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Planar Graphs Without Pairwise Adjacent 3-, 4-, 5-, and 6-cycle are 4-choosable

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Abstract. Xu and Wu proved that if every 5-cycle of a planar graph G is not simultaneously adjacent to 3-cycles and 4-cycles, then G is 4-choosable. In this paper, we improve this result as follows. If G is a planar graph without pairwise adjacent 3-, 4-, 5-, and 6-cycle, then G is 4-choosable.

1. Introduction

Every graph in this paper is finite, simple, and undirected. The concept of choosability was introduced by Vizing in 1976 [12] and by Erdős, Rubin, and Taylor in 1979 [5], independently. A *k-assignment* L of a graph G assigns a list $L(v)$ (a set of colors) with $|L(v)| = k$ to each vertex v. A graph G is *L-colorable* if there is a proper coloring f where $f(v) \in L(v)$. If G is L-colorable for any k-assignment L, then we say G is *k-choosable*.

It is known that every planar graphs is 4-colorable [1, 2]. Thomassen [11] proved that every planar graph is 5-choosable. Meanwhile, Voight [13] presented an example of non 4-choosable planar graph. Additionally, Gutner [8] showed that determining whether a given planar graph 4-choosable is NP-hard. Since every planar graph without 3-cycle always has a vertex of degree at most 3, it is 4-choosable. More conditions for a planar graph to be 4-choosable are investigated. It is shown that a planar graph is 4-choosable if it has no 4-cycles [10], 5-cycles [14], 6-cycles [7], 7-cycles [6], intersecting 3-cycles [15], intersecting 5-cycles [9], or 3-cycles adjacent to 4-cycles [3, 4]. Xu and Wu [16] proved that if every 5-cycle of a planar graph G is not simultaneously adjacent to 3-cycles and 4-cycles, then G is 4-choosable. In this paper, we improve this result as follows.

Theorem 1.1. *If G is a planar graph without pairwise adjacent 3-, 4-, 5-, and 6-cycle, then G is 4-choosable.*

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2. Preliminaries

First, we introduce some definitions and notation.

Let G be a plane graph. We use $V(G)$, $E(G)$, and $F(G)$ for the vertex set, the edge set, and the face set respectively. We use $B(f)$ to denote a boundary of a face f. A *wheel* W_n is an n-vertex graph formed by connecting a single vertex (*hub*) to all vertices (*external vertices*) of an $(n-1)$-cycle. A k-vertex (k^+-vertex, k^--vertex, respectively) is a vertex of degree k (at least k, at most k, respectively). The same notations are applied to faces.

A $(d_1, d_2, \ldots, d_k)$-*face* f is a face of degree k where vertices on f have degree $d_1, d_2, \ldots,$ d_k in a cyclic order. A $(d_1, d_2, \ldots, d_k)$-*vertex* v is a vertex of degree k where faces incident to v have degree $d_1, d_2, \ldots, d_k$ in a cyclic order. Note that some face may appear more than one time in the order.

An *extreme* face is a bounded face that shares a vertex with the unbounded face. An *inner* face is a bounded face that is not an extreme face. A $(3, 5, 3, 5^+)$-vertex v is called a *flaw* 4-*vertex* if v is incident to a poor inner 5-face and two inner 3-faces. A $(3, 5, 3, 5^+)$-vertex v is called a *pseudo flaw* 4-*vertex* if v is incident to a poor inner 5-face and at least one extreme 3-face.

We say xy is a *chord* in an embedding cycle C if $x, y \in V(C)$ but $xy \in E(G) - E(C)$. An *internal chord* is a chord inside C while *external chord* is a chord outside C. A *triangular chord* is a chord e such that two edges in C and e form a 3-cycle. A graph $C(m, n)$ is obtained from a cycle $x_1x_2 \ldots x_{m+n-2}$ with an internal chord x_1x_m.

A graph $C(l, m, n)$ is obtained from a cycle $x_1x_2 \ldots x_{l+m+n-4}$ with internal chords x_1x_l and x_1x_{l+m-2}. A graph $C(m, n, p, q)$ can be defined similarly. We use $\mathrm{int}(C)$ and $\mathrm{ext}(C)$ to denote the graphs induced by vertices inside and outside a cycle C, respectively. A cycle C is a *separating cycle* if $\mathrm{int}(C)$ and $\mathrm{ext}(C)$ are not empty.

Let L be a list assignment of G and let H be an induced subgraph of G. Suppose $G - H$ has an L-coloring ϕ on $G - H$ where L is restricted to $G - H$. For a vertex $v \in H$, let $L''(v)$ be a set of colors used on the neighbors of v by ϕ. We define the *residual list assignment* L' of H by $L'(v) = L(v) - L''(v)$. One can see that if $G - H$ has an L-coloring ϕ and H has an L'-coloring, then G has an L-coloring.

The following is a fact on list colorings that we use later.

Lemma 2.1. [5] *Let L be a 2-assignment. A cycle C_n is L-colorable if and only if n is even or L does not assign the same list to all vertices.*

Let $\mathcal{A}$ denote the family of planar graphs without pairwise adjacent 3-, 4-, 5-, and 6-cycle.

Next, we explore some properties of graphs in $\mathcal{A}$ which are helpful in a proof of the main results.

Lemma 2.2. *Every graph G in $\mathcal{A}$ does not contain each of the followings:*

(1) $C(3,3,4)$, (2) $C(3,3,5)$, (3) $C(3,4,4^-)$, (4) $C(4,3,5)$,

(5) *W_5 that shares exactly one edge with a 6^--cycle.*

Proof. Let $C(l,m,n)$ be obtained from a cycle $x_1x_2\dots x_{l+m+n-4}$ with internal chords x_1x_l and x_1x_{l+m-2}.

(1) Suppose G contains $C(3,3,4)$. Then we have four pairwise adjacent cycles $x_1x_2x_3$, $x_1x_2x_3x_4$, $x_1x_3x_4x_5x_6$, and $x_1x_2x_3x_4x_5x_6$, contrary to $G \in \mathcal{A}$.

(2) Suppose G contains $C(3,3,5)$. Then we have four pairwise adjacent cycles $x_1x_3x_4$, $x_1x_2x_3x_4$, $x_1x_4x_5x_6x_7$, and $x_1x_3x_4x_5x_6x_7$, contrary to $G \in \mathcal{A}$.

(3) Suppose G contains $C(3,4,3)$. Then we have four pairwise adjacent cycles $x_1x_2x_3$, $x_1x_3x_4x_5$, $x_1x_2x_3x_4x_5$, and $x_1x_2x_3x_4x_5x_6$, contrary to $G \in \mathcal{A}$. Suppose G contains $C(3,4,4)$. Then we have four pairwise adjacent cycles $x_1x_2x_3$, $x_1x_3x_4x_5$, $x_1x_2x_3x_4x_5$, and $x_1x_3x_4x_5x_6x_7$, contrary to $G \in \mathcal{A}$.

(4) Suppose G contains $C(4,3,5)$. Then we have four pairwise adjacent cycles $x_1x_4x_5$, $x_1x_2x_3x_4$, $x_1x_2x_3x_4x_5$, and $x_1x_4x_5x_6x_7x_8$, contrary to $G \in \mathcal{A}$.

(5) Let the hub of W_5 be q and let external vertices be r, s, u, and v in a cyclic order. Suppose there is a cycle uvw. Then we have four pairwise adjacent cycles vwu, $vwuq$, $vwusq$, and $vwusqr$, contrary to $G \in \mathcal{A}$. Suppose there is a cycle $uvwx$. Then we have four pairwise adjacent cycles usq, $usqv$, $usqrv$, and $usqvwx$, contrary to $G \in \mathcal{A}$. Suppose there is a cycle $uvwxy$. Then we have four pairwise adjacent cycles uqv, $uqrv$, $uqsrv$, and $uqvwxy$, contrary to $G \in \mathcal{A}$. Suppose there is a cycle $uvwxyz$. Then we have four pairwise adjacent cycles uvq, $uvqs$, $uvqrs$, and $uvwxyz$, contrary to $G \in \mathcal{A}$. □

Lemma 2.3. *If C is a 6-cycle with a triangular chord, then C has exactly one chord.*

Proof. Let $C = tuvxyz$ with a chord tv. Suppose to the contrary that C has another chord e. By symmetry, it suffices to assume that $e = ux, uy, tx, ty$, or xz. If $e = ux$, then we have four pairwise adjacent cycles tuv, $tuxv$, $tvxyz$, and $tuvxyz$, contrary to $G \in \mathcal{A}$. If $e = uy$, then we have four pairwise adjacent cycles tuv, $uvxy$, $tvxyz$, and $tuvxyz$, contrary to $G \in \mathcal{A}$. If $e = tx$, then we have four pairwise adjacent cycles tuv, $tuvx$, $tvxyz$, and $tuvxyz$, contrary to $G \in \mathcal{A}$. If $e = ty$, then we have four pairwise adjacent cycles tuv, $tvxy$, $tvxyz$, and $tuvxyz$, contrary to $G \in \mathcal{A}$. If $e = xz$, then we have four pairwise adjacent cycles tuv, $tvxz$, $tvxyz$, and $tuvxyz$, contrary to $G \in \mathcal{A}$. Thus C has exactly one chord. □

3. Structure

To prove Theorem 1.1, we prove a stronger result as follows.

Theorem 3.1. *If $G \in \mathcal{A}$ with a 4-assignment L, then each precoloring of a 3-cycle in G can be extended to an L-coloring of G.*

We consider (G, C_0) and a 4-assignment L where C_0 is a precolored 3-cycle as a minimal counterexample to Theorem 3.1. Embed G in the plane.

Lemma 3.2. *G has no separating 3-cycles.*

Proof. Suppose to the contrary that there exists a separating 3-cycle C in G. By symmetry, we assume $V(C_0) \subseteq V(C) \cup \text{int}(C)$. By the minimality of G, a precoloring of C_0 can be extended to $V(C) \cup \text{int}(C)$. After C is colored, then again the coloring of C can be extended to $\text{ext}(C)$. Thus we have an L-coloring of G, a contradiction. □

So we may assume that a minimal counterexample (G, C_0) has no separating 3-cycles, and C_0 is the boundary of the unbounded face D of G in the rest of this paper.

Lemma 3.3. *Each vertex in $\text{int}(C_0)$ has degree at least four.*

Proof. Suppose otherwise that there exists a 3^--vertex v in $\text{int}(C_0)$. By the minimality of (G, C_0), $(G - v, C_0)$ has an L-coloring. One can see that the residual list $L'(v)$ is not empty. Thus we can color v and thus extend a coloring to G, a contradiction. □

Lemma 3.4. *For faces in G, each of the followings holds.*

(1) *The boundary of a bounded 6^--face is a cycle.*

(2) *If a bounded k_1-face f and a bounded k_2-face g are adjacent where $k_1 + k_2 \leq 8$, then $B(f) \cup B(g) = C(k_1, k_2)$.*

(3) *If a bounded 4-face f and a bounded 5-face g are adjacent, then $B(f) \cup B(g)$ is $C(4, 5)$ or a configuration as in Figure 3.1 where tuy is C_0.*

(4) *If bounded 5-faces f and g are adjacent, then $B(f) \cup B(g)$ is $C(5, 5)$ or a configuration as in Figure 3.2.*

Proof. (1) One can observe that a boundary of a 5^--face is always a cycle. Consider a bounded 6-face f. If $B(f)$ is not a cycle, then a boundary closed walk is in a form of $uvwxywu$. By Lemma 3.3, u or x has degree at least 4. Consequently, uvw or xyw is a separating 3-cycle, contrary to Lemma 3.2.

(2) It suffices to show that such f and g share exactly two vertices. Let $B(f) = uvw$ and $B(g) = vwx$. If $u = x$, then f or g is the unbounded face, a contradiction.

Let $B(f) = uvw$ and $B(g) = vwxy$. If $u = x$ or y, then $d(w) = 2$ or $d(v) = 2$, contrary to Lemma 3.3.

Let $B(f) = uvw$ and $B(g) = vwxyz$. If $u = x$ or z, then $d(w) = 2$ or $d(v) = 2$, contrary to Lemma 3.3. If $u = y$, then vyz or wxy is a separating 3-cycle, contrary to Lemma 3.2.

Let $B(f) = stuv$ and $B(g) = uvwx$. If $s = w$, then $d(v) = 2$, contrary to Lemma 3.3. If $s = x$, then utx or vwx is a separating 3-cycle, contrary to Lemma 3.2. The remaining cases are similar.

(3) Let $B(f) = stuv$ and $B(g) = uvwxy$. It suffices to show that $V(B(f)) \cap V(B(g)) = \{u, v\}$ or $\{u, v, x\}$ where $x = s$ or t. If $t = w$, then uvw is a separating 3-cycle, contrary to Lemma 3.2. If $t = x$, then tuy is C_0, otherwise tuy is a separating cycle, contrary to Lemma 3.2. If $t = y$, then $d(u) = 2$, contrary to Lemma 3.3. The remaining cases are similar.

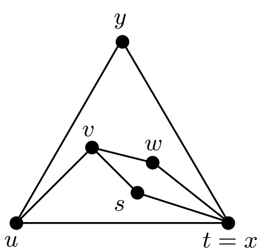

Figure 3.1: A graph F is formed by a 4-face and a 5-face with $tuy = C_0$.

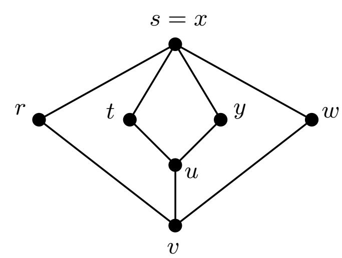

Figure 3.2: A graph H is formed by two adjacent 5-faces with but is not $C(5, 5)$.

(4) Let $B(f) = rstuv$ and $B(g) = uvwxy$. It suffices to show that $V(B(f)) \cap V(B(g)) = \{u, v\}$ or $\{u, v, x = s\}$. If $r = w$, then $d(v) = 2$, contrary to Lemma 3.3. If $B(f) \cap B(g) = \{u, v, r = x\}$, then vwx, $uvxy$, $uvwxy$, and $stuvwx$ are four pairwise adjacent cycles, contrary to $G \in \mathcal{A}$. If $B(f) \cap B(g) = \{u, v, r = x, s = y\}$, then rvs, $rvus$, $rvuts$, and $rstuvw$ are four pairwise adjacent cycles, contrary to $G \in \mathcal{A}$, then uts or vwx is a separating 3-cycle, contrary to Lemma 3.2. If $B(f) \cap B(g) = \{u, v, r = y\}$, then ruv is a separating 3-cycle, contrary to Lemma 3.2. If $B(f) \cap B(g) = \{u, v, s = w\}$, then rvw, $tuvw$, $uvwxy$, and $rwxyuv$ are four pairwise adjacent cycles, contrary to $G \in \mathcal{A}$. The remaining cases are similar. □

Lemma 3.5. *If a k-vertex v is incident to bounded faces $f_1, \ldots, f_k$ in a cyclic order and d_i is a degree of a face f_i for each $i \in \{1, \ldots, k\}$, then each of the followings holds.*

(1) $(d_1, d_2, d_3) \neq (3, 3, 4)$, (2) $(d_1, d_2, d_3) \neq (3, 3, 5)$,

(3) $(d_1, d_2, d_3) \neq (3, 4, 4^-)$, (4) $(d_1, d_2, d_3) \neq (4, 3, 5)$,

(5) *Let H be W_5 such that a hub and each two vertices of consecutive external vertices form a boundary of an inner 3-face. Then H is not adjacent to a boundary of a 6^--face other than these 3-faces.*

Proof. Let $F = B_1 \cup B_2 \cup B_3$ where B_i denote $B(f_i)$.

(1) Suppose $(d_1, d_2, d_3) = (3, 3, 4)$. Let $B_1 = rsv$, $B_2 = vst$, and $B_3 = vtxy$. It follows from Lemma 3.4(2) that $V(B_1) \cap V(B_2) = \{s, v\}$ and $V(B_2) \cap V(B_3) = \{t, v\}$. If $r = x$, then stx or vxy is a separating 3-cycle, contrary to Lemma 3.2. If $r = y$, then $d(v) = 3$, contrary to Lemma 3.3. Thus $V(B_1) \cap V(B_3) = \{v\}$. Altogether we have $F = C(3, 3, 4)$, contrary to Lemma 2.2(1).

(2) Suppose $(d_1, d_2, d_3) = (3, 3, 5)$. Let $B_1 = rsv$, $B_2 = vst$, and $B_3 = vtxyz$. It follows from Lemma 3.4(2) that $V(B_1) \cap V(B_2) = \{s, v\}$ and $V(B_2) \cap V(B_3) = \{t, v\}$. We have $C = stxyzv$ is a 6-cycle with a triangular chord tv. If $r \in \{x, y, z\}$, then C has another chord, contrary to Lemma 2.3. Thus $V(B_1) \cap V(B_3) = \{v\}$. Altogether we have $F = C(3, 3, 5)$, contrary to Lemma 2.2(2).

(3) Suppose $(d_1, d_2, d_3) = (3, 4, 3)$. Let $B_1 = rsv$, $B_2 = vstu$, and $B_3 = vuw$. It follows from Lemma 3.4(2) that $V(B_1) \cap V(B_2) = \{s, v\}$ and $V(B_2) \cap V(B_3) = \{u, v\}$. If $r = w$, then $d(v) = 3$, contrary to Lemma 3.3. Thus $V(B_1) \cap V(B_3) = \{v\}$. Altogether we have $F = C(3, 4, 3)$, contrary to Lemma 2.2(3).

Suppose $(d_1, d_2, d_3) = (3, 4, 4)$. Let $B_1 = rsv$, $B_2 = vstu$, and $B_3 = uvxy$. It follows from Lemma 3.4(2) that $V(B_1) \cap V(B_2) = \{s, v\}$ and $V(B_2) \cap V(B_3) = \{u, v\}$. If $r = x$, then $d(v) = 3$, contrary to Lemma 3.3. If $r = y$, then vuy is a separating 3-cycle, contrary to Lemma 3.2. Thus $V(B_1) \cap V(B_3) = \{v\}$. Altogether we have $F = C(3, 4, 4)$, contrary to Lemma 2.2(3).

(4) Suppose $(d_1, d_2, d_3) = (4, 3, 5)$. Let $B_1 = qrsv$, $B_2 = vst$, and $B_3 = vtxyz$. It follows from Lemma 3.4(2) that $V(B_1) \cap V(B_2) = \{s, v\}$ and $V(B_2) \cap V(B_3) = \{t, v\}$. We have $C = stxyzv$ is a 6-cycle with a triangular chord tv. If $\{q, r\}$ and $\{x, y, z\}$ are not disjoint, then C has another chord or $q = z$. The former contradicts Lemma 2.3 and the latter yields $d(v) = 3$, contrary to Lemma 3.3. Thus $V(B_1) \cap V(B_3) = \{v\}$. Altogether we have $F = C(4, 3, 5)$, contrary to Lemma 2.2(2).

(5) Let v be a hub and let w, x, y, z be external vertices of H in the cyclic order. Suppose to the contrary that H is adjacent to a face f with $B(f) = wxq, wxqr, wxqrs$, or $wxqrst$. Now we have $\{w, x\} \subseteq V(H) \cap V(B(f))$. By Lemma 2.2(5), $V(H) \cap V(B(f)) \neq \{w, x\}$. If $q = y$, then $d(x) = 3$, contrary to Lemma 3.3. If $r = y$, then $vwxqyz$ is a 6-cycle with four triangular chords, contrary to Lemma 2.3. If $s = y$, then vxw, $vxwz$,

$vxwzy$, and $vxqryz$ are four pairwise adjacent cycles, contrary to $G \in \mathcal{A}$. If $t = y$, then vxw, $vxwz$, $vxwzy$, and $vxqrsy$ are four pairwise adjacent cycles, contrary to $G \in \mathcal{A}$. The remaining cases lead to similar contradictions. Thus f is not a 6^--face. □

Lemma 3.6. *Let $C(m,n)$ in $\mathrm{int}(C_0)$ be obtained from a cycle $C = x_1 \dots x_{m+n-2}$ with a chord x_1x_m and $d(x_1) \le 5$. If C has at most one additional chord e and e is not $x_{m-1}x_{m+1}$ or x_1x_k where $k \ne m$, then there exists $i \in \{2, \dots, m+n-2\}$ with $d(x_i) \ge 5$.*

Proof. Suppose to the contrary that G has such C with $d(x_i) \le 4$ for each $i \in \{2, \dots, m+n-2\}$. By minimality, there exists an L-coloring for $G - C$. Considering the residual list $L'(x_i)$ for each $x_i \in V(C)$, we have $|L'(x_m)| \ge 3$ and $|L'(x_i)| \ge 2$ for each $x_i \in V(C)$.

Case 1. C has exactly one chord. Assume that $\{1,2\} \subseteq L'(x_1)$.

Case 1.1. Assume $\{1,2\} \subseteq L'(x_i)$ for each x_i where $i \ne m$. We can color vertices in a path $C - x_m$ with colors 1 and 2. Finally, we assign an available to x_m to complete a coloring.

Case 1.2. Assume that there are adjacent vertices x_k and x_{k+1} in $C - x_m$ such that $\{1,2\} \subseteq L'(x_k)$ but $\{1,2\} \nsubseteq L(x_{k+1})$ where $k \le m$. Assign a color in $L'(x_k)$ to x_k such that $|L'(x_{k+1})| \ge 2$. Apply L'-coloring to $x_{k-1}, x_{k-2}, \dots, x_1, x_{m+n-2}, x_{m+n-3}, \dots, x_{k+2}$ in this order. Consequently, $|L'(x_{k+1})| \ge 1$ and thus we can complete an L-coloring.

Case 2. C has exactly one more chord e such that e is not $x_{m-1}x_{m+1}$ or x_1x_k where $k \ne m$. Let $e = x_sx_t$. By symmetry, we may assume that $s < t$ and $s < m-1$. Since $|L'(x_s)| \ge 3$, we can apply an L'-coloring to x_s such that $|L'(x_{s+1})| \ge 2$. Apply L'-coloring to $x_{s-1}, x_{s-2}, \dots, x_1, x_{m+n-2}, x_{m+n-3}, \dots, x_{s+2}$ in this order. Consequently, $|L'(x_{s+1})| \ge 1$ and thus we can complete an L-coloring. □

Corollary 3.7. *If v is a flaw vertex, then we have the followings.*

(1) *v is incident to exactly one poor 5-face.*

(2) *Each 3-face that is incident to v is a semi-rich face.*

Proof. Let v be incident to inner faces f_1, f_2, f_3, f_4 in a cyclic order where f_1 and f_3 are inner 3-faces, f_2 is an inner poor 5-face, and f_4 is a 5^+-face. By Lemma 3.4, $B(f_1) \cup B(f_2)$ and $B(f_2) \cup B(f_3)$ are $C(3,5)$. It follows from Lemmas 3.2 and 3.3 that a 6-cycle C in such $C(3,5)$ has at most one external chord and such chord (if it exists) is not a triangular chord. By Lemma 3.6, some vertex in $B(f_1) \cup B(f_2)$ and in $B(f_2) \cup B(f_3)$ has degree at least 5. Since f_2 is a poor face, some vertex in $B(f_1)$ and in $B(f_3)$ has degree at least 5

(1) If f_4 is also a poor 5-face, then f_1 is a poor face, contrary to the observation above.

(2) By observation above, f_1 and f_3 are not poor 3-faces. Since f_2 is a poor face, we obtain that f_1 and f_3 are not rich faces. □

Lemma 3.8. *If H in Figure 3.2 is in* $\mathrm{int}(C_0)$ *and contains a 5^--vertex v, then there is another vertex of H with degree at least 5 in G.*

Proof. First, we show that H is an induced subgraph. Suppose to the contrary that there is an edge e joining vertices in $V(H)$ such that $e \notin E(H)$. If $e = ty$, then tuy is a separating 3-cycle. If $e = ux$, then stu is a separating 3-cycle. If $e = sv$, then rsv is a separating 3-cycle. If $e = rw$, then rvw is a separating 3-cycle. All consequences contradicts Lemma 3.2. Thus H is an induced subgraph.

Suppose to the contrary that $d(v) \leq 5$ but each of remaining vertices has degree at most 4. By minimality, $G - H$ has an L-coloring where L is restricted to $G - H$. Consider a residual list assignment L' on H. Since L is a 4-assignment, we have $|L'(s)| = 4$, $|L'(u)| \geq 3$, and $|L'(v)|, |L'(r)|, |L'(t)|, |L'(y)|, |L'(w)| \geq 2$. We begin by choosing a color c from $L'(u)$ such that $|L'(y) - c| \geq 2$. Then we choose colors of v, r, w, t, s, and y in this order, we obtain an L'-coloring on H. Thus we can extend an L-coloring to G, a contradiction. □

Corollary 3.9. *Let v be a k-vertex in* $\mathrm{int}(C_0)$ *with consecutive incident faces $f_1, \ldots, f_k$ where $k \leq 5$. If f_1 and f_2 are inner 5^--faces, then there exists $w \in B(f_1) \cup B(f_2)$ such that $w \neq v$ and $d(w) \geq 5$.*

Proof. It follows from Lemmas 3.2 and 3.4 that that $B(f_1) \cup B(f_2)$ is a graph H as in Figure 3.2 or $C(s,t)$ where $s = d(f_1)$ and $t = d(f_2)$. The former case is proved by Lemma 3.8. Assume $B(f_1) \cup B(f_2) = C(s,t)$. It follows from Lemmas 3.2 and 3.3 that a cycle C in the above $C(s,t)$ has at most one external chord and such chord (if it exists) is not a triangular chord. Use Lemma 3.6 to complete the proof. □

Corollary 3.10. *If v is a 5-vertex in which each incident face is a 5^--face, then v is incident to at least three faces that are rich or extreme.*

Proof. Suppose to the contrary that v is incident to three faces that are neither rich nor extreme. Consequently, v is incident to consecutive inner faces 5^--faces f and g such that each vertex in $B(f) \cup B(g)$ except v have degree 4. This contradicts Corollary 3.9. □

Lemma 3.11. *Let $C(l_1, \ldots, l_k)$ in* $\mathrm{int}(C_0)$ *be obtained from a cycle $C = x_1 \ldots x_m$ with $k-1$ internal chords sharing a common endpoint x_1. Suppose x_1 is not incident to other chords while x_2 or x_m is not incident to any chord. If $d(x_1) \leq k+2$, then there exists $i \in \{2, 3, \ldots, m\}$ such that $d(x_i) \geq 5$.*

Proof. By symmetry, we assume x_m is not an endpoint of any chord in C. Suppose to the contrary that $d(x_i) \leq 4$ for each $i = 2, 3, \ldots, m$. By the minimality of G, the subgraph $G - \{x_1, \ldots, x_m\}$ has an L-coloring where L is restricted to $G - \{x_1, \ldots, x_m\}$. Consider a residual list assignment L' on $x_1, \dots, x_m$. Since L is a 4-assignment, we have $|L'(x_1)| \geq 3$ and $|L'(v)| \geq 3$ for each $v \in V(C)$ with an edge x_1v and $|L'(x_i)| \geq 2$ for each of the remaining vertices x_i in $V(C)$. Since x_m is not an endpoint of a chord in C, we can choose a color c from $L'(x_1)$ such that $|L'(x_m) - c| \geq 2$. By choosing colors of $x_2, x_3, \dots, x_m$ in this order, we obtain an L'-coloring on G'. Thus we can extend an L-coloring to G, a contradiction. □

Corollary 3.12. *Let v be a 6-vertex with consecutive inner incident faces $f_1, \dots, f_6$ and let $F = B_1 \cup B_2 \cup B_3 \cup B_4$ where B_i denote $B(f_i)$. If $f_1 \dots f_4$ are inner faces and $(d(f_1), d(f_2), d(f_3), d(f_4)) = (5, 3, 5, 3)$, then there exists $w \in V(F) - \{v\}$ with $d(w) \geq 5$.*

Proof. By Lemma 3.11, it suffices to show that $F = C(5, 3, 5, 3)$. Let cycles $B_1 = vqrst$, $B_2 = vtu$, $B_3 = vuwxy$, and $B_4 = vyz$. Using Lemma 3.4, we have that $V(B_1) \cap V(B_2) = \{v, t\}$, $V(B_2) \cap V(B_3) = \{v, u\}$, and $V(B_3) \cap V(B_4) = \{v, y\}$. It suffices to show that $V(B_1) \cap V(B_3) = \{v\} = V(B_4) \cap (V(B_1) \cup V(B_2))$.

Suppose to the contrary that $V(B_1) \cap V(B_3) \neq \{v\}$. Consider a 6-cycle $vtuwxy$ with a triangular chord uv. If $s = u, w, x$, or y, then $vtuwxy$ has another chord, contrary to Lemma 2.3. Thus $s \notin V(B_1) \cap V(B_3)$. Similarly each of q, w, and y is not in $V(B_1) \cap V(B_3)$. The only remaining possibility is that $r = x$. Suppose this holds. Then vyz, $vyxq$, $vyxwu$, and $vyrstu$ are four pairwise adjacent cycles, contrary to $G \in \mathcal{A}$. Thus $V(B_1) \cap V(B_3) = \{v\}$ which implies $B_1 \cup B_2 \cup B_3 = C(5, 3, 5)$. As a consequence, we have $vqrstu$ and $vtuwxy$ are 6-cycles with a triangular chord.

If there is a vertex $b \in V(B_4) \cap (V(B_1) \cup V(B_2))$ such that $b \neq v$, then $vqrstu$ or $vtuwxy$ has another chord, contrary to Lemma 2.3. This completes the proof. □

Corollary 3.13. *Let v be a 4-vertex incident to four inner 3-faces. If all four neighbors of v are 5^--vertices, then at least three of them are 5-vertices.*

Proof. Let w, x, y, z be neighbor of v in a cyclic order. Let cycles $B_1 = vwx$ and $B_2 = vxy$. Note that w and y are not adjacent, otherwise vwy is a separating 3-cycle, contrary to Lemma 3.2. Similarly, x and z are not adjacent.

Suppose to the contrary that there are at least two 4-vertices among w, x, y, and z. If those two 4-vertices are not adjacent, say w and y, then $B_1 \cup B_2$ contradicts Lemma 3.6. Thus we assume that w and x are 4-vertices.

Let H be the graph induced by v and its neighbors. By minimality of G, the graph $G - H$ has an L-coloring where L is restricted to $G - H$. Consider a residual list assignment L' on H. Since L is a 4-assignment, we have $|L'(y)|, |L'(z)| \geq 2$, $|L'(w)|, |L'(x)| \geq 3$, and $|L'(v)| = 4$. It suffices to assume that equalities holds for these list sizes. We aim to show that H has an L'-coloring, and thus an L-coloring can be extended to G, a contradiction.

Case 1. There is a color t in $L'(v) - (L'(y) \cup L'(z))$. We begin by choosing t for v. Each of the residual lists of w, x, y, z now has sizes at least 2. By Lemma 2.1, an even cycle is 2-choosable, thus H has an L'-coloring.

Case 2. $L'(v) - (L'(y) \cup L'(z)) = \emptyset$. This implies $L'(y) \cap L'(z) = \emptyset$. Choose $t \in L'(v) - L'(w)$ for v. If $t \in L'(y)$, then $t \notin L'(z)$ and we can color y, x, z, and w in this order, otherwise we can color z, y, x, and w in this order. Thus H has an L'-coloring. This contradiction completes the proof. □

4. Proof of Theorem 3.1

Let the initial charge of a vertex u in G be $\mu(u) = 2d(u) - 6$, let the initial charge of a bounded face f in G be $\mu(f) = d(f) - 6$, and let the initial charge of the unbounded face D be $\mu(D) = d(D) + 6$. Then by Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ and by the Handshaking lemma, we have

$$\sum_{u \in V(G)} \mu(u) + \sum_{f \in F(G)} \mu(f) = 0.$$

Now we design the discharging rule transferring charge from one element to another to provide a new charge $\mu^*(x)$ for all $x \in V(G) \cup F(G)$. The total of new charges remains 0. If the final charge $\mu^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$ and $\mu^*(D) > 0$, then we get a contradiction and complete the proof.

Before we establish a discharging rule, some definitions are required.

A 4-vertex is a *special* 4-*vertex* if it is incident to two consecutive inner 3-faces. A graph $C(3,3,3)$ in $\text{int}(C_0)$ is called a *trio*. A vertex that is not in any trio is called a *good* vertex. We call a vertex v incident to a face f in a trio T a *bad* (*worse*, *worst*, respectively) vertex of f if v is incident to exactly one (two, three, respectively) 3-face(s) in T. We call a face f in a trio T a *bad* (*worse*, *worst*, respectively) face of a vertex v if v is a bad (worse, worst, respectively) vertex of f in T. A *good* face f of a vertex v is a 3-face incident to v such that f is not in a trio. For our purpose, we regard an external vertex of W_5 as a worse vertex of its incident 3-faces in W_5.

Let $w(v \to f)$ be the charge transferred from a vertex v to an incident face f. From now on, a vertex v is in $\text{int}(C_0)$ unless stated otherwise. The discharging rules are as follows.

(R1) Let f be an inner 3-face that is not adjacent to another 3-face.

(R1.1) For a 4-vertex v,

$$w(v \to f) = \begin{cases} \frac{9}{10} & \text{if } v \text{ is flaw,} \\ 1 & \text{otherwise.} \end{cases}$$

(R1.2) For a 5^+-vertex v,

$$w(v \to f) = \begin{cases} \frac{6}{5} & \text{if } f \text{ is a } (4,4,5^+)\text{-face,} \\ 1 & \text{otherwise.} \end{cases}$$

(R2) Let f be an inner 3-face that is adjacent to another 3-face.

(R2.1) For a 4-vertex v,

$$w(v \to f) = \begin{cases} \frac{1}{2} & \text{if } v \text{ is incident to four internal 3-faces,} \\ 1 & \text{if } f \text{ is a good, bad, or worse face of } v, \\ \frac{2}{3} & \text{if } f \text{ is a worst face of } v. \end{cases}$$

(R2.2) For a 5-vertex v,

$$w(v \to f) = \begin{cases} 1 & \text{if } f \text{ is a good or worst face of } v, \\ \frac{5}{4} & \text{if } f \text{ is a worse face of } v, \\ \frac{3}{2} & \text{if } f \text{ is a bad face of } v. \end{cases}$$

(R2.3) For a 6^+-vertex v,

$$w(v \to f) = \begin{cases} 1 & \text{if } f \text{ is a good or worst face of } v, \\ \frac{3}{2} & \text{if } f \text{ is a bad or worse face of } v. \end{cases}$$

(R3) Let f be an inner 4-face.

(R3.1) For a 4-vertex v, let $w(v \to f) = \frac{1}{3}$.

(R3.2) For a 5^+-vertex v,

$$w(v \to f) = \begin{cases} 1 & \text{if } f \text{ is a } (4,4,4,5^+)\text{-face,} \\ \frac{2}{3} & \text{if } f \text{ is rich.} \end{cases}$$

(R4) Let f be an inner 5-face.

(R4.1) For a 4-vertex v,

$$w(v \to f) = \begin{cases} \frac{1}{5} & \text{if } v \text{ is flaw and } f \text{ is a poor 5-face,} \\ \frac{1}{4} & \text{if } v \text{ is pseudo flaw and } f \text{ is a poor 5-face,} \\ \frac{1}{3} & \text{if } v \text{ is incident to at most one 3-face,} \\ 0 & \text{otherwise.} \end{cases}$$

(R4.2) For a 5^+-vertex v,

$$w(v \to f) = \begin{cases} 1 & \text{if } f \text{ is a } (4,4,4,4,5^+)\text{-face adjacent to five 3-faces,} \\ \frac{2}{3} & \text{if } f \text{ is a } (4,4,4,4,5^+)\text{-face adjacent to at least one } 4^+\text{-face} \\ & \text{other than } f, \\ \frac{1}{t} & \text{if } f \text{ is a rich face with } t \text{ incident } 5^+\text{-vertices.} \end{cases}$$

(R5) Let f be an inner 3-face. If f is adjacent to a 7^+-face g, we let $w(g \to f) = \frac{1}{8}$.

(R6) The unbounded face D gets $\mu(v)$ from each incident vertex.

(R7) Let f be an extreme face.

$$w(x \to f) = \begin{cases} 3 & \text{if } f \text{ is a 3-face incident to a special 4-vertex and } x = D, \\ \frac{5}{2} & \text{if } f \text{ is a 3-face not incident to a special 4-vertex} \\ & \text{such that } B(f) \text{ shares an edge with } C_0 \text{ and } x = D, \\ 2 & \text{if } f \text{ is a 4- or 5-face and } x = D, \\ 2 & \text{if } f \text{ is a 3-face not incident to a special 4-vertex} \\ & \text{such that } B(f) \text{ shares exactly one vertex with } C_0 \text{ and } x = D, \\ \frac{1}{2} & \text{if } f \text{ is a 3-face incident to a vertex } x \text{ in } \mathrm{int}(C_0) \\ & \text{but } x \text{ is not a special 4-vertex,} \\ 0 & \text{otherwise.} \end{cases}$$

(R8) After (R1) to (R7), redistribute the total of charges of 3-faces in the same cluster of at least three adjacent inner 3-faces (trio or W_5) equally among its 3-faces.

It remains to show that resulting $\mu^*(x) \geq 0$ for all $x \in V(G) \cup F(G)$. Let v be a k-vertex incident to faces $f_1, \ldots, f_k$ in a cyclic order. By (R6), we only consider v in $\mathrm{int}(C_0)$. Consider the following cases.

(1) v is a 4-vertex.

(1.1) A vertex v is incident to a 3-face that is adjacent to another 3-face.

(1.1.1) v is incident to at least two consecutive 3-faces.
Assume v is incident to four 3-faces. If v is not adjacent to a vertex in $V(C_0)$, then v is incident to four inner 3-faces. Thus $\mu^*(v) \geq \mu(v) - 4 \times \frac{1}{2} = 0$ by (R2.1). If v is adjacent to exactly one vertex in $V(C_0)$, then v is incident to exactly two inner 3-faces which are good faces of v. Thus

$\mu^*(v) \geq \mu(v) - 2 \times 1 = 0$ by (R2.1) and (R7). Observe that two endpoints of an edge in the boundary of an incident 3-face of v cannot be both in $V(C_0)$ by Lemma 2.2(5). If v is adjacent to at least two vertices in $V(C_0)$, then each incident face of v is an extreme 3-face by the observation above. Thus $\mu^*(v) \geq \mu(v) - 4 \times \frac{1}{2} = 0$ by (R7).

Assume v is incident to exactly three 3-faces, say f_1, f_2, and f_3, then f_4 is a 6^+-face by Lemma 3.5(1), (2). If v is incident to three inner 3-faces, then $\mu^*(v) \geq \mu(v) - 3 \times \frac{2}{3} = 0$ by (R2.1). If v is incident to exactly two inner 3-faces and those two are consecutive, then v is a special 4-vertex, and thus $\mu^*(v) \geq \mu(v) - 2 \times 1 = 0$ by (R2.1). If v is incident to exactly two inner 3-faces but they are not consecutive, then $\mu^*(v) \geq \mu(v) - \frac{1}{2} > 0$ by (R7). If v is incident to at most one inner 3-face, then $\mu^*(v) \geq \mu(v) - 1 - 2 \times \frac{1}{2} = 0$ by (R2.1) and (R7).

Assume v is incident to exactly two 3-faces, say f_1 and f_2, then f_3 and f_4 are 6^+-faces by Lemma 3.5(1), (2). Thus $\mu^*(v) \geq \mu(v) - 2 \times 1 = 0$ by (R2.1) and (R7).

(1.1.2) v has no adjacent incident 3-faces.

Let f_1 be a 3-face adjacent to another 3-cycle. It follows from Lemma 3.5(1) and (2) that f_2 and f_4 are 6^+-faces. Then $w(v \to f_1) \leq 1$ by (R2.1) and (R7), and $w(v \to f_3) \leq 1$ by (R2.1), (R3.1), (R4.1), and (R7). Thus $\mu^*(v) \geq \mu(v) - 2 \times 1 = 0$.

(1.2) v is not incident to a 3-face that is adjacent to another 3-face and v is adjacent to at most one 3-face.

Using the fact that $w(v \to f_i) \leq 1$ for a 3-face f_i by (R1.1) and (R7), and $w(v \to f_i) \leq \frac{1}{3}$ for each 4^+-face f_i by (R3.1), (R4.1), and (R7), we obtain that $\mu^*(v) \geq \mu(v) - 1 - 3 \times \frac{1}{3} = 0$.

(1.3) v is not incident to a 3-face that is adjacent to another 3-face and v is adjacent to two 3-faces.

Consequently, v is incident to exactly two 3-faces, say f_1 and f_3. It follows from Lemma 3.5(3) that f_2 and f_4 are 5^+-faces. Assume v is flaw. Consequently, v is incident to exactly one poor 5-face, say f_2 by Corollary 3.7(1), and f_1 and f_3 are semi-rich 3-faces by Corollary 3.7(2). It follows that $w(v \to f_i) = \frac{9}{10}$ for $i = 1$ and 3 by (R1.1), $w(v \to f_2) \leq \frac{1}{5}$ and $w(v \to f_4) = 0$ by (R4.1) and (R7). Thus $\mu^*(v) \geq \mu(v) - 2 \times \frac{9}{10} - \frac{1}{5} = 0$.

Assume v is not flaw. If f_1 and f_3 are inner faces, then each of f_2 and f_4 is an extreme 5-face or a 6^+-face by the definition. Thus $\mu^*(v) = \mu(v) - 2 \times 1 = 0$ by (R1.1). If at least one of f_1 and f_3 is an extreme 3-face, then $\mu^*(v) =$

$\mu(v) - 1 - \frac{1}{2} - 2 \times \frac{1}{4} = 0$ by (R1.1), (R4.1), and (R7).

(2) A 5-vertex v is incident to a 3-face that is adjacent to another 3-face.

(2.1) v has at least two consecutive incident 3-faces.

If v is incident to four 3-faces say f_1, f_2, f_3, and f_4, then one can see that $B(f_1) \cup B(f_2) \cup B(f_3) \cup B(f_4) = C(3,3,3,3)$. But $C(3,3,3,3)$ contains four pairwise adjacent cycles that contradict $G \in \mathcal{A}$. Thus v is incident to at most three consecutive 3-faces.

If v incident to consecutive three 3-faces say f_1, f_2, and f_3, then f_4 and f_5 are 6^+-faces by Lemma 3.5(1) and (2). Thus $\mu^*(v) = \mu(v) - 3 \times 1 > 0$ by (R2.2) and (R7).

If v incident to exactly two consecutive 3-faces say f_1 and f_2, then f_3 and f_5 are 6^+-faces by Lemma 3.5(1) and (2). Consequently, $w(v \to f_i) \leq \frac{5}{4}$ for $i = 1$ and 2, and $w(v \to f_4) \leq \frac{3}{2}$ by (R2.2), (R3.2), (R4.2), and (R7). Thus $\mu^*(v) \geq \mu(v) - 2 \times \frac{5}{4} - \frac{3}{2} = 0$.

(2.2) v is not incident to consecutive 3-faces.

Let f_1 be a 3-face adjacent to another 3-face. It follows from Lemma 3.5(1) and (2) that f_2 and f_5 are 6^+-faces. By (R2.2) and (R7), $w(v \to f_1) \leq \frac{3}{2}$. If neither f_3 nor f_4 are 3-faces, then $w(v \to f_i) \leq 1$ for $i = 3$ and 4 by (R3.2), (R4.2), and (R7). Thus $\mu^*(v) \geq \mu(v) - \frac{3}{2} - 2 \times 1 > 0$.

Now assume that f_3 is a 3-face. By the condition of (2.2), f_4 is a 4^+-face which implies $w(v \to f_4) \leq 1$ by (R3.2), (R4.2), and (R7). If f_3 is adjacent to another 3-face, then f_4 is a 6^+-face by Lemma 3.5(1) and (2). Moreover, $w(v \to f_3) \leq \frac{3}{2}$ by (R2.2) and (R7). Thus $\mu^*(v) \geq \mu(v) - 2 \times \frac{3}{2} > 0$. If f_3 is not adjacent to another 3-face, then $w(v \to f_3) \leq \frac{6}{5}$ by (R2.2) and (R7). Thus $\mu^*(v) \geq \mu(v) - \frac{3}{2} - \frac{6}{5} > 0$.

(3) A 5-vertex v is not incident to a 3-face that is adjacent to another 3-face and v is incident to at least one 6^+-face. Consequently, v is incident to at most two 3-faces.

(3.1) v is incident to at least two 6^+-faces.

Recall that $w(v \to f_i) \leq \frac{6}{5}$ for each 3-face f_i by (R1.2) and (R7), and $w(v \to f_i) \leq 1$ for each k-face f_i where $k = 4, 5$ by (R3.2), (R4.2), and (R7). If v is incident to t 3-faces, then there are at most $3 - t$ faces f with $d(f) = 4$ or 5. Thus $\mu^*(v) \geq \mu(v) - t \times \frac{6}{5} - (3 - t) \times 1 > 0$ by $t \leq 3$.

(3.2) v is incident to exactly one 6^+-face and incident to at most one 3-face.

If v has no incident 3-faces, then v has all incident faces f except one 6^+-face has $d(f) = 4$ or 5. Thus $\mu^*(v) \geq \mu(v) - 4 \times 1 = 0$ by (R3.2), (R4.2), and (R7).

Assume v is incident to exactly one 3-face, say f_1. By Lemma 3.5(3), v is not a $(3,4,4,4,6^+)$- or a $(3,4,4,6^+,4)$-face. Consequently, v has at least one incident 5-face f_j. Moreover, f_j is adjacent to at least one 4^+-face. We have $w(v \to f_1) \leq \frac{6}{5}$ by (R1.2) and (R7), $w(v \to f_j) \leq \frac{2}{3}$ by (R4.2) and (R7), and $w(v \to f_i) \leq 1$ for each remaining k-face f_i where $k = 4, 5$ by (R3.2), (R4.2), and (R7). Thus $\mu^*(v) \geq \mu(v) - \frac{6}{5} - \frac{2}{3} - 2 \times 1 > 0$.

(3.3) v is incident to exactly one 6^+-face and incident to exactly two 3-faces.

By symmetry and using Lemma 3.5(3) and (4), we have that v is either a $(3,5,3,5,6^+)$-, $(3,5,5,3,6^+)$- or $(3,5,4,3,6^+)$-vertex.

Assume v is a $(3,5,3,5,6^+)$- or $(3,5,5,3,6^+)$-vertex. Applying Corollary 3.9 to $B(f_2) \cup B(f_3)$, v has an incident 5-face f_j which is rich or extreme. Recall that $w(v \to f_i) \leq \frac{6}{5}$ for each 3-face f_i by (R1.2) and (R7), $w(v \to f_j) \leq \frac{1}{2}$ by (R4.2) and (R7), and $w(v \to f_i) \leq 1$ for the remaining 5-face f_i by (R4.2) and (R7). Thus $\mu^*(v) \geq \mu(v) - 2 \times \frac{6}{5} - \frac{1}{2} - 1 > 0$.

Assume v is a $(3,5,4,3,6^+)$-vertex. Applying Corollary 3.9 to $B(f_1) \cup B(f_2)$, we obtain that f_1 or f_2 is rich or extreme. In the former case, $w(v \to f_1) \leq 1$ by (R1.2) and (R7), and $w(v \to f_2) \leq \frac{2}{3}$ by (R4.2) and (R7). In the latter case, $w(v \to f_1) \leq \frac{6}{5}$ by (R1.2) and (R7), and $w(v \to f_2) \leq \frac{1}{2}$ by (R4.2) and (R7). Combining with $w(v \to f_3) \leq 1$ by (R3.2) and (R7) and $w(v \to f_4) \leq \frac{6}{5}$ by (R1.2) and (R7), we have $\mu^*(v) \geq \mu(v) - 2 \times 1 - \frac{2}{3} - \frac{6}{5} > 0$ or $\mu^*(v) \geq \mu(v) - 2 \times \frac{6}{5} - \frac{1}{2} - 1 > 0$.

(4) A 5-vertex v is not incident to a 3-face that is adjacent to another 3-face and v is not incident to a 6^+-face. Consequently, v is incident to at most two 3-faces. Using Corollary 3.10, we have that v has at least three incident faces that are rich or extreme.

(4.1) v has no incident 3-faces.

If f has an extreme face f_i, then $w(v \to f_i) = 0$ by (R7) and $w(v \to f_i) \leq 1$ for each remaining f_i by (R3.2), (R4.2), and (R7). Thus $\mu^*(v) \geq \mu(v) - 4 \times 1 = 0$.

If f has t rich faces, then $\mu^*(v) \geq \mu(v) - t \times \frac{2}{3} - (5 - t) \times 1 \geq 0$ by (R3.2), (R4.2), (R7), and $t \geq 3$.

(4.2) v is incident to exactly one 3-face, say f_1. It follows from Lemma 3.5(3) that v has at most two incident 4-faces.

(4.2.1) v has no incident 4-faces.

We have that $w(v \to f_1) \leq \frac{6}{5}$ by (R1.2) and (R7) and $w(v \to f_i) \leq \frac{2}{3}$ for each 5-face f_i by (R4.2) and (R7). Thus $\mu^*(v) \geq \mu(v) - \frac{6}{5} - 4 \times \frac{2}{3} > 0$.

(4.2.2) v has exactly one incident 4-face.

It follows from Lemma 3.5(4) that v is a $(3,5,4,5,5)$-face. Recall that $w(v \to f_1) \le \frac{6}{5}$ by (R1.2) and (R7), $w(v \to f_3) \le 1$ by (R3.2) and (R7), and $w(v \to f_i) \le \frac{2}{3}$ for each remaining f_i by (R4.2) and (R7). If f_3 is rich or extreme, then $w(v \to f_3) \le \frac{2}{3}$ by (R3.2) and (R7). Thus $\mu^*(v) \ge \mu(v) - \frac{6}{5} - 4 \times \frac{2}{3} > 0$. If f_3 is neither rich nor extreme, then f_2 and f_4 are rich or extreme by Corollary 3.9. Consequently, $w(v \to f_i) \le \frac{1}{2}$ for $i = 2$ or 4 by (R4.2) and (R7). Thus $\mu^*(v) \ge \mu(v) - \frac{6}{5} - 1 - 2 \times \frac{1}{2} - \frac{2}{3} > 0$.

(4.2.3) v has exactly two incident 4-faces.

It follows from Lemma 3.5(3) and (4) that v is a $(3,4,5,5,4)$- or a $(3,5,4,4,5)$-face. Moreover, v has at least three incident faces that are rich or extreme by Corollary 3.10. Consequently, we have (i) f_1 and at least one 4-face f_i are rich or extreme, (ii) f_1 and two 5^+-faces are rich or extreme, (iii) a 4-face and two 5-faces are rich or extreme, or (iv) two 4-faces and a 5-face are rich or extreme.

Recall that $w(v \to f_1) \le \frac{6}{5}$ by (R1.2) and (R7), $w(v \to f_i) \le 1$ for each 4-face f_i by (R3.2) and (R7), and $w(v \to f_i) \le \frac{2}{3}$ for each 5-face f_i by (R4.2) and (R7). Additionally, $w(v \to f_1) \le 1$ if f_1 is rich or extreme by (R1.2) and (R7), $w(v \to f_i) \le \frac{2}{3}$ for each rich or extreme 4-face f_i by (R3.2) and (R7), and $w(v \to f_i) \le \frac{1}{2}$ for each rich or extreme 5-face f_i by (R4.2) and (R7).

If f_1 and a 4-face f_i are rich or extreme, then $\mu^*(v) \ge \mu(v) - 2\times 1 - 3 \times \frac{2}{3} = 0$. If f_1 and two 5^+-faces are rich or extreme, then $\mu^*(v) \ge \mu(v) - 1 - 2 \times 1 - 2 \times \frac{1}{2} = 0$. If a 4-face and two 5^+-faces are rich or extreme, then $\mu^*(v) \ge \mu(v) - \frac{6}{5} - 1 - \frac{2}{3} - 2 \times \frac{1}{2} > 0$. If two 4-faces and a 5-face are rich or extreme, then $\mu^*(v) \ge \mu(v) - \frac{6}{5} - 3 \times \frac{2}{3} - \frac{1}{2} > 0$.

(4.3) v is incident to exactly two 3-faces, say f_1 and f_3.

It follows from Lemma 3.5(3) and (4) that v has no incident 4-faces. This implies v is a $(3,5,3,5,5)$-vertex. Recall that $w(v \to f_i) \le \frac{6}{5}$ for each 3-face f_i by (R1.2) and (R7), and $w(v \to f_i) \le 1$ for each 5-face f_i by (R4.2) and (R7). Furthermore, $w(v \to f_i) \le 1$ for each rich 3-face f_i by (R1.2) and (R7), and $w(v \to f_i) \le \frac{1}{2}$ for each rich 5-face f_i by (R4.2) and (R7). Furthermore, $w(v \to f_i) = \frac{1}{2}$ for each extreme 3-face (f_i) by (R7), and $w(v \to f_i) = 0$ for each extreme 5-face f_i by (R7).

If f_1 or f_3 is an extreme 3-face, then $\mu^*(v) \ge \mu(v) - \frac{6}{5} - \frac{1}{2} - 3 \times \frac{2}{3} > 0$. If f_2, f_4, or f_5 is an extreme 3-face, then $\mu^*(v) \ge \mu(v) - 2 \times \frac{6}{5} - 2 \times \frac{2}{3} > 0$. Thus we assume that all incident faces of v are inner faces.

If each incident 5-face is rich, then $\mu^*(v) \geq \mu(v) - 2 \times \frac{6}{5} - 3 \times \frac{1}{2} > 0$. If f_2 is not rich, then f_1 and f_3 are rich by Corollary 3.9. Consequently, f_4 and f_5 are also rich. Thus $\mu^*(v) \geq \mu(v) - 3 \times 1 - 2 \times \frac{1}{2} = 0$. If f_4 is not rich, then f_3 and f_5 are rich by Corollary 3.9. Consequently, f_2 is also rich. Thus $\mu^*(v) \geq \mu(v) - \frac{6}{5} - 1 - \frac{2}{3} - 2 \times \frac{1}{2} > 0$. The case that f_5 is not rich is similar.

(5) A 6-vertex v is incident to a 3-face that is adjacent to another 3-face.

(5.1) v is incident to at least two consecutive 3-faces.

Let $f_1, \ldots, f_k$ be consecutive 3-faces. Similar to Case (2.1), we have $k \leq 3$. It follows from Lemma 3.5(1) and (2) that v is a $(3, 3, 6^+, k_4, k_5, 6^+)$- or $(3, 3, 3, 6^+, k_5, 6^+)$-face. Since $w(v \to f_i) \leq \frac{3}{2}$ for each 5^--face f_i by (R2.3), (R3.2), (R4.2), and (R7), Thus $\mu^*(v) \geq \mu(v) - 4 \times \frac{3}{2} = 0$.

(5.2) v has no adjacent incident 3-faces.

Let f_1 be a 3-face adjacent to another 3-face. It follows from Lemma 3.5(1) and (2) that f_2 and f_6 are 6^+-faces. Similar to Case (5.1), we obtain that $\mu^*(v) \geq \mu(v) - 4 \times \frac{3}{2} = 0$.

(6) A 6-vertex v is not incident to a 3-face that is adjacent to another 3-face. Consequently, v is incident to at most three 3-faces.

(6.1) v is incident to at least one 6^+-face.

Recall that $w(v \to f_i) \leq \frac{6}{5}$ for each 3-face f_i by (R1.2) and (R7), and $w(v \to f_i) \leq \frac{3}{2}$ for each k-face f_i where $k = 4$ or 5 by (R3.2) and (R4.2). Thus $\mu^*(v) \geq \mu(v) - t \times \frac{6}{5} - (5 - t) \times 1 > 0$ where $t \leq 3$ is the number of incident 3-faces.

(6.2) v has no incident 6^+-face.

(6.2.1) v has no incident 3-faces.

By (R3.2), (R4.2), and (R7), we have $\mu^*(v) \geq \mu(v) - 6 \times 1 = 0$.

(6.2.2) v has exactly one incident 3-face, say f_1.

It follows from Lemma 3.5(3) that v is not a $(3, 4, 4, 4, 4, 4)$-vertex. Consequently, v has s 5-faces where $t \geq 1$. Note that each incident face of v is adjacent to another 4^+-face. It follows that $w(v \to f_i) \leq \frac{2}{3}$ for each 5-face f_i by (R4.2) and (R7). Recall that $w(v \to f_1) \leq \frac{6}{5}$ by (R1.2) and (R7), and $w(v \to f_i) \leq 1$ for each 4-face f. Thus $\mu^*(v) \geq \mu(v) - \frac{6}{5} - s \times \frac{2}{3} - (5 - s) \times 1 > 0$.

(6.2.3) v has exactly two incident 3-faces. Consequently, v is a $(3, k_2, 3, k_4, k_5, k_6)$- or $(3, k_2, k_3, 3, k_5, k_6)$-vertex.

Assume v is a $(3, k_2, 3, k_4, k_5, k_6)$-face. Then $k_2 = 5$ by Lemma 3.5(3). This implies $k_4 = k_6 = 5$ by Lemma 3.5(4). Since v is a $(3, 5, 3, 5, 4^+, 5)$-vertex, we have $w(v \to f_i) \le \frac{6}{5}$ for $i = 1$ and 3 by (R1.2) and (R7), $w(v \to f_i) \le 1$ for $i = 2$ and 5 by (R3.2),(R4.2) and (R7), and $w(v \to f_i) \le \frac{2}{3}$ for $i = 4$ and 6 by (R4.2) and (R7). Thus $\mu^*(v) \ge \mu(v) - 2 \times \frac{6}{5} - 2 \times 1 - 2 \times \frac{2}{3} > 0$.
Assume v is a $(3, k_2, k_3, 3, k_5, k_6)$-vertex. It follows from Lemma 3.5(4) that $\{k_2, k_6\} \neq \{4, 5\}$. If $k_2 = k_6 = 4$, then $k_3 = k_5 = 5$ by Lemma 3.5(3). Consequently, we may assume that v is a $(3, 4, 5, 3, 5, 4)$- and $(3, 5, 5, 3, 5, 5)$-vertex. Recall that $w(v \to f_i) \le \frac{6}{5}$ for $i = 1$ and 4 by (R1.2) and (R7), $w(v \to f_i) \le 1$ for each 4-face f_i by (R3.2) and (R7), and $w(v \to f_i) \le \frac{2}{3}$ for each 5-face f_i by (R4.2) and (R7). Thus a $(3, 4, 5, 3, 5, 4)$-vertex has $\mu^*(v) \ge \mu(v) - 2 \times \frac{6}{5} - 2 \times 1 - 2 \times \frac{2}{3} > 0$, and a $(3, 5, 5, 3, 5, 5)$-vertex has $\mu^*(v) \ge \mu(v) - 2 \times \frac{6}{5} - 4 \times \frac{2}{3} > 0$.

(6.2.4) v has exactly three incident 3-faces. Consequently, v is a $(3, 5, 3, 5, 3, 5)$-vertex by Lemma 3.5(3).
Assume v is incident to at least one extreme 5-face. Consequently, $\mu^*(v) \ge \mu(v) - 3 \times \frac{6}{5} - 2 \times 1 > 0$ by (R1.2), (R4.2), and (R7).
Assume v is not incident to an extreme 5-face. Consequently, each incident face of v is an inner face. It follows from Corollary 3.12 that each union of the boundaries of four consecutive incident faces has a 5^+-vertex other than v. Consequently, two incident 5-faces of v has at least two incident 5^+-vertices, or v has one incident 5-face with at least three incident 5^+-vertices. Thus $\mu^*(v) \ge \mu(v) - 3 \times \frac{6}{5} - 2 \times \frac{1}{2} - 1 > 0$, or $\mu^*(v) \ge \mu(v) - 3 \times \frac{6}{5} - 2 \times 1 - \frac{1}{3} > 0$ by (R1.2), (R4.2), and (R7).

(7) v is a k-vertex where $k \ge 7$.

(7.1) A vertex v is incident to a 3-face that is adjacent to another 3-face. Then v is incident to at least two 6^+-faces by Lemma 3.5(1) and (2). Thus $\mu^*(v) \ge \mu(v) - (k - 2) \times \frac{3}{2} > 0$ by (R2.3), (R3.2), (R4.2), and (R7).

(7.2) A vertex v is not incident to a 3-face that is adjacent to another 3-face. Consequently v is incident to t 3-faces where $t \le k/2$. Thus $\mu^*(v) \ge \mu(v) - t \times \frac{6}{5} - (k - t) \times 1 > 0$ by (R1.2), (R3.2), (R4.2), and (R7).

(8) An inner 3-face f is not adjacent to another 3-face.

If f has no incident flaw 4-vertices, then $\mu^*(f) \ge \mu(f) + 3 \times 1 = 0$ by (R1.1) and (R1.2). If f has an incident flaw vertex, then f is a $(4, 4, 5^+)$-face by Corollary 3.7(2). Recall that $w(v \to f) \ge \frac{9}{10}$ for an incident 4-vertex v by (R1.1), and $w(v \to f) \ge \frac{6}{5}$ for an incident 5^+-vertex v by (R1.2). Thus $\mu^*(f) \ge \mu(f) + 2 \times \frac{9}{10} + \frac{6}{5} = 0$.

(9) An inner 3-face f is adjacent to another 3-face. Note that we use only (R2) to calculate a new charge.

(9.1) A face f is not in a trio. Then $\mu^*(f) \geq \mu(f) + 3 \times 1 = 0$.

(9.2) A face f is in a trio T but not in W_5 formed by four inner 3-faces.

Let f_1, f_2, and f_3 be 3-faces in the same trio T. Define $\mu(T) := \mu(f_1) + \mu(f_2) + \mu(f_3) = -9$ and $\mu^*(T) := \mu^*(f_1) + \mu^*(f_2) + \mu^*(f_3)$. By (R8), it suffices to prove that $\mu^*(T) \geq 0$.

(9.2.1) A worst vertex is a 5^+-vertex. Then $\mu^*(T) \geq \mu(T) + 9 \times 1 = 0$.

(9.2.2) A worst vertex is a 4-vertex and each worse vertex is a 4-vertex. Then two bad vertices are 5^+-vertices by Corollary 3.9. Thus $\mu^*(T) \geq \mu(T) + 3 \times \frac{2}{3} + 2 \times \frac{3}{2} + 4 \times 1 = 0$.

(9.2.3) A worst vertex is a 4-vertex and one of worse vertices is a 5-vertex. Then Corollary 3.9 yields that the other worse vertex or at least one bad vertex is a 5^+-vertex. Thus $\mu^*(T) \geq \mu(T) + 3 \times \frac{2}{3} + 4 \times \frac{5}{4} + 2 \times 1 = 0$ or $\mu^*(T) \geq \mu(T) + 3 \times \frac{2}{3} + 2 \times \frac{5}{4} + \frac{3}{2} + 3 \times 1 = 0$, respectively.

(9.2.4) A worst vertex is a 4-vertex and one of worse vertices is a 6^+-vertex. Then $\mu^*(T) \geq \mu(T) + 3 \times \frac{2}{3} + 2 \times \frac{3}{2} + 4 \times 1 = 0$.

(9.3) A face f is in W_5 formed by four inner 3-faces incident to v.

Let f_1, f_2, f_3, and f_4 be 3-faces in the same W_5. Define $\mu(W_5) := \mu(f_1) + \mu(f_2) + \mu(f_3) + \mu(f_4) = -12$ and $\mu^*(W_5) := \mu^*(f_1) + \mu^*(f_2) + \mu^*(f_3) + \mu^*(f_4)$. By (R8), it suffices to prove that $\mu^*(W_5) \geq 0$. Note that each 3-face in W_5 is adjacent to a 7^+-face by Lemma 3.5(5). Thus W_5 always obtains $4 \times \frac{1}{8}$ from four 7^+-faces by (R5).

(9.3.1) Each vertex of W_5 is a 5^--vertex. Then at least three of them are 5-vertices by Corollary 3.13. Thus $\mu^*(W_5) \geq \mu(W_5) + 6 \times \frac{5}{4} + 2 \times 1 + 4 \times \frac{1}{2} + 4 \times \frac{1}{8} = 0$.

(9.3.2) Exactly one vertex of W_5 is a 6^+-vertex. Then one of the remaining vertices is a 5^+-vertex by Corollary 3.9. Thus $\mu^*(W_5) = \mu(W_5) + 2 \times \frac{3}{2} + 2 \times \frac{5}{4} + 4 \times 1 + 4 \times \frac{1}{2} + 4 \times \frac{1}{8} = 0$.

(9.3.3) At least two vertices of W_5 are 6^+-vertices. Then $\mu^*(W_5) \geq \mu(W_5) + 4 \times \frac{3}{2} + 4 \times 1 + 4 \times \frac{1}{2} + 4 \times \frac{1}{8} > 0$.

(10) f is an inner 4-face.

We claim that f is a $(4^+, 4^+, 4^+, 5^+)$-face. Suppose to the contrary that f is a $(4, 4, 4, 4)$-face. By the minimality of G, there is an L-coloring of $G - B(f)$ where L is restricted to $G - B(f)$. After the coloring, each vertex of $B(f)$ has at least two legal colors. By Lemma 2.1, we can extend an L-coloring to G, a contradiction.

If f is a $(4,4,4,5^+)$-face, then $\mu^*(f) \geq \mu(f) + 3 \times \frac{1}{3} + 1 = 0$ by (R3). If f is a $(4^+,4^+,5^+,5^+)$- or $(4^+,5^+,4^+,5^+)$-face, then f is a rich face and thus $\mu^*(f) \geq \mu(f) + 2 \times \frac{1}{3} + 2 \times \frac{2}{3} = 0$ by (R3).

(11) f is an inner 5-face.

(11.1) f is a poor 5-face, that is f is a $(4,4,4,4,4)$-face.

It follows from Lemma 3.5(2) that each incident 4-vertex of f is incident to at most two 3-faces. If an incident vertex v of f is incident to at most one 3-face, then $w(v \to f) = \frac{1}{3}$ by (R4.1). If an incident vertex v of f is incident to two 3-faces, then v is a flaw vertex or a pseudo flaw vertex, and thus $w(v \to f) \geq \frac{1}{5}$ by (R4.1). Thus $\mu^*(f) \geq \mu(f) + 5 \times \frac{1}{5} = 0$.

(11.2) f is a $(4,4,4,4,5^+)$-face.

(11.2.1) f is adjacent to at least one 4^+-face g. It follows from (R4.2) that $w(v \to f) = \frac{2}{3}$ for an incident 5^+-vertex v of f. Consider a 4-vertex $u \in V(B(f)) \cap V(B(g))$. It follows from Lemma 3.5(2) that u is incident to at most one 3-face. Consequently, $w(u \to f) = \frac{1}{3}$ by (R4.1). Thus $\mu^*(f) \geq \mu(f) + \frac{2}{3} + \frac{1}{3} = 0$.

(11.2.2) f is adjacent to five 3-faces. Then $\mu^*(f) = \mu(f) + 1 = 0$ by (R4.2).

(11.3) f is a rich face with t incident 5^+-vertices. Then $\mu^*(f) \geq \mu(f) + t \times \frac{1}{t} = 0$ by (R4.2).

(12) f is an inner 6^+-face.

If f is a 6-face, then $\mu^*(f) = \mu(f) = 0$. If f is a k-face where $k \geq 7$, then $\mu^*(f) \geq \mu(f) - k \times \frac{1}{8} > 0$ by (R5).

(13) f is an extreme face.

It follows from (R7) that $w(D \to f) = 3$ if a 3-face f is adjacent to a special 4-vertex. Consequently $\mu^*(f) = \mu(f) + 3 = 0$. Thus we assume f is a 3-face not incident to a special 4-vertex, a 4-face, or a 5-face.

(13.1) f is a 3-face that shares exactly one vertex, say u, with C_0. It follows from (R7) that $w(D \to f) = 2$ and $w(v \to f) = \frac{1}{2}$ for each incident vertex v in $\mathrm{int}(C_0)$. Thus $\mu^*(f) = \mu(f) + 2 + 2 \times \frac{1}{2} = 0$.

(13.2) f is a 3-face that shares an edge with C_0. It follows from (R7) that $w(D \to f) = \frac{5}{2}$ and $w(v \to f) = \frac{1}{2}$ for an incident vertex v in $\mathrm{int}(C_0)$. Thus $\mu^*(f) = \mu(f) + \frac{5}{2} + \frac{1}{2} = 0$.

(13.3) f is a 4- or 5-face. Then $\mu^*(f) \geq \mu(f) + 2 \geq 0$ by (R7).

(14) D is the unbounded face.

If a 3-face is incident to a special 4-vertex, then we call it a *special* 3-face, otherwise we call it a *non-special* 3-face.

Let f_3^*, f_3', f' be the number of special 3-faces sharing an incident vertex with D, non-special 3-faces sharing exactly one incident edge with D, non-special 3-faces sharing exactly one incident vertex with D or 4- or 5-faces sharing incident vertices with D, respectively. Let $E(C_0, V(G)-C_0)$ be the set of edges between C_0 and $V(G)-C_0$, and let $e(C_0, V(G)-C_0)$ be its size. Let $E^*(C_0, V(G)-C_0)$ be the set of edges between C_0 and $V(G)-C_0$ that are incident with special 3-faces, and let $e^*(C_0, V(G)-C_0)$ be its size. Let $E'(C_0, V(G)-C_0) = E(C_0, V(G)-C_0) - E^*(C_0, V(G)-C_0)$, and let $e'(C_0, V(G)-C_0)$ be its size.

Then by (R6) and (R7),

$$\begin{aligned}\mu^*(D) &= 3 + 6 + \sum_{v\in C_0}(2d(v)-6) - 3f_3^* - \frac{5}{2}f_3' - 2f' \\ &= 9 + 2\sum_{v\in C_0}(d(v)-2) - 2\times 3 - 3f_3^* - \frac{5}{2}f_3' - 2f' \\ &= 3 - \frac{1}{2}f_3' + 2e(C_0, V(G)-C_0) - 3f_3^* - 2f_3' - 2f' \\ &= 3 - \frac{1}{2}f_3' + (2e^*(C_0, V(G)-C_0) - 3f_3^*) \\ &\quad + (2e'(C_0, V(G)-C_0) - 2f_3' - 2f').\end{aligned}$$

So we may consider that each edge in $E(C_0, V(G)-C_0)$ gives a charge of 2 to D. It follows from Lemma 2.2(1),(2),(5) and Lemma 3.4(2) that an edge in $E^*(C_0, V(G)-C_0)$ is not incident to an extreme non-special 3-face, and not incident to an extreme 4- or 5-face. Moreover, an extreme special 3-face f share incident edges with at most one another extreme special 3-face. Consider an extreme special 3-face f that does not share incident edges with other extreme special 3-faces. By the observation above, f contributes 2 to $e^*(C_0, V(G)-C_0)$ and 1 to f_3^*. Consider two extreme special 3-faces f and g that share an incident edge. By the observation above, f and g contribute 3 to $e(C_0, V(G)-C_0)$ and 2 to f_3^*. Altogether, $2e^*(C_0, V(G)-C_0) - 3f_3^* \geq 0$. Similarly, $2e'(C_0, V(G)-C_0) - 2f_3' - 2f' \geq 0$. Note that $f_3' \leq 3$. Thus $\mu^*(D) > 0$.

This completes the proof.

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