta)(\gamma_2+\gamma_1 q)}{pq-1}+\frac{2\delta}{N}\frac{(q+1)}{pq-1}.
\end{aligned}$$

The choice of δ gives

$$\delta>1-\frac{pq-1}{(\gamma_2+\gamma_1 q)p}\quad\Longrightarrow\quad p\sigma_2=\frac{(1-\delta)(\gamma_2+\gamma_1 q)}{pq-1}p<1,$$

and

$$\delta>1-\frac{pq-1}{(\gamma_1+\gamma_2 p)q}\quad\Longrightarrow\quad q\sigma_1=\frac{(1-\delta)(\gamma_1+\gamma_2 p)}{pq-1}q<1.$$

It is easy to check that

$$\begin{gathered}
s_1>q,\quad s_2>p,\quad ps_1>s_2,\quad qs_2>s_1,\quad s_1>r_1>1,\quad s_2>r_2>1,\\
\frac{N}{2}\gamma_1\left(\frac{1}{r_1}-\frac{1}{s_1}\right)q<1,\quad \frac{N}{2}\gamma_2\left(\frac{1}{r_2}-\frac{1}{s_2}\right)p<1,
\end{gathered}$$

and

$$\frac{N}{2}\left(\frac{p}{s_2}-\frac{1}{s_1}\right)=\delta=\frac{N}{2}\left(\frac{q}{s_1}-\frac{1}{s_2}\right).$$

One can easily verify that

$$\delta>\frac{pq(\gamma_1-1)+1+p\gamma_2}{[\gamma_1 q+\gamma_2]p}\quad\Longleftrightarrow\quad\left(\gamma_2-\frac{N}{2}\gamma_2\left(\frac{q}{s_1}-\frac{1}{s_2}\right)-q\sigma_1\right)p>-1.$$

Let $(u_0, v_0) \in L^{r_1}(\mathbb{R}^N) \times L^{r_2}(\mathbb{R}^N)$. Let $u \in C([0, T_{\max}); L^{s_1}(\mathbb{R}^N))$ and $v \in C([0, T_{\max}); L^{s_2}(\mathbb{R}^N))$. For $t \in [0, T_{\max})$, from (1.1), we have

$$\begin{aligned}\|u(t,\cdot)\|_{s_1} \le{}& \big\|\widetilde{E}_{\gamma_1,1}(t)u_0\big\|_{s_1} + \big\|t\widetilde{E}_{\gamma_1,2}(t,\cdot)\big\|_{s_1}\\ &+ \int_0^t (t-\tau)^{\gamma_1-1}\big\|\widetilde{E}_{\gamma_1,\gamma_1}(t-\tau)|v(\tau,\cdot)|^p\big\|_{s_1}\,d\tau,\end{aligned} \tag{4.2}$$

$$\begin{aligned}\|v(t,\cdot)\|_{s_2} \le{}& \big\|\widetilde{E}_{\gamma_2,1}(t)v_0\big\|_{s_2} + \big\|t\widetilde{E}_{\gamma_2,2}(t,\cdot)\big\|_{s_2}\\ &+ \int_0^t (t-\tau)^{\gamma_2-1}\big\|\widetilde{E}_{\gamma_2,\gamma_2}(t-\tau)|u(\tau,\cdot)|^q\big\|_{s_2}\,d\tau.\end{aligned} \tag{4.3}$$

Applying Lemmas 2.1 and 2.2, we get

$$\begin{aligned}\|u(t,\cdot)\|_{s_1} \le{}& t^{-\sigma_1}\|u_0\|_{r_1} + t^{-\sigma_1}\|u_1\|_{\dot{\mathcal{H}}_{r_1}^{-\frac{2}{\gamma_1}}}\\ &+ C\int_0^t (t-\tau)^{\gamma_1-1}(t-\tau)^{-\frac{N}{2}\gamma_1\left(\frac{p}{s_2}-\frac{1}{s_1}\right)}\|v(\tau,\cdot)\|_{s_2}^p\,d\tau,\end{aligned} \tag{4.4}$$

$$\begin{aligned}\|v(t,\cdot)\|_{s_2} \le{}& t^{-\sigma_2}\|v_0\|_{r_2} + t^{-\sigma_1}\|v_1\|_{\dot{\mathcal{H}}_{r_2}^{-\frac{2}{\gamma_2}}}\\ &+ C\int_0^t (t-\tau)^{\gamma_2-1}(t-\tau)^{-\frac{N}{2}\gamma_2\left(\frac{q}{s_1}-\frac{1}{s_2}\right)}\|u(\tau,\cdot)\|_{s_1}^q\,d\tau.\end{aligned} \tag{4.5}$$

Using (4.5) into (4.4), we obtain

$$\begin{aligned}&\|u(t,\cdot)\|_{s_1}\\ &\le \Big(\|u_0\|_{r_1} + \|u_1\|_{\dot{\mathcal{H}}_{r_1}^{-\frac{2}{\gamma_1}}}\Big)t^{-\sigma_1} + C\int_0^t (t-\tau)^{\gamma_1-1}(t-\tau)^{-\frac{N}{2}\gamma_1\left(\frac{p}{s_2}-\frac{1}{s_1}\right)}\,d\tau\\ &\quad\times\left(\Big(\|v_0\|_{r_2} + \|v_1\|_{\dot{\mathcal{H}}_{r_2}^{-\frac{2}{\gamma_2}}}\Big)t^{-\sigma_2} + C\int_0^t (t-\tau)^{\gamma_2-1}(t-\tau)^{-\frac{N}{2}\gamma_2\left(\frac{q}{s_1}-\frac{1}{s_2}\right)}\|u(t,\cdot)\|_{s_1}^q\,d\tau\right)^p,\end{aligned}$$

provided that $1-\frac{1}{\gamma_1} < \frac{N}{2}\big(\frac{q}{s_1}-\frac{1}{s_2}\big) < 1$ and $1-\frac{1}{\gamma_2} < \frac{N}{2}\big(\frac{p}{s_2}-\frac{1}{s_1}\big) < 1$ which are indeed satisfied.

Hence

$$\begin{aligned}&\|u(t,\cdot)\|_{s_1}\\ &\le \Big(\|u_0\|_{r_1} + \|u_1\|_{\dot{\mathcal{H}}_{r_1}^{-\frac{2}{\gamma_1}}}\Big)t^{-\sigma_1}\\ &\quad+ C\int_0^t (t-\tau)^{\gamma_1-1-\frac{N}{2}\gamma_1\left(\frac{p}{s_2}-\frac{1}{s_1}\right)}\tau^{-p\sigma_2}\,d\tau\Big(\|v_0\|_{r_2} + \|v_1\|_{\dot{\mathcal{H}}_{r_2}^{-\frac{2}{\gamma_2}}}\Big)^p\\ &\quad+ C\int_0^t (t-\tau)^{\gamma_1-1-\frac{N}{2}\gamma_1\left(\frac{p}{s_2}-\frac{1}{s_1}\right)}\tau^{\left(\gamma_2-\frac{N}{2}\gamma_2\left(\frac{q}{s_1}-\frac{1}{s_2}\right)-q\sigma_1\right)p}\big(\tau^{\sigma_1}\|u(\tau,\cdot)\|_{s_1}\big)^{pq}\,d\tau.\end{aligned} \tag{4.6}$$

Multiplying both sides of (4.6) by t^{σ_1} with $\sigma_1 = \frac{(1-\delta)(\gamma_1+\gamma_2 p)}{pq-1}$, we get

$$
\begin{aligned}
& t^{\sigma_1}\|u(t,\cdot)\|_{s_1} \\
& \le \|u_0\|_{r_1} + \|u_1\|_{\dot{\mathcal{H}}_{r_1}^{-\frac{2}{\gamma_1}}} \\
& \quad + Ct^{\sigma_1}\int_0^t (t-\tau)^{\gamma_1-1-\frac{N}{2}\gamma_1\left(\frac{p}{s_2}-\frac{1}{s_1}\right)}\tau^{-p\sigma_2}\,d\tau\left(\|v_0\|_{r_2} + \|v_1\|_{\dot{\mathcal{H}}_{r_2}^{-\frac{2}{\gamma_2}}}\right)^p \\
& \quad + Ct^{\sigma_1}\int_0^t (t-\tau)^{\gamma_1-1-\frac{N}{2}\gamma_1\left(\frac{p}{s_2}-\frac{1}{s_1}\right)}\tau^{\left(\gamma_2-\frac{N}{2}\gamma_2\left(\frac{q}{s_1}-\frac{1}{s_2}\right)-q\sigma_1\right)p}\left(\tau^{\sigma_1}\|u(\tau,\cdot)\|_{s_1}\right)^{pq}d\tau.
\end{aligned}
\tag{4.7}
$$

Since $\gamma_1 - 1 - \frac{N}{2}\gamma_1\left(\frac{p}{s_2}-\frac{1}{s_1}\right) > -1$, $\left(\gamma_2 - \frac{N}{2}\gamma_2\left(\frac{q}{s_1}-\frac{1}{s_2}\right) - q\sigma_1\right)p > -1$, we have

$$
\begin{aligned}
t^{\sigma_1}\|u(t,\cdot)\|_{s_1} \le{}& \|u_0\|_{r_1} + \|u_1\|_{\dot{\mathcal{H}}_{r_1}^{-\frac{2}{\gamma_1}}} + Ct^{\sigma_1+\gamma_1-\frac{N}{2}\gamma_1\left(\frac{p}{s_2}-\frac{1}{s_1}\right)-p\sigma_2}\left(\|v_0\|_{r_2}^p + \|v_1\|^p_{\dot{\mathcal{H}}_{r_2}^{-\frac{2}{\gamma_2}}}\right) \\
& + Ct^{\sigma_1+\gamma_1-\frac{N}{2}\gamma_1\left(\frac{p}{s_2}-\frac{1}{s_1}\right)+\left(\gamma_2-\frac{N}{2}\gamma_2\left(\frac{q}{s_1}-\frac{1}{s_2}\right)-q\sigma_1\right)p}\left(\sup_{0\le\tau\le t}\tau^{\sigma_1}\|u(\tau,\cdot)\|_{s_1}\right)^{pq}.
\end{aligned}
$$

Note that

$$
\begin{gathered}
\sigma_1 = \frac{N}{2}\gamma_1\left(\frac{1}{r_1}-\frac{1}{s_1}\right), \\
\sigma_1 + \gamma_1 - \frac{N}{2}\gamma_1\left(\frac{p}{s_2}-\frac{1}{s_1}\right) - p\sigma_2 = 0, \\
\sigma_1 + \gamma_1 - \frac{N}{2}\gamma_1\left(\frac{p}{s_2}-\frac{1}{s_1}\right) + \left(\gamma_2 - \frac{N}{2}\gamma_2\left(\frac{q}{s_1}-\frac{1}{s_2}\right) - q\sigma_1\right)p = 0, \\
\sigma_1 + \gamma_1 - \gamma_1\delta + (\gamma_2 - \gamma_2\delta - q\sigma_1)p = 0.
\end{gathered}
$$

Define $f(t) = \sup_{0\le\tau\le t}\tau^{\sigma_1}\|u(\tau,\cdot)\|_{s_1}$, $t \in [0, T_{\max})$. So we deduce from (4.7) that

$$
f(t) \le C\left(\|u_0\|_{r_1} + \|u_1\|_{\dot{\mathcal{H}}_{r_1}^{-\frac{2}{\gamma_1}}} + \|v_0\|_{r_2}^p + \|v_1\|^p_{\dot{\mathcal{H}}_{r_2}^{-\frac{2}{\gamma_2}}} + f(t)^{pq}\right)
\tag{4.8}
$$

for all $t \in (0, T_{\max})$. Setting

$$
A = \|u_0\|_{r_1} + \|u_1\|_{\dot{\mathcal{H}}_{r_1}^{-\frac{2}{\gamma_1}}} + \|v_0\|_{r_2}^p + \|v_1\|^p_{\dot{\mathcal{H}}_{r_2}^{-\frac{2}{\gamma_2}}}.
$$

Now if we take A small enough such that $A < (2C)^{\frac{pq}{1-pq}}$, then it follows by continuity argument that (4.8) implies

$$
f(t) \le 2CA \quad \text{for all } t \in [0, T_{\max}).
\tag{4.9}
$$

Indeed, if (4.9) is not true. That is to say $f(t_0) > 2CA$ holds true for some $t_0 \in (0, T_{\max})$. By the intermediate value theorem since f is continuous, non-decreasing and $f(0) = 0$, there exists $t_1 \in (0, t_0)$ such that $f(t_1) = 2CA$. From (4.8), we get

$$
2CA = f(t_1) \le C(A + f(t_1)^{pq}),
$$

from which, it yields

$$2CA \le C(A + (2CA)^{pq}),$$

which is equivalent to

$$A \ge (2C)^{\frac{pq}{1-pq}}.$$

This is a contradiction. Therefore, it follows that

$$f(t) \le 2CA \quad \text{for any } t \in [0, T_{\max}).$$

Thus

$$t^{\sigma_1}\|u(t,\cdot)\|_{s_1} \le C \quad \text{for any } t \in [0, T_{\max}). \tag{4.10}$$

Similarly, we obtain

$$t^{\sigma_2}\|v(t,\cdot)\|_{s_2} \le C \quad \text{for any } t \in [0, T_{\max}). \tag{4.11}$$

Now, from (4.2), (4.3) and Lemma 2.2, we can easily see that

$$\|u(t,\cdot)\|_\infty,\ \|v(t,\cdot)\|_\infty \le C \quad \text{for any } t \in [0,1]. \tag{4.12}$$

On the other hand, since s_1 and s_2 satisfy

$$\frac{(1-\delta)(p+1)s_1}{(pq-1)s_2}\gamma_2 < 1, \quad \frac{(1-\delta)(q+1)s_2}{(pq-1)s_1}\gamma_2 < 1,$$

it follows from (4.2), (4.3), Lemmas 2.1 and 2.2 that

$$\begin{aligned}
\|u(t,\cdot)\|_{s_1} &\le \big\|\widetilde{E}_{\gamma_1,1}(t)u_0\big\|_{s_1} + t\big\|\widetilde{E}_{\gamma_2,2}(t)u_1\big\|_{s_1} \\
&\quad + \int_0^t (t-\tau)^{\gamma_1-1}\big\|\widetilde{E}_{\gamma_1,\gamma_1}(t-\tau)|v(\tau,\cdot)|^p\big\|_{s_1}\,d\tau \\
&\le C\|u_0\|_{s_1} + t\|u_1\|_{s_1} + C\int_0^t (t-\tau)^{\gamma_1-1}\||v(\tau,\cdot)|^p\|_{s_1}\,d\tau \\
&\le C\|u_0\|_{s_1} + \|u_1\|_{s_1} + C\sup_{\tau\in(0,t)}\|v(\tau)\|_\infty^{p-\frac{s_2}{s_1}}\int_0^t (t-\tau)^{\gamma_1-1}\|v(\tau,\cdot)\|_{s_2}^{\frac{s_2}{s_1}}\,d\tau \\
&\le C\|u_0\|_{s_1} + \|u_1\|_{s_1} + C\sup_{\tau\in(0,t)}\|v(\tau)\|_\infty^{p-\frac{s_2}{s_1}}\int_0^t \|v(\tau,\cdot)\|_{s_2}^{\frac{s_2}{s_1}}\,d\tau
\end{aligned} \tag{4.13}$$

for all $t \in [0,1]$. Hence $\|u(t,\cdot)\|_{s_1} \le C$ for any $t \in [0,1]$. Analogously,

$$\|v(t,\cdot)\|_{s_2} \le C \quad \text{for all } t \in [0,1]. \tag{4.14}$$

From (4.10), (4.11), (4.13), (4.14) and Lemma 2.3, we conclude that

$$\begin{cases} \|u(t,\cdot)\|_{s_1} \le C(t+1)^{-\frac{(1-\delta)(\gamma_1+p\gamma_2)}{pq-1}}, \\ \|u(t,\cdot)\|_{s_2} \le C(t+1)^{-\frac{(1-\delta)(\gamma_2+q\gamma_1)}{pq-1}} \end{cases} \tag{4.15}$$

for all $t \in [0, T_{\max})$.

Second step: L^∞-global existence estimates of (u,v) in $L^\infty(\mathbb{R}^N) \times L^\infty(\mathbb{R}^N)$.

Let s_1, s_2 be as in (4.1). Since $p \le q$, we have

$$\frac{Np}{2s_2} \le \frac{Nq}{2s_1}.$$

We further assume, for some $\xi > q$ and $w > p$, that $u(t) \in L^w(\mathbb{R}^N)$, $v(t) \in L^\xi(\mathbb{R}^N)$, and

$$\begin{cases} \|u(t,\cdot)\|_w \le C(1+t^{k_1}), & t \in [0, T_{\max}), \\ \|v(t,\cdot)\|_\xi \le C(1+t^{k_2}), & t \in [0, T_{\max}) \end{cases} \tag{4.16}$$

holds true for some positive constants k_1 and k_2. Then, by (4.2), (4.3) and Lemma 2.2, we have

$$\|u(t,\cdot)\|_\infty \le \big\|\widetilde{E}_{\gamma_1,1}(t)u_0\big\|_\infty + t\big\|\widetilde{E}_{\gamma_2,2}(t)u_1\big\|_\infty + \int_0^t (t-\tau)^{\gamma_1-1-\frac{N\gamma_1 p}{2\xi}} \|v(\tau,\cdot)\|_\xi^p \, d\tau, \tag{4.17}$$

$$\|v(t,\cdot)\|_\infty \le \big\|\widetilde{E}_{\gamma_2,1}(t)v_0\big\|_\infty + t\big\|\widetilde{E}_{\gamma_2,2}(t)u_1\big\|_\infty + \int_0^t (t-\tau)^{\gamma_2-1-\frac{N\gamma_2 q}{2w}} \|u(\tau,\cdot)\|_w^q \, d\tau \tag{4.18}$$

for all $t \in [0, T_{\max})$. If one can find ξ and w such that

$$\frac{Np}{2\xi} < 1 \quad \text{or} \quad \frac{Nq}{2w} < 1, \tag{4.19}$$

then the L^∞-estimates of (u,v) is obtained. In fact, if $\frac{Np}{2\xi} < 1$, in view of (4.16), it yields from (4.17) that

$$\begin{aligned} \|u(t,\cdot)\|_\infty &\le \big\|\widetilde{E}_{\gamma_1,1}(t)u_0\big\|_\infty + C \max_{\tau\in[0,t]} \|v(\tau,\cdot)\|_\xi^p t^{\left(1-\frac{Np}{2\xi}\right)\gamma_1} \\ &\le C\left(1 + t^{\left(1-\frac{Np}{2\xi}\right)\gamma_1 + pk_2}\right), \end{aligned} \tag{4.20}$$

and by taking $w = \infty$ in (4.18), we get

$$\begin{aligned} \|v(t,\cdot)\|_\infty &\le \big\|\widetilde{E}_{\gamma_2,1}(t)v_0\big\|_\infty + t\big\|\widetilde{E}_{\gamma_2,2}(t)u_1\big\|_\infty + \int_0^t (t-\tau)^{\gamma_2-1} \|u(\tau,\cdot)\|_\infty^q \, d\tau \\ &\le \big\|\widetilde{E}_{\gamma_2,1}(t)v_0\big\|_\infty + t\big\|\widetilde{E}_{\gamma_2,2}(t)u_1\big\|_\infty \\ &\quad + \int_0^t (t-\tau)^{\gamma_2-1}\left(1 + t^{\left(1-\frac{Np}{2\xi}\right)\gamma_1+pk_2}\right)^q d\tau \\ &\le C\left(1 + t^{\gamma_2 + \left[\left(1-\frac{Np}{2\xi}\right)\gamma_1 + pk_2\right]q}\right). \end{aligned} \tag{4.21}$$

These estimates show that $T_{\max} = \infty$, and

$$u, v \in L^\infty_{\mathrm{loc}}([0,\infty); L^\infty(\mathbb{R}^N)). \tag{4.22}$$

In a similar manner, we can establish the case $\frac{Nq}{2w} < 1$. To find appropriate ξ and w, we note that (4.20) and (4.21) hold by taking $\xi = s_1$ or $w = s_2$ if $\frac{Nq}{2s_1} < 1$ or $\frac{Np}{2s_2} < 1$; this is certainly the case when $N \le 2$ with $s_1 > q$ and $s_2 > p$.

Thus it remains to deal with the case $N > 2$, $\frac{Nq}{2s_1} \ge 1$ and $\frac{Np}{2s_2} \ge 1$. We do this via an iterative process. Define $s_1' = s_1$, $s_1'' = s_2$, since $s_1' > q$ and $s_1'' > p$, using the Hölder inequality and Lemmas 2.1 and 2.2, we get from (4.2), (4.3) that

$$\|u(t,\cdot)\|_{s_2'} \le \left\|\widetilde{E}_{\gamma_1,1}(t)u_0\right\|_{s_2'} + t\left\|\widetilde{E}_{\gamma_1,2}(t)u_1\right\|_{s_2'} + \int_0^t (t-\tau)^{\gamma_1-1-\frac{N\gamma_1}{2}\left(\frac{p}{s_2''}-\frac{1}{s_2'}\right)} \|v(\tau,\cdot)\|_{s_2''}^p \, d\tau,$$

$$\|v(t,\cdot)\|_{s_2''} \le \left\|\widetilde{E}_{\gamma_2,1}(t)v_0\right\|_{s_2''} + t\left\|\widetilde{E}_{\gamma_2,2}(t)u_1\right\|_{s_2''} + \int_0^t (t-\tau)^{\gamma_2-1-\frac{N\gamma_2}{2}\left(\frac{q}{s_1'}-\frac{1}{s_2''}\right)} \|u(\tau,\cdot)\|_{s_1'}^q \, d\tau,$$

where s_2' and s_2'' are such that

$$\frac{N}{2}\left(\frac{p}{s_1''} - \frac{1}{s_2'}\right) < 1, \quad \frac{N}{2}\left(\frac{q}{s_1'} - \frac{1}{s_2''}\right) < 1.$$

This can be shown by taking

$$\frac{1}{s_2'} = \frac{p}{s_1''} - \frac{2}{N} + \eta, \quad \frac{1}{s_2''} = \frac{q}{s_1'} - \frac{2}{N} + \eta,$$

where $0 < \eta < \frac{2(1-\delta)}{N}$ with $\delta > 1 - \frac{1}{\gamma_1}$. Namely

$$\frac{N}{2}\left(\frac{p}{s_1''} - \frac{1}{s_2'}\right) = \frac{N}{2}\left(\frac{q}{s_1'} - \frac{1}{s_2''}\right) = 1 - \frac{N}{2}\eta > 1 - \frac{1}{\gamma_1}.$$

Observe that, since $\delta > 1 - \frac{pq-1}{q(p+1)\gamma_2} > 1 - \frac{1}{\gamma_2}$, we have

$$\begin{aligned} &1 - \frac{1}{\gamma_1} < \frac{N}{2}\left(\frac{p}{s_1''} - \frac{1}{s_2'}\right) < 1, \qquad 1 - \frac{1}{\gamma_2} < \frac{N}{2}\left(\frac{q}{s_1'} - \frac{1}{s_2''}\right) < 1, \\ &\frac{1}{s_1'} - \frac{1}{s_2'} = \frac{2}{N}(1-\delta) - \eta > 0, \qquad \frac{1}{s_1''} - \frac{1}{s_2''} = \frac{2}{N}(1-\delta) - \eta > 0, \end{aligned} \tag{4.23}$$

and hence $s_2' > s_1' > q$ and $s_2'' > s_1'' > p$.

Next, define the sequences $\{s_i'\}_{i\ge1}$ and $\{s_i''\}_{i\ge1}$, iteratively, as follows

$$\frac{1}{s_i'} = \frac{p}{s_{i-1}''} - \frac{2}{N} + \eta, \quad \frac{1}{s_i''} = \frac{q}{s_{i-1}'} - \frac{2}{N} + \eta, \quad i \ge 3. \tag{4.24}$$

Then

$$\begin{aligned} \frac{1}{s_i'} - \frac{1}{s_{i+1}'} &= p\left(\frac{1}{s_{i-1}''} - \frac{1}{s_i''}\right) = pq\left(\frac{1}{s_{i-2}'} - \frac{1}{s_{i-1}'}\right), \\ \frac{1}{s_i''} - \frac{1}{s_{i+1}''} &= q\left(\frac{1}{s_{i-1}'} - \frac{1}{s_i'}\right) = pq\left(\frac{1}{s_{i-2}''} - \frac{1}{s_{i-1}''}\right). \end{aligned}$$

Since $pq > 1$, in view of (4.23), we get

$$\frac{1}{s'_i} > \frac{1}{s'_{i+1}}, \quad \frac{1}{s''_i} > \frac{1}{s''_{i+1}}, \quad i \geq 1, \tag{4.25}$$

and

$$\lim_{i\to+\infty}\left(\frac{1}{s'_i} - \frac{1}{s'_{i+1}}\right) = \lim_{i\to+\infty}\left(\frac{1}{s''_i} - \frac{1}{s''_{i+1}}\right) = +\infty. \tag{4.26}$$

Now, we ensure that there exists i_0 such that

$$\frac{p}{s''_{i_0}} < \frac{2}{N} \quad \text{or} \quad \frac{q}{s'_{i_0}} < \frac{2}{N}. \tag{4.27}$$

On the contrary, that is, $\frac{p}{s''_i} \geq \frac{2}{N}$ and $\frac{q}{s'_i} \geq \frac{2}{N}$ for all $i \geq 1$. Then, by (4.24), we see that $s'_i > 0$, $s''_i > 0$ for all $i \geq 1$ and hence, by (4.25),

$$q < s'_1 < \cdots < s'_i < \cdots, \quad p < s''_1 < \cdots < s''_i < \cdots$$

which contradicts (4.26).

Let i_0 be the smallest number satisfying (4.27). Notice that $i_0 \geq 2$. Without loss of generality, we assume that

$$\frac{p}{s''_{i_0}} < \frac{2}{N}, \ \frac{p}{s''_i} \geq \frac{2}{N} \ \text{ for any } 1 \leq i \leq i_0 - 1, \quad \frac{q}{s'_i} \geq \frac{2}{N} \ \text{ for any } 1 \leq i \leq i_0. \tag{4.28}$$

It then follows from (4.24) that

$$s'_i > 0 \ \text{ for any } 1 \leq i \leq i_0, \quad s''_i > 0 \ \text{ for any } 1 \leq i \leq i_0 + 1,$$

which together with (4.25) leads to

$$q < \cdots < s'_{i_0-1} < s'_{i_0}, \quad p < \cdots < s''_{i_0} < s''_{i_0+1}.$$

Now, from (4.24), we have, for all $i \geq 2$,

$$\frac{N}{2}\left(\frac{p}{s''_{i-1}} - \frac{1}{s'_i}\right) = 1 - \frac{N}{2}\eta = \frac{N}{2}\left(\frac{q}{s'_{i-1}} - \frac{1}{s''_i}\right).$$

Now, let us deal with the boundedness of $(u(t,\cdot), v(t,\cdot))$ in $L^{s'_i}(\mathbb{R}^N) \times L^{s''_i}(\mathbb{R}^N)$. Using the Hölder inequality, Lemmas 2.1 and 2.2, it follows from (3.1)–(3.2), inductively, that

$$\begin{aligned} \|u(t,\cdot)\|_{s'_i} &\leq \left\|\widetilde{E}_{\gamma_1,1}(t)u_0\right\|_{s'_i} + t\left\|\widetilde{E}_{\gamma_2,2}(t)u_1\right\|_{s'_i} \\ &\quad + C\int_0^t (t-\tau)^{\gamma_1 - 1 - \frac{N}{2}\gamma_1\left(\frac{p}{s''_{i-1}} - \frac{1}{s'_i}\right)} \|v(\tau,\cdot)\|^p_{s''_{i-1}}\, d\tau \\ &\leq C\|u_0\|_{s'_i} + t\|u_1\|_{s'_i} + C\int_0^t (t-\tau)^{\gamma_1 - 1 - \gamma_1\left(1 - \frac{N}{2}\eta\right)} \|v(\tau,\cdot)\|^p_{s''_{i-1}}\, d\tau \end{aligned} \tag{4.29}$$

for any $2 \le i \le i_0$, $t \in (0, T_{\max})$ and

$$\begin{aligned} \|v(t,\cdot)\|_{s''_i} &\le \big\|\widetilde{E}_{\gamma_2,1}(t)v_0\big\|_{s''_i} + t\big\|\widetilde{E}_{\gamma_2,2}(t)v_1\big\|_{s''_i} \\ &\quad + C\int_0^t (t-\tau)^{\gamma_2-1+\frac{N}{2}\gamma_2\left(\frac{q}{s'_{i-1}}-\frac{1}{s''_i}\right)} \|u(\tau,\cdot)\|^q_{s'_{i-1}}\, d\tau \\ &\le C\|v_0\|_{s''_i} + C\|v_1\|_{s''_i} + C\int_0^t (t-\tau)^{\gamma_2-1-\gamma_2\left(1-\frac{N\eta}{2}\right)} \|u(\tau,\cdot)\|^q_{s'_{i-1}}\, d\tau \end{aligned} \tag{4.30}$$

for any $t \in (0, T_{\max})$ and for any $2 \le i \le i_0 + 1$.

It clearly follows from (4.29) and (4.30) that $u(t) \in L^{s'_i}(\mathbb{R}^N)$, $v(t) \in L^{s''_i}(\mathbb{R}^N)$:

$$\begin{cases} u(t,\cdot) \in L^{s'_i}(\mathbb{R}^N), & \|u(t,\cdot)\|_{s'_i} \le C(1+t^{a_i}), \quad 1 \le \forall\, i \le i_0,\ t \in (0, T_{\max}), \\ v(t,\cdot) \in L^{s''_i}(\mathbb{R}^N), & \|v(t,\cdot)\|_{s''_i} \le C(1+t^{b_i}), \quad 1 \le \forall\, i \le i_0+1,\ t \in (0, T_{\max}) \end{cases} \tag{4.31}$$

for some positive constants a_i, b_i. Since $\frac{Np}{2s_{i''_0}} < 1$, taking $s_2 = s''_{i_0}$, (4.19) holds. In consequence, we get $T_{\max} = +\infty$ and that (4.22) holds.

Third step: L^∞-decay estimates.

Let

$$\sigma_1 = \frac{(1-\delta)(p\gamma_2+\gamma_1)}{(pq-1)}, \quad \sigma_2 = \frac{(1-\delta)(q\gamma_1+\gamma_2)}{(pq-1)}.$$

If $\frac{pN}{2s_2} < 1$, by taking $\xi = s_2$ in (4.18) and using (4.15), we get

$$\begin{aligned} \|u(t,\cdot)\|_\infty &\le Ct^{-\frac{N\gamma_1}{2r_1}}\|u_0\|_{r_1} + Ct^{1-\frac{N\gamma_1}{2m}}\|u_1\|_m \\ &\quad + C\int_0^t (t-\tau)^{\gamma_1-1-\frac{N\gamma_1}{2}\frac{p}{s_2}}\tau^{-p\sigma_2}\, d\tau. \end{aligned} \tag{4.32}$$

From (3.3) with $pq > 2q+3$, we get $\frac{N}{2r_1} < 1$ and for any m depending on N such that $\frac{N}{2} < m < \frac{N\gamma_1}{2}$, $N \ge 2$, we infer that

$$1 - \frac{N\gamma_1}{2m} < 0 \quad \text{and} \quad \frac{N}{2m} < 1.$$

On the other hand, since

$$p\sigma_2 < 1, \quad \gamma_1 - \frac{N\gamma_1}{2}\frac{p}{s_2} - p\sigma_2 = -\frac{[\gamma_1 + \gamma_1 p\delta + (1-\delta)p\gamma_2]}{pq-1},$$

and

$$\frac{\gamma_1 + \gamma_1 p\delta + p\gamma_2(1-\delta)}{pq-1} = \frac{N\gamma_1}{2r_1}, \tag{4.33}$$

it follows from (4.32) and (4.33) that

$$\|u(t,\cdot)\|_\infty \le Ct^{-\frac{N}{2r_1}\gamma_1} + Ct^{1-\frac{N}{2m}\gamma_1} + Ct^{-\frac{[\gamma_1+\gamma_1 p\delta+(1-\delta)p\gamma_2]}{pq-1}}. \tag{4.34}$$

Therefore, we have from (4.31), (4.34) and Lemma 2.3 that

$$\|u(t,\cdot)\|_\infty \le C(1+t)^{-\min\left\{\frac{N}{2r_1}\gamma_1, \frac{N}{2m}\gamma_1-1\right\}} \quad \text{for any } t \ge 0.$$

Similarly, for $\frac{qN}{2s_1} < 1$ we find that

$$\|v(t,\cdot)\|_\infty \le C(1+t)^{-\min\left\{\frac{N}{2r_2}\gamma_2, 1-\frac{N}{2m}\gamma_2\right\}} \quad \text{for any } t \ge 0. \tag{4.35}$$

Also, (4.34) holds as $pN/(2s_2) \le qN/(2s_1)$.

In particular, if $pq > \gamma_2(q+1)+1$, we can choose $\delta > 1 - \frac{pq-1}{q(p+1)\gamma_2}$ and $\delta \approx 1 - \frac{pq-1}{q(p+1)\gamma_2}$ such that $qN/(2s_1) < 1$. Therefore, the estimates (4.34) and (4.35) hold. It is useful to note that $N \le 2$ implies $qN/(2s_1) < 1$ and $qN/(2s_1) < 1$ implies $pq > \gamma_2(q+1)+1$.

It remains to consider the following two cases:

▷ $N > 2$, $\frac{Np}{2s_2} < 1$ and $\frac{Nq}{2s_1} \ge 1$.

Let

$$\sigma' = \frac{\gamma_1 + \gamma_1 p\delta + (1-\delta)p\gamma_2}{pq-1}.$$

For positive μ such that $\mu < \min\{\sigma', \sigma_1\}$ and $q\mu < 1$; Since $N > 2$ and $q > 1$, we can choose $k > 0$ such that $k > \frac{qN}{2}$ and $q\mu + \frac{qN\gamma_2}{2k} > \gamma_2$. Since $s_1 \le qN/2$, we have $k > s_1$. By the interpolation inequality,

$$\|u(t)\|_k \le \|u(t)\|_\infty^{(k-s_1)/k} \|u(t)\|_{s_1}^{s_1/k} \le Ct^{-\sigma'(k-s_1)/k} t^{-\sigma_1 s_1/k} \quad \text{for any } t > 0.$$

Therefore, by (4.10), (4.34), we have

$$\|u(t)\|_k \le Ct^{-\mu} \quad \text{for all } t > 0.$$

Consequently, for any $t > 0$,

$$\begin{aligned} \|v(t)\|_\infty &\le \left\|\widetilde{E}_{\gamma_2,1}(t)v_0\right\|_\infty + t\left\|\widetilde{E}_{\gamma_2,2}(t)v_1\right\|_\infty + C\int_0^t (t-\tau)^{\gamma_2-1-\frac{Nq}{2k}\gamma_2} \|u(\tau)\|_k^q \, d\tau \\ &\le Ct^{-\frac{N}{2r_2}\gamma_2}\|v_0\|_{r_2} + Ct^{1-\frac{N}{2r_2}\gamma_2}\|v_1\|_{r_2} + C\int_0^t (t-\tau)^{\gamma_2-1-\frac{N\gamma_2 q}{2k}} \tau^{-q\mu}\, d\tau \\ &\le C\left(t^{-\frac{N}{2}\gamma_2} + t^{1-\frac{N}{2r_2}\gamma_2} + t^{\gamma_2-\frac{N\gamma_2 q}{2k}-q\mu}\right) \\ &\le Ct^{-\alpha}, \end{aligned} \tag{4.36}$$

where $\alpha = \min\left\{\frac{N}{2r_2}\gamma_2 - 1, -\gamma_2 + \frac{N\gamma_2 q}{2k} + q\mu\right\} > 0$,

$$k > s_1, \quad q\mu < 1, \quad k > q, \quad \gamma_2 - \frac{Nq\gamma_2}{2k} > 0, \quad \gamma_2 - \frac{Nq\gamma_2}{2k} - q\mu < 0.$$

From (4.12) and (4.36), we infer that

$$\|v(t)\|_\infty \le C(1+t)^{-\alpha} \quad \text{for all } t \ge 0.$$

In case $p = 1$ and $q^2 > 1 + 4q$, we can choose $\delta > (1 + 3q)/(p + 1)q\gamma_2 = (1 + 3q)/(2\gamma_2 q)$ and $\delta \approx (1 + 3q)/(2\gamma_2 q)$ such that $N/(2s_2) < 1$. Thus we obtain the estimate (4.34).

▷ $N > 2$, $qN/(2s_1) \geq 1$, $pN/(2s_2) \geq 1$, $q \geq p > 1$ and $\gamma_1 \leq \gamma_2$.

This case needs a careful handling and we need to restrict further the choice of δ. As $\sqrt{\frac{(p+1)q\gamma_1}{(q+1)p}} < \gamma_1 \leq \gamma_2 < 2$, $pq > 1$, it follows that $1 - \frac{pq-1}{q(p+1)\gamma_2} < 1 - \frac{(pq-1)}{p(q+1)\gamma_1^2}$. We can select δ such that

$$1 - \frac{pq-1}{q(p+1)\gamma_2} < \delta < \min\left\{\frac{N(pq-1)}{2(p+1)q}, 1 - \frac{pq-1}{p(q+1)\gamma_1^2}\right\}.$$

Then we get immediately that $p\sigma_2 > 1/\gamma_1 > 1/q\gamma_1$ and $q\sigma_1 > 1/\gamma_1 > 1/p\gamma_2$.

Further, we notice that there exist $\varepsilon \in (0, 1)$ and $\beta < 1$ close to 1 such that

$$p\sigma_2 - \varepsilon > \frac{1}{\gamma_1} > \frac{1}{q\gamma_1}, \quad q\sigma_1 - \varepsilon > \frac{1}{\gamma_2} > \frac{1}{p\gamma_2}, \quad \text{and} \quad \frac{1}{\gamma_1} < \beta - \varepsilon. \tag{4.37}$$

Letting $\eta = 2\varepsilon(1-\delta)/N$, we find the integer i_0 as in Step 2, and, without loss of generality, assume that (4.28) holds. We choose β in addition to (4.37) satisfying

$$\gamma_1 < \gamma_1 \frac{pN}{2s''_{i_0}} + \beta, \quad \text{since } 1 - \frac{1}{\gamma_1} < \frac{pN}{2s''_{i_0}}.$$

As

$$\delta < \frac{N(pq-1)}{2(p+1)q} \leq \frac{N(pq-1)}{2(q+1)p}, \quad \text{and} \quad \beta < 1,$$

we have

$$\beta + \frac{(p+1)q\delta}{(pq-1)} < 1 + \frac{N}{2}, \quad \beta + \frac{(q+1)p\delta}{(pq-1)} < 1 + \frac{N}{2}. \tag{4.38}$$

For $2 \leq i \leq i_0 - 1$, define r'_{i+1} and r''_{i+1}, inductively, as follows:

$$\frac{1}{r'_2} = \frac{1}{s'_2} + \frac{2}{N}(p\sigma_2 - \varepsilon(1-\delta)), \qquad \frac{1}{r''_2} = \frac{1}{s''_2} + \frac{2}{N}(q\sigma_1 - \varepsilon(1-\delta)),$$
$$\frac{1}{r'_{i+1}} = \frac{1}{s'_{i+1}} + \frac{2}{N}(\beta - \varepsilon(1-\delta)), \qquad \frac{1}{r''_{i+1}} = \frac{1}{s''_{i+1}} + \frac{2}{N}(\beta - \varepsilon(1-\delta)).$$

It is clear that $r'_i, r''_i > 0$ and $r'_i < s'_i$, $r''_i < s''_i$ for all $2 \leq i \leq i_0$. A simple calculation shows that $r'_i, r''_i > 1$.

As s'_i and s''_i are increasing in i for $1 \leq i \leq i_0$, we have

$$\begin{aligned}\frac{1}{r'_{i+1}} &< \frac{1}{s'_2} + \frac{2}{N}(\beta - \varepsilon(1-\delta)) \\ &= \frac{p}{s''_1} - \frac{2}{N} + \frac{2}{N}\varepsilon(1-\delta) + \frac{2}{N}(\beta - \varepsilon(1-\delta)) \\ &= \frac{2}{N}\left(\frac{p(q+1)\delta}{pq-1} + \beta - 1\right) < 1\end{aligned}$$

from (4.38), i.e., $r'_{i+1} > 1$.

Similarly, we can find that $r''_{i+1} > 1$.

From (4.22) and (4.31), we infer that there exists a positive constant $\widehat{C}$ such that, for any $0 \le t \le 1$,

$$\|u(t)\|_\infty,\ \|v(t)\|_\infty,\ \|u(t)\|_{k_1},\ \|v(t)\|_{k_2} \le \widehat{C}, \quad s'_1 \le k_1 \le s'_{i_0}, \quad s''_1 \le k_2 \le s''_{i_0}.$$

Further, since $1 - \eta N/2 = 1 - \varepsilon(1-\delta)$ and $p\sigma_2 < 1$, using (4.29), (4.30), (4.10) and (4.11), we arrive at the estimate

$$\begin{aligned}\|u(t,\cdot)\|_{s'_2} &\le \big\|\widetilde{E}_{\gamma_1,1}(t)u_0\big\|_{s'_2} + t\big\|\widetilde{E}_{\gamma_1,2}(t)u_1\big\|_{s'_2}\\ &\quad + C\int_0^t (t-\tau)^{\gamma_1-1-\gamma_1(1-\varepsilon(1-\delta))}\|u(\tau,\cdot)\|^p_{s''_1}\,d\tau,\end{aligned}$$

from which, we get

$$\begin{aligned}\|u(t,\cdot)\|_{s'_2} &\le Ct^{-\frac{N}{2}\gamma_1\left(\frac{1}{r'_2}-\frac{1}{s'_2}\right)}\|u_0\|_{r'_2} + t^{-\frac{N}{2}\gamma_1\left(\frac{1}{r'_2}-\frac{1}{s'_2}\right)}\|u_1\|_{\dot{\mathcal{H}}^{-\frac{2}{\gamma_1}}_{r'_2}}\\ &\quad + C\int_0^t (t-\tau)^{\gamma_1-1-\gamma_1(1-\varepsilon(1-\delta))}\tau^{-p\sigma_2}\,d\tau.\end{aligned}$$

Therefore

$$\begin{aligned}\|u(t,\cdot)\|_{s'_2} &\le Ct^{-\gamma_1(p\sigma_2-\varepsilon(1-\delta))}\|u_0\|_{r'_2} + t^{-\gamma_1(p\sigma_2-\varepsilon(1-\delta))}\|u_1\|_{\dot{\mathcal{H}}^{-\frac{2}{\gamma_1}}_{r'_2}}\\ &\quad + C\int_0^t (t-\tau)^{\gamma_1-1-\gamma_1(1-\varepsilon(1-\delta))}\tau^{-p\sigma_2}\,d\tau\\ &\le Ct^{-\gamma_1(p\sigma_2-\varepsilon(1-\delta))} \quad \text{for any } t>0.\end{aligned}$$

Similarly,

$$\|v(t,\cdot)\|_{s''_2} \le Ct^{-\gamma_2(q\sigma_1-\varepsilon(1-\delta))} \quad \text{for any } t>0.$$

In view of (4.37) and $\beta < 1$, thanks to Lemma 2.3, for any $t > 0$, we conclude that

$$\|u(t,\cdot)\|_{s'_2} \le Ct^{-\gamma_1\beta/q} \quad \text{and} \quad \|v(t,\cdot)\|_{s''_2} \le Ct^{-\gamma_2\beta/p}. \tag{4.39}$$

An iterative argument leads to

$$\|u(t,\cdot)\|_{s'_{i_0}} \le Ct^{-\gamma_1(\beta-\varepsilon(1-\delta))} \le Ct^{-\beta/q}, \quad \|v(t,\cdot)\|_{s''_{i_0}} \le Ct^{-\gamma_2(\beta-\varepsilon(1-\delta))} \le Ct^{-\beta/p}$$

for any $t \ge 1$. Therefore, by (4.17) and (4.18), we have

$$\begin{aligned}\|u(t,\cdot)\|_\infty &\le Ct^{-\frac{N}{2r_1}\gamma_1}\|u_0\|_{r_1} + Ct^{1-\frac{N}{2m}\gamma_1}\|u_1\|_m + C\int_0^t (t-\tau)^{\gamma_1-1-\gamma_1\frac{pN}{2s''_{i_0}}}\|v(\tau,\cdot)\|^p_{s''_{i_0}}\,d\tau\\ &\le Ct^{-\frac{N}{2r_1}\gamma_1}\|u_0\|_{r_1} + Ct^{1-\frac{N}{2m}\gamma_1}\|u_1\|_m + C\int_0^t (t-\tau)^{\gamma_1-1-\gamma_1\frac{pN}{2s''_{i_0}}}\tau^{-\beta}\,d\tau.\end{aligned}$$

So

$$\|u(t,\cdot)\|_\infty \leq C\left(t^{-\frac{N}{2r_1}\gamma_1} + t^{1-\frac{N}{2m}\gamma_1} + t^{\gamma_1-\gamma_1\frac{pN}{2s''_{i_0}}-\beta}\right) \leq Ct^{-\widetilde{\sigma}},$$

where $\widetilde{\sigma} = \min\left\{\frac{N}{2r_1}\gamma_1, \frac{N}{2m}\gamma_1 - 1, \gamma_1\frac{pN}{2s''_{i_0}} - \gamma_1 + \beta\right\} > 0$ from (4.39).

In view of the fact that $\frac{Nq}{2s_1} \geq 1$, we can make use of the arguments similar to the ones employed for the case $\frac{Np}{2s_2} < 1$ and $\frac{Nq}{2s_1} \geq 1$ to obtain $\|v(t,\cdot)\|_\infty \leq Ct^{-\widehat{\sigma}}$ for some $\widehat{\sigma} > 0$ and for every $t > 0$. This completes the proof. □

Remark 4.1. In the particular case: $N > 2$, $qN/(2s_1) \geq 1$, $pN/(2s_2) \geq 1$, $q > p = 1$ and $q \leq 3$, using the above method, we obtain

$$\|u(t,\cdot)\|_\infty \leq Ct^{-\widetilde{\sigma}} \quad \text{for any } t > 0,$$

where $\widetilde{\sigma} = \min\left\{\frac{N}{2}\gamma_1, \frac{N}{2m}\gamma_1 - 1, \frac{pN}{2s''_{i_0}}\gamma_1 - \gamma_1 + \gamma_2(\beta - \varepsilon(1-\delta))\right\}$. Here, $\varepsilon > 0$ can be arbitrarily small, and β can be arbitrarily close to 1. However, since s''_{i_0} depends on ε and s''_{i_0} is decreasing in ε, it is not clear that $\widetilde{\sigma}$ positive.

Proof of Theorem 3.6. The proof proceeds by contradiction. Suppose that (u,v) is a mild solution of (1.1) which exists globally in time. Set

$$\varphi(t,x) = \varphi_1(x)\varphi_2(t),$$

where $\varphi_1(x) = \Phi^l\left(\frac{|x|}{T^\lambda}\right)$ with $\Phi \in C_0^\infty(\mathbb{R})$, $0 \leq \Phi(z) \leq 1$, that satisfies

$$\Phi(z) = \begin{cases} 1 & \text{if } |z| \leq 1, \\ 0 & \text{if } |z| > 2 \end{cases} \quad \text{and} \quad \varphi_2(t) = \begin{cases} \left(1 - \frac{t}{T}\right)^l & \text{if } t \leq T, \\ 0 & \text{if } t > T, \end{cases}$$

where $l > \max\left\{1, \frac{q}{q-1}\gamma_1 - 1, \frac{p}{p-1}\gamma_2 - 1\right\}$ and $\lambda > 0$ to be determined later.

We set $Q_T := R^N \times [0,T]$. From Definition 3.4, we have

$$\begin{aligned}
&\int_{Q_T} uD^{\gamma_1}_{t|T}\varphi(t,x)\,dxdt - \int_{Q_T} u\Delta\varphi(t,x)\,dxdt \\
&= \int_{\mathbb{R}^N} u_0(x)(D^{\gamma_1-1}_{t|T}\varphi)(0,\cdot)\,dx + \int_{Q_T} u_1(x)D^{\gamma_1-1}_{t|T}\varphi(t,x)\,dxdt + \int_{Q_T} |v(t,x)|^p\varphi(t,x)\,dxdt, \\
&\int_{Q_T} vD^{\gamma_2}_{t|T}\varphi(t,x)\,dxdt - \int_{Q_T} v\Delta\varphi(t,x)\,dxdt \\
&= \int_{\mathbb{R}^N} v_0(x)(D^{\gamma_2-1}_{t|T}\varphi)(0,\cdot)\,dx + \int_{Q_T} v_1(x)D^{\gamma_2-1}_{t|T}\varphi(t,x)\,dxdt + \int_{Q_T} |u(t,x)|^q\varphi(t,x)\,dxdt.
\end{aligned}$$

On the other hand, we have from the definition of φ that

$$\begin{aligned}
&\int_{Q_T} u\varphi_1(x) D^{\gamma_1}_{t|T}\varphi_2(t)\,dxdt - \int_{Q_T} u\varphi_2(t)\Delta\varphi_1(x)\,dxdt \\
&= \int_{\mathbb{R}^N} u_0(x)\varphi_1(x)(D^{\gamma_1-1}_{t|T}\varphi_2)(0,\cdot)\,dx + \int_{Q_T} u_1\varphi_1(x) D^{\gamma_1-1}_{t|T}\varphi_2(t)\,dxdt \\
&+ \int_{Q_T} |v(t,x)|^p \varphi_1(x)\varphi_2(t)\,dxdt,
\end{aligned} \tag{4.40}$$

and

$$\begin{aligned}
&\int_{Q_T} v\varphi_1(x) D^{\gamma_2}_{t|T}\varphi_2(t)\,dxdt - \int_{Q_T} v\varphi_2(t)\Delta\varphi_1(x)\,dxdt \\
&= \int_{\mathbb{R}^N} v_0(x)\varphi_1(x)(D^{\gamma_2-1}_{t|T}\varphi_2)(0,\cdot)\,dx + \int_{Q_T} v_1(x)\varphi_1(x) D^{\gamma_2-1}_{t|T}\varphi_2(t)\,dxdt \\
&+ \int_{Q_T} |u(t,x)|^q \varphi_1(x)\varphi_2(t)\,dxdt.
\end{aligned}$$

Applying Hölder's inequality with exponents q and $q' = \frac{q}{q-1}$ to the right-hand side of (4.40), we get

$$\begin{aligned}
\int_{Q_T} u\varphi_1(x) D^{\gamma_1}_{t|T}\varphi_2(t)\,dxdt &= \int_{Q_T} u|\varphi_2(t)|^{\frac{1}{q}} |\varphi_1(x)|^{1-\frac{1}{q}+\frac{1}{q}} |\varphi_2(t)|^{-\frac{1}{q}} D^{\gamma_1}_{t|T}\varphi_2(t)\,dxdt \\
&\le \mathcal{I}^{\frac{1}{q}} \widetilde{\mathcal{A}},
\end{aligned}$$

where we have set

$$\mathcal{I} := \int_{Q_T} |u|^q \varphi_1(x)\varphi_2\,dxdt,$$

$$\widetilde{\mathcal{A}} := \left(\int_{Q_T} \left|D^{\gamma_1}_{t|T}\varphi_2(t)\right|^{q'} |\varphi_2(t)|^{-\frac{q'}{q}} |\varphi_1(x)|^{\left(1-\frac{1}{q}\right)q'}\,dxdt\right)^{\frac{1}{q'}},$$

$$\begin{aligned}
&\int_{Q_T} u\Delta\varphi_1(x)\varphi_2(t)\,dxdt \\
&\le \mathcal{I}^{\frac{1}{q}} \left(\int_{Q_T} |\Delta\varphi_1(x)|^{q'} |\varphi_1(x)|^{-\frac{q'}{q}} |\varphi_2(t)|^{\left(1-\frac{1}{q}\right)q'}\,dxdt\right)^{\frac{1}{q'}} \\
&\le C\mathcal{I}^{\frac{1}{q}} \left(\int_{\operatorname{supp}(\Delta\varphi_1)} \varphi_1^{1-q'}(x)|\Delta\varphi_1(x)|^{q'} |\varphi_1^{l}(x)|^{-\frac{q'}{q}}\,dx \int_0^T |\varphi_2(t)|^{\left(1-\frac{1}{q}\right)q'}\,dt\right)^{\frac{1}{q'}}.
\end{aligned}$$

Collecting the above estimates, we obtain

$$\begin{aligned}
&CT^{(1-\gamma_1)} \int_{\mathbb{R}^N} u_0(x)\varphi_1(x)\,dx + CT^{2-\gamma_1} \int_{\mathbb{R}^N} u_1(x)\varphi_1(x)\,dx + \mathcal{J} \\
&\le \mathcal{I}^{\frac{1}{q}} \widetilde{\mathcal{A}} + \mathcal{I}^{\frac{1}{q}} \left(\int_{\mathbb{R}^N} |\Delta\varphi_1(x)|^{q'} |\varphi_1(x)|^{-\frac{q'}{q}}\,dx \int_0^T |\varphi_2(t)|^{\left(1-\frac{1}{q}\right)q'}\,dt\right)^{\frac{1}{q'}},
\end{aligned} \tag{4.41}$$

where we have set

$$\mathcal{J} := \int_{Q_T} |v|^p \varphi_1(x)\varphi_2(t)\, dxdt.$$

Similarly, we obtain

$$\begin{aligned}
&\mathcal{I} + T^{(1-\gamma_2)} \int_{\mathbb{R}^N} v_0\varphi_1(x)\, dx + CT^{2-\gamma_2} \int_{\mathbb{R}^N} v_1(x)\varphi_1(x)\, dx \\
&\le \mathcal{J}^{\frac{1}{p}} \left(\int_{Q_T} |D^{\gamma_2}_{t|T}\varphi_2(t)|^{p'} |\varphi_2(t)|^{-\frac{p'}{p}} |\varphi_1(x)|^{\left(1-\frac{1}{p}\right)p'}\, dxdt \right)^{\frac{1}{p'}} \\
&\quad + \mathcal{J}^{\frac{1}{p}} \left(\int_{\mathbb{R}^N} |\Delta\varphi_1(x)|^{p'} |\varphi_1(x)|^{-\frac{p'}{p}}\, dx \int_0^T |\varphi_2(t)|^{\left(1-\frac{1}{p}\right)p'}\, dt \right)^{\frac{1}{p'}},
\end{aligned} \tag{4.42}$$

where $pp' = p + p'$. Consequently,

$$\mathcal{J} + CT^{1-\gamma_1} \int_{\mathbb{R}^N} u_0(x)\varphi_1(x)\, dx + CT^{2-\gamma_1} \int_{\mathbb{R}^N} u_1(x)\varphi_1(x)\, dx \le \mathcal{A}\mathcal{I}^{\frac{1}{q}},$$

and

$$\mathcal{I} + CT^{1-\gamma_2} \int_{\mathbb{R}^N} v_0(x)\varphi_1(x)\, dx + CT^{2-\gamma_2} \int_{\mathbb{R}^N} v_1(x)\varphi_1(x)\, dx \le \mathcal{B}\mathcal{J}^{\frac{1}{p}}$$

with

$$\begin{aligned}
\mathcal{A} &= \left(\int_{Q_T} |D^{\gamma_1}_{t|T}\varphi_2(t)|^{q'} |\varphi_2(t)|^{-\frac{q'}{q}} |\varphi_1(x)|^{\left(1-\frac{1}{q}\right)q'}\, dxdt \right)^{\frac{1}{q'}} \\
&\quad + \left(\int_{Q_T} |\Delta\varphi_1(x)|^{q'} |\varphi_1(x)|^{-\frac{q'}{q}} |\varphi_2(t)|^{\left(1-\frac{1}{q}\right)q'}\, dxdt \right)^{\frac{1}{q'}} \\
&\le CT^{(-q'\gamma_1+1+N\lambda)\frac{1}{q'}} + CT^{(-2\lambda q'+1+N\lambda)\frac{1}{q'}},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B} &= \left(\int_{Q_T} |D^{\gamma_2}_{t|T}\varphi_2(t)|^{p'} |\varphi_2(t)|^{-\frac{p'}{p}} |\varphi_1(x)|^{\left(1-\frac{1}{p}\right)p'}\, dxdt \right)^{\frac{1}{p'}} \\
&\quad + \left(\int_{Q_T} |\Delta\varphi_1(x)|^{p'} |\varphi_1^l(x)|^{-\frac{p'}{p}} |\varphi_2(t)|^{\left(1-\frac{1}{p}\right)p'}\, dxdt \right)^{\frac{1}{p'}} \\
&\le CT^{(-\gamma_2 p'+1+N\lambda)\frac{1}{p'}} + CT^{(-2\lambda p'+1+N\lambda)\frac{1}{p'}}.
\end{aligned}$$

Using inequalities (4.41) and (4.42), we can write

$$\mathcal{J} + CT^{2-\gamma_1} \int_{\mathbb{R}^N} u_1(x)\varphi_1(x)\, dx \le \mathcal{A}\,\mathcal{B}^{\frac{1}{q}}\mathcal{J}^{\frac{1}{pq}},$$

and

$$\mathcal{I} + CT^{2-\gamma_2} \int_{\mathbb{R}^N} v_1(x)\varphi_1(x)\, dx \le \mathcal{B}\,\mathcal{A}^{\frac{1}{p}}\mathcal{I}^{\frac{1}{pq}}.$$

Now, applying Young's inequality to the right hand side of the above estimates, we get

$$(pq-1)\mathcal{J}+CpqT^{2-\gamma_1}\int_{\mathbb{R}^N}u_1(x)\varphi_1(x)\,dx\le(pq-1)(\mathcal{A}\mathcal{B}^{\frac{1}{q}})^{\frac{pq}{pq-1}},$$

and

$$(pq-1)\mathcal{I}+CpqT^{2-\gamma_2}\int_{\mathbb{R}^N}v_1(x)\varphi_1(x)\,dx\le(pq-1)(\mathcal{B}\mathcal{A}^{\frac{1}{p}})^{\frac{pq}{pq-1}}.$$

At this stage, we set $x=T^{\lambda}y$, $t=T\tau$, with $\lambda>0$ to be chosen later. Then we have

$$\begin{aligned}\mathcal{A}\mathcal{B}^{\frac{1}{q}}\le C&\left(T^{\left(-\gamma_1q'+1+N\lambda\right)\frac{1}{q'}}+T^{\left(-2\lambda q'+1+N\lambda\right)\frac{1}{q'}}\right)\\&\times\left(T^{\left(-\gamma_2p'+1+N\lambda\right)\frac{1}{p'q}}+T^{\left(-2\lambda p'+1+N\lambda\right)\frac{1}{p'q}}\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}\mathcal{A}^{\frac{1}{p}}\le C&\left(T^{\left(-\gamma_2p'+1+N\lambda\right)\frac{1}{p'}}+T^{\left(-2\lambda p'+1+N\lambda\right)\frac{1}{p'}}\right)\\&\times\left(T^{\left(-q'\gamma_1+1+N\lambda\right)\frac{1}{q'}}+T^{\left(-2\lambda q'+1+N\lambda\right)\frac{1}{q'}}\right)^{\frac{1}{p}}.\end{aligned}$$

We choose $\lambda=\frac{\gamma_1}{2}$ so that $\big(-q'\gamma_1+1+N\lambda\big)\frac{1}{q'}=\big(-2\lambda q'+1+N\lambda\big)\frac{1}{q'}\big)$. Therefore, we have

$$\int_{\mathbb{R}^N}u_1(x)\varphi_1(x)\,dx\le CT^{\delta_1},\tag{4.43}$$

and

$$\int_{\mathbb{R}^N}v_1(x)\varphi_1(x)\,dx\le CT^{\delta_2},\tag{4.44}$$

where

$$\begin{aligned}\delta_1=\max\bigg\{&\bigg[\Big(-q'\gamma_1+1+\frac{\gamma_1}{2}N\Big)\frac{1}{q'}+\Big(-p'\gamma_2+1+N\frac{\gamma_1}{2}\Big)\frac{1}{p'q}\bigg]\frac{pq}{pq-1}+\gamma_1-2,\\&\bigg[\Big(-q'\gamma_1+1+\frac{\gamma_1}{2}N\Big)\frac{1}{q'}+\Big(-p'\gamma_1+1+N\frac{\gamma_1}{2}\Big)\frac{1}{qp'}\bigg]\frac{pq}{pq-1}+\gamma_1-2\bigg\},\end{aligned}$$

and

$$\begin{aligned}\delta_2=\max\bigg\{&\bigg[\Big(-\gamma_2p'+1+N\frac{\gamma_1}{2}\Big)\frac{1}{p'}+\Big(-q'\gamma_1+1+N\frac{\gamma_1}{2}\Big)\frac{1}{pq'}\bigg]\frac{pq}{pq-1}+\gamma_2-2,\\&\bigg[\Big(-\gamma_1p'+1+N\frac{\gamma_1}{2}\Big)\frac{1}{p'}+\Big(-q'\gamma_1+1+N\frac{\gamma_1}{2}\Big)\frac{1}{pq'}\bigg]\frac{pq}{pq-1}+\gamma_2-2\bigg\}.\end{aligned}$$

The condition (3.3) leads to either $\delta_1<0$ or $\delta_2<0$. Then, as $T\to\infty$, the right-hand side of (4.43) (resp. (4.44)) tends to zero and the left-hand side converges to $\int_{\mathbb{R}^N}u_1(x)\,dx>0$ (resp. $\int_{\mathbb{R}^N}v_1(x)\,dx>0$), which is contradiction.

We repeat the same argument with $\lambda=\frac{\gamma_2}{2}$ to conclude the proof of Theorem 3.6. □

Remark 4.2. In the single equation case, when $\gamma_1 = \gamma_2 = \gamma$, we recover the case studied by [3]. In the system case, when $\gamma_1, \gamma_2 \to 1$ with $(u_1, v_1) \equiv (0, 0)$, we recover the classical Fujita exponent that studied by [13]. Moreover, letting $\gamma_1, \gamma_2 \to 2$, we have shown in particular for $q \geq p > 1$ and $N = 3$ blow-up in finite time result holds for $\frac{3}{2} < \frac{q+1}{pq-1}$, which is subcritical exponent for the classical system of wave equations (see [8, 9]).

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