TAIWANESE JOURNAL OF MATHEMATICS
Vol. 26, No. 1, pp. 17–32, February 2022
DOI: 10.11650/tjm/210702

Maximal Density of Integral Sets with Missing Differences and the Kappa Values

Ram Krishna Pandey and Anshika Srivastava*

Abstract. Let M be a given set of positive integers. A set S of nonnegative integers is said to be an *M-set* if $a, b \in S$ implies $a-b \notin M$. In an unpublished problem collection, Motzkin asked to find *maximal upper asymptotic density*, denoted by $\mu(M)$, of M-sets. The first published work on $\mu(M)$ is due to Cantor and Gordon in 1973, in which, they found the exact value of $\mu(M)$ when $|M| \leq 2$. In fact, this is the only general case, in which, we have a closed formulae for $\mu(M)$. If $|M| \geq 3$, then the exact value of $\mu(M)$ is not known for the general set M. In the past six decades or so, several attempts have been given to study $\mu(M)$ but $\mu(M)$ has been found exactly or estimated only in very few cases. In this paper, we study $\mu(M)$ for the families $M = \{a, a+1, x\}$ and $M = \{a, a+1, x, y\}$, where $y - x \leq 2$ and $y > x > a+1$. Our results in the case of $M = \{a, a+1, x\}$ also give counterexamples to a conjecture of Carraher. Although, different counterexamples to this conjecture, were already given by Liu and Robinson in 2020. We also relate our results with the already know results for the families $M = \{1, 2, x, x+2\}$ and $M = \{2, 3, x, x+2\}$.

1. Introduction

Let S be a set of nonnegative integers and let $S(x)$ denote the number of elements $n \in S$ such that $1 \leq n \leq x$, $x \in \mathbb{R}$. The upper and lower asymptotic densities of S, denoted respectively by $\overline{\delta}(S)$ and $\underline{\delta}(S)$, are defined by

$$\overline{\delta}(S) := \limsup_{x \to \infty} \frac{S(x)}{x} \quad \text{and} \quad \underline{\delta}(S) := \liminf_{x \to \infty} \frac{S(x)}{x}.$$

If $\overline{\delta}(S) = \underline{\delta}(S) = \delta(S)$, then we say that the density of the set S is $\delta(S)$. Now let M be a given set of positive integers. Then S is said to be an M-set if $a, b \in S$ implies $a - b \notin M$. In an unpublished problem collection, Motzkin asked how dense can an M-set be? More precisely, Motzkin asked to determine the maximal density of M-sets, defined by

$$\mu(M) := \sup_S \overline{\delta}(S),$$

Received November 14, 2020; Accepted July 13, 2021.
Communicated by Daphne Der-Fen Liu.
2020 *Mathematics Subject Classification.* 11B05, 11B39.
Key words and phrases. upper density, M-sets, lonely runner conjecture.
*Corresponding author.

where the supremum is taken over all M-sets S.

Since we have $\mu(M) = \mu(tM)$ for any positive integer t [1, Theorem 2], it is sufficient to consider the case $\gcd(M) = 1$. Cantor and Gordon [1] found $\mu(M)$ in the case $|M| \le 2$. They also proved that if M is a finite set then there is a periodic M-set whose density is equal to $\mu(M)$.

For a real number x, let $\|x\|$ denote the distance of x from the nearest integer, i.e., $\|x\| = \min\{x - \lfloor x \rfloor, \lceil x \rceil - x\}$. The following remark due to Haralambis [14] gives some equivalent definitions for the *kappa value of* M, denoted by, $\kappa(M)$.

Remark 1.1. Let $M = \{m_1, m_2, \ldots, m_n\}$ and

$$d_1(M) = \sup_{x\in(0,1)} \min_i \|xm_i\|, \quad d_2(M) = \sup_{(k,m)=1} (1/m) \min_i |km_i|_m,$$
$$d_3(M) = \max_{\substack{m=m_r+m_s\\ 1\le k\le m/2}} (1/m) \min_i |km_i|_m.$$

Then $d_1(M) = d_2(M) = d_3(M) = \kappa(M)$.

We also define for a real number t and a set M of real numbers, $\|tM\| := \inf\{\|tm\| : m \in M\}$.

The parameter $\kappa(M)$ which serves as a lower bound for $\mu(M)$ [1, Theorem 1], is related to the "lonely runner conjecture". The lonely runner conjecture is a long standing open conjecture on the diophantine approximations, which was first posed by Wills [26] and then independently by Cusick [11]. The conjecture, in our context, may simply be read as: if M is a finite set of positive integers with $|M| = m$, then $\kappa(M) \ge 1/(m + 1)$. Our focus is not on the conjecture in the present article. For the current developments and all related references on the conjecture, one can see the recent paper by Tao [25]. Some old and important results on $\kappa(M)$ may be found in [3–9].

Cantor and Gordon [1] proved that if $M = \{a\}$, then $\mu(M) = \kappa(M) = 1/2$; and if $M = \{a, b\}$ with $\gcd(a, b) = 1$, then $\mu(M) = \kappa(M) = \frac{\lfloor (a+b)/2 \rfloor}{a+b}$. However, if $|M| > 3$, then there exist sets M (see [1, 16]) such that $\mu(M) > \kappa(M)$. In case $|M| = 3$, it is still unknown that whether or not $\mu(M) = \kappa(M)$? For $|M| \ge 3$, the problem was studied extensively by several authors in different contexts. Most of the important and relevant results may be found in [1, 12–14, 16–21, 23, 24]. Haralambis [14] gave expressions for $\mu(M)$ for most members of the families $\{1, x, y\}$ and $\{1, 2, x, y\}$. For the general 3-element set M, Gupta [12] gave a lower bound for $\mu(M)$ and proved that in some specific cases these bounds are the exact values. Rabinowitz and Proulx [22] gave a lower bound for $\mu(M)$ when $M = \{x, y, x + y\}$ and conjectured that the bound is the exact value. Liu and Zhu [16] confirmed their conjecture and completely determined the values of $\kappa(M)$ and $\mu(M)$ in this case. They also established that $\mu(M) = \kappa(M)$ in this case. They further computed the value of $\kappa(M)$ for $M = \{x, y, y - x, x + y\}$, $y > x$ and gave a better lower bound than $\kappa(M)$ for $\mu(M)$ when both x and y are odd. In the rest of the cases, Liu and Zhu [16] exactly computed $\mu(M)$ as well as $\kappa(M)$. Collister and Liu [10] has discussed $\kappa(M)$ and $\mu(M)$ for $M = \{2, 3, x, y\}$ with $|x - y| \leq 6$. Very recently, Liu and Robinson [15, Theorems 1, 2, 3] have found $\mu(M)$ and $\kappa(M)$ and proved their equality for most of the remaining cases of the family $\{1, x, y\}$. Their results also provide the counterexamples for the two conjectures of Carraher et al. [2].

In this paper, we study maximal density problem for certain three and four-element sets M. In Section 2, we investigate the values of $\mu(M)$ and $\kappa(M)$ for the 3-element set $M = \{a, a+1, x\}$. In Section 3, we investigate $\mu(M)$ and $\kappa(M)$ for the 4-element sets $M = \{a, a+1, x, x+1\}$ and $M = \{a, a+1, x, x+2\}$. In this study, we relate our results with the already known results for the families $M = \{1, 2, x, x+2\}$ [14] and $M = \{2, 3, x, x+2\}$ [10]. Our Corollary 2.6 for $a = 1$, provides the counterexamples to the conjecture [2, Conjecture 29]. Although, different counterexamples to this conjecture were already provided by Liu and Robinson [15].

2. The family $M = \{a, a+1, x\}$

For the sake of completeness, we mention below, Lemmas 2.1 and 2.2, which respectively, give sharp lower and upper bounds for $\mu(M)$.

Lemma 2.1. [1] *Let $M = \{m_1, m_2, \ldots\}$. Then*

$$\mu(M) \geq \kappa(M) := \sup_{\gcd(c,m)=1} (1/m) \min_{k \geq 1} |cm_k|_m,$$

where for an integer x and a positive integer m, $|x|_m = |r|$ if $x \equiv r \pmod{m}$ with $0 \leq |r| \leq m/2$ and the supremum is taken over all pairs of relatively prime positive integers c and m.

Lemma 2.2. [14] *Let $S(n) = |\{0, 1, 2, \ldots, n\} \cap S|$. Let α be a real number, $\alpha \in [0, 1]$. If for any M-set S with $0 \in S$ there exists a positive integer k such that $S(k) \leq (k+1)\alpha$, then $\mu(M) \leq \alpha$.*

In this section, first we give a general lower bound for $\kappa(M)$ in Theorem 2.3. Then, in Theorem 2.4, we prove that this lower bound turns out to be the exact value for both of $\mu(M)$ and $\kappa(M)$ for certain families of $\{a, a+1, x\}$. Later, we improve the lower bound for $\kappa(M)$ presented in Theorem 2.3, in a series of propositions, for certain families of $\{a, a+1, x\}$.

Theorem 2.3. *Let $M = \{a, a+1, x\}$ with $x > a+1$. If $x = k(2a+1) - r$, where $0 \leq r \leq 2a$ and $k \geq 1$, then*

$$\kappa(M) \geq \begin{cases} \frac{ak}{x+a} & \text{if } 0 \leq r \leq a, \\ \frac{ak+a+1-r}{x+a+1} & \text{if } a+1 \leq r \leq 2a. \end{cases}$$

Proof. Consider the following two cases.

Case 1*:* $0 \le r \le a$. Let $t = k$. Then we have

$$\begin{aligned} at &\equiv ak \pmod{x+a}, \\ (a+1)t &= (a+1)k = \big(a - r + k(2a+1)\big) - (ak + a - r) \\ &= (a+x) - (ak + a - r) \equiv -(ak + a - r) \pmod{x+a}, \\ xt &\equiv -at \equiv -ak \pmod{x+a}. \end{aligned}$$

Since $x+a = (ak+k)+(ak+(a-r))$ and $0 \le r \le a$, we have $ak \le \min\{ak+(a-r), ak+k\} \le (x+a)/2$.

Case 2*:* $a+1 \le r \le 2a$. Let $t = k$. Then we have

$$\begin{aligned} at &\equiv ak \pmod{x+a+1}, \\ (a+1)t &= (a+1)k = (x+a+1) - a(k+1) + k(2a+1) - x - 1 \\ &= (x+a+1) - \big(a(k+1) - r + 1\big) \equiv -\big(a(k+1) - r + 1\big) \pmod{x+a+1}, \\ xt &\equiv -(a+1)t \pmod{x+a+1}. \end{aligned}$$

Since $x+a+1 = (ak+k)+(a(k+1)-r+1) = ak+(a(k+1)-r+1+k)$ and $a+1 \le r \le 2a$, we have $a(k+1) - r + 1 \le \min\{ak, a(k+1) - r + 1 + k\} \le (x+a+1)/2$.

So, using $d_3(M)$ for $\kappa(M)$, we get

$$\kappa(M) \ge \begin{cases} \frac{ak}{x+a} & \text{if } 0 \le r \le a, \\ \frac{ak+a+1-r}{x+a+1} & \text{if } a+1 \le r \le 2a. \end{cases}$$

This completes the proof of the theorem. □

Theorem 2.4. *Let $M = \{a, a+1, x\}$ with $x > a+1$. If $x = k(2a+1) - r$, where $0 \le r \le 2a$ and $k \ge 1$, then*

$$\kappa(M) = \mu(M) = \begin{cases} \frac{ak}{x+a} & \text{if } 0 \le r \le a, \\ \frac{ak+a+1-r}{x+a+1} & \text{if } a+1 \le r \le 2a \end{cases}$$

provided

$$k \ge \begin{cases} a - r - 1 & \text{if } 0 \le r \le \lfloor \frac{a-1}{2} \rfloor, \\ a - r & \text{if } \lceil \frac{a}{2} \rceil \le r \le a, \\ r - a - 1 & \text{if } a+1 \le r \le 2a. \end{cases}$$

Proof. From Theorem 2.3, we have

$$\mu(M) \ge \begin{cases} \frac{ak}{x+a} & \text{if } 0 \le r \le a, \\ \frac{ak+a+1-r}{x+a+1} & \text{if } a+1 \le r \le 2a. \end{cases}$$

If possible, let $\mu(M)$ be strictly greater than the above mentioned bound. Then by Lemma 2.2 there exists an M-set S such that for any $n \geq 0$,

$$S(n) > \begin{cases} \frac{ak}{x+a}(n+1) & \text{if } 0 \leq r \leq a, \\ \frac{ak+a+1-r}{x+a+1}(n+1) & \text{if } a+1 \leq r \leq 2a. \end{cases} \tag{2.1}$$

Now using

$$k \geq \begin{cases} a-r-1 & \text{if } 0 \leq r \leq \lfloor \frac{a-1}{2} \rfloor, \\ a-r & \text{if } \lceil \frac{a}{2} \rceil \leq r \leq a, \\ r-a-1 & \text{if } a+1 \leq r \leq 2a \end{cases}$$

in (2.1), we can show that $S(2a-1) \geq a$. On the other hand, $|S \cap [0, 2a]| \leq a$, as S is an $\{a, a+1\}$-set. So $|S \cap [0, 2a-1]| = a$. To obtain $S \cap [0, 2a-1] = [0, a-1]$, we assume on contrary that $S \cap [a, 2a-1] \neq \emptyset$.

Let $a+i$ be the largest element of $[a, 2a-1]$ which is in S, i.e., for $i < j < a$, $a+j \notin S$. Now we have the following conditions on an $\{a, a+1\}$-set S.

- $S \cap [0, 2a-1] = S \cap [0, a+i]$,
- $a+i \in S \;\Rightarrow\; i, i-1 \notin S$,
- $S(0) \geq 1$, gives $0 \in S \;\Rightarrow\; a, a+1 \notin S$ and $i \geq 2$,
- S has at most one element of $\{l, l+a+1\}$.

We partition $[0, a+i]$ into disjoint sets $\{0, a, a+1\}, \{1, a+2\}, \{2, a+3\}, \ldots, \{i-2, a+i-1\}, \{i-1, i, a+i\}$, and $\{i+1, i+2, \ldots, a-1\}$. This gives

$$\begin{aligned} |S \cap [0, a+i]| &\leq 1 + \underbrace{1 + 1 + \cdots + 1}_{(i-2) \text{ times}} + 1 + (a-1-i-1+1) \\ &= 1 + i - 2 + 1 + a - i - 1 = a - 1. \end{aligned}$$

This means S contains at most $a-1$ elements from the set $[0, a+i]$, which is a contradiction as $|S \cap [0, 2a-1]| = |S \cap [0, a+i]| = a$. Hence $S \cap [0, 2a-1] = [0, a-1]$. Now we consider the following two cases.

Case 1: $0 \leq r \leq a$. From (2.1), we have $S\big(k(2a+1) + a - r - 1\big) > ak$. Moreover, since S is an $\{a, a+1\}$-set, we have $S\big(k(2a+1) - 1\big) \leq ak$. Therefore, at least one element of the set $\{k(2a+1), k(2a+1)+1, k(2a+1)+2, \ldots, k(2a+1)+a-r-1\}$ must be in S. But this is not true since for $0 \leq r \leq a-1$ we have $\{r, r+1, \ldots, a-1\} \subset S$ and $x = k(2a+1) - r \in M$. So

$$\mu(M) = \kappa(M) = \frac{ka}{k(2a+1)+a-r} = \frac{ak}{x+a}.$$

Case 2: $a+1 \le r \le 2a$. From (2.1), we have $S\big(k(2a+1)+a-r\big) > ak+a+1-r$. Moreover, since S is an $\{a, a+1\}$-set, we have $S(k(2a+1)-(2a+2)) \le ak-a$. Therefore, at least $(2a-r+2)$ elements of the set $\{k(2a+1)-(2a+1), k(2a+1)-2a, k(2a+1)-(2a-1), \ldots, k(2a+1)+a-r-1, k(2a+1)+a-r\}$ must be in S. But, since $\{0,1,2,\ldots,a-1\} \subset S$ and $x = k(2a+1)-r \in M$, elements from the set $\{k(2a+1)-r, k(2a+1)-r+1, \ldots, k(2a+1)+a-r-1\}$ cannot be in S. Furthermore, since $|(k(2a+1)+a-r)-(k(2a+1)-r-1)| = a+1 \in M$, we have exactly one element of $\{k(2a+1)+a-r, k(2a+1)-r-1\}$ in S. Therefore,

$$\begin{aligned} &S \cap [k(2a+1)-(2a+1), k(2a+1)+a-r] \\ &= S \cap \big([k(2a+1)-(2a+1), k(2a+1)-r-2] \cup \{k(2a+1)+a-r, k(2a+1)-r-1\}\big) \end{aligned}$$

can contain at most $(2a-r+1)$ elements. This is a contradiction. So

$$\mu(M) = \kappa(M) = \frac{ak+a+1-r}{k(2a+1)+a-r+1} = \frac{ak+a+1-r}{x+a+1}.$$

This completes the proof of the theorem. $\square$

Corollary 2.5. *Let $M = \{a, a+1, x\}$, where $x = k(2a+1)-r > a+1$ with $0 \le r \le 2a$ and $k \ge 1$. Let $r \in \{a-1, a, a+1, a+2\}$. Then*

$$\mu(M) = \kappa(M) = \begin{cases} \frac{ak}{x+a} & \text{if } 0 \le r \le a, \\ \frac{ak+a+1-r}{x+a+1} & \text{if } a+1 \le r \le 2a. \end{cases}$$

Proof. The proof follows immediately from Theorem 2.4. $\square$

Corollary 2.6. *Let $M = \{a, a+1, x\}$, where $x = k(2a+1)-r > a+1$ with $0 \le r \le 2a$ and $k \ge 1$. Let $a \in \{1,2,3,4\}$. Then*

$$\mu(M) = \kappa(M) = \begin{cases} \frac{ak}{x+a} & \text{if } 0 \le r \le a, \\ \frac{ak+a+1-r}{x+a+1} & \text{if } a+1 \le r \le 2a. \end{cases}$$

Proof. It follows from Theorem 2.4 that if $a \in \{1,2,3,4\}$ and $M \notin \big\{\{3,4,7\}, \{4,5,7\}, \{4,5,8\}, \{4,5,9\}, \{4,5,10\}, \{4,5,18\}\big\}$, then $\mu(M) = \kappa(M)$. However, for these few exceptions of M the $\mu(M)$ and $\kappa(M)$ may directly be computed to obtain

$$\mu(M) = \kappa(M) = \begin{cases} \frac{ak}{x+a} & \text{if } 0 \le r \le a, \\ \frac{ak+a+1-r}{x+a+1} & \text{if } a+1 \le r \le 2a. \end{cases}$$

For the sake of the convenience of the reader, we verify the above relation for $M = \{3,4,7\}$. Clearly, we have $a = 3$, $x = 7$. Hence $k = 1$ and $r = 0$. From Theorem 2.3, we have

$\mu(M) \geq \kappa(M) \geq ak/(x+a) = 3/10$. To prove the reverse inequality, assume to the contrary that $\mu(M) > 3/10$. Then by Lemma 2.2 there exists an M-set S such that for any $n \geq 0$, $S(n) > \frac{3}{10}(n+1)$. This implies, for instance, $|S \cap [0,9]| = S(9) \geq 4$. But no subset of $S \cap [0,9]$ has this property. Therefore $\mu(M) \leq 3/10$. Combining, we have $3/10 \geq \mu(M) \geq \kappa(M) \geq ak/(x+a) = 3/10$. Hence $\mu(M) = \kappa(M) = 3/10$. □

From Corollary 2.6, we have if $a \leq 4$, then the lower bound established in Theorem 2.3 for $\kappa(M)$ is the exact value of $\kappa(M)$. Furthermore, it also gives that $\mu(M) = \kappa(M)$. However, this is not always true for $a \geq 5$. Following three propositions give some families of M in which the lower bound established in Theorem 2.3 for $\kappa(M)$ have been improved.

Proposition 2.7. *Let $M = \{a, a+1, 2a-1\}$, where $a = 5k+i$ with $k \geq 1$. Then*

$$\kappa(M) \geq \begin{cases} \frac{a-k}{2a+1} & \text{if } 0 \leq i \leq 2, \\ \frac{a+k}{3a-1} & \text{if } 3 \leq i \leq 4. \end{cases}$$

Proof. Consider the following cases.

Case 1*:* $0 \leq i \leq 2$. Let $t = 2k+1$. Then we have

$$\begin{aligned} at = 2ak + a = k(2a+1) + (a-k) &\equiv a-k \pmod{2a+1}, \\ (a+1)t = (k+1)(2a+1) - (a-k) &\equiv -(a-k) \pmod{2a+1}, \\ (2a-1)t = (2a+1)(2k+1) - 2(2k+1) &\equiv -2(2k+1) \pmod{2a+1}. \end{aligned}$$

We have $\min\{a-k, 4k+2\} = \min\{4k+i, 4k+2\} = 4k+i = a-k < (2a+1)/2$.

Case 2*:* $3 \leq i \leq 4$. Let $t = 3k+1$. Then we have

$$\begin{aligned} at &= k(3a-1) + (a+k) \equiv a+k \pmod{3a-1}, \\ (a+1)t &= (k+1)(3a-1) - 2a + 4k + 2 = (k+1)(3a-1) - 2(a-(2k+1)) \\ &\equiv -2(a-(2k+1)) \pmod{3a-1}, \\ (2a-1)t &\equiv -at \pmod{3a-1}. \end{aligned}$$

For $3 \leq i \leq 4$, we have that $a+k \leq 2(a-(2k+1)) \leq (3a-1)/2$. Therefore,

$$\kappa(M) \geq \begin{cases} \frac{a-k}{2a+1} & \text{if } 0 \leq i \leq 2, \\ \frac{a+k}{3a-1} & \text{if } 3 \leq i \leq 4. \end{cases}$$

This completes the proof of the proposition. □

Proposition 2.8. *Let $M = \{a, a+1, 2a+3\}$, where $a = 5k+i \geq 6$ with $k \geq 1$. Then*

$$\kappa(M) \geq \begin{cases} \frac{a-k}{2a+1} & \text{if } 1 \leq i \leq 2, \\ \frac{a+2+k}{3a+4} & \text{if } 3 \leq i \leq 5. \end{cases}$$

Proof. Consider the following cases.

Case 1*:* $1 \le i \le 2$. Let $t = 2k+1$. Then we have

$$\begin{aligned}
at &= 2ak + a = k(2a+1) + (a-k) \equiv a-k \pmod{2a+1},\\
(a+1)t &\equiv -at \equiv -(a-k) \pmod{2a+1},\\
(2a+3)t &= (2a+1)(2k+1) + 2(2k+1) \equiv 2(2k+1) \pmod{2a+1}.
\end{aligned}$$

For $1 \le i \le 2$, we have that $a-k \le 4k+2 \le (2a+1)/2$.

Case 2*:* $3 \le i \le 5$. Let $t = 3k+2$. Then we have

$$\begin{aligned}
at &= k(3a+4) + 2(a-2k) \equiv 2(a-2k) \pmod{3a+4},\\
(a+1)t &= (k+1)(3a+3) - (a+1) = (k+1)(3a+4) - (a+k+2)\\
&\equiv -(a+k+2) \pmod{3a+4},\\
(2a+3)t &\equiv -(a+1)t \equiv (a+k+2) \pmod{3a+4}.
\end{aligned}$$

For $3 \le i \le 5$, we have that $a+k+2 \le 2(a-2k) \le (3a+4)/2$. Therefore,

$$\kappa(M) \ge \begin{cases} \frac{a-k}{2a+1} & \text{if } 1 \le i \le 2,\\ \frac{a+2+k}{3a+4} & \text{if } 3 \le i \le 5. \end{cases}$$

This completes the proof of the proposition. □

Proposition 2.9. *Let $M = \{a, a+1, x\}$. Let $a = 3k+i \ge 6$ with $k \ge 1$ and $x = n(2a+1) \pm r$, where $1 \le n \le (k-1)$ and $(i,r) \in \{(0,k), (1,k), (1,k+1), (2,k+1)\}$. Then $\kappa(M) \ge (a-1)/(2a+1)$.*

Proof. Note that

$$3a = (2a+1) + (a-1), \quad 3(a+1) = 2(2a+1) - (a-1)$$

and

$$3(n(2a+1) \pm r) = 3n(2a+1) \pm 3r.$$

We see that, in each case, $a-1 \le 3r \le a+2$. Therefore $\mu(M) \ge \kappa(M) \ge \left\| \frac{3}{2a+1} M \right\| = (a-1)/(2a+1)$. Hence the proposition holds. □

3. The families $M = \{a, a+1, x, y\}$

In this section, we study $\kappa(M)$ and $\mu(M)$ when $y = x+1$ and $y = x+2$ with $a < x-1$. For $M = \{a, a+1, x, x+1\}$, first we give a general lower bound for $\kappa(M)$ in Theorem 3.1. Then, in Theorem 3.2, we prove that this lower bound turns out to be the exact value for both of $\mu(M)$ and $\kappa(M)$ for certain families of $\{a, a+1, x, x+1\}$. For $M = \{a, a+1, x, x+2\}$, we give general lower bounds for $\kappa(M)$ in Theorem 3.5 and in a series of propositions.

Theorem 3.1. *Let* $M = \{a, a+1, x, x+1\}$*, where* $x = k(2a+1) + r > a+1$ *with* $0 \le r \le 2a$*. Then*

$$\kappa(M) \ge \begin{cases} \frac{ak+r}{x+a+1} & \text{if } 0 \le r \le a-1, \\ \frac{ak+a}{x+a+1} & \text{if } a \le r \le 2a. \end{cases}$$

Proof. Let $t = k+1$. So we have

$$\begin{aligned} at &\equiv ak + a \pmod{x+a+1}, \\ (a+1)t &= (a+1)(k+1) = (x+a+1) - (x - (a+1)k) \\ &= (x+a+1) - (ak+r) \equiv -(ak+r) \pmod{x+a+1}, \\ xt &= x(k+1) = xk + (r + k(2a+1)) \\ &= k(x+a+1) + (ak+r) \equiv ak + r \pmod{x+a+1}, \\ (x+1)t &= (x+1)(k+1) = (k+1)(x+a+1) - a(k+1) \\ &\equiv -(ak+a) \pmod{x+a+1}. \end{aligned}$$

We see that, if $0 \le r \le a-1$, then

$$\min\left\{|at|_{(x+a+1)}, |(a+1)t|_{(x+a+1)}, |xt|_{(x+a+1)}, |(x+1)t|_{(x+a+1)}\right\} = ak + r,$$

and if $a \le r \le 2a$, then

$$\min\left\{|at|_{(x+a+1)}, |(a+1)t|_{(x+a+1)}, |xt|_{(x+a+1)}, |(x+1)t|_{(x+a+1)}\right\} = ak + a.$$

Therefore,

$$\kappa(M) \ge \begin{cases} \frac{ak+r}{x+a+1} & \text{if } 0 \le r \le a-1, \\ \frac{ak+a}{x+a+1} & \text{if } a \le r \le 2a. \end{cases} \qquad \square$$

Theorem 3.2. *Let* $M = \{a, a+1, x, x+1\}$*, where* $x = k(2a+1) + r$ *with* $0 \le r \le 2a$*. Then*

$$\kappa(M) = \mu(M) = \begin{cases} \frac{ak+r}{x+a+1} & \text{if } 0 \le r \le a-1, \\ \frac{ak+a}{x+a+1} & \text{if } a \le r \le 2a \end{cases}$$

provided

$$k \ge \begin{cases} a - r - 1 & \text{if } 0 \le r \le a-1, \\ r - a - 1 & \text{if } a \le r \le \lfloor \frac{3a}{2} \rfloor, \\ r - a - 2 & \text{if } \lceil \frac{3a+1}{2} \rceil \le r \le 2a. \end{cases}$$

Proof. From Theorem 3.1, we have

$$\mu(M) \ge \begin{cases} \frac{ak+r}{x+a+1} & \text{if } 0 \le r \le a-1, \\ \frac{ak+a}{x+a+1} & \text{if } a \le r \le 2a. \end{cases}$$

We first prove that equality holds when $r = 0$. If possible, let $\mu(M) > ak/(x + a + 1) = ak/[k(2a + 1) + a + 1]$. Then there exists an M-set S such that for any $n \geq 0$, $S(n) > \frac{ak}{k(2a+1)+a+1}(n+1)$. This gives $S\big(k(2a+1)+a\big) > ak$ and if $k \geq a-1$, then $S \cap [0, 2a-1] = [0, a-1]$. Moreover, since S is an $\{a, a+1\}$-set, we have $S\big(k(2a+1)-1\big) \leq ak$. Therefore, at least one element of the set $\{k(2a + 1), k(2a + 1) + 1, \ldots, k(2a + 1) + a\}$ must be in S. But this is not true as $k(2a + 1), k(2a + 1) + 1 \in M$ and $\{0, 1, 2, \ldots, a - 1\} \subset S$.

Now let $1 \leq r \leq 2a$ and k be as it is stated in the theorem. Then using Theorem 2.4, we have

$$\mu(M) \leq \begin{cases} \mu(\{a, a+1, x\}) = \frac{ak+r}{x+a+1} & \text{if } 1 \leq r \leq a-1, \\ \mu(\{a, a+1, x+1\}) = \frac{ak+a}{x+a+1} & \text{if } a \leq r \leq 2a. \end{cases}$$

This completes the proof of the theorem. □

Corollary 3.3. *Let $M = \{a, a + 1, x, x + 1\}$, where $x = k(2a + 1) + r > a + 1$ with $0 \leq r \leq 2a$. Let $r \in \{a, a \pm 1, a - 2\}$. Then, $\kappa(M) = \mu(M)$.*

Proof. The proof follows immediately from Theorem 3.2. □

Corollary 3.4. *Let $M = \{a, a + 1, x, x + 1\}$, where $x = k(2a + 1) + r$ with $0 \leq r \leq 2a$. Let $a \in \{1, 2, 3, 4\}$. Then*

$$\kappa(M) = \mu(M) = \begin{cases} \frac{ak+r}{x+a+1} & \text{if } 0 \leq r \leq a-1, \\ \frac{ak+a}{x+a+1} & \text{if } a \leq r \leq 2a. \end{cases}$$

Proof. It follows from Theorem 3.2 that if $a \in \{1, 2, 3, 4\}$ and

$$M \notin \big\{\{3,4,6,7\}, \{3,4,7,8\}, \{4,5,6,7\}, \{4,5,7,8\}, \{4,5,8,9\}, \{4,5,9,10\}, \{4,5,10,11\}, \{4,5,17,18\}, \{4,5,18,19\}\big\},$$

then $\mu(M) = \kappa(M)$. However, for the above exceptions of the set M, one can directly obtain $\mu(M)$ and $\kappa(M)$ from their respective definitions that

$$\kappa(M) = \mu(M) = \begin{cases} \frac{ak+r}{x+a+1} & \text{if } 0 \leq r \leq a-1, \\ \frac{ak+a}{x+a+1} & \text{if } a \leq r \leq 2a. \end{cases}$$

Hence the corollary holds.

For the sake of the convenience of the reader, we verify the above relation for $M = \{3, 4, 6, 7\}$. Clearly, we have $a = 3$, $x = 6$. Hence $k = 0$ and $r = 6$. From Theorem 2.4, we have $\mu(M) \geq \kappa(M) \geq (ak+a)/(x+a+1) = 3/10$. To prove the reverse inequality, assume to the contrary that $\mu(M) > 3/10$. Then by Lemma 2.2 there exists an M-set S such that for any $n \geq 0$, $S(n) > \frac{3}{10}(n + 1)$. This implies, for instance, $|S \cap [0, 9]| = S(9) \geq 4$. But no subset of $S \cap [0, 9]$ has this property. Therefore $\mu(M) \leq 3/10$. Combining, we have $3/10 \geq \mu(M) \geq \kappa(M) \geq (ak + a)/(x + a + 1) = 3/10$. Hence $\mu(M) = \kappa(M) = 3/10$. □

Theorem 3.5. *Let* $M = \{a, a+1, x, x+2\}$ *with* $x > a+1$, *where* $x+a+2 = (k+1)(2a+2)+r$ *with* $0 \le r < 2a+2 \le (2k+3)+r$. *Then*

$$\kappa(M) \ge \max\left\{\frac{ak+a}{x+a+2}, \frac{a(2k+1)+r-2}{2x+2}\right\}.$$

Proof. Since $x > a+1$, we have $k \ge 0$.

For $t_1 = k+1$, noting that $x+a+2 = (k+1)(2a+2)+r$, we have

$$at_1 = ak+a, \quad (a+1)t_1 = (a+1)(k+1) \le \frac{1}{2}(x+a+2),$$
$$xt_1 = k(x+a+2) + (ak+a+r) \equiv ak+a+r \equiv -(ak+a+2k+2) \pmod{x+a+2},$$
$$(x+2)t_1 \equiv -a(k+1) \pmod{x+a+2}.$$

It follows that

$$\min\left\{|at_1|_{(x+a+2)}, |(a+1)t_1|_{(x+a+2)}, |xt_1|_{(x+a+2)}, |(x+2)t_1|_{(x+a+2)}\right\} = ak+a.$$

Hence

$$\kappa(M) \ge \frac{ak+a}{x+a+2}.$$

For $t_2 = 2k+3$, noting that $x+1 = (a+1)(2k+1)+r$, we have

$$at_2 = a(2k+1)+r-2+(2a+2-r) \le x+1,$$
$$(a+1)t_2 \equiv -\big((x+1)-(2a+2-r)\big) \pmod{2x+2},$$
$$(x+1)-(2a+2-r) = a(2k+1)+r-2+(2k+3+r-2a-2) \ge a(2k+1)+r-2,$$
$$xt_2 = (2x+2)(k+1)+x-2(k+1) \equiv a(2k+1)+r-2 \pmod{2x+2},$$
$$(x+2)t_2 \equiv -xt_2 \equiv -\big(a(2k+1)+r-2\big) \pmod{2x+2}.$$

If $k \ge 1$, then $a(2k+1)+r-2 \ge 3a+r-2 > 0$. If $r \ge 2$, then $a(2k+1)+r-2 \ge a > 0$. If $k = 0$ and $r \le 1$, then $x+1 = (a+1)(2k+1)+r \le a+2$, i.e., $x \le a+1$, a contradiction. So, in all cases, we have $0 < a(2k+1)+r-2 < x+1$. It follows that

$$\min\left\{|at_2|_{(2x+2)}, |(a+1)t_2|_{(2x+2)}, |xt_2|_{(2x+2)}, |(x+2)t_2|_{(2x+2)}\right\} = a(2k+1)+r-2.$$

Hence

$$\kappa(M) \ge \frac{a(2k+1)+r-2}{2x+2}.$$

This completes the proof of the theorem. □

The remaining case, i.e., $M = \{a, a+1, x, x+2\}$ with $x+a+2 = (k+1)(2a+2)+r$ and $0 \le r < (2k+3)+r < 2a+2$ is dealt with the following three propositions.

Proposition 3.6. *Let* $M = \{a, a+1, x, x+2\}$ *with* $2a \neq x = n(2a+1) - r'$, *where* $1 \leq r' \leq a - n + 2$. *Then*

$$\kappa(M) \geq \frac{na}{x+a+2}.$$

Proof. Let $t = n$. Then we have

$$\begin{aligned}
at &\equiv an \pmod{x+a+2}, \\
(a+1)t &\equiv (a+1)n \pmod{x+a+2}, \\
xt &= (x+a+2)n - (a+2)n \equiv (x+a+2) - (a+2)n \pmod{x+a+2}, \\
(x+2)t &= (x+a+2)n - an \equiv -an \pmod{x+a+2}.
\end{aligned}$$

Since $x+a+2 = n(2a+1) - r' + a + 2 = n(a+2) + (na + a + 2 - n - r')$ and $1 \leq r' \leq a - n + 2$, we have

$$\min\left\{|at|_{(x+a+2)}, |(a+1)t|_{(x+a+2)}, |xt|_{(x+a+2)}, |(x+2)t|_{(x+a+2)}\right\} = na.$$

Therefore $\kappa(M) \geq na/(x+a+2)$. □

Proposition 3.7. *Let* $M = \{a, a+1, x, x+2\}$ *with* $x = n(2a+1) + r'$, *where* $0 \leq r' \leq a - (n+2)$. *Then*

$$\kappa(M) \geq \frac{na + r'}{x+a+1}.$$

Proof. Let $t = n + 1$. Then we have

$$\begin{aligned}
at &= (x+a+1) - (na + r' + (n+1)) \equiv -(na + r' + (n+1)) \pmod{x+a+1}, \\
(a+1)t &= (x+a+1) - (na + r') \equiv -(na + r') \pmod{x+a+1}, \\
xt &= (x+a+1)(n+1) - (a+1)(n+1) \equiv (na + r') \pmod{x+a+1}, \\
(x+2)t &= (x+a+1)(n+1) - (n+1)(a-1) \equiv -(n+1)(a-1) \pmod{x+a+1}.
\end{aligned}$$

Since $x+a+1 = n(2a+1) + r' + a + 1 = (n+1)(a+1) + na + r' = (n+1)(a-1) + 2(n+1) + na + r'$ and $0 \leq r' \leq a - (n+2)$, we have

$$\min\left\{|at|_{(x+a+1)}, |(a+1)t|_{(x+a+1)}, |xt|_{(x+a+1)}, |(x+2)t|_{(x+a+1)}\right\} = na + r'.$$

Therefore $\kappa(M) \geq (na + r')/(x+a+1)$. □

Proposition 3.8. *Let* $M = \{a, a+1, 2a, 2a+2\}$ *with* $a \geq 2$. *Then*

$$\frac{1}{3} \geq \mu(M) \geq \kappa(M) \geq \begin{cases} \frac{2a}{3(2a+1)} & \text{if } a \equiv 0 \pmod{3}, \\ \frac{1}{3} & \text{if } a \equiv 1 \pmod{3}, \\ \frac{a}{3a+2} & \text{if } a \equiv 2 \pmod{3}. \end{cases}$$

Proof. Clearly, $\mu(M) \le \mu(\{a, 2a\}) = 1/3$. We prove the reverse inequality in the following cases.

Case 1*:* $a \equiv 0 \pmod 3$. Take $t = (2a+3)/[3(2a+1)]$. Then

$$\begin{aligned} at &= \frac{a(2a+3)}{3(2a+1)} = \frac{a}{3} + \frac{2a}{3(2a+1)}, \\ (a+1)t &= \frac{(a+1)(2a+3)}{3(2a+1)} = \frac{a+3}{3} - \frac{2a}{3(2a+1)}, \\ 2at &= \frac{2a(2a+3)}{3(2a+1)} = \frac{2a+3}{3} - \frac{2a+3}{3(2a+1)}, \\ (2a+2)t &= \frac{(2a+2)(2a+3)}{3(2a+1)} = \frac{2a+3}{3} + \frac{2a+3}{3(2a+1)}. \end{aligned}$$

Therefore $\mu(M) \ge \kappa(M) \ge \|tM\| = 2a/[3(2a+1)]$.

Case 2*:* $a \equiv 1 \pmod 3$. Clearly, M does not contain any multiple of 3. Hence $\mu(M) \ge \kappa(M) \ge 1/3$.

Case 3*:* $a \equiv 2 \pmod 3$. Take $t = 1/(3a+2)$. Then

$$at = \frac{a}{3a+2}, \quad (a+1)t = \frac{a+1}{3a+2}, \quad 2at = 1 - \frac{a+2}{3a+2}, \quad (2a+2)t = 1 - \frac{a}{3a+2}.$$

Therefore $\mu(M) \ge \kappa(M) \ge \|tM\| = a/(3a+2)$. □

4. Concluding remarks

1. Let $M = \{a, a+1, x, x+2\}$. If $a = 1$, that is, $M = \{1, 2, x, x+2\}$, then Haralambis [14] proved that

$$\mu(M) = \kappa(M) = \begin{cases} \frac{x}{3(x+1)} & \text{if } x \equiv 0 \pmod 3, \\ \frac{x+2}{3(x+4)} & \text{if } x \equiv 1 \pmod 3, \\ \frac{1}{3} & \text{if } x \equiv 2 \pmod 3. \end{cases}$$

We see that for $a = 1$, the lower bound established in Theorem 3.5 for $\kappa(M)$ is strict. Whereas, for $a = 2$, Collister and Liu [10] computed the exact value of $\kappa(M)$ and they proved that if $M = \{2, 3, x, x+2\}$ with $x + 4 = 6\beta + r$, then

$$\kappa(M) = \begin{cases} \frac{\lfloor \frac{x+4}{3} \rfloor}{x+4} & \text{if } 0 \le r \le 2, \\ \frac{\lfloor \frac{2x+1}{3} \rfloor}{2x+2} & \text{if } 3 \le r \le 5. \end{cases}$$

Moreover, $\kappa(M) = \mu(M)$, if $r \ne 3$. So we see that for $M = \{2, 3, x, x+2\}$, the lower bound established in Theorem 3.5 for $\kappa(M)$ is the exact value.

2. If we extend the family of sets $M = \{a, a+1, x, x+1\}$ to

$$M' = M \cup \{y : y \equiv \pm a \text{ or } \pm (a+1) \pmod{x+a+1}\},$$

where $x \equiv r \pmod{2a+1}$ with $0 \le r \le 2a$, in Theorems 3.1, 3.2, Corollaries 3.3, and 3.4, then we get $\kappa(M) = \kappa(M')$ and $\mu(M) = \mu(M')$.

Acknowledgments

The authors wish to thank to the anonymous referees for providing their valuable comments to improve the paper. We specially thank to the referee during second revision for providing a stronger version of our result in the form of Theorem 3.5 and many other relevant corrections.

References

[1] D. G. Cantor and B. Gordon, *Sequences of integers with missing differences*, J. Combinatorial Theory Ser. A **14** (1973), 281–287.

[2] J. M. Carraher, D. Galvin, S. G. Hartke, A. J. Radcliffe and D. Stolee, *On the independence ratio of distance graphs*, Discrete Math. **339** (2016), no. 12, 3058–3072.

[3] Y. G. Chen, *On a conjecture about Diophantine approximations I*, Acta Math. Sinica **33** (1990), no. 5, 712–717.

[4] ———, *On a conjecture in Diophantine approximations II*, J. Number Theory **37** (1991), no. 2, 181–198.

[5] ———, *On a conjecture in Diophantine approximations III*, J. Number Theory **39** (1991), no. 1, 91–103.

[6] ———, *On a conjecture in Diophantine approximations IV*, J. Number Theory **43** (1993), no. 2, 186–197.

[7] ———, *View-obstruction problems and Kronecker's theorem*, Proc. Amer. Math. Soc. **123** (1995), no. 11, 3279–3284.

[8] ———, *View-obstruction problems and Kronecker's theorem II*, Sci. China Ser. A **39** (1996), no. 4, 347–358.

[9] Y.-G. Chen and T. W. Cusick, *The view-obstruction problem for n-dimensional cubes*, J. Number Theory **74** (1999), no. 1, 126–133.

[10] D. Collister and D. D.-F. Liu, *Study of $\kappa(D)$ for $D = \{2, 3, x, y\}$*, in: *Combinatorial Algorithms*, 250–261, Lecture Notes in Comput. Sci. **8986**, Springer, Cham, 2015.

[11] T. W. Cusick, *View-obstruction problems in n-dimensional geometry*, J. Combinatorial Theory Ser. A **16** (1974), 1–11.

[12] S. Gupta, *Sets of integers with missing differences*, J. Combin. Theory Ser. A **89** (2000), no. 1, 55–69.

[13] S. Gupta and A. Tripathi, *Density of M-sets in arithmetic progression*, Acta Arith. **89** (1999), no. 3, 255–257.

[14] N. M. Haralambis, *Sets of integers with missing differences*, J. Combinatorial Theory Ser. A **23** (1977), no. 1, 22–33.

[15] D. D.-F. Liu and G. Robinson, *Sequences of integers with three missing separations*, European J. Combin. **85** (2020), 103056, 11 pp.

[16] D. D.-F. Liu and X. Zhu, *Fractional chromatic number and circular chromatic number for distance graphs with large clique size*, J. Graph Theory **47** (2004), no. 2, 129–146.

[17] ———, *Fractional chromatic number of distance graphs generated by two-interval sets*, European J. Combin. **29** (2008), no. 7, 1733–1743.

[18] R. K. Pandey, *Maximal upper asymptotic density of sets of integers with missing differences from a given set*, Math. Bohem. **140** (2015), no. 1, 53–69.

[19] R. K. Pandey and A. Tripathi, *A note on a problem of Motzkin regarding density of integral sets with missing differences*, J. Integer Seq. **14** (2011), no. 6, Article 11.6.3, 8 pp.

[20] ———, *On the density of integral sets with missing differences from sets related to arithmetic progressions*, J. Number Theory **131** (2011), no. 4, 634–647.

[21] ———, *A note on the density of M-sets in geometric sequence*, Ars Combin. **119** (2015), 221–224.

[22] J. H. Rabinowitz and V. K. Proulx, *An asymptotic approach to the channel assignment problem*, SIAM J. Algebraic Discrete Methods **6** (1985), no. 3, 507–518.

[23] A. Srivastava, R. K. Pandey and O. Prakash, *On the maximal density of integral sets whose differences avoiding the weighted Fibonacci numbers*, Integers **17** (2017), Paper No. A48, 19 pp.

[24] ———, *Motzkin's maximal density and related chromatic numbers*, Unif. Distrib. Theory **13** (2018), no. 1, 27–45.

[25] T. Tao, *Some remarks on the lonely runner conjecture*, Contrib. Discrete Math. **13** (2018), no. 2, 1–31.

[26] J. M. Wills, *Zwei Sätze über inhomogene diophantische Approximation von Irrationalzahlen*, Monatsh. Math. **71** (1967), 263–269.

Ram Krishna Pandey
Department of Mathematics, Indian Institute of Technology Roorkee, Uttarakhand, 247667, India
E-mail address: ram.pandey@ma.iitr.ac.in

Anshika Srivastava
Department of Mathematics, School of Applied Sciences, Kalinga Institute of Industrial Technology, Bhubaneswar, Odisha, 751024, India
E-mail address: anshikasme@gmail.com