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A Note on High-dimensional D. H. Lehmer Problem

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Abstract. By using the properties of trigonometric sums and the estimates of n-dimensional Kloosterman sums, we study the high-dimensional D. H. Lehmer problem over incomplete intervals. First we generalize the previous results in [11] by presenting some sharp asymptotic formulae. Then with the aid of a more elementary method, we improve the error terms in a straight-forward manner.

1. Introduction

Let $q > 2$ be an odd integer. If an integer b with $0 < b < q$ and $(q, b) = 1$ has an inverse $\overline{b}$ modulo q with $2 \nmid (b + \overline{b})$, then b is called a D. H. Lehmer number. Denote the number of all Lehmer numbers modulo q by $M(1, q)$, D. H. Lehmer (see [3, Problem F12, p. 251]) asked whether anything non-trivial could be said about $M(1, p)$, where p is an odd prime. Zhang [16] proved that

$$M(1,q) = \frac{\phi(q)}{2} + O(q^{1/2} d^2(q) \ln^2 q),$$

where $\phi(q)$, $d(q)$ are the Euler and the divisor functions, respectively. Then Xu and Zhang [12] studied the mean square value of the error term of D. H. Lehmer problem over the incomplete interval $[1, (q-1)/2]$, which proved that the bound of the error term is best to the possible. While for $q = p$ being an odd prime, Cohen and Trudgian [2] recently made Zhang's work explicit, that is,

$$\left| M(1,p) - \frac{p-1}{2} \right| < \frac{1}{2} p^{1/2} \log^2 p.$$

They [2] also connected the problem with primitive roots.

In [4,5], Khan and Shparlinski studied the maximal difference between an integer and its inverse

$$M(q) = \max\{|a - \overline{a}| : 1 \le a \le q, (a,q) = 1\},$$

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and proved

$$q - M(q) = o(q^{3/4+\epsilon})$$

for any $\epsilon > 0$. Then Xu [10] studied the distribution of the difference of an integer and its m-th power modulo q over incomplete intervals $[1, [\lambda q]]$ with $0 < \lambda \le 1$. While Zhang and Liu [13] gave an identity for $2k$-th power of the non-negative least residue $r_p(q)$ and Fermat quotient $q_p(q)$ of an integer q.

Now let s be a nonnegative integer, and a, b be integers with $0 < a, b < q$. Zhang [17] investigated the distribution of $2s$-th power of $|a-b|$, and obtained that

$$\sum_{a=1}^{q}{}' \sum_{\substack{b=1 \\ ab\equiv 1 \ (\bmod q) \\ 2\nmid(a+b)}}^{q}{}' (a-b)^{2s} = \frac{\phi(q) q^{2s}}{(2s+1)(2s+2)} + O(4^s q^{(2s+1/2)} d^2(q) \ln^2 q).$$

In addition, Lu and Yi [6,7] generalized the problems in [16,17]. Let $n \ge 2$ be a fixed integer, c and $q \ge 3$ be integers with $(n, q) = (c, q) = 1$. For $0 < \lambda_1, \lambda_2 \le 1$, they obtained

$$\sum_{a=1}^{[\lambda_1 q]}{}' \sum_{\substack{b=1 \\ ab\equiv c \ (\bmod q) \\ n\nmid(a+b)}}^{[\lambda_2 q]}{}' 1 = \left(1 - \frac{1}{n}\right) \lambda_1 \lambda_2 \phi(q) + O(q^{1/2} d^6(q) \ln^2 q).$$

Let N be a positive integer and $\alpha > 0$. For any integer M, they also derived

$$\sum_{a=M+1}^{M+N}{}' \sum_{\substack{b=M+1 \\ ab\equiv c \ (\bmod q) \\ n\mid(a+b)}}^{M+N}{}' |a-b|^{\alpha} = \frac{2\phi(q)}{(\alpha+1)(\alpha+2) n q^2} N^{\alpha+2} + O(q^{1/2+\varepsilon} N^{\alpha} (N q^{-1} + 1)).$$

In fact, they studied the more general case.

Another important direction is concerning the high-dimensional case. Let $q > 2$ and c are two integers with $(q, c) = 1$, k be a fixed positive integer, then for any nonnegative integers s, Zhang and Zhang [15] considered the distribution of $2s$-th power of $|b_1 \cdots b_k - b_{k+1}|$, and obtained the asymptotic formula

$$\sum_{b_1=1}^{q}{}' \cdots \sum_{b_k=1}^{q}{}' \sum_{\substack{b_{k+1}=1 \\ b_1\cdots b_k b_{k+1}\equiv c \ (\bmod q)}}^{q}{}' (b_1 \cdots b_k - b_{k+1})^{2s} = \frac{\phi^k(q) q^{2sk}}{(2s+1)^k} + O(4^s q^{(2s+1)k-1/2} d^2(q) \ln q).$$

Soon afterwards, Shparlinski [8] improved the error term to $O(2sq^{(2s+1)s-s+1})$. Almost at the same time, Alkan, Stan and Zaharescu [1] considered $k+1$-tuples numbers with product congruent to 1 modulo q, and the parity condition is replaced by linear congruences with respect to more general moduli.

Now let m be a nonnegative integer, and k be a fixed positive integer. For any positive integer t with $t \leq k$, it is natural to ask what about the distribution of m-th power of $(b_1 \cdots b_t - b_{t+1} \cdots b_{k+1})$ over incomplete intervals. To be specific, let $0 < \lambda_1, \lambda_2, \dots, \lambda_{k+1} \leq 1$ be real numbers and $\mathbf{w} = (\lambda_1, \lambda_2, \dots, \lambda_{k+1})$. Let $q \geq \max\big\{[1/\lambda_j] : j = 1, 2, \dots, k+1\big\}$ be a positive integer, and a, n are integers coprime to q. Define

$$N(a, k+1, t, \mathbf{w}, q, m, n) = \sideset{}{'}\sum_{b_1=1}^{[\lambda_1 q]} \sideset{}{'}\sum_{b_2=1}^{[\lambda_2 q]} \cdots \sideset{}{'}\sum_{\substack{b_{k+1}=1 \\ b_1 b_2 \cdots b_{k+1} \equiv a \ (\mathrm{mod}\ q) \\ n \nmid (b_1 + b_2 + \cdots + b_{k+1})}}^{[\lambda_{k+1} q]} (b_1 \cdots b_t - b_{t+1} \cdots b_{k+1})^m.$$

It is obvious that by taking $t = k$, $m = 2s$ and $n = 2$, the above reduces to the case considered by Xu and Zhang [11]. By applying the properties of trigonometric sums and the estimates of high-dimensional Kloosterman sums, they [11] obtained the asymptotic formula

$$N(a, k+1, k, \mathbf{w}, q, 2s, 2) = C(k, \mathbf{w}, s)\phi^k(q) q^{2ks} + O(4^{s\varepsilon_k} q^{(2s+1)k-1/2} d^2(q) \ln^2 q),$$

where

$$C(k, \mathbf{w}, s) = \begin{cases} \frac{(\lambda_1 \cdots \lambda_k)^{2s+1} \lambda_{k+1}}{2(2s+1)^k} & \text{if } k \geq 2, \\ \frac{\lambda_1^{2s+2} + \lambda_2^{2s+2} - (\lambda_1 - \lambda_2)^{2s+2}}{4(s+1)(2s+1)} & \text{if } k = 1 \end{cases}$$

and $\varepsilon_k = \frac{1}{2}\big(1 - (-1)^{[1/k]}\big)$.

Actually, the error terms for $N(a, k+1, k, \mathbf{w}, q, 0, 2)$ can be improved to the best possible in some special cases like $\mathbf{w} = (1/2, 1/2, \dots, 1/2)$ and $\mathbf{w} = (1/4, 1/4, \dots, 1/4)$, which are shown in [11, 14] by studying the mean square value of the error terms. It should be pointed out that the latter case holds with a strict condition. While for other cases, the methods in [11, 14] failed to improve the error terms.

This article is outlined in the following way: we first generalize the previous results in [11] by presenting some sharp asymptotic formulae for $N(a, k+1, t, \mathbf{w}, q, m, n)$. Then with the aid of a more elementary method, we improve the error terms in a straight-forward manner.

We will prove the following

Theorem 1.1. *For any nonnegative integer m, we have the asymptotic formulae*

$$N(a, k+1, t, \mathbf{w}, q, m, n) = \begin{cases} \big(1 - \frac{1}{n}\big) A(k+1, t, \mathbf{w}, m)\phi^k(q) q^{mt} & \\ \quad + O(q^{mt+k-1/2} d^2(q) \ln^2 q) & \text{if } (k+1)/2 < t \leq k, \\ \big(1 - \frac{1}{n}\big) B(k+1, t, \mathbf{w}, m)\phi^k(q) q^{m(k-t+1)} & \\ \quad + O(q^{m(k-t+1)+k-1/2} d^2(q) \ln^2 q) & \text{if } 1 \leq t < (k+1)/2, \\ \big(1 - \frac{1}{n}\big) D(k+1, t, \mathbf{w}, m)\phi^k(q) q^{mt} & \\ \quad + O(2^m q^{mt+k-1/2} d^2(q) \ln^2 q) & \text{if } t = (k+1)/2, \end{cases}$$

where

$$A(k+1,t,\mathbf{w},m)=\frac{(\lambda_1\cdots\lambda_t)^{m+1}(\lambda_{t+1}\cdots\lambda_{k+1})}{(m+1)^t},$$
$$B(k+1,t,\mathbf{w},m)=\frac{(-1)^m(\lambda_1\cdots\lambda_t)(\lambda_{t+1}\cdots\lambda_{k+1})^{m+1}}{(m+1)^{k-t+1}},$$
$$D(k+1,t,\mathbf{w},m)=\sum_{j=0}^{m}C_m^j(-1)^j\frac{(\lambda_1\cdots\lambda_t)^{m-j+1}(\lambda_{t+1}\cdots\lambda_{k+1})^{j+1}}{((m-j+1)(j+1))^{(k+1)/2}}$$

are computable constants.

Taking $\mathbf{w}=\mathbf{1}=(1,1,\ldots,1)$ in Theorem 1.1, we can get the following result about the high-dimensional D. H. Lehmer problem over complete intervals.

Corollary 1.2. *For any nonnegative integer m, we have*

$$\begin{aligned}&N(a,k+1,t,\mathbf{1},q,m,n)\\&=\sideset{}{'}\sum_{b_1=1}^{q}\sideset{}{'}\sum_{b_2=1}^{q}\cdots\sideset{}{'}\sum_{\substack{b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a \pmod q\\ n\nmid(b_1+b_2+\cdots+b_{k+1})}}^{q}(b_1\cdots b_t-b_{t+1}\cdots b_{k+1})^m\\&=\begin{cases}\left(1-\frac{1}{n}\right)\frac{\phi^k(q)q^{mt}}{(m+1)^t}+O(q^{mt+k-1/2}d^2(q)\ln^2 q) & \text{if } (k+1)/2<t\le k,\\ \left(1-\frac{1}{n}\right)\frac{(-1)^m\phi^k(q)q^{m(k-t+1)}}{(m+1)^{k-t+1}}+O(q^{m(k-t+1)+k-1/2}d^2(q)\ln^2 q) & \text{if } 1\le t<(k+1)/2,\\ \left(1-\frac{1}{n}\right)\sum_{j=0}^{m}\frac{C_m^j(-1)^j}{((m-j+1)(j+1))^{(k+1)/2}}\phi^k(q)q^{mt}\\ \quad +O(2^mq^{mt+k-1/2}d^2(q)\ln^2 q) & \text{if } t=(k+1)/2.\end{cases}\end{aligned}$$

While taking $n=2$ in Theorem 1.1, we have

Corollary 1.3. *For any nonnegative integer m, we have the asymptotic formulae*

$$\begin{aligned}&N(a,k+1,t,\mathbf{w},q,m,2)\\&=\sideset{}{'}\sum_{b_1=1}^{[\lambda_1 q]}\sideset{}{'}\sum_{b_2=1}^{[\lambda_2 q]}\cdots\sideset{}{'}\sum_{\substack{b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a \pmod q\\ 2\nmid(b_1+b_2+\cdots+b_{k+1})}}^{[\lambda_{k+1} q]}(b_1\cdots b_t-b_{t+1}\cdots b_{k+1})^m\\&=\begin{cases}\frac{A(k+1,t,\mathbf{w},m)}{2}\phi^k(q)q^{mt}+O(q^{mt+k-1/2}d^2(q)\ln^2 q) & \text{if } (k+1)/2<t\le k,\\ \frac{B(k+1,t,\mathbf{w},m)}{2}\phi^k(q)q^{m(k-t+1)}+O(q^{m(k-t+1)+k-1/2}d^2(q)\ln^2 q) & \text{if } 1\le t<(k+1)/2,\\ \frac{D(k+1,t,\mathbf{w},m)}{2}\phi^k(q)q^{mt}+O(2^mq^{mt+k-1/2}d^2(q)\ln^2 q) & \text{if } t=(k+1)/2.\end{cases}\end{aligned}$$

Letting $t=k\ge 2$ in Corollary 1.3, our result recovers the main result of [11]. What's more, taking $k=t=1$ in Theorem 1.1, we have

Corollary 1.4. *For any nonnegative even integer m, we have*

$$\begin{aligned}N(a,2,1,\mathbf{w},q,m,n) &= \sideset{}{'}\sum_{\substack{b_1=1\\ b_1b_2\equiv a \ (\mathrm{mod}\ q)\\ n\nmid(b_1+b_2)}}^{[\lambda_1 q]}\sideset{}{'}\sum_{b_2=1}^{[\lambda_2 q]} (b_1-b_2)^m\\ &= \left(1-\frac{1}{n}\right)\frac{\lambda_1^{m+2}+\lambda_2^{m+2}-(\lambda_1-\lambda_2)^{m+2}}{(m+1)(m+2)}\phi(q)q^m\\ &\quad + O(2^m q^{m+1/2}d^2(q)\ln^2 q).\end{aligned}$$

Taking $n=2$, $\lambda_1=\lambda_2=1$ in Corollary 1.4, we can immediately obtain the result of [17]. While taking $m=0$, we can also obtain the result of [6].

With the notation $\sigma_m(\lambda,q)=\sum'^{[\lambda q]}_{a=1} a^m$, we will give stronger error terms for $N(a,k+1,t,\mathbf{w},q,m,n)$ in the cases $\mathbf{w}=(\lambda_1,\lambda_2,\ldots,\lambda_{k+1})$ with at least one $\lambda_j=1$ $(j=1,2,\ldots,k+1)$. Define

$$N_j(a,k+1,t,\mathbf{w},q,m,n)=\sideset{}{'}\sum_{\substack{b_1=1\\ b_1b_2\cdots b_{k+1}\equiv a\ (\mathrm{mod}\ q)\\ n\nmid(b_1+b_2+\cdots+b_{k+1})}}^{[\lambda_1 q]}\sideset{}{'}\sum_{b_2=1}^{[\lambda_2 q]}\cdots\sideset{}{'}\sum_{b_j=1}^{q}\cdots\sideset{}{'}\sum_{b_{k+1}=1}^{[\lambda_{k+1}q]}(b_1\cdots b_t-b_{t+1}\cdots b_{k+1})^m.$$

Then we have

Theorem 1.5. *For any nonnegative integer m, we have*

$$\begin{aligned}N_j(a,k+1,t,\mathbf{w},q,m,n) &= \left(1-\frac{1}{n}\right)\prod_{u=1}^{t}\sigma_m(\lambda_u,q)\prod_{\substack{v=t+1\\ v\neq j}}^{k+1}\sigma_0(\lambda_v,q)+O(q^{2k+(m-2)t+1})\\ &\quad + O(q^{mt+k/2}d_{k+1}(q)\ln^{k+1}q) \quad \text{if } (k+1)/2<t<j,\end{aligned}$$

$$\begin{aligned}&N_j(a,k+1,t,\mathbf{w},q,m,n)\\ &= \left(1-\frac{1}{n}\right)(-1)^m\prod_{\substack{u=1\\ u\neq j}}^{t}\sigma_0(\lambda_u,q)\prod_{v=t+1}^{k+1}\sigma_m(\lambda_v,q)+O(q^{2k+(m-2)(k-t+1)+1})\\ &\quad + O(q^{m(k-t+1)+k/2}d_{k+1}(q)\ln^{k+1}q) \quad \text{if } j<t<(k+1)/2,\end{aligned}$$

and

$$\begin{aligned}&N_j(a,k+1,t,\mathbf{w},q,m,n)\\ &= \left(1-\frac{1}{n}\right)\left(\prod_{u=1}^{(k+1)/2}\sigma_m(\lambda_u,q)\prod_{\substack{v=(k+3)/2\\ v\neq j}}^{k+1}\sigma_0(\lambda_v,q)+E(k+1,t,\mathbf{w},m)\phi^k(q)q^{m(k+1)/2}\right)\\ &\quad + O(2^m q^{m(k+1)/2+k-1/2}d^2(q)\ln^2 q) \quad \text{if } t=(k+1)/2<j,\end{aligned}$$

where

$$E(k+1,t,\mathbf{w},m)=\sum_{j=0}^{m}C_m^j(-1)^j\frac{(\lambda_1\cdots\lambda_t)^{m-j+1}(\lambda_{t+1}\cdots\lambda_k)^{j+1}}{((m-j+1)(j+1))^{(k+1)/2}}$$

is a computable constant.

Note. Let q be fixed positive integer, then for any nonnegative integer m, from the property of Möbius function, we have

$$\sigma_m(\lambda,q)=\sideset{}{'}\sum_{a=1}^{[\lambda q]}a^m=\sum_{d|q}\mu(d)\sum_{a=1}^{[\lambda q]/d}(ad)^m=\frac{[\lambda q]^{m+1}}{m+1}\frac{\phi(q)}{q}+O([\lambda q]^m 2^{\omega(q)}).$$

This shows the bounds of error terms in the first two cases of Theorem 1.5 are sharper than those of [11]. However, our method does not improve the bound of the error term in the case $t=(k+1)/2$. So how to improve it is still an open problem.

Taking $\mathbf{w}=\mathbf{1}$ in Theorem 1.5, we may have

Corollary 1.6. *For any nonnegative integer m, we have*

$$\begin{aligned}
&N(a,k+1,t,\mathbf{1},q,m,n)\\
&=\sideset{}{'}\sum_{b_1=1}^{q}\sideset{}{'}\sum_{b_2=1}^{q}\cdots\sideset{}{'}\sum_{\substack{b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a\ (\bmod q)\\ n\nmid(b_1+b_2+\cdots+b_{k+1})}}^{q}(b_1\cdots b_t-b_{t+1}\cdots b_{k+1})^m\\
&=\begin{cases}
\left(1-\frac{1}{n}\right)\phi^{k-t}(q)(\sigma_m(q))^t+O(q^{2k+(m-2)t+1})\\
\quad +O(q^{mt+k/2}d_{k+1}(q)\ln^{k+1}q) & \text{if } \frac{k+1}{2}<t,\\
\left(1-\frac{1}{n}\right)(-1)^m\phi^{t-1}(q)(\sigma_m(q))^{k-t+1}+O(q^{2k+(m-2)(k-t+1)+1})\\
\quad +O(q^{m(k-t+1)+k/2}d_{k+1}(q)\ln^{k+1}q) & \text{if } t<\frac{k+1}{2},\\
\left(1-\frac{1}{n}\right)\left(\phi^{(k-1)/2}(q)(\sigma_m(q))^{(k+1)/2}+E(k+1,t,\mathbf{1},m)\phi^k(q)q^{m(k+1)/2}\right)\\
\quad +O(2^m q^{m(k+1)/2+k-1/2}d^2(q)\ln^2 q) & \text{if } t=\frac{k+1}{2},
\end{cases}
\end{aligned}$$

where $\sigma_m(q)=\sigma_m(1,q)$.

It is obvious to see that the bounds of error terms in the first two cases are sharper than those in Corollary 1.2.

Taking $t=k\geq 2$ in Corollary 1.6, we can immediately obtain the following

Corollary 1.7. *For any positive integer m, we have*

$$\begin{aligned} N(a,k+1,k,\mathbf{1},q,m,n) &= \sideset{}{'}\sum_{b_1=1}^{q}\sideset{}{'}\sum_{b_2=1}^{q}\cdots\sideset{}{'}\sum_{\substack{b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a\ (\mathrm{mod}\ q)\\ n\nmid(b_1+b_2+\cdots+b_{k+1})}}^{q} (b_1\cdots b_k-b_{k+1})^m \\ &= \left(1-\frac{1}{n}\right)(\sigma_m(q))^k+O(q^{(m+1/2)k}d_{k+1}(q)\ln^{k+1}q). \end{aligned}$$

Taking $m=0$ in Corollary 1.7, we have

Corollary 1.8.

$$\begin{aligned} N(a,k+1,t,\mathbf{1},q,0,n) &= \sideset{}{'}\sum_{b_1=1}^{q}\sideset{}{'}\sum_{b_2=1}^{q}\cdots\sideset{}{'}\sum_{\substack{b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a\ (\mathrm{mod}\ q)\\ n\nmid(b_1+b_2+\cdots+b_{k+1})}}^{q} 1 \\ &= \left(1-\frac{1}{n}\right)\phi^k(q)+O(q^{k/2}d_{k+1}(q)\ln^{k+1}q). \end{aligned}$$

From the discussions in references [11] and [14], the bound in Corollary 1.8 is close to the best possible. Taking $k=1$, $n=2$ in Corollary 1.8, we can immediately obtain the result of [16].

2. Several lemmas

To prove Theorems 1.1 and 1.5, we need the following lemmas.

Lemma 2.1. *Let q and t be integers with $q>2$ and $t\geq 0$. Let y and l be integers with $1\leq y\leq q$ and $1\leq l\leq n$. Let $0<\lambda\leq 1$ be a real number. For any given integer $n\geq 2$, we have*

$$K(-y,t,l)=\sum_{c=1}^{[\lambda q]}c^t\mathrm{e}\left(c\frac{-yn+ql}{qn}\right)=\begin{cases}\frac{(\lambda q)^{t+1}}{t+1}+O((\lambda q)^t) & \text{if } qn\mid(-yn+ql),\\ O\Big(\frac{(\lambda q)^t}{\left|\sin\frac{\pi(-yn+ql)}{qn}\right|}\Big) & \text{if } qn\nmid(-yn+ql).\end{cases}$$

Proof. If $qn\nmid(-yn+ql)$, then

$$\begin{aligned} &\sum_{c=1}^{[\lambda q]}c^t\mathrm{e}\left(c\frac{-yn+ql}{qn}\right)\left(1-\mathrm{e}\left(\frac{-yn+ql}{qn}\right)\right)\\ &=\sum_{c=1}^{[\lambda q]}c^t\mathrm{e}\left(c\frac{-yn+ql}{qn}\right)-\sum_{c=1}^{[\lambda q]}c^t\mathrm{e}\left(\frac{(c+1)(-yn+ql)}{qn}\right)\\ &=e\left(\frac{-yn+ql}{qn}\right)-[\lambda q]^t\mathrm{e}\left(\frac{([\lambda q]+1)(-yn+ql)}{qn}\right) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{c=1}^{[\lambda q]-1}(c+1)^t\mathrm{e}\left(\frac{(c+1)(-yn+ql)}{qn}\right)-\sum_{c=1}^{[\lambda q]-1}c^t\mathrm{e}\left(\frac{(c+1)(-yn+ql)}{qn}\right)\\
&\ll 1+[\lambda q]^t+\sum_{c=1}^{[\lambda q]-1}((c+1)^t-c^t)\\
&\ll [\lambda q]^t.
\end{aligned}
$$

Since

$$\left|\left(1-\mathrm{e}\left(\frac{-yn+ql}{qn}\right)\right)\right|=\left|2\sin\frac{\pi(-yn+ql)}{qn}\right|,$$

we have

$$K(-y,t,l)=O\left(\frac{(\lambda q)^t}{|\sin\frac{\pi(-yn+ql)}{qn}|}\right).$$

If $qn \mid (-yn+ql)$, then we get

$$K(-y,t,l)=\sum_{c=1}^{[\lambda q]}c^t=\frac{(\lambda q)^{t+1}}{t+1}+O((\lambda q)^t).$$

This proves Lemma 2.1. □

Lemma 2.2. *For each prime p and any $\mathbf{y}\in\mathbb{Z}^{k+1}$, there exists a unique integer $r\geq 0$ such that $\mathbf{y}=p^r\mathbf{x}$ for some $\mathbf{x}\in\mathbb{Z}^{k+1}-p\mathbb{Z}^{k+1}$. Let t denote the number of components of $\mathbf{x}$ which are divisible by p such that $0\leq t\leq k$. For each $\alpha\geq 0$, define*

$$(\mathbf{y};p^\alpha)_k=p^{\sigma_k(\mathbf{y};p^\alpha)_k}$$

where

$$\sigma_k(\mathbf{y};p^\alpha)_k=\begin{cases}\alpha & \text{if } r\geq\alpha,\\ r & \text{if } r<\alpha-1,\\ r & \text{if } r=\alpha-1 \text{ and } t=0,\\ r-1+\frac{2(t-1)}{k} & \text{if } r=\alpha-1 \text{ and } 1\leq t\leq k.\end{cases}$$

For any integer $q\geq 1$, we now define

$$(\mathbf{y};q)_k=\prod_{p^\alpha\|q}(\mathbf{y};p^\alpha)_k.$$

Then for all integers $k,q\geq 1$ and all $\mathbf{y}\in\mathbb{Z}^{k+1}$, we have the upper bound

$$|S_k(\mathbf{y};q)|\leq q^{k/2}(\mathbf{y};q)_k^{k/2}d_{k+1}(q),$$

where $d_{k+1}(n)$ is the $k+1$-th divisor function (i.e., the number of solutions of the equation $n_1n_2\cdots n_{k+1}=n$ in positive integers $n_1,n_2,\ldots,n_{k+1}$).

Proof. See [9]. □

Lemma 2.3. *Let q, k, n, a be positive integers with $q > 2$ and $(n, q) = (a, q) = 1$, let $0 < \lambda_1, \lambda_2, \ldots, \lambda_{k+1} \le 1$ be real numbers. Then for any nonnegative integers $t_1, t_2, \ldots, t_{k+1}$, we have*

$$\mathop{{\sum}'}_{b_1=1}^{[\lambda_1 q]}\mathop{{\sum}'}_{b_2=1}^{[\lambda_2 q]}\cdots\mathop{{\sum}'}_{\substack{b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a \ (\mathrm{mod}\ q)}}^{[\lambda_{k+1} q]} b_1^{t_1}b_2^{t_2}\cdots b_{k+1}^{t_{k+1}} = \frac{\lambda_1^{t_1+1}\lambda_2^{t_2+1}\cdots\lambda_{k+1}^{t_{k+1}+1}\phi^k(q)q^{t_1+t_2+\cdots+t_{k+1}}}{(t_1+1)(t_2+1)\cdots(t_{k+1}+1)} + O(q^{t_1+t_2+\cdots+t_{k+1}+k-1/2}d^2(q)\ln^2 q),$$

$$\mathop{{\sum}'}_{b_1=1}^{[\lambda_1 q]}\mathop{{\sum}'}_{b_2=1}^{[\lambda_2 q]}\cdots\mathop{{\sum}'}_{\substack{b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a \ (\mathrm{mod}\ q)\\ n\nmid(b_1+b_2+\cdots+b_{k+1})}}^{[\lambda_{k+1} q]} b_1^{t_1}b_2^{t_2}\cdots b_{k+1}^{t_{k+1}} = \left(1-\frac{1}{n}\right)\frac{\lambda_1^{t_1+1}\cdots\lambda_k^{t_k+1}\lambda_{k+1}^{t_{k+1}+1}\phi^k(q)q^{t_1+t_2+\cdots+t_{k+1}}}{(t_1+1)(t_2+1)\cdots(t_{k+1}+1)} + O(q^{t_1+t_2+\cdots+t_{k+1}+k-1/2}d^2(q)\ln^2 q),$$

and

$$\sum_{l=1}^{n-1}\mathop{{\sum}'}_{b_1=1}^{[\lambda_1 q]}\mathop{{\sum}'}_{b_2=1}^{[\lambda_2 q]}\cdots\mathop{{\sum}'}_{\substack{b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a \ (\mathrm{mod}\ q)}}^{[\lambda_{k+1} q]}\left(\mathrm{e}\left(\frac{b_1 l}{n}\right)b_1^{t_1}\right)\left(\mathrm{e}\left(\frac{b_2 l}{n}\right)b_2^{t_2}\right)\cdots\left(\mathrm{e}\left(\frac{b_{k+1} l}{n}\right)b_{k+1}^{t_{k+1}}\right)$$
$$= O(q^{t_1+t_2+\cdots+t_{k+1}+k/2}d_{k+1}(q)\ln^{k+1} q).$$

Proof. (i) For the first part of Lemma 2.3, see Lemma 2.6 in [11].

(ii) For the second part, first we have

$$\mathop{{\sum}'}_{b_1=1}^{[\lambda_1 q]}\mathop{{\sum}'}_{b_2=1}^{[\lambda_2 q]}\cdots\mathop{{\sum}'}_{\substack{b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a \ (\mathrm{mod}\ q)\\ n\nmid(b_1+b_2+\cdots+b_{k+1})}}^{[\lambda_{k+1} q]} b_1^{t_1}b_2^{t_2}\cdots b_{k+1}^{t_{k+1}}$$
$$= \mathop{{\sum}'}_{b_1=1}^{[\lambda_1 q]}\mathop{{\sum}'}_{b_2=1}^{[\lambda_2 q]}\cdots\mathop{{\sum}'}_{\substack{b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a \ (\mathrm{mod}\ q)}}^{[\lambda_{k+1} q]} b_1^{t_1}b_2^{t_2}\cdots b_{k+1}^{t_{k+1}} - \mathop{{\sum}'}_{b_1=1}^{[\lambda_1 q]}\mathop{{\sum}'}_{b_2=1}^{[\lambda_2 q]}\cdots\mathop{{\sum}'}_{\substack{b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a \ (\mathrm{mod}\ q)\\ n\mid(b_1+b_2+\cdots+b_{k+1})}}^{[\lambda_{k+1} q]} b_1^{t_1}b_2^{t_2}\cdots b_{k+1}^{t_{k+1}}.$$

Thus we only need to focus on the second sum. By using the trigonometric identity

$$\sum_{a=1}^{q}\mathrm{e}\left(\frac{au}{q}\right) = \begin{cases} q & \text{if } q \mid u, \\ 0 & \text{if } q \nmid u, \end{cases}$$

we obtain

$$\begin{aligned}
&\mathop{{\sum}'}_{\substack{b_1=1\\ b_1b_2\cdots b_{k+1}\equiv a \ (\bmod q)\\ n|(b_1+b_2+\cdots+b_{k+1})}}^{[\lambda_1 q]}\mathop{{\sum}'}_{b_2=1}^{[\lambda_2 q]}\cdots\mathop{{\sum}'}_{b_{k+1}=1}^{[\lambda_{k+1} q]} b_1^{t_1}b_2^{t_2}\cdots b_{k+1}^{t_{k+1}}\\
&=\frac{1}{n}\mathop{{\sum}'}_{\substack{b_1=1\\ b_1b_2\cdots b_{k+1}\equiv a \ (\bmod q)}}^{[\lambda_1 q]}\mathop{{\sum}'}_{b_2=1}^{[\lambda_2 q]}\cdots\mathop{{\sum}'}_{b_{k+1}=1}^{[\lambda_{k+1} q]} b_1^{t_1}b_2^{t_2}\cdots b_{k+1}^{t_{k+1}}\sum_{l=1}^{n}\mathrm{e}\left(\frac{b_1+b_2+\cdots+b_{k+1}}{n}l\right)\\
&=\frac{1}{nq^{k+1}}\mathop{{\sum}'}_{\substack{b_1=1\\ b_1b_2\cdots b_{k+1}\equiv a \ (\bmod q)}}^{q}\mathop{{\sum}'}_{b_2=1}^{q}\cdots\mathop{{\sum}'}_{b_{k+1}=1}^{q}\mathop{{\sum}'}_{c_1=1}^{[\lambda_1 q]}\cdots\mathop{{\sum}'}_{c_k=1}^{[\lambda_k q]}\mathop{{\sum}'}_{c_{k+1}=1}^{[\lambda_{k+1} q]} c_1^{t_1}c_2^{t_2}\cdots c_{k+1}^{t_{k+1}}\sum_{l=1}^{n}\mathrm{e}\left(\frac{c_1+c_2+\cdots+c_{k+1}}{n}l\right)\\
&\quad\times\sum_{y_1,\ldots,y_{k+1}=1}^{q}\mathrm{e}\left(\frac{y_1(b_1-c_1)+\cdots+y_k(b_k-c_k)+y_{k+1}(b_{k+1}-c_{k+1})}{q}\right)\\
&=\frac{1}{nq^{k+1}}\sum_{y_1,\ldots,y_{k+1}=1}^{q}\mathop{{\sum}'}_{b_1=1}^{q}\mathop{{\sum}'}_{b_2=1}^{q}\cdots\mathop{{\sum}'}_{b_k=1}^{q}\mathrm{e}\left(\frac{y_1b_1+\cdots+y_kb_k+y_{k+1}a\overline{b_1\cdots b_k}}{q}\right)\\
&\quad\times\sum_{l=1}^{n}\left(\sum_{c_1=1}^{[\lambda_1 q]}c_1^{t_1}\mathrm{e}\left(c_1\frac{-y_1n+ql}{qn}\right)\right)\cdots\left(\sum_{c_{k+1}=1}^{[\lambda_{k+1} q]}c_{k+1}^{t_{k+1}}\mathrm{e}\left(c_{k+1}\frac{-y_{k+1}n+ql}{qn}\right)\right)\\
&=\frac{1}{nq^{k+1}}\sum_{y_1,\ldots,y_{k+1}=1}^{q}\sum_{l=1}^{n}S(y_1,\ldots,y_k,y_{k+1}a;q)K(-y_1,t_1,l)\cdots K(-y_k,t_k,l)K(-y_{k+1},t_{k+1},l)\\
&=\frac{1}{nq^{k+1}}S(q,\ldots,q,qa;q)K(-q,t_1,n)\cdots K(-q,t_k,n)K(-q,t_{k+1},n)\\
&\quad+\frac{1}{nq^{k+1}}\sum_{l=1}^{n-1}S(q,\ldots,q,qa;q)K(-q,t_1,l)\cdots K(-q,t_k,l)K(-q,t_{k+1},l)\\
&\quad+\sum_{r=1}^{k}\frac{C_k^r}{q^{k+1}}E_1+\sum_{r=1}^{k}\frac{C_k^{r-1}}{q^{k+1}}E_2\\
&\quad+\frac{1}{nq^{k+1}}\sum_{y_1,\ldots,y_{k+1}=1}^{q-1}\sum_{l=1}^{n}S(y_1,\ldots,y_k,y_{k+1}a;q)K(-y_1,t_1,l)\cdots K(-y_k,t_k,l)\\
&\qquad\times K(-y_{k+1},t_{k+1},l)\\
&:=\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4+\Sigma_5,
\end{aligned}$$

where $C_k^r=\frac{k!}{r!(k-r)!}$ $(1\le r\le k)$, $K(-y,t,l)$ and $S(y_1,\ldots,y_k,y_{k+1}a;q)=S_k(\mathbf{y};q)$ are defined on the above, and

$$E_1=\sum_{y_1=1}^{q-1}\cdots\sum_{y_r=1}^{q-1}\sum_{l=1}^{n}S(y_1,\ldots,y_r,q,\ldots,q,qa;q)K(-y_1,t_1,l)\cdots K(-y_r,t_r,l)$$

$$\times K(-q,t_{r+1},l)\cdots K(-q,t_k,l)K(-q,t_{k+1},l),$$

$$E_2=\sum_{y_1=1}^{q-1}\cdots\sum_{y_{r-1}=1}^{q-1}\sum_{y_{k+1}=1}^{q-1}\sum_{l=1}^{n}S(y_1,\dots,y_{r-1},q,\dots,q,y_{k+1}a;q)$$
$$\times K(-y_1,t_1,l)\cdots K(-y_{r-1},t_{r-1},l)K(-q,t_r,l)\cdots K(-q,t_k,l)K(-y_{k+1},t_{k+1},l).$$

For the contribution of Σ_1, from Lemma 2.1, we obtain

$$\begin{aligned}\Sigma_1&=\frac{1}{nq^{k+1}}S(q,\dots,q,qa;q)K(-q,t_1,n)\cdots K(-q,t_k,n)K(-q,t_{k+1},n)\\&=\frac{\phi^k(q)}{nq^{k+1}}\left(\frac{(\lambda_1 q)^{t_1+1}}{t_1+1}+O((\lambda_1 q)^{t_1})\right)\cdots\left(\frac{(\lambda_k q)^{t_k+1}}{t_k+1}+O((\lambda_k q)^{t_k})\right)\\&\quad\times\left(\frac{(\lambda_{k+1} q)^{t_{k+1}+1}}{t_{k+1}+1}+O((\lambda_{k+1} q)^{t_{k+1}})\right)\\&=\frac{\lambda_1^{t_1+1}\cdots\lambda_k^{t_k+1}\lambda_{k+1}^{t_{k+1}+1}\phi^k(q)q^{t_1+\cdots+t_k+t_{k+1}}}{n(t_1+1)\cdots(t_k+1)(t_{k+1}+1)}+O(q^{t_1+\cdots+t_k+t_{k+1}+k-1}).\end{aligned}$$

For the contribution of Σ_2, by applying the Jordan inequality

$$\frac{2}{\pi}\le\frac{\sin x}{x}\quad\text{if } |x|\le\frac{\pi}{2},$$

we can get the estimate

$$\begin{aligned}\Sigma_2&=\frac{1}{nq^{k+1}}\sum_{l=1}^{n-1}S(q,\dots,q,qa;q)K(-q,t_1,l)\cdots K(-q,t_k,l)K(-q,t_{k+1},l)\\&\ll\frac{\phi^k(q)}{nq^{k+1}}\sum_{l=1}^{n-1}\frac{q^{t_1}}{|\sin\frac{\pi(-qn+ql)}{qn}|}\cdots\frac{q^{t_k}}{|\sin\frac{\pi(-qn+ql)}{qn}|}\frac{q^{t_{k+1}}}{|\sin\frac{\pi(-qn+ql)}{qn}|}\\&\ll n^k q^{t_1+\cdots+t_k+t_{k+1}-1}.\end{aligned}$$

For the contributions of Σ_3 and Σ_4, we need to estimate E_1 and E_2. Now we estimate E_1. Noting that $qn\mid(-yn+ql)$ if and only if $y=q$ and $l=n$, then

$$\begin{aligned}E_1&=\sum_{y_1=1}^{q-1}\cdots\sum_{y_r=1}^{q-1}\sum_{l=1}^{n}S(y_1,\dots,y_r,q,\dots,q,qa;q)K(-y_1,t_1,l)\cdots K(-y_r,t_r,l)\\&\quad\times K(-q,t_{r+1},l)\cdots K(-q,t_k,l)K(-q,t_{k+1},l)\\&=\sum_{y_1=1}^{q-1}\cdots\sum_{y_r=1}^{q-1}S(y_1,\dots,y_r,q,\dots,q,qa;q)K(-y_1,t_1,n)\cdots K(-y_r,t_r,n)\\&\quad\times K(-q,t_{r+1},n)\cdots K(-q,t_k,n)K(-q,t_{k+1},n)\\&\quad+\sum_{y_1=1}^{q-1}\cdots\sum_{y_r=1}^{q-1}\sum_{l=1}^{n-1}S(y_1,\dots,y_r,q,\dots,q,qa;q)K(-y_1,t_1,l)\cdots K(-y_r,t_r,l)\end{aligned}$$

$$
\begin{aligned}
&\times K(-q,t_{r+1},l)\cdots K(-q,t_k,l)K(-q,t_{k+1},l)\\
=&\sum_{y_1=1}^{q-1}\cdots\sum_{y_r=1}^{q-1}\sideset{}{'}\sum_{b_1=1}^{q}\cdots\sideset{}{'}\sum_{b_r=1}^{q}\sideset{}{'}\sum_{b_{r+1}=1}^{q}\cdots\sideset{}{'}\sum_{b_k=1}^{q}\mathrm{e}\left(\frac{y_1b_1+\cdots+y_rb_r}{q}\right)\\
&\times K(-y_1,t_1,n)\cdots K(-y_r,t_r,n)K(-q,t_{r+1},n)\cdots K(-q,t_k,n)K(-q,t_{k+1},n)\\
&+\sum_{y_1=1}^{q-1}\cdots\sum_{y_r=1}^{q-1}\sideset{}{'}\sum_{b_1=1}^{q}\cdots\sideset{}{'}\sum_{b_r=1}^{q}\sideset{}{'}\sum_{b_{r+1}=1}^{q}\cdots\sideset{}{'}\sum_{b_k=1}^{q}\mathrm{e}\left(\frac{y_1b_1+\cdots+y_rb_r}{q}\right)\\
&\times\sum_{l=1}^{n-1}K(-y_1,t_1,l)\cdots K(-y_r,t_r,l)K(-q,t_{r+1},l)\cdots K(-q,t_k,l)K(-q,t_{k+1},l)\\
\ll&\,\phi^{k-r}(q)q^{r/2}d^r(q)\sum_{y_1=1}^{q-1}\cdots\sum_{y_r=1}^{q-1}\frac{(y_1,q)^{1/2}q^{t_1}}{\left|\sin\left(\frac{\pi(-y_1n+qn)}{qn}\right)\right|}\cdots\frac{(y_r,q)^{1/2}q^{t_r}}{\left|\sin\left(\frac{\pi(-y_rn+qn)}{qn}\right)\right|}\\
&\times\left(\frac{(\lambda_{r+1}q)^{t_{r+1}+1}}{t_{r+1}+1}+O((\lambda_{r+1}q)^{t_{r+1}})\right)\cdots\left(\frac{(\lambda_{k+1}q)^{t_{k+1}+1}}{t_{k+1}+1}+O((\lambda_{k+1}q)^{t_{k+1}})\right)\\
&+\phi^{k-r}(q)q^{r/2}d^r(q)\sum_{y_1=1}^{q-1}\cdots\sum_{y_r=1}^{q-1}\sum_{l=1}^{n-1}\frac{(y_1,q)^{1/2}q^{t_1}}{\left|\sin\left(\frac{\pi(-y_1n+ql)}{qn}\right)\right|}\cdots\frac{(y_r,q)^{1/2}q^{t_r}}{\left|\sin\left(\frac{\pi(-y_rn+ql)}{qn}\right)\right|}\\
&\times\frac{q^{t_{r+1}}}{\left|\sin\left(\frac{\pi(-qn+ql)}{qn}\right)\right|}\cdots\frac{q^{t_{k+1}}}{\left|\sin\left(\frac{\pi(-qn+ql)}{qn}\right)\right|}\\
\ll&\,q^{t_1+\cdots+t_k+t_{k+1}+2k-r/2+1}d^r(q)\sum_{y_1=1}^{q-1}\cdots\sum_{y_r=1}^{q-1}\frac{(y_1,q)^{1/2}\cdots(y_r,q)^{1/2}}{y_1\cdots y_r}\\
&+n^rq^{t_1+\cdots+t_k+t_{k+1}+k+r/2}d^r(q)\sum_{y_1=1}^{q-1}\cdots\sum_{y_r=1}^{q-1}\sum_{l=1}^{n-1}\frac{(y_1,q)^{1/2}\cdots(y_r,q)^{1/2}}{|-y_1n+ql|\cdots|-y_rn+ql|}\\
\ll&\,q^{t_1+\cdots+t_k+t_{k+1}+2k-r/2+1}d^{2r}(q)\ln^r q\\
&+q^{t_1+\cdots+t_k+t_{k+1}+k+r/2}d^r(q)\sum_{y_1=1}^{q-1}\cdots\sum_{y_r=1}^{q-1}\sum_{l=1}^{n-1}\frac{(y_1,q)^{1/2}\cdots(y_r,q)^{1/2}}{y_1\cdots y_r}\\
\ll&\,q^{t_1+\cdots+t_k+t_{k+1}+2k-r/2+1}d^{2r}(q)\ln^r q.
\end{aligned}
$$

Then we estimate E_2. From Lemmas 2.1 and 2.2, we have

$$
\begin{aligned}
E_2=&\sum_{y_1=1}^{q-1}\cdots\sum_{y_{r-1}=1}^{q-1}\sum_{y_{k+1}=1}^{q-1}S(y_1,\dots,y_{r-1},q,\dots,q,y_{k+1}a;q)\\
&\times K(-y_1,t_1,n)\cdots K(-y_{r-1},t_{r-1},n)K(-q,t_r,n)\cdots K(-q,t_k,n)K(-y_{k+1},t_{k+1},n)\\
&+\sum_{y_1=1}^{q-1}\cdots\sum_{y_{r-1}=1}^{q-1}\sum_{y_{k+1}=1}^{q-1}\sum_{l=1}^{n-1}S(y_1,\dots,y_{r-1},q,\dots,q,y_{k+1}a;q)\\
&\times K(-y_1,t_1,l)\cdots K(-y_{r-1},t_{r-1},l)K(-q,t_r,l)\cdots K(-q,t_k,l)K(-y_{k+1},t_{k+1},l)
\end{aligned}
$$

$$\ll \sum_{y_1=1}^{q-1}\cdots\sum_{y_{r-1}=1}^{q-1}\sum_{y_{k+1}=1}^{q-1} q^{k/2}(\mathbf{y};q)_k^{k/2} d_{k+1}(q)\frac{q^{t_1}}{\left|\sin\left(\frac{\pi(-y_1n+qn)}{qn}\right)\right|}\cdots\frac{q^{t_{r-1}}}{\left|\sin\left(\frac{\pi(-y_{r-1}n+qn)}{qn}\right)\right|}$$
$$\times\frac{q^{t_{k+1}}}{\left|\sin\left(\frac{\pi(-y_{k+1}n+qn)}{qn}\right)\right|}\left(\frac{(\lambda_r q)^{t_r+1}}{t_r+1}+O((\lambda_r q)^{t_r})\right)\cdots\left(\frac{(\lambda_k q)^{t_k+1}}{t_k+1}+O((\lambda_k q)^{t_k})\right)$$
$$+\sum_{y_1=1}^{q-1}\cdots\sum_{y_{r-1}=1}^{q-1}\sum_{y_{k+1}=1}^{q-1} q^{k/2}(\mathbf{y};q)_k^{k/2} d_{k+1}(q)\sum_{l=1}^{n-1}\frac{q^{t_1}}{\left|\sin\left(\frac{\pi(-y_1n+ql)}{qn}\right)\right|}$$
$$\times\cdots\frac{q^{t_{r-1}}}{\left|\sin\left(\frac{\pi(-y_{r-1}n+ql)}{qn}\right)\right|}\frac{q^{t_{k+1}}}{\left|\sin\left(\frac{\pi(-y_{k+1}n+ql)}{qn}\right)\right|}\frac{q^{t_r}}{\left|\sin\left(\frac{\pi(-qn+ql)}{qn}\right)\right|}\cdots\frac{q^{t_k}}{\left|\sin\left(\frac{\pi(-qn+ql)}{qn}\right)\right|}$$
$$\ll q^{t_1+\cdots+t_k+t_{k+1}+3k/2+1}d_{k+1}(q)\sum_{y_1=1}^{q-1}\cdots\sum_{y_{r-1}=1}^{q-1}\sum_{y_{k+1}=1}^{q-1}\frac{(y_1,\ldots,y_{r-1},y_{k+1},q)^{k/2}}{y_1\cdots y_{r-1}y_{k+1}}$$
$$+q^{t_1+\cdots+t_k+t_{k+1}+k/2+r}d_{k+1}(q)$$
$$\times\sum_{y_1=1}^{q-1}\cdots\sum_{y_{r-1}=1}^{q-1}\sum_{y_{k+1}=1}^{q-1}\sum_{l=1}^{n-1}\frac{(y_1,\ldots,y_{r-1},y_{k+1},q)^{k/2}}{|-y_1n+ql|\cdots|-y_{r-1}n+ql||-y_{k+1}n+ql|l^{k-r+1}}$$
$$\ll q^{t_1+\cdots+t_k+t_{k+1}+3k/2+1}d_{k+1}(q)\sum_{d|q}\sum_{s_1=1}^{(q-1)/d}\cdots\sum_{s_{r-1}=1}^{(q-1)/d}\sum_{s_r=1}^{(q-1)/d}\frac{d^{(k-2r)/2}}{s_1\cdots s_{r-1}s_r}$$
$$+q^{t_1+\cdots+t_k+t_{k+1}+k/2+r}d_{k+1}(q)\sum_{y_1=1}^{q-1}\cdots\sum_{y_{r-1}=1}^{q-1}\sum_{y_{k+1}=1}^{q-1}\sum_{l=1}^{n-1}\frac{(y_1,\ldots,y_{r-1},y_{k+1},q)^{k/2}}{y_1\cdots y_{r-1}y_{k+1}n^r l^{k-r+1}}$$
$$\ll q^{t_1+\cdots+t_k+t_{k+1}+2k-r+1}d_{k+1}(q)d(q)\ln^r q.$$

Finally, for the contribution of Σ_5, we have

$$\Sigma_5=\frac{1}{nq^{k+1}}\sum_{y_1,\ldots,y_{k+1}=1}^{q-1}\sum_{l=1}^{n}S(y_1,\ldots,y_k,y_{k+1}a;q)$$
$$\times K(-y_1,t_1,l)\cdots K(-y_k,t_k,l)K(-y_{k+1},t_{k+1},l)$$
$$\ll\frac{1}{nq^{k+1}}\sum_{y_1=1}^{q-1}\cdots\sum_{y_k=1}^{q-1}\sum_{y_{k+1}=1}^{q-1}\sum_{l=1}^{n}q^{k/2}(\mathbf{y};q)_k^{k/2}d_{k+1}(q)$$
$$\times\frac{(\lambda_1 q)^{t_1}}{\left|\sin\left(\frac{\pi(-y_1n+ql)}{qn}\right)\right|}\cdots\frac{(\lambda_k q)^{t_k}}{\left|\sin\left(\frac{\pi(-y_kn+ql)}{qn}\right)\right|}\frac{(\lambda_{k+1} q)^{t_{k+1}}}{\left|\sin\left(\frac{\pi(-y_{k+1}n+ql)}{qn}\right)\right|}$$
$$\ll q^{t_1+\cdots+t_k+t_{k+1}+k/2}d_{k+1}(q)\sum_{y_1=1}^{q-1}\cdots\sum_{y_k=1}^{q-1}\sum_{y_{k+1}=1}^{q-1}$$
$$\times\sum_{l=1}^{n}\frac{(y_1,\ldots,y_k,y_{k+1},q)^{k/2}n^{k+1}}{|-y_1n+ql|\cdots|-y_kn+ql||-y_{k+1}n+ql|}$$

$$\ll q^{t_1+\cdots+t_k+t_{k+1}+k/2} d_{k+1}(q) \sum_{y_1=1}^{q-1} \cdots \sum_{y_k=1}^{q-1} \sum_{y_{k+1}=1}^{q-1} \sum_{l=1}^{n} \frac{(y_1,\ldots,y_k,y_{k+1},q)^{k/2}}{y_1\cdots y_k y_{k+1}}$$
$$\ll q^{t_1+\cdots+t_k+t_{k+1}+k/2} d_{k+1}(q) \sum_{d|q} \sum_{s_1=1}^{(q-1)/d} \cdots \sum_{s_k=1}^{(q-1)/d} \sum_{s_{k+1}=1}^{(q-1)/d} \frac{1}{d^{(k+2)/2} s_1 \cdots s_k s_{k+1}}$$
$$\ll q^{t_1+\cdots+t_k+t_{k+1}+k/2} d_{k+1}(q) \ln^{k+1} q.$$

Collecting the contributions of all terms, we immediately deduce

$$\mathop{\sum\nolimits'}_{b_1=1}^{[\lambda_1 q]} \mathop{\sum\nolimits'}_{b_2=1}^{[\lambda_2 q]} \cdots \mathop{\sum\nolimits'}_{\substack{b_{k+1}=1 \\ b_1 b_2\cdots b_{k+1}\equiv a \ (\mathrm{mod}\ q) \\ n|(b_1+b_2+\cdots+b_{k+1})}}^{[\lambda_{k+1} q]} b_1^{t_1} b_2^{t_2} \cdots b_{k+1}^{t_{k+1}}$$
$$= \frac{\lambda_1^{t_1+1}\cdots\lambda_k^{t_k+1}\lambda_{k+1}^{t_{k+1}+1}\phi^k(q) q^{t_1+\cdots+t_k+t_{k+1}}}{n(t_1+1)\cdots(t_k+1)(t_{k+1}+1)} + O(q^{t_1+\cdots+t_k+t_{k+1}+k-1/2} d^2(q)\ln^2 q),$$

which implies the second part of Lemma 2.3.

(iii) For the third part of Lemma 2.3, similarly, we have

$$\sum_{l=1}^{n-1} \mathop{\sum\nolimits'}_{b_1=1}^{[\lambda_1 q]} \mathop{\sum\nolimits'}_{b_2=1}^{[\lambda_2 q]} \cdots \mathop{\sum\nolimits'}_{\substack{b_{k+1}=1 \\ b_1 b_2\cdots b_{k+1}\equiv a \ (\mathrm{mod}\ q)}}^{[\lambda_{k+1} q]} \left(\mathrm{e}\left(\frac{b_1 l}{n}\right) b_1^{t_1}\right)\left(\mathrm{e}\left(\frac{b_2 l}{n}\right) b_2^{t_2}\right)\cdots\left(\mathrm{e}\left(\frac{b_{k+1} l}{n}\right) b_{k+1}^{t_{k+1}}\right)$$
$$= \frac{1}{q^{k+1}} \mathop{\sum\nolimits'}_{b_1=1}^{q} \cdots \mathop{\sum\nolimits'}_{b_k=1}^{q} \mathop{\sum\nolimits'}_{\substack{b_{k+1}=1 \\ b_1\cdots b_k b_{k+1}\equiv a \ (\mathrm{mod}\ q)}}^{q} \mathop{\sum\nolimits'}_{c_1=1}^{\lambda_1 q} \cdots \mathop{\sum\nolimits'}_{c_k=1}^{\lambda_k q} \mathop{\sum\nolimits'}_{c_{k+1}=1}^{\lambda_{k+1} q} \left(\mathrm{e}\left(\frac{c_1 l}{n} c_1^{t_1}\right)\right)\cdots\left(\mathrm{e}\left(\frac{c_{k+1} l}{n} c_{k+1}^{t_{k+1}}\right)\right)$$
$$\times \sum_{y_1,\ldots,y_k,y_{k+1}=1}^{q} \mathrm{e}\left(\frac{y_1(b_1-c_1)+\cdots+y_k(b_k-c_k)+y_{k+1}(b_{k+1}-c_{k+1})}{q}\right)$$
$$= \frac{1}{q^{k+1}} \sum_{y_1,\ldots,y_k,y_{k+1}=1}^{q} \mathop{\sum\nolimits'}_{b_1=1}^{q} \cdots \mathop{\sum\nolimits'}_{b_k=1}^{q} \mathrm{e}\left(\frac{y_1 b_1+\cdots+y_k b_k + y_{k+1} a\overline{b_1\cdots b_k}}{q}\right)$$
$$\times \sum_{l=1}^{n-1}\left(\sum_{c_1=1}^{\lambda_1 q} c_1^{t_1}\mathrm{e}\left(c_1\frac{-y_1 n+ql}{qn}\right)\right)\cdots\left(\sum_{c_k=1}^{\lambda_k q} c_k^{t_k}\mathrm{e}\left(c_k\frac{-y_k n+ql}{qn}\right)\right)$$
$$\times \left(\sum_{c_{k+1}=1}^{\lambda_{k+1} q} c_{k+1}^{t_{k+1}}\mathrm{e}\left(c_{k+1}\frac{-y_{k+1} n+ql}{qn}\right)\right)$$
$$= \frac{1}{q^{k+1}} \sum_{y_1,\ldots,y_k,y_{k+1}=1}^{q} \sum_{l=1}^{n-1} S(y_1,\ldots,y_k,y_{k+1}a;q)$$
$$\times K(-y_1,t_1,l)\cdots K(-y_k,t_k,l)K(-y_{k+1},t_{k+1},l).$$

Then using the same method as above, we have

$$\sum_{l=1}^{n-1}\sideset{}{'}\sum_{b_1=1}^{[\lambda_1 q]}\sideset{}{'}\sum_{b_2=1}^{[\lambda_2 q]}\cdots\sideset{}{'}\sum_{\substack{b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a\ (\bmod q)}}^{[\lambda_{k+1} q]}\left(\mathrm{e}\left(\frac{b_1 l}{n}\right)b_1^{t_1}\right)\left(\mathrm{e}\left(\frac{b_2 l}{n}\right)b_2^{t_2}\right)\cdots\left(\mathrm{e}\left(\frac{b_{k+1} l}{n}\right)b_{k+1}^{t_{k+1}}\right)$$
$$= O(q^{t_1+t_2+\cdots+t_{k+1}+k/2}d_{k+1}(q)\ln^{k+1} q).$$

This proves Lemma 2.3. □

Lemma 2.4. *Let q, k, t, m be positive integers with $q > 2$, and a an integer with $(a, q) = 1$. Then*

$$\sideset{}{'}\sum_{b_1=1}^{[\lambda_1 q]}\cdots\sideset{}{'}\sum_{b_k=1}^{[\lambda_k q]}\sideset{}{'}\sum_{\substack{b_{k+1}=1\\ b_1\cdots b_k b_{k+1}\equiv a\ (\bmod q)}}^{q}\sum_{j=1}^{m} C_m^j (b_1\cdots b_t)^{m-j}(b_{t+1}\cdots b_{k+1})^j = O(q^{2k+(m-2)t+1})$$

if $(k+1)/2 < t < k+1$, and

$$\sideset{}{'}\sum_{b_1=1}^{q}\cdots\sideset{}{'}\sum_{b_k=1}^{[\lambda_k q]}\sideset{}{'}\sum_{\substack{b_{k+1}=1\\ b_1\cdots b_k b_{k+1}\equiv a\ (\bmod q)}}^{[\lambda_{k+1} q]}\sum_{j=1}^{m} C_m^j (b_1\cdots b_t)^{j}(b_{t+1}\cdots b_{k+1})^{m-j} = O(q^{2k+(m-2)(k-t+1)+1})$$

if $t < (k+1)/2$.

Proof. Note that

$$(1 + q^{k-2t+1})^m - 1 = O(mq^{k-2t+1})$$

holds for $t > (k + 1)/2$, we have

$$\sideset{}{'}\sum_{b_1=1}^{[\lambda_1 q]}\cdots\sideset{}{'}\sum_{b_k=1}^{[\lambda_k q]}\sideset{}{'}\sum_{\substack{b_{k+1}=1\\ b_1\cdots b_k b_{k+1}\equiv a\ (\bmod q)}}^{q}\sum_{j=1}^{m} C_m^j (b_1\cdots b_t)^{m-j}(b_{t+1}\cdots b_{k+1})^j$$
$$= O\left(\sideset{}{'}\sum_{b_1=1}^{[\lambda_1 q]}\cdots\sideset{}{'}\sum_{b_k=1}^{[\lambda_k q]}\sideset{}{'}\sum_{\substack{b_{k+1}=1\\ b_1\cdots b_k b_{k+1}\equiv a\ (\bmod q)}}^{q}\sum_{j=1}^{m} C_m^j q^{t(m-j)+(k-t+1)j}\right)$$
$$= O\left(q^{mt}\sum_{j=1}^{m} C_m^j q^{(k-2t+1)j}\sideset{}{'}\sum_{b_1=1}^{[\lambda_1 q]}\cdots\sideset{}{'}\sum_{b_k=1}^{[\lambda_k q]} 1\right)$$
$$= O\left(q^{mt}((1+q^{k-2t+1})^m - 1)\prod_{u=1}^{k}\sigma_0(\lambda_u, q)\right)$$
$$= O(q^{2k+(m-2)t+1}).$$

If $t < (k+1)/2$, similarly we have

$$\sideset{}{'}\sum_{\substack{b_1=1 \\ b_1\cdots b_k b_{k+1}\equiv a \ (\mathrm{mod}\ q)}}^{q} \cdots \sideset{}{'}\sum_{b_k=1}^{[\lambda_k q]} \sideset{}{'}\sum_{b_{k+1}=1}^{[\lambda_{k+1} q]} \sum_{j=1}^{m} C_m^j (b_1\cdots b_t)^j (b_{t+1}\cdots b_{k+1})^{m-j} = O(q^{2k+(m-2)(k-t+1)+1}).$$

This completes the proof of Lemma 2.4. □

3. Proofs of theorems

Proof of Theorem 1.1. First for $m > 0$, we have

$$\begin{aligned}
&N(a, k+1, t, \mathbf{w}, q, m, n) \\
&= \sideset{}{'}\sum_{\substack{b_1=1 \\ b_1 b_2\cdots b_{k+1}\equiv a \ (\mathrm{mod}\ q) \\ n\nmid(b_1+b_2+\cdots+b_{k+1})}}^{[\lambda_1 q]} \sideset{}{'}\sum_{b_2=1}^{[\lambda_2 q]} \cdots \sideset{}{'}\sum_{b_{k+1}=1}^{[\lambda_{k+1} q]} (b_1\cdots b_t - b_{t+1}\cdots b_{k+1})^m \\
&= \sum_{j=0}^{m} C_m^j (-1)^j \sideset{}{'}\sum_{\substack{b_1=1 \\ b_1 b_2\cdots b_{k+1}\equiv a \ (\mathrm{mod}\ q) \\ n\nmid(b_1+b_2+\cdots+b_{k+1})}}^{[\lambda_1 q]} \sideset{}{'}\sum_{b_2=1}^{[\lambda_2 q]} \cdots \sideset{}{'}\sum_{b_{k+1}=1}^{[\lambda_{k+1} q]} (b_1\cdots b_t)^{m-j} (b_{t+1}\cdots b_{k+1})^j \\
&= \sum_{j=0}^{m} C_m^j (-1)^j \Bigg(\left(1-\frac{1}{n}\right) \frac{(\lambda_1\cdots\lambda_t)^{m-j+1}(\lambda_{t+1}\cdots\lambda_{k+1})^{j+1}}{(m-j+1)^t (j+1)^{k-t+1}} \phi^k(q) q^{mt+(k-2t+1)j} \\
&\quad + O(q^{mt+(k-2t+1)j+k-1/2} d^2(q)\ln^2 q) \Bigg),
\end{aligned}$$

where we have used Lemma 2.3.

We can separate it into three cases $t > (k+1)/2$, $t < (k+1)/2$, and $t = (k+1)/2$. If $t > (k+1)/2$, noting the fact that the main term only comes from the item $j = 0$ and the other items $0 < j \le m$ are all error terms, we have

$$\begin{aligned}
&N(a, k+1, t, \mathbf{w}, q, m, n) \\
&= \left(1-\frac{1}{n}\right) \frac{(\lambda_1\cdots\lambda_t)^{m+1}(\lambda_{t+1}\cdots\lambda_{k+1})}{(m+1)^t} \phi^k(q) q^{mt} + O(q^{mt+k-1/2} d^2(q)\ln^2 q).
\end{aligned}$$

If $t < (k+1)/2$, similarly we have

$$\begin{aligned}
N(a, k+1, t, \mathbf{w}, q, m, n) &= \left(1-\frac{1}{n}\right) \frac{(-1)^m(\lambda_1\cdots\lambda_t)(\lambda_{t+1}\cdots\lambda_{k+1})^{m+1}}{(m+1)^{k-t+1}} \phi^k(q) q^{m(k-t+1)} \\
&\quad + O(q^{m(k-t+1)+k-1/2} d^2(q)\ln^2 q).
\end{aligned}$$

While if $t = (k+1)/2$, for the reason that all items $0 \le j \le m$ are the main terms, we have

$$\begin{aligned}&N(a, k+1, t, \mathbf{w}, q, m, n)\\&= \sum_{j=0}^{m} C_m^j (-1)^j \left(1 - \frac{1}{n}\right) \frac{(\lambda_1 \cdots \lambda_t)^{m-j+1} (\lambda_{t+1} \cdots \lambda_{k+1})^{j+1}}{((m-j+1)(j+1))^{(k+1)/2}} \phi^k(q) q^{mt}\\&\quad + O(2^m q^{mt+k-1/2} d^2(q) \ln^2 q).\end{aligned}$$

Then for $m = 0$, we have

$$N(a, k+1, t, \mathbf{w}, q, m, n) = \frac{1}{2} \mathop{{\sum}'\sum'\cdots\sum'}_{\substack{b_1=1\ b_2=1 \quad b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a \ (\mathrm{mod}\ q)}}^{[\lambda_1 q]\ [\lambda_2 q] \quad [\lambda_{k+1} q]} 1 - \frac{1}{2} \mathop{{\sum}'\sum'\cdots\sum'}_{\substack{b_1=1\ b_2=1 \quad b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a \ (\mathrm{mod}\ q)}}^{[\lambda_1 q]\ [\lambda_2 q] \quad [\lambda_{k+1} q]} (-1)^{b_1+b_2+\cdots+b_{k+1}}.$$

Thus Theorem 1.1 holds again by appealing to Lemma 2.3. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.5. Noting that $(k+1)/2 < t < k+1$ and $\mathbf{w} = (\lambda_1, \lambda_2, \dots, \lambda_{k+1})$ with at least one $\lambda_j = 1$ $(j > t)$, we can assume $\lambda_{k+1} = 1$ (without loss of generality). Then we have

$$\begin{aligned}&N_{k+1}(a, k+1, t, \mathbf{w}, q, m, n)\\
&= \mathop{{\sum}'\cdots\sum'\sum'}_{\substack{b_1=1 \quad b_k=1\ b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a \ (\mathrm{mod}\ q)\\ n\nmid(b_1+b_2+\cdots+b_{k+1})}}^{[\lambda_1 q] \quad [\lambda_k q] \quad q} (b_1\cdots b_t - b_{t+1}\cdots b_{k+1})^m\\
&= \mathop{{\sum}'\cdots\sum'\sum'}_{\substack{b_1=1 \quad b_k=1\ b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a \ (\mathrm{mod}\ q)}}^{[\lambda_1 q] \quad [\lambda_k q] \quad q} (b_1\cdots b_t - b_{t+1}\cdots b_{k+1})^m\\
&\quad - \mathop{{\sum}'\cdots\sum'\sum'}_{\substack{b_1=1 \quad b_k=1\ b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a \ (\mathrm{mod}\ q)\\ n\mid(b_1+b_2+\cdots+b_{k+1})}}^{[\lambda_1 q] \quad [\lambda_k q] \quad q} (b_1\cdots b_t - b_{t+1}\cdots b_{k+1})^m\\
&= \mathop{{\sum}'\cdots\sum'\sum'}_{\substack{b_1=1 \quad b_k=1\ b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a \ (\mathrm{mod}\ q)}}^{[\lambda_1 q] \quad [\lambda_k q] \quad q} (b_1\cdots b_t - b_{t+1}\cdots b_{k+1})^m\\
&\quad - \frac{1}{n}\mathop{{\sum}'\cdots\sum'\sum'}_{\substack{b_1=1 \quad b_k=1\ b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a \ (\mathrm{mod}\ q)}}^{[\lambda_1 q] \quad [\lambda_k q] \quad q} (b_1\cdots b_t - b_{t+1}\cdots b_{k+1})^m\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{n}\mathop{{\sum}'}_{b_1=1}^{[\lambda_1 q]}\cdots\mathop{{\sum}'}_{b_k=1}^{[\lambda_k q]}\mathop{{\sum}'}_{\substack{b_{k+1}=1\\ b_1b_2\cdots b_{k+1}\equiv a\ (\bmod\ q)}}^{q}(b_1\cdots b_t-b_{t+1}\cdots b_{k+1})^m\sum_{l=1}^{n-1}\mathrm{e}\left(\frac{b_1+\cdots+b_{k+1}}{n}l\right)\\
&=\left(1-\frac{1}{n}\right)\mathop{{\sum}'}_{b_1=1}^{[\lambda_1 q]}\cdots\mathop{{\sum}'}_{b_k=1}^{[\lambda_k q]}\mathop{{\sum}'}_{\substack{b_{k+1}=1\\ b_1\cdots b_kb_{k+1}\equiv a\ (\bmod\ q)}}^{q}(b_1\cdots b_t)^m\\
&\quad+\left(1-\frac{1}{n}\right)\sum_{j=1}^{m}C_m^j(-1)^j\mathop{{\sum}'}_{b_1=1}^{[\lambda_1 q]}\cdots\mathop{{\sum}'}_{b_k=1}^{[\lambda_k q]}\mathop{{\sum}'}_{\substack{b_{k+1}=1\\ b_1\cdots b_kb_{k+1}\equiv a\ (\bmod\ q)}}^{q}(b_1\cdots b_t)^{m-j}(b_{t+1}\cdots b_{k+1})^j\\
&\quad-\frac{1}{n}\sum_{j=0}^{m}C_m^j(-1)^j\sum_{l=1}^{n-1}\mathop{{\sum}'}_{b_1=1}^{[\lambda_1 q]}\cdots\mathop{{\sum}'}_{b_k=1}^{[\lambda_k q]}\mathop{{\sum}'}_{\substack{b_{k+1}=1\\ b_1\cdots b_kb_{k+1}\equiv a\ (\bmod\ q)}}^{q}\left(\mathrm{e}\left(\frac{b_1l}{n}\right)b_1^{m-j}\right)\cdots\left(\mathrm{e}\left(\frac{b_tl}{n}\right)b_t^{m-j}\right)\\
&\quad\times\left(\mathrm{e}\left(\frac{b_{t+1}l}{n}\right)b_{t+1}^{j}\right)\cdots\left(\mathrm{e}\left(\frac{b_{k+1}l}{n}\right)b_{k+1}^{j}\right)\\
&=\left(1-\frac{1}{n}\right)\mathop{{\sum}'}_{b_1=1}^{[\lambda_1 q]}\cdots\mathop{{\sum}'}_{b_k=1}^{[\lambda_k q]}(b_1\cdots b_t)^m+O(q^{mt+k/2}d_{k+1}(q)\ln^{k+1}q)\\
&\quad+O\left(\mathop{{\sum}'}_{b_1=1}^{[\lambda_1 q]}\cdots\mathop{{\sum}'}_{b_k=1}^{[\lambda_k q]}\mathop{{\sum}'}_{\substack{b_{k+1}=1\\ b_1\cdots b_kb_{k+1}\equiv a\ (\bmod\ q)}}^{q}\sum_{j=1}^{m}C_m^j(b_1\cdots b_t)^{m-j}(b_{t+1}\cdots b_{k+1})^j\right)\\
&=\left(1-\frac{1}{n}\right)\prod_{u=1}^{t}\sigma_m(\lambda_u,q)\prod_{\substack{v=t+1\\ v\neq j}}^{k+1}\sigma_0(\lambda_v,q)+O(q^{2k+(m-2)t+1})+O(q^{mt+k/2}d_{k+1}(q)\ln^{k+1}q),
\end{aligned}
$$

where we have used Lemmas 2.3 and 2.4.

If $t<(k+1)/2$ and $\mathbf{w}=(\lambda_1,\lambda_2,\ldots,\lambda_{k+1})$ with at least one $\lambda_j=1$ $(j<t)$, similarly we have

$$
\begin{aligned}
N_1(a,k+1,t,\mathbf{w},q,m,n)&=\left(1-\frac{1}{n}\right)(-1)^m\prod_{\substack{u=1\\ u\neq j}}^{t}\sigma_0(\lambda_u,q)\prod_{v=t+1}^{k+1}\sigma_m(\lambda_v,q)\\
&\quad+O(q^{2k+(m-2)(k-t+1)+1})+O(q^{m(k-t+1)+k/2}d_{k+1}(q)\ln^{k+1}q).
\end{aligned}
$$

If $t=(k+1)/2$ and $\mathbf{w}=(\lambda_1,\lambda_2,\ldots,\lambda_{k+1})$ with at least one $\lambda_j=1$ $(j>t)$, we have

$$
\begin{aligned}
&N_{k+1}(a,k+1,t,\mathbf{w},q,m,n)\\
&=\left(1-\frac{1}{n}\right)\mathop{{\sum}'}_{b_1=1}^{[\lambda_1 q]}\cdots\mathop{{\sum}'}_{b_k=1}^{[\lambda_k q]}\mathop{{\sum}'}_{\substack{b_{k+1}=1\\ b_1\cdots b_kb_{k+1}\equiv a\ (\bmod\ q)}}^{q}(b_1\cdots b_t)^m
\end{aligned}
$$

$$
\begin{aligned}
&+\left(1-\frac{1}{n}\right)\sum_{j=1}^{m}C_m^j(-1)^j\sideset{}{'}\sum_{\substack{b_1=1\\ b_1\cdots b_kb_{k+1}\equiv a\ (\bmod q)}}^{[\lambda_1 q]}\cdots\sideset{}{'}\sum_{b_k=1}^{[\lambda_k q]}\sideset{}{'}\sum_{b_{k+1}=1}^{q}(b_1\cdots b_t)^{m-j}(b_{t+1}\cdots b_{k+1})^j\\
&-\frac{1}{n}\sum_{j=0}^{m}C_m^j(-1)^j\sum_{l=1}^{n-1}\sideset{}{'}\sum_{\substack{b_1=1\\ b_1\cdots b_kb_{k+1}\equiv a\ (\bmod q)}}^{[\lambda_1 q]}\cdots\sideset{}{'}\sum_{b_k=1}^{[\lambda_k q]}\sideset{}{'}\sum_{b_{k+1}=1}^{q}\left(\mathrm{e}\left(\frac{b_1 l}{n}\right)b_1^{m-j}\right)\cdots\left(\mathrm{e}\left(\frac{b_t l}{n}\right)b_t^{m-j}\right)\\
&\times\left(\mathrm{e}\left(\frac{b_{t+1} l}{n}\right)b_{t+1}^{j}\right)\cdots\left(\mathrm{e}\left(\frac{b_{k+1} l}{n}\right)b_{k+1}^{j}\right)\\
=&\left(1-\frac{1}{n}\right)\sideset{}{'}\sum_{b_1=1}^{[\lambda_1 q]}\cdots\sideset{}{'}\sum_{b_k=1}^{[\lambda_k q]}(b_1\cdots b_t)^m+O(q^{mt+k/2}d_{k+1}(q)\ln^{k+1}q)\\
&+\left(1-\frac{1}{n}\right)\sum_{j=1}^{m}C_m^j(-1)^j\left(\frac{(\lambda_1\cdots\lambda_t)^{m-j+1}(\lambda_{t+1}\cdots\lambda_k)^{j+1}}{(m-j+1)^t(j+1)^{k-t+1}}\phi^k(q)q^{mt+(k-2t+1)j}\right.\\
&\left.+O(q^{mt+(k-2t+1)j+k-1/2}d^2(q)\ln^2 q)\right)\\
=&\left(1-\frac{1}{n}\right)\prod_{u=1}^{(k+1)/2}\sigma_m(\lambda_u,q)\prod_{\substack{v=(k+3)/2\\ v\neq j}}^{k+1}\sigma_0(\lambda_v,q)+O(2^m q^{m(k+1)/2+k-1/2}d^2(q)\ln^2 q)\\
&+\left(1-\frac{1}{n}\right)\sum_{j=0}^{m}C_m^j(-1)^j\frac{(\lambda_1\cdots\lambda_t)^{m-j+1}(\lambda_{t+1}\cdots\lambda_k)^{j+1}}{((m-j+1)(j+1))^{(k+1)/2}}\phi^k(q)q^{m(k+1)/2},
\end{aligned}
$$

where we have used Lemmas 2.3 and 2.4. This completes the proof of Theorem 1.5. □

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