TAIWANESE JOURNAL OF MATHEMATICS
Vol. 26, No. 1, pp. 33–47, February 2022
DOI: 10.11650/tjm/210802

A Generalization of Piatetski–Shapiro Sequences

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Abstract. We consider a generalization of Piatetski–Shapiro sequences in the sense of Beatty sequences, which is of the form $(\lfloor \alpha n^c + \beta \rfloor)_{n=1}^{\infty}$ with real numbers $\alpha \geq 1$, $c > 1$ and β. We show there are infinitely many primes in the generalized Piatetski–Shapiro sequence with $c \in (1, 14/13)$. Moreover, we prove there are infinitely many Carmichael numbers composed entirely of the primes from the generalized Piatetski–Shapiro sequences with $c \in (1, 64/63)$.

1. Introduction

The *Piatetski–Shapiro sequences* are sequences of the form

$$\mathcal{N}^{(c)} := (\lfloor n^c \rfloor)_{n=1}^{\infty}, \quad c > 1,\ c \notin \mathbb{N}.$$

Such sequences have been named in honor of Piatetski–Shapiro, who proved [30] that $\mathcal{N}^{(c)}$ contains infinitely many primes if $c \in (1, 12/11)$. More precisely, for such c he showed that the counting function

$$\pi^{(c)}(x) := \#\{\text{prime } p \leq x : p \in \mathcal{N}^{(c)}\}$$

satisfies the asymptotic relation

$$\pi^{(c)}(x) \sim \frac{x^{1/c}}{\log x} \quad \text{as } x \to \infty.$$

The range for c in which it is known that $\mathcal{N}^{(c)}$ contains infinitely many primes has been extended many times over the years [9, 18–23, 26] and the above formula is currently known to hold for all $c \in (1, 2817/2426)$ thanks to Rivat and Sargos [31]. Rivat and Wu [32] also showed that there are infinitely many Piatetski–Shapiro primes for $c \in (1, 243/205)$. The same result is expected to hold for all larger values of c. We remark that if $c \in (0, 1)$ then $\mathcal{N}^{(c)}$ contains all natural numbers, hence all primes in particular. More recent research related to Piatetski–Shapiro sequences can be found in [1, 4–7, 15, 24, 25, 27–29, 33] and references therein.

Received April 1, 2021; Accepted August 11, 2021.
Communicated by Liang-Chung Hsia.
2020 *Mathematics Subject Classification.* 11B83, 11N13, 11L07.
Key words and phrases. Piatetski–Shapiro sequences, arithmetic progression, exponential sums.
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For fixed real numbers α, β the associated *non-homogeneous Beatty sequence* is the sequence of integers defined by

$$\mathcal{B}_{\alpha,\beta} := (\lfloor \alpha n + \beta \rfloor)_{n=1}^{\infty},$$

where $\lfloor t \rfloor$ denotes the integer part of any $t \in \mathbb{R}$. Such sequences are also called *generalized arithmetic progressions*. If α is irrational, it follows from a classical exponential sum estimate of Vinogradov [35] that $\mathcal{B}_{\alpha,\beta}$ contains infinitely many prime numbers; in fact, one has the asymptotic relation

$$\#\{\text{prime } p \le x : p \in \mathcal{B}_{\alpha,\beta}\} \sim \alpha^{-1}\pi(x), \quad x \to \infty,$$

where $\pi(x)$ is the prime counting function. More recent literatures related to prime numbers and Beatty sequences can be found in [10–12, 15–17].

It is interesting to generalize the Piatetski–Shapiro sequences in the sense of Beatty sequences, since both Piatetski–Shapiro sequences and Beatty sequences produce infinitely many primes. Let $\alpha \ge 1$ and β be real numbers. We investigate the following generalized Piatetski–Shapiro sequences

$$\mathcal{N}_{\alpha,\beta}^{(c)} = (\lfloor \alpha n^c + \beta \rfloor)_{n=1}^{\infty}.$$

Note that the special case $\mathcal{N}_{1,0}^{(c)}$ is the normal Piatetski–Shapiro sequences. Let

$$\pi(x; d, a) := \#\{p \le x : p \equiv a \bmod d\}$$

and

$$\pi_{\alpha,\beta,c}(x; d, a) := \#\{p \le x : p \in \mathcal{N}_{\alpha,\beta}^{(c)} \text{ and } p \equiv a \bmod d\}.$$

We prove the following theorem.

Theorem 1.1. *Let $\alpha \ge 1$ and β be real numbers. Let $c \in (1, 14/13)$.*

$$\begin{aligned}\pi_{\alpha,\beta,c}(x; d, a) = {} & \alpha^{-1/c} c^{-1} x^{1/c-1} \pi(x; d, a) \\ & + \alpha^{-1/c} c^{-1} (1 - c^{-1}) \int_2^x u^{1/c-2} \pi(u; d, a)\, du + O\big(x^{3/(5c)+13/35+\varepsilon}\big).\end{aligned}$$

Note that

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

We conclude that

Corollary 1.2. *Let $\alpha \ge 1$ and β be real numbers. Let $c \in (1, 14/13)$. Let*

$$\pi_{\alpha,\beta,c}(x) := \#\{p \le x : p \in \mathcal{N}_{\alpha,\beta}^{(c)}\}.$$

Then

$$\pi_{\alpha,\beta,c}(x) = \frac{x^{1/c}}{\alpha^{1/c} \log x} + O\left(\frac{x^{1/c}}{\log^2 x}\right). \tag{1.1}$$

We clarify that (1.1) can be proved by a similar argument to the proof of Piatetski–Shapiro prime number theorem. The key point is to estimate

$$\sum_{1\le h\le H}\left|\sum_{N\le n\le N_1}\Lambda(n)\mathbf{e}(\theta h n^{\gamma})\right|,$$

where H, N, N_1, θ, γ are positive numbers such that $H\ge 1$, $N_1\le 2N$, $\theta<1$, $\gamma<1$. Since the function $\theta h n^{\gamma}$ is smooth enough to apply the method of exponent pairs, the constant θ does not play a big role in the estimation of exponential sums. We expect that all the methods for Piatetski–Shapiro prime number theorem should work for estimating $\pi_{\alpha,\beta,c}(x)$. However, in this paper, we mean to give a first result. For the sake of simplicity, we do not give more discussion to the prime counting function $\pi_{\alpha,\beta,c}(x)$.

In the end, we prove a theorem related to *Carmichael numbers*, which are the composite natural numbers N with the property that $N\mid(a^N-a)$ for every integer a. In 1994, Alford, Granville and Pomerance [2] proved there exist infinitely many Carmichael numbers. Their proof relies on the arithmetic properties of primes. Since we show the arithmetic properties of the primes in $\mathcal{N}^{(c)}_{\alpha,\beta}$, we are able to prove the following result by a similar method of [2].

Theorem 1.3. *For every $c\in(1,64/63)$, there are infinitely many Carmichael numbers composed entirely of the primes from the set $\mathcal{N}^{(c)}_{\alpha,\beta}$.*

2. Preliminaries

2.1. Notation

We denote by $\lfloor t\rfloor$ and $\{t\}$ the integer part and the fractional part of t, respectively. As is customary, we put

$$\mathbf{e}(t):=e^{2\pi i t}.$$

We make considerable use of the sawtooth function defined by

$$\psi(t):=t-\lfloor t\rfloor-\frac{1}{2}=\{t\}-\frac{1}{2},\quad t\in\mathbb{R}.$$

The letter p always denotes a prime. For the generalized Piatetski–Shapiro sequence $(\lfloor\alpha n^c+\beta\rfloor)_{n=1}^{\infty}$, we denote $\gamma:=c^{-1}$ and $\theta:=\alpha^{-\gamma}$. We use notation of the form $m\sim M$ as an abbreviation for $M<m\le 2M$.

Throughout the paper, implied constants in symbols O, $\ll$ and $\gg$ may depend (where obvious) on the parameters α, c, ε but are absolute otherwise. For given functions F and G, the notations $F\ll G$, $G\gg F$ and $F=O(G)$ are all equivalent to the statement that the inequality $|F|\le C|G|$ holds with some constant $C>0$.

2.2. Technical lemmas

We need the following well-known approximation of Vaaler [34].

Lemma 2.1. *For any $H \geq 1$ there are numbers a_h, b_h such that*

$$\left|\psi(t) - \sum_{0<|h|\leq H} a_h\,\mathbf{e}(th)\right| \leq \sum_{|h|\leq H} b_h\,\mathbf{e}(th), \quad a_h \ll \frac{1}{|h|},\ b_h \ll \frac{1}{H}.$$

Next, we recall the following identity for the von Mangoldt function Λ, which is due to Vaughan.

Lemma 2.2. *Let $U, V \geq 1$ be real parameters. For any $n > U$ we have*

$$\Lambda(n) = -\sum_{k|n} a(k) + \sum_{\substack{cd=n\\ d\leq V}} (\log c)\mu(d) - \sum_{\substack{kc=n\\ k>1\\ c>U}} \Lambda(c)b(k),$$

where

$$a(k) = \sum_{\substack{cd=k\\ c\leq U\\ d\leq V}} \Lambda(c)\mu(d) \quad \text{and} \quad b(k) = \sum_{\substack{d|k\\ d\leq V}} \mu(d).$$

Proof. See Davenport [13, p. 139]. □

The Vaughan's identity gives a decomposition for sums of the form

$$S(f) := \sum_{X<n\leq X'} \Lambda(n)f(n)$$

where f is any complex-valued function, and $X' \sim X$. Let $N_1 \leq 2N$. A Type I sum is a sum of the form

$$S_I(K,L) := \sum_{\substack{k\sim K\\ N<kl\leq N_1}} \sum_{l\sim L} a_k f(kl)$$

where $|a_k| \ll k^{\varepsilon}$ for every $\varepsilon > 0$. A Type II sum is a sum of the form

$$S_{II}(K,L) := \sum_{\substack{k\sim K\\ N<kl\leq N_1}} \sum_{l\sim L} a_k b_l f(kl)$$

where $|a_k| \ll k^{\varepsilon}$ and $|b_l| \ll l^{\varepsilon}$ for every $\varepsilon > 0$.

Lemma 2.3. *Suppose that every Type I sum with $K \ll X^{3/7}$ satisfies the bound*

$$S_I \ll B(X)$$

and every Type II sum with $X^{3/7} \ll K \ll X^{1/2}$ satisfies the bound

$$S_{II} \ll B(X).$$

Then

$$S(f) \ll B(X) X^{\varepsilon}.$$

Proof. The lemma can be deduced by the Vaughan's identity (Lemma 2.2) since $S(f)$ can be written as a linear decomposition of Type I and Type II sums. A detailed proof of a similar lemma can be found in [3, Lemma 2]. □

Lemma 2.4. *For a bounded function g and $N' \sim N$ we have*

$$\sum_{N<p\le N'} g(p) \ll \frac{1}{\log N} \max_{N_1 \le 2N} \left| \sum_{N<n\le N_1} \Lambda(n) g(n) \right| + N^{1/2}.$$

Proof. See [14, p. 48]. □

Lemma 2.5. *Let*

$$L(Q) := \sum_{j=1}^{J} C_j Q^{c_j} + \sum_{k=1}^{K} D_k Q^{-d_k},$$

where $C_j, c_j, D_k, d_k > 0$. Then

(1) *For any $Q \ge Q' > 0$ there exists $Q_1 \in [Q', Q]$ such that*

$$L(Q_1) \ll \sum_{j=1}^{J} \sum_{k=1}^{K} (C_j^{d_k} D_k^{c_j})^{1/(c_j+d_k)} + \sum_{j=1}^{J} C_j (Q')^{c_j} + \sum_{k=1}^{K} D_k Q^{-d_k}.$$

(2) *For any $Q > 0$ there exists $Q_1 \in (0, Q]$ such that*

$$L(Q_1) \ll \sum_{j=1}^{J} \sum_{k=1}^{K} (C_j^{d_k} D_k^{c_j})^{1/(c_j+d_k)} + \sum_{k=1}^{K} D_k Q^{-d_k}.$$

Proof. The proof of the first assertion is in [14, Lemma 2.4]. The proof of the second assertion is similar. □

Lemma 2.6. *Let f be twice continuously differentiable on a subinterval $\mathcal{I}$ of $(N, 2N]$. Suppose that for some $\lambda > 0$, the inequalities*

$$\lambda \ll |f''(t)| \ll \lambda, \quad t \in \mathcal{I}$$

hold, where the implied constants are independent of f and λ. Then

$$\sum_{n \in \mathcal{I}} \mathbf{e}(f(n)) \ll N \lambda^{1/2} + \lambda^{-1/2}.$$

Proof. See [14, Theorem 2.2]. □

Lemma 2.7. *Suppose $|a_k| \le 1$ for all $k \sim K$. Fix $\gamma \in (0,1)$, $\mu, \rho \in \mathbb{R}$, $\mu \neq 0$ and $m \in \mathbb{N}$. Then for any $K \ll N^{3/7}$ the Type I sum*

$$S_I = \sum_{\substack{k\sim K \\ N<kl\le N_1}} \sum_{l\sim L} a_k \, \mathbf{e}(\mu m k^\gamma l^\gamma + \rho k l)$$

satisfies the bound

$$S_I \ll m^{1/2} N^{3/7+\gamma/2} + m^{-1/2} N^{1-\gamma/2}.$$

Proof. Writing $f(l) = \mu m k^\gamma l^\gamma + \rho k l$ we see that

$$|f''(l)| = |\mu m \gamma(\gamma-1) k^\gamma l^{\gamma-2}| \asymp m K^\gamma L^{\gamma-2}.$$

Using Lemma 2.6 it follows that

$$\sum_{\substack{l\sim L \\ N<kl\le N_1}} a_k \, \mathbf{e}(\mu m k^\gamma l^\gamma + \rho k l) \ll m^{1/2} K^{\gamma/2} L^{\gamma/2} + m^{-1/2} K^{-\gamma/2} L^{1-\gamma/2}.$$

Since $|a_k| \le 1$ for all $k \sim K$ we see that

$$\begin{aligned} S_I &= \sum_{k\sim K} \sum_{\substack{l\sim L \\ N<kl\le N_1}} a_k \, \mathbf{e}(\mu m k^\gamma l^\gamma + \rho k l) \\ &\ll m^{1/2} K^{1+\gamma/2} L^{\gamma/2} + m^{-1/2} K^{1-\gamma/2} L^{1-\gamma/2} \\ &\ll m^{1/2} N^{3/7+\gamma/2} + m^{-1/2} N^{1-\gamma/2}. \end{aligned}$$

□

We need the following lemma to bound the Type II sum.

Lemma 2.8. *Let $1 < Q < L$. If f is a function of the form $f(n) = \mathbf{e}(g(n))$, then any Type II sum satisfies*

$$|S_{II}|^2 \ll N^2 Q^{-1} + N Q^{-1} \sum_{0<|q|<Q} \sum_{l\sim L} |S(q,l)|,$$

where

$$S(q,l) = \sum_{k\in\mathcal{I}(q,l)} \mathbf{e}(g(kl) - g(k(l+q)))$$

for a certain subinterval $\mathcal{I}(q,l)$ of $(K, 2K]$.

Proof. See the proof of [14, Lemma 4.13]. □

Lemma 2.9. *Suppose* $|a_k| \le 1$ *and* $|b_l| \le 1$ *for* $(k,l) \sim (K,L)$. *Fix* $\gamma \in (0,1)$, $\mu, \rho \in \mathbb{R}$, $\mu \neq 0$ *and* $m \in \mathbb{N}$. *For any* K *in the range* $N^{3/7} \ll K \ll N^{1/2}$, *the Type II sum*

$$S_{II} = \sum_{\substack{k \sim K \\ N<kl\le N_1}} \sum_{l \sim L} a_k b_l \, \mathbf{e}(\mu m k^\gamma l^\gamma + \rho k l)$$

satisfies the bound

$$S_{II} \ll m^{1/6} N^{16/21+\gamma/6} + N^{25/28} + m^{-1/4} N^{1-\gamma/4}.$$

Proof. We assume that $KL \asymp N$. By Lemma 2.8 we have

$$|S_{II}|^2 \ll K^2L^2Q^{-1} + KLQ^{-1} \sum_{l \sim L} \sum_{0<|q|\le Q} |S(q;l)|,$$

where

$$S(q;l) = \sum_{k \in I(q;l)} \mathbf{e}(f(k)) \quad \text{and} \quad f(k) = \mu k^\gamma (l^\gamma - (l+q)^\gamma) - \rho k q,$$

and each $I(q;n)$ is a certain subinterval in the set of numbers $k \sim K$. Since

$$\begin{aligned} |f''(k)| &= |\mu m \gamma (1-\gamma) k^{\gamma-2} (l^\gamma - (l+q)^\gamma)| \\ &\asymp m K^{\gamma-2} L^{\gamma-1} |q|, \end{aligned}$$

it follows from Lemma 2.6 that

$$S(q;l) \ll (mK^{\gamma-2}L^{\gamma-1}|q|)^{1/2} K + (mK^{\gamma-2}L^{\gamma-1}|q|)^{-1/2}.$$

Inserting the bound to $|S_{II}|^2$ and summing over l and q, we derive that

$$|S_{II}|^2 \ll K^2L^2Q^{-1} + m^{1/2}K^{1+\gamma/2}L^{3/2+\gamma/2}Q^{1/2} + m^{-1/2}K^{2-\gamma/2}L^{5/2-\gamma/2}Q^{-1/2}.$$

By Lemma 2.5 we have

$$\begin{aligned} |S_{II}|^2 &\ll m^{1/3}K^{4/3+\gamma/3}L^{5/3+\gamma/3} + K^{3/2}L^2 + K^2L + m^{-1/2}N^{2-\gamma/2} \\ &\ll m^{1/3}K^{-1/3}N^{5/3+\gamma/3} + K^{-1/2}N^2 + KN + m^{-1/2}N^{2-\gamma/2}. \end{aligned}$$

We have $N^{3/7} \ll K \ll N^{1/2}$, the proof is done. □

We use the following lemma, which provides a characterization of the numbers that occur in the Piatetski–Shapiro sequence $\mathcal{N}^{(c)}$.

Lemma 2.10. *A natural number* m *has the form* $\lfloor n^c \rfloor$ *if and only if* $\mathcal{X}^{(c)}(m) = 1$, *where* $\mathcal{X}^{(c)}(m) := \lfloor -m^\gamma \rfloor - \lfloor -(m+1)^\gamma \rfloor$. *Moreover,*

$$\mathcal{X}^{(c)}(m) = \gamma m^{\gamma-1} + \psi(-(m+1)^\gamma) - \psi(-m^\gamma) + O(m^{\gamma-2}).$$

In particular, for any $c \in (1, 2817/2426)$ the results of [31] yield the estimate

$$\pi^{(c)}(x) = \sum_{p \le x} \mathcal{X}^{(c)}(p) = \frac{x^\gamma}{c \log x} + O\left(\frac{x^\gamma}{\log^2 x}\right).$$

3. Proof of Theorem 1.1

Recall that $\gamma = c^{-1}$ and $\theta = \alpha^{-\gamma}$. A prime p equals $\lfloor \alpha n^c + \beta \rfloor$ if and only if

$$p \le \alpha n^c + \beta < p + 1,$$

which is equivalent to

$$\theta(p-\beta)^\gamma \le n < \theta(p+1-\beta)^\gamma,$$

except for the case that $p = \lfloor \beta \rfloor$. Then

$$\pi_{\alpha,\beta,c}(x; d, a) = \sum_{\substack{p \le x \\ p \equiv a \bmod d}} \big(\lfloor -\theta(p-\beta)^\gamma \rfloor - \lfloor -\theta(p+1-\beta)^\gamma \rfloor \big) = \textstyle\sum_1 + \sum_2 + O(1),$$

where

$$\textstyle\sum_1 := \theta\gamma \displaystyle\sum_{\substack{p \le x \\ p \equiv a \bmod d}} (p-\beta)^{\gamma-1},$$

and

$$\textstyle\sum_2 := \displaystyle\sum_{\substack{p \le x \\ p \equiv a \bmod d}} \big(\psi(-\theta(p-\beta+1)^\gamma) - \psi(-\theta(p-\beta)^\gamma) \big).$$

A partial summation gives

$$\textstyle\sum_1 = \theta\gamma x^{\gamma-1}\pi(x; d, a) + \theta\gamma(1-\gamma) \displaystyle\int_2^x u^{\gamma-2}\pi(u; d, a)\, du + O(x^{\gamma-1} + 1),$$

which is the main term. Let $N \le x$ and $N_1 \le 2N$. We estimate $\sum_2$ by considering

$$S := \sum_{\substack{N < n \le N_1 \\ n \equiv a \bmod d}} \Lambda(n) \big(\psi(-\theta(n-\beta+1)^\gamma) - \psi(-\theta(n-\beta)^\gamma) \big) = S_1 + O(S_2)$$

by Lemmas 2.1 and 2.4, where

$$S_1 := \sum_{\substack{N < n \le N_1 \\ n \equiv a \bmod d}} \Lambda(n) \sum_{0 < |h| \le H} a_h \big(\mathbf{e}(\theta h (n-\beta+1)^\gamma) - \mathbf{e}(\theta h (n-\beta)^\gamma) \big)$$

and

$$S_2 := \sum_{\substack{N < n \le N_1 \\ n \equiv a \bmod d}} \Lambda(n) \sum_{|h| \le H} b_h \big(\mathbf{e}(\theta h (n-\beta+1)^\gamma) + \mathbf{e}(\theta h (n-\beta)^\gamma) \big)$$

for some $H \ge 1$. We consider the upper bound of S_1 firstly. By partial summation (see [14, p. 48]), we have

$$S_1 \ll N^{\gamma-1} \max_{N_1 \le 2N} \sum_{1 \le h \le H} \Bigg| \sum_{\substack{N < n \le N_1 \\ n \equiv a \bmod d}} \Lambda(n)\mathbf{e}(\theta h n^\gamma) \Bigg|. \tag{3.1}$$

Note that

$$\sum_{\substack{N<n\le N_1\\ n\equiv a \bmod d}} \Lambda(n)\mathbf{e}(\theta h n^{\gamma}) = \frac{1}{d}\sum_{m=1}^{d}\sum_{N<n\le N_1}\Lambda(n)\mathbf{e}\left(\theta h n^{\gamma} + \frac{(n-a)m}{d}\right).$$

Hence we need to bound

$$T := \sum_{N<n\le N_1}\Lambda(n)\mathbf{e}\left(\theta h n^{\gamma} + \frac{nm}{d}\right).$$

We apply Lemmas 2.3, 2.7 and 2.9 with

$$(\mu, m, \rho) \to \left(\theta, h, \frac{m}{d}\right)$$

and obtain

$$TN^{-\varepsilon} \ll h^{1/2}N^{3/7+\gamma/2} + h^{1/6}N^{16/21+\gamma/6} + N^{25/28} + h^{-1/4}N^{1-\gamma/4}, \tag{3.2}$$

for ε being a small positive number.

Now we work on the upper bound of S_2. The contribution from $h=0$ is

$$2b_0 \sum_{\substack{N<n\le N_1\\ n\equiv a \bmod d}} \Lambda(n) \ll \frac{b_0 N}{\varphi(d)} \ll H^{-1}N, \tag{3.3}$$

where the function $\varphi(d)$ is the Euler's totient function and $b_h \ll H^{-1}$. Taking into account that

$$(n-\beta+1)^{\gamma} = n^{\gamma} + O(n^{\gamma-1})$$

and $\gamma - 1 < 0$, the contribution from $h \neq 0$ is

$$\ll H^{-1}\max_{N_1\le 2N}\sum_{0<h\le H}\left|\sum_{\substack{N<n\le N_1\\ n\equiv a \bmod d}}\Lambda(n)\mathbf{e}(\theta h n^{\gamma})\right|. \tag{3.4}$$

The right-hand side of (3.4) can be estimated by the same method of (3.2). Therefore, inserting (3.2) into (3.1) and (3.4), and combining with (3.3), it follows that

$$\begin{aligned} SN^{-\varepsilon} &\ll S_1 + S_2 \\ &\ll H^{3/2}N^{3\gamma/2-4/7} + H^{7/6}N^{7\gamma/6-5/21} + HN^{\gamma-3/28} + H^{3/4}N^{3\gamma/4} \\ &\quad + H^{1/2}N^{3/7+\gamma/2} + H^{1/6}N^{16/21+\gamma/6} + N^{25/28} + H^{-1/4}N^{1-\gamma/4} + H^{-1}N \end{aligned}$$

holds for any $H \ge 1$. By Lemma 2.5, we get that

$$SN^{-\varepsilon} \ll N^{3\gamma/5+13/35} + N^{7\gamma/13+3/7} + N^{\gamma/2+25/56} + N^{\gamma/3+13/21} + N^{\gamma/7+13/14} + N^{25/28}.$$

Note that $\sum_1 \ll x^{\gamma}$, so we need that $S \ll x^{\gamma-\varepsilon}$. Hence

$$\gamma > \max\left(\frac{13}{14}, \frac{25}{28}\right) = \frac{13}{14},$$

and

$$S \ll x^{3\gamma/5+13/35+\varepsilon}.$$

4. Sketch of proof of Theorem 1.3

We sketch the proof of Theorem 1.3 because the idea of the proof is close to the proof in [2], the proof of [6, Theorem 7] or the proof of Theorem 1.1. We only give the changes that are necessary for our Theorem 1.3.

We set

$$\vartheta(x;d,a) := \sum_{\substack{p\le x\\ p\equiv a \bmod d}} \log p$$

and consider a weighted counting function

$$\begin{aligned}\vartheta_{\alpha,\beta,c}(x;d,a) &:= \sum_{\substack{p\le x\\ p\in\mathcal{N}^{(c)}_{\alpha,\beta}\\ p\equiv a \bmod d}} \log p\\ &= \sum_{\substack{p\le x\\ p\equiv a \bmod d}} \left(\lfloor -\theta(p-\beta)^{\gamma}\rfloor - \lfloor -\theta(p+1-\beta)^{\gamma}\rfloor\right)\log p.\end{aligned}$$

By a similar argument as in the proof of Theorem 1.1, we conclude that

Theorem 4.1. *Let $\alpha\ge 1$ and β be real numbers. Let $c\in(1,14/13)$. Then*

$$\begin{aligned}\vartheta_{\alpha,\beta,c}(x;d,a) &= \alpha^{-1/c}\gamma x^{\gamma-1}\vartheta(x;d,a)\\ &\quad + \alpha^{-1/c}\gamma(1-\gamma)\int_2^x u^{\gamma-2}\vartheta(u;d,a)\,du + O(x^{3\gamma/5+13/35+\varepsilon}).\end{aligned}$$

The Brun–Titchmarsh theorem states that for $d<x^{1-\varepsilon}$, there is some $C>0$ such that

$$\pi(x;d,a)\le \frac{Cx}{\varphi(d)\log x}.$$

We also give a Brun–Titchmarsh bound for the primes in the generalized Piatetski–Shapiro sequences.

Corollary 4.2. *Let $\alpha\ge 1$ and β be real numbers. Let $c\in(1,14/13)$ and $A\in(0,-13/35+2\gamma/5)$. There is a number $C=C(\alpha,c,A)>0$ such that*

$$\pi_{\alpha,\beta,c}(x;d,a)\le \frac{Cx^{\gamma}}{\varphi(d)\log x}$$

if $(a,d)=1$ and $1\le d\le x^A$.

Proof. Let $\varepsilon>0$ be chosen so that

$$\max\left(2A\gamma, \frac{3}{5}\gamma+\frac{13}{35}+\varepsilon\right)\le \gamma - A-\varepsilon.$$

Then by Theorem 1.1 we have

$$\pi_{\alpha,\beta,c}(x;d,a) \ll \alpha^{-1/c}\gamma x^{\gamma-1}\pi(x;d,a) + \alpha^{-1/c}\gamma(1-\gamma)\int_2^x u^{\gamma-2}\pi(u;d,a)\,du + x^{\gamma-A-\varepsilon}$$

where the implied constant depends on c, α, A. If $1 \le d \le x^A$, then

$$x^{\gamma-A-\varepsilon} \ll \frac{x^{\gamma-A}}{\log x} \le \frac{x^{\gamma}}{\varphi(d)\log x}.$$

Applying the Brun–Titchmarsh theorem, we prove Corollary 4.2. □

The following statement analogous to [2, Theorem 2.1] and [6, Lemma 28] is important in the construction of Carmichael numbers.

Lemma 4.3. *Let $\alpha \ge 1$ and β be real numbers. Let $c \in (1, 14/13)$ and $B \in (0, -13/35 + 3\gamma/5)$. There exist numbers $\eta > 0$, x_0 and D such that for all $x \ge x_0$ there is a set $\mathcal{D}(x)$ consisting of at most D integers such that*

$$\left|\vartheta_{\alpha,\beta,c}(x;d,a) - \frac{\theta x^{\gamma}}{\varphi(d)}\right| \le \frac{\theta x^{\gamma}}{2\varphi(d)}$$

provided that

(1) *d is not divisible by any element of $\mathcal{D}(x)$;*

(2) *$1 \le d \le x^B$;*

(3) *$\gcd(a,d) = 1$.*

Every number in $\mathcal{D}(x)$ exceeds $\log x$, and all, but at most one, exceeds x^{η}.

Sketch of proof. We set

$$\vartheta_c(x;d,a) := \sum_{\substack{p \le x \\ p \in \mathcal{N}^{(c)} \\ p \equiv a \bmod d}} \log p.$$

By Theorem 26 in [6], we conclude that

$$\vartheta_{\alpha,\beta,c}(x;d,a) \sim \theta\vartheta_c(x;d,a).$$

Replacing the factor $17/39 + 7\gamma/13 + \varepsilon$ in the proof of Lemma 28 in [6] by $13/35 + 3\gamma/5 + \varepsilon$ in this case, the proof of Lemma 4.3 in this paper is a straightforward reworking of the proof of Lemma 28 in [6]. □

By Lemma 4.3, we extend [2, Theorem 3.1] to the setting of the primes in the generalized Piatetski–Shapiro sequence.

Lemma 4.4. *Let $\alpha \geq 1$ and β be real numbers. Let $c \in (1, 14/13)$ and let A, B, B_1 be positive real numbers such that $B_1 < B < A < -13/35 + 2\gamma/5$. Let $C > 0$ have the property described in Corollary 4.2. There exists a number x_2 such that if $x \geq x_2$ and L is a squarefree integer not divisible by any prime q exceeding $x^{(A-B)/2}$ and for which*

$$\sum_{\text{prime } q|L} \frac{1}{q} \leq \frac{1-A}{16C},$$

then there is a positive integer $k \leq x^{1-B}$ with $\gcd(k, L) = 1$ such that

$$\begin{aligned}&\#\{d|L : dk+1 \leq x \text{ and } p = dk+1 \text{ is a prime in } \mathcal{N}^{(c)}_{\alpha,\beta}\}\\ &\geq \frac{2^{-D-2}(x^{1-B+B_1})^{\gamma-1}}{\log x}\#\{d|L : x^{B_1} \leq d \leq x^B\},\end{aligned}$$

where D is chosen as in Lemma 4.3.

Sketch of proof. We follow the proof of [6, Lemma 29] and use the notation of [2, Theorem 3.1]. By the same argument we have

$$\pi_{\alpha,\beta,c}(dx^{1-B}; d, 1) \geq \frac{\theta}{2}\frac{(dx^{1-B})^\gamma}{\phi(d)\log x}, \quad d|L',\ 1 \leq d \leq x^B$$

and

$$\pi_{\alpha,\beta,c}(dx^{1-B}; dq, 1) \leq \frac{4\theta C}{q(1-A)}\frac{(dx^{1-B})^\gamma}{\phi(d)\log x}, \quad 1 \leq d \leq x^B$$

for every prime q dividing L'. The rest of the proof stays the same as the proof of [6, Lemma 29] by considering primes in $\mathcal{N}^{(c)}_{\alpha,\beta}$ instead of primes in $\mathcal{N}^{(c)}$. □

Let $\pi(x, y)$ be the number of those primes for which $p-1$ is free of prime factors exceeding y. Let $\mathcal{E}$ be the set of numbers E in the range $0 < E < 1$ for which

$$\pi(x, x^{1-E}) \geq x^{1+o(1)}, \quad x \to \infty,$$

where the function implied by $o(1)$ depends only on E. By a similar argument as in [6, pp. 64–66], we conclude the following statement.

Lemma 4.5. *Let $\alpha \geq 1$ and β be real numbers. Let $c \in (1, 49/48)$. Let B, B_1 be positive real numbers such that $B_1 < B < -13/35 + 2\gamma/5$. For any $E \in \mathcal{E}$ there is a number x_3 depending on c, B, B_1, E and ε, such that for any $x \geq x_1$ there are at least $x^{EB+(1-B+B_1)(\gamma-1)-\varepsilon}$ Carmichael numbers up to x composed solely of primes from $\mathcal{N}^{(c)}_{\alpha,\beta}$.*

Taking B and B_1 arbitrarily close to $-13/35 + 2\gamma/5$, Lemma 4.5 implies that there are infinitely many Carmichael numbers composed entirely of the primes from $\mathcal{N}^{(c)}_{\alpha,\beta}$ with

$$\left(-\frac{13}{35} + \frac{2}{5}\gamma\right)E + \gamma - 1 > 0.$$

Taking $E = 0.7039$ from [8], we eventually have $\gamma > 63/64$.

Acknowledgments

The authors thank the referees for their valuable comments. The authors also thank Prof. Yuan Yi and Prof. Yaming Lu for several helpful discussions. The first author is supported in part by the National Natural Science Foundation of China (No. 11901447), the China Postdoctoral Science Foundation (No. 2019M653576) and the Natural Science Foundation of Shaanxi Province (No. 2020JQ-009). The second author is supported in part by the National Natural Science Foundation of China (No. 11971381, No. 11701447, No. 11871317 and No. 11971382).

References

[1] Y. Akbal and A. M. Güloğlu, *Waring–Goldbach problem with Piatetski–Shapiro primes*, J. Théor. Nombres Bordeaux **30** (2018), no. 2, 449–467.

[2] W. R. Alford, A. Granville and C. Pomerance, *There are infinitely many Carmichael numbers*, Ann. of Math. (2) **139** (1994), no. 3, 703–722.

[3] R. C. Baker, *Sums of two relatively prime cubes*, Acta Arith. **129** (2007), no. 2, 103–146.

[4] ______, *The intersection of Piatetski–Shapiro sequences*, Mathematika **60** (2014), no. 2, 347–362.

[5] R. C. Baker and W. D. Banks, *Character sums with Piatetski–Shapiro sequences*, Q. J. Math. **66** (2015), no. 2, 393–416.

[6] R. C. Baker, W. D. Banks, J. Brüdern, I. E. Shparlinski and A. J. Weingartner, *Piatetski–Shapiro sequences*, Acta Arith. **157** (2013), no. 1, 37–68.

[7] R .C. Baker, W. D. Banks, Z. V. Guo and A. M. Yeager, *Piatetski–Shapiro primes from almost primes*, Monatsh. Math. **174** (2014), no. 3, 357–370.

[8] R. C. Baker and G. Harman, *Shifted primes without large prime factors*, Acta Arith. **83** (1998), no. 4, 331–361.

[9] R. C. Baker, G. Harman and J. Rivat, *Primes of the form* $[n^c]$, J. Number Theory **50** (1995), no. 2, 261–277.

[10] R. C. Baker and L. Zhao, *Gaps between primes in Beatty sequences*, Acta Arith. **172** (2016), no. 3, 207–242.

[11] W. D. Banks and V. Z. Guo, *Consecutive primes and Beatty sequences*, J. Number Theory **191** (2018), 158–174.

[12] W. D. Banks and I. E. Shparlinski, *Prime numbers with Beatty sequences*, Colloq. Math. **115** (2009), no. 2, 147–157.

[13] H. Davenport, *Multiplicative Number Theory*, Second edition, Graduate Texts in Mathematics **74**, Springer-Verlag, New York, 1980.

[14] S. W. Graham and G. Kolesnik, *Van der Corput's Method of Exponential Sums*, London Mathematical Society Lecture Note Series **126**, Cambridge University Press, Cambridge, 1991.

[15] V. Z. Guo, *Piatetski–Shapiro primes in a Beatty sequence*, J. Number Theory **156** (2015), 317–330.

[16] G. Harman, *Primes in intersections of Beatty sequences*, J. Integer Seq. **18** (2015), no. 7, Article 15.7.3, 12 pp.

[17] ——, *Primes in Beatty sequences in short intervals*, Mathematika **62** (2016), no. 2, 572–586.

[18] D. R. Heath-Brown, *The Pjateckiĭ–Šapiro prime number theorem*, J. Number Theory **16** (1983), no. 2, 242–266.

[19] C. H. Jia, *On Pjateckiĭ–Šapiro prime number theorem II*, Sci. China Ser. A **36** (1993), no. 8, 913–926.

[20] ——, *On Pjateckiĭ–Šapiro prime number theorem*, Chinese Ann. Math. Ser. B **15** (1994), no. 1, 9–22.

[21] G. A. Kolesnik, *The distribution of primes in sequences of the form* $[n^c]$, Mat. Zametki **2** (1967), 117–128.

[22] ——, *Primes of the form* $[n^c]$, Pacific J. Math. **118** (1985), no. 2, 437–447.

[23] A. Kumchev, *On the distribution of prime numbers of the form* $[n^c]$, Glasg. Math. J. **41** (1999), no. 1, 85–102.

[24] A. Kumchev and Z. Petrov, *A hybrid of two theorems of Piatetski–Shapiro*, Monatsh. Math. **189** (2019), no. 2, 355–376.

[25] J. Li and M. Zhang, *Hua's theorem with the primes in Piatetski–Shapiro prime sets*, Int. J. Number Theory **14** (2018), no. 1, 193–220.

[26] H. Q. Liu and J. Rivat, *On the Pjateckiĭ–Šapiro prime number theorem*, Bull. London Math. Soc. **24** (1992), no. 2, 143–147.

[27] K. Liu, I. E. Shparlinski and T. Zhang, *Squares in Piatetski–Shapiro sequences*, Acta Arith. **181** (2017), no. 3, 239–252.

[28] Y. M. Lu, *An additive problem on Piatetski–Shapiro primes*, Acta Math. Sin. (Engl. Ser.) **34** (2018), no. 2, 255–264.

[29] M. Mirek, *Roth's theorem in the Piatetski–Shapiro primes*, Rev. Mat. Iberoam. **31** (2015), no. 2, 617–656.

[30] I. I. Pyateckiĭ–Šapiro, *On the distribution of prime numbers in sequences of the form* $[f(n)]$, Mat. Sbornik N.S. **33** (1953), 559–566.

[31] J. Rivat and P. Sargos, *Nombres premiers de la forme* $\lfloor n^c \rfloor$, Canad. J. Math. **53** (2001), no. 2, 414–433.

[32] J. Rivat and J. Wu, *Prime numbers of the form* $[n^c]$, Glasg. Math. J. **43** (2001), no. 2, 237–254.

[33] L. Spiegelhofer, *Piatetski–Shapiro sequences via Beatty sequences*, Acta Arith. **166** (2014), no. 3, 201–229.

[34] J. D. Vaaler, *Some extremal functions in Fourier analysis*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 2, 183–216.

[35] I. M. Vinogradov, *A new estimate of a certain sum containing primes*, Rec. Math. Moscou, n. Ser. **2** (44) (1937), no. 5, 783–792. English translation: *New estimations of trigonometrical sums containing primes*, C. R. (Dokl.) Acad. Sci. URSS, n. Ser. **17** (1937), 165–166.

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