

Existence of Solutions for a Class of Fractional Kirchhoff-type Systems in \mathbb{R}^N with Non-standard Growth

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Abstract. This paper is concerned with the existence and multiplicity of nontrivial solutions for a class of Kirchhoff-type systems in \mathbb{R}^N involving the fractional pseudo-differential operators defined as the generalizations of the $p(x)$ -Laplace operator. Our main tools come from a direct variational methods, the Mountain Pass Theorem, the symmetric Mountain Pass Theorem and the Fountain Theorem in critical point theory. The obtained results of this note significantly contribute to the study of Kirchhoff-type systems in the sense that our situation covers not only differential operators of fractional order but also nonhomogeneous differential operators in Sobolev spaces with variable exponent.

1. Introduction

In this paper, we aim to investigate the existence and multiplicity of weak solutions for the following fractional Kirchhoff type elliptic system

$$(1.1) \quad \begin{cases} M_1(I_{s,p(x,y)}(u))(-\Delta)_{p(x,\cdot)}^s(u) + |u|^{\bar{p}(x)-2}u = \frac{\partial F}{\partial u}(x, u, v) & \text{in } \mathbb{R}^N, \\ M_2(I_{s,q(x,y)}(v))(-\Delta)_{q(x,\cdot)}^s(v) + |v|^{\bar{q}(x)-2}v = \frac{\partial F}{\partial v}(x, u, v) & \text{in } \mathbb{R}^N, \\ (u, v) \in W^{s,p(x,y)}(\mathbb{R}^N) \times W^{s,q(x,y)}(\mathbb{R}^N), \end{cases}$$

where

$$I_{s,r(x,y)}(w) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w(x) - w(y)|^{r(x,y)}}{r(x,y)|x - y|^{N+sr(x,y)}} dx dy \quad \text{for } w \in W^{s,r(x,y)}(\mathbb{R}^N),$$

and $p, q: \mathbb{R}^N \times \mathbb{R}^N \rightarrow]1, +\infty[$ are symmetric continuous functions such that

$$(1.2) \quad 1 < p^- \leq p(x, y) \leq p^+ < +\infty, \quad 1 < q^- \leq q(x, y) \leq q^+ < +\infty,$$

where

$$\begin{aligned} p^- &= \inf_{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N} p(x, y), & p^+ &= \sup_{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N} p(x, y), \\ q^- &= \inf_{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N} q(x, y), & q^+ &= \sup_{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N} q(x, y), \end{aligned}$$

Received August 15, 2020; Accepted May 17, 2021.

Communicated by Cheng-Hsiung Hsu.

2020 *Mathematics Subject Classification.* Primary: 35J48; Secondary: 35J50, 35R11.

Key words and phrases. elliptic systems, generalized fractional Sobolev spaces, fountain theorem, mountain pass theorem, fractional $p(x)$ -Kirchhoff type systems.

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and $\bar{p}(x) = p(x, x)$, and $\bar{q} = q(x, x)$.

The critical fractional Sobolev exponent is given by

$$m_s^*(x) = \begin{cases} \frac{Nm(x,x)}{N-sm(x,x)} & \text{if } N > sm(x, x), \\ +\infty & \text{if } N \leq sm(x, x). \end{cases}$$

$M_i: [0, +\infty) \rightarrow (0, +\infty)$, $i = 1, 2$, are continuous functions satisfying the following conditions:

(M₀) There exist $m_i > 0$, $i = 1, 2$, such that

$$M_i(t) \geq m_i \quad \text{for all } t \geq 0.$$

(M₁) There exists $\theta \in (0, 1)$ such that

$$\widehat{M}_i(t) \geq \theta t M_i(t) \quad \text{for all } t \geq 0,$$

where $\widehat{M}_i(t) = \int_0^t M_i(s) ds$.

The function F is assumed to satisfy Caratheodory conditions and be L^∞ in $x \in \mathbb{R}^N$ and C^1 in $u, v \in \mathbb{R}$. For $s \in (0, 1)$, $(-\Delta)_{p(x,\cdot)}^s$ is the fractional $p(x)$ -Laplacian operator defined as

$$(-\Delta)_{p(x,y)}^s u(x) = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y))}{|x - y|^{N+sp(x,y)}} dy \quad \text{for } x \in \mathbb{R}^N,$$

which is a pseudo-differential operator allowing to introduce the non-integer derivative order. This operator is a fractional version of the so-called $p(x)$ -Laplacian operator which is given by $(-\Delta)_{p(x)} u = -\operatorname{div}(|\nabla u|^{p(x)-2} \nabla u)$.

The problem (1.1) is related to the stationary version of the Kirchhoff equation

$$(1.3) \quad \rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0,$$

presented by Kirchhoff [22] in 1883, is an extension of the classical d'Alembert's wave equation by considering the changes in the length of the string during vibrations. In (1.3), L is the length of string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension. The Kirchhoff's model takes into account the length changes of the string produced by transverse vibrations. Some interesting results can be found, for example in [10]. On the other hand, Kirchhoff-type boundary value problems model several physical and biological systems where u describes a process which depend on the average of itself, as for example, the population density. We refer the reader to [2, 18, 26] for some related works.

Recently, a great attention has been given to the study of the fractional Lebesgue and Sobolev spaces and their generalizations to variable exponents (see [21, 24]). We refer to Di Nezza, Palatucci and Valdinoci [24] for a comprehensive introduction to the study of nonlocal problems. In the context of non homogeneous materials (such that electrorheological fluids and smart fluids), the use of Lebesgue and Sobolev spaces L^p and $W^{s,p}$ seems to be inadequate, which leads to the study of variable exponent Lebesgue and Sobolev spaces $L^{p(x)}$ and $W^{s,p(x,\cdot)}$. Moreover, the study of problems which involves the fractional $p(x, \cdot)$ -Laplacian and the corresponding nonlocal elliptic equations constitutes a promising domain of research in which many mathematicians had contributed (see for example [1, 7, 11, 12, 14, 21]). Furthermore, this type of problems arise in many physical phenomena such as conservation laws, ultra-materials and water waves, optimization, population dynamics, soft thin films, mathematical finance, phases transitions, stratified materials, anomalous diffusion, crystal dislocation, semipermeable membranes, flames propagation, ultra-relativistic limits of quantum mechanics, we refer the reader to [9, 24] for details.

For the problems involving fractional Kirchhoff type, we refer the reader to the works [1, 3–6], where the authors use different methods to establish the existence of solutions.

In the local case ($s = 1$) and when $M_1 = M_2 = 1$, Xu et al. [28] have shown the existence and multiplicity of solutions for the following elliptic system with nonstandard growth conditions in \mathbb{R}^N :

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = \frac{\partial F}{\partial u}(x, u, v) & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(|\nabla v|^{q(x)-2}\nabla v) + |v|^{q(x)-2}v = \frac{\partial F}{\partial v}(x, u, v) & \text{in } \mathbb{R}^N, \\ (u, v) \in X, \end{cases}$$

where $X = W^{1,p(x)}(\mathbb{R}^N) \times W^{1,q(x)}(\mathbb{R}^N)$, $N \geq 2$, $p(\cdot), q(\cdot)$ are functions on \mathbb{R}^N , the function F is assumed to verify Carathéodory conditions and L^∞ in $x \in \mathbb{R}^N$ and C^1 in $u, v \in \mathbb{R}$.

In our context, Dai [13] considered the following nonlocal elliptic systems of gradient type with nonstandard growth conditions

$$\begin{cases} -M_1 \left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \frac{\partial F}{\partial u}(x, u, v) & \text{in } \Omega, \\ -M_2 \left(\int_\Omega \frac{1}{q(x)} |\nabla v|^{q(x)} dx \right) \operatorname{div}(|\nabla v|^{q(x)-2}\nabla v) = \frac{\partial F}{\partial v}(x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^N with a smooth boundary $\partial\Omega$. $M_1(\cdot), M_2(\cdot)$ are continuous functions and $F: \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be continuous in $x \in \Omega$ and of class C^1 in $u, v \in \mathbb{R}$.

The main purpose of this paper is to prove the existence and multiplicity of weak solutions for problem (1.1). We know that in the study of fractional problems in \mathbb{R}^N

involving the $p(x, \cdot)$ -Laplacian operator, the main difficulty arises from the lack of compactness. To overcome this difficulty we will include the method of weight functions. By the critical point theory, we present here two type of results for problem (1.1). These results correspond to the “sublinear” and the “superlinear” cases. Our paper is motivated by [4, 8, 13, 28] and organized as follows: in Section 2 we recall some notations and properties of fractional Lebesgue and Sobolev spaces with variable exponents. In order to prove the main theorems in Section 3 some useful lemmas are given.

2. Preliminaries and basic assumptions

In this section, we recall some necessary properties of variable exponent spaces. For more details we refer to [17, 23, 25], and the references therein. Consider the set

$$C_+(\mathbb{R}^N) = \{r \in C(\mathbb{R}^N) : r(x) > 1, \forall x \in \mathbb{R}^N\}.$$

For any $r \in C_+(\mathbb{R}^N)$, we define the generalized Lebesgue space $L^{r(\cdot)}(\mathbb{R}^N)$ as

$$L^{r(x)}(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^N} |u(x)|^{r(x)} dx < +\infty \right\},$$

this space equipped with the *Luxemburg* norm

$$\|u\|_{r(x)} = \|u\|_{L^{r(x)}(\mathbb{R}^N)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} \left| \frac{u(x)}{\lambda} \right|^{r(x)} dx \leq 1 \right\}$$

is a separable reflexive Banach space.

Let $\widehat{r} \in C_+(\mathbb{R}^N)$ be the conjugate exponent of r , i.e., $\frac{1}{r(x)} + \frac{1}{\widehat{r}(x)} = 1$. Then we have the following Hölder-type inequality.

Lemma 2.1 (Hölder’s inequality). *If $u \in L^{r(\cdot)}(\mathbb{R}^N)$ and $v \in L^{\widehat{r}(x)}(\mathbb{R}^N)$, so*

$$\left| \int_{\mathbb{R}^N} uv dx \right| \leq \left(\frac{1}{r^-} + \frac{1}{\widehat{r}^-} \right) \|u\|_{r(x)} \|v\|_{\widehat{r}(x)} \leq 2 \|u\|_{r(x)} \|v\|_{\widehat{r}(x)}.$$

The modular of $L^{r(x)}(\mathbb{R}^N)$ is defined by

$$\rho_{r(x)} : L^{r(x)}(\mathbb{R}^N) \rightarrow \mathbb{R}, \quad u \mapsto \rho_{r(x)}(u) = \int_{\mathbb{R}^N} |u(x)|^{r(x)} dx.$$

Proposition 2.2. [16, 23] *Let $u \in L^{r(x)}(\mathbb{R}^N)$, then we have*

- (1) $\|u\|_{r(x)} < 1$ (resp. $= 1, > 1$) $\iff \rho_{r(x)}(u) < 1$ (resp. $= 1, > 1$).
- (2) $\|u\|_{r(x)} < 1 \implies \|u\|_{r(x)}^{r^+} \leq \rho_{r(\cdot)}(u) \leq \|u\|_{r(x)}^{r^-}$.
- (3) $\|u\|_{r(x)} > 1 \implies \|u\|_{r(x)}^{r^-} \leq \rho_{r(\cdot)}(u) \leq \|u\|_{r(x)}^{r^+}$.

Proposition 2.3. *If $u, u_k \in L^{r(x)}(\mathbb{R}^N)$ and $k \in \mathbb{N}$, then the following assertions are equivalent*

- (1) $\lim_{k \rightarrow +\infty} \|u_k - u\|_{r(x)} = 0.$
- (2) $\lim_{k \rightarrow +\infty} \rho_{r(x)}(u_k - u) = 0.$
- (3) $u_k \rightarrow u$ in measure in \mathbb{R}^N and $\lim_{k \rightarrow +\infty} \rho_{r(x)}(u_k) = \rho_{r(x)}(u).$

Now, let's introduce our fundamental space. Let $s \in (0, 1)$, and let $r: \mathbb{R}^N \times \mathbb{R}^N \rightarrow (1, +\infty)$ be a Lipschitz continuous variable exponent satisfying (1.2). We define the usual fractional Sobolev space with variable exponent as

$$W^{s,r(x,y)}(\mathbb{R}^N) = \left\{ u \in L^{\bar{r}(x)}(\mathbb{R}^N) : \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{r(x,y)}}{|x - y|^{N+sr(x,y)}} dx dy < \infty \right\},$$

which is equipped with the norm

$$\|u\|_{W^{s,r(x,y)}(\mathbb{R}^N)} = \|u\|_{\bar{r}(x)} + [u]_{s,r(x,y)},$$

where $\bar{r}(x) = r(x, x)$ and $[u]_{s,r(x,y)}$ is a Gagliardo seminorm with variable exponent which is defined by

$$[u]_{s,r(x,y)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{r(x,y)}}{\lambda^{r(x,y)} |x - y|^{N+sr(x,y)}} dx dy \leq 1 \right\}.$$

The space $(W^{s,r(x,y)}(\mathbb{R}^N), \|\cdot\|_{W^{s,r(x,y)}(\mathbb{R}^N)})$ is separable and reflexive Banach space (see, [7, Lemma 3.1]).

For any $u \in W^{s,r(x,y)}(\mathbb{R}^N)$, define the modular function $\rho_{r(\cdot,\cdot)}: W^{s,r(x,y)}(\mathbb{R}^N) \rightarrow \mathbb{R}$

$$\rho_{r(x,y)}(u) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{r(x,y)}}{|x - y|^{N+sr(x,y)}} dx dy + \int_{\mathbb{R}^N} |u(x)|^{\bar{r}(x)} dx,$$

and its norm

$$\|u\|_{s,r(x,y)} = \|u\|_{\rho_{r(x,y)}} = \inf \left\{ \lambda > 0 : \rho_{r(x,y)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

It is easy to see that $\|\cdot\|_{s,r(x,y)}$ is a norm which is equivalent to the norm $\|\cdot\|_{W^{s,r(x,y)}(\mathbb{R}^N)}$. Throughout this paper we will use the norm $\|\cdot\|_{s,r(x,y)}$.

Lemma 2.4. *Let $r: \mathbb{R}^N \times \mathbb{R}^N \rightarrow]1, +\infty[$ be a continuous variable exponent and $s \in]0, 1[$. Let $u, u_k \in W^{s,r(x,y)}(\mathbb{R}^N)$, then we have*

- (1) $\|u\|_{s,r(x,y)} < 1$ (resp. $= 1, > 1$) $\iff \rho_{p(\cdot,\cdot)}(u) < 1$ (resp. $= 1, > 1$),

- (2) $\|u\|_{s,r(x,y)} < 1 \implies \|u\|_{s,r(x,y)}^{q^+} \leq \rho_{p(\cdot,\cdot)}(u) \leq \|u\|_{s,r(x,y)}^{q^-}$,
- (3) $\|u\|_{s,r(x,y)} > 1 \implies \|u\|_{s,r(x,y)}^{q^-} \leq \rho_{p(\cdot,\cdot)}(u) \leq \|u\|_{s,r(x,y)}^{q^+}$,
- (4) $\lim_{k \rightarrow +\infty} \|u_k - u\|_{s,r(x,y)} = 0 \iff \lim_{k \rightarrow +\infty} \rho_{p(\cdot,\cdot)}(u_k - u) = 0$.

Theorem 2.5. [14, 21] *Let Ω be a open bounded subset of \mathbb{R}^N and $s \in (0, 1)$. Let $r: \bar{\Omega} \times \bar{\Omega} \rightarrow (1, +\infty)$ be a continuous variable exponent with $sr^+ < N$ for all $(x, y) \in \bar{\Omega} \times \bar{\Omega}$. Let $q: \bar{\Omega} \rightarrow (1, +\infty)$ be a continuous variable exponent such that*

$$r_s^*(x) = \frac{Nr(x, x)}{N - r(x, x)} > q(x) \geq q^- > 1, \quad \forall x \in \bar{\Omega}.$$

Then the space $W^{s,r(x,y)}(\Omega)$ is continuously embedded in $L^{q(x)}(\Omega)$. Moreover, this embedding is compact. In other words, there exists a constant $C = C(N, s, p, q, \Omega) > 0$ such that for every $u \in W^{s,r(x,y)}(\Omega)$,

$$\|u\|_{L^{q(x)}(\Omega)} \leq C \|u\|_{W^{s,r(x,y)}(\Omega)}.$$

Now, let $\omega: \mathbb{R}^N \rightarrow \mathbb{R}$ be a function satisfying the following condition

(ω_0) $\omega \in C(\mathbb{R}^N, \mathbb{R})$ such that $\omega(x) \geq 0$ for all $x \in \mathbb{R}^N$ and $\omega \neq 0$.

For $q \in C_+(\mathbb{R}^N)$, define

$$L_\omega^{q(x)}(\mathbb{R}^N) = \left\{ u: \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^N} \omega(x) |u(x)|^{q(x)} dx < +\infty \right\},$$

which endowed with the *Luxemburg* norm

$$\|u\|_{q(x),\omega} = \|u\|_{L_\omega^{q(x)}(\mathbb{R}^N)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} \omega(x) \left| \frac{u(x)}{\lambda} \right|^{q(x)} dx \leq 1 \right\}.$$

It is easy to see that $\rho_{q(x),\omega} = \int_{\mathbb{R}^N} \omega(x) |u(x)|^{q(x)} dx$ is a semimodular. Moreover, $(L_\omega^{q(x)}(\mathbb{R}^N), \|u\|_{q(x),\omega})$ is a Banach space.

Lemma 2.6. [27] *The following assertions are equivalent:*

- (i) $\lim_{n \rightarrow \infty} \|u_n\|_{q(x),\omega} = 0$;
- (ii) $\lim_{n \rightarrow \infty} \rho_{q(x),\omega}(u_n) = 0$.

Lemma 2.7. [4, 19] *Suppose that (1.2) holds. Let $h \in C_+(\mathbb{R}^N)$ with $1 < h^- \leq h(x) \leq h^+ < r_s^*(x)$ for all $x \in \mathbb{R}^N$. Assume that $\omega \in L^{\frac{n(\cdot)}{n(\cdot)-h(\cdot)}}$ such that*

$$\bar{r}(x) \leq n(x) \leq r_s^*(x) \text{ for all } x \in \mathbb{R}^N, \quad \text{and} \quad \inf_{x \in \mathbb{R}^N} (n(x) - h(x)) > 0.$$

Then, the embedding $W^{s,r(x,y)}(\mathbb{R}^N) \hookrightarrow L_\omega^{h(x)}(\mathbb{R}^N)$ is continuous. Moreover, if $n^+ < r_s^(x)$ for all $x \in \mathbb{R}^N$, the embedding $W^{s,r(x,y)}(\mathbb{R}^N) \hookrightarrow L_\omega^{h(x)}(\mathbb{R}^N)$ is compact.*

Lemma 2.8. *Let $r \in C_+(\mathbb{R}^N \times \mathbb{R}^N)$ satisfy (1.2), $h \in C_+(\mathbb{R}^N)$ and let ω be a weight function. Then for any $u \in W^{s,r(x,y)}(\mathbb{R}^N)$ there exist $\bar{h} \in [h^-, h^+]$ and a constant $c > 0$ such that*

$$\int_{\mathbb{R}^N} \omega(x)|u|^h(x) dx \leq c\|u\|_{s,r(x,y)}^{\bar{h}}.$$

Proposition 2.9 (Fountain Theorem). *Let X be a Banach space with the norm $\|\cdot\|_X$ and let X_j be a sequence of subspaces of X with $\dim X_j < \infty$ for each $j \in \mathbb{N}$. Further, $X = \overline{\bigoplus_{j \in \mathbb{N}} X_j}$, the closure of the direct sum of all X_j . Set $\mathbb{Y}_k = \bigoplus_{j=1}^k X_j$, $\mathbb{Z}_k = \overline{\bigoplus_{j=k}^\infty X_j}$. Assume that $\Psi \in C^1(X, \mathbb{R})$ satisfies the (PS) condition and $\Psi(-u) = \Psi(u)$. For every $k \in \mathbb{N}$, suppose that there exist $R_k > r_k > 0$ such that*

$$(A1) \inf_{\substack{u \in \mathbb{Z}_k \\ \|u\|_X = r_k}} \Psi(u) \rightarrow +\infty \text{ as } k \rightarrow \infty.$$

$$(A2) \max_{\substack{u \in \mathbb{Y}_k \\ \|u\|_X = R_k}} \Psi(u) \leq 0.$$

Then Ψ has an unbounded sequence of critical values.

Remark 2.10. Since X is a separable and reflexive space, there exist $\{e_i\}_{i=1}^\infty \subset X$ and $\{f_i\}_{i=1}^\infty \subset X^*$ such that

$$f_i(e_j) = \delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Hence $X = \overline{\text{span}}\{e_i : i = 1, 2, \dots\}$ and $X^* = \overline{\text{span}}\{f_i : i = 1, 2, \dots\}$. For $k = 1, 2, \dots$, we define

$$X = \overline{\text{span}}\{e_i : i = 1, 2, \dots\}, \quad \mathbb{Y}_k = \bigoplus_{i=0}^k X_i, \quad \mathbb{Z}_k = \bigoplus_{i=k}^\infty X_i.$$

Lemma 2.11. [27] *Let $r \in C_+(\mathbb{R}^N)$ such that $1 < r^- \leq r(x) \leq r^+ < \min\{p_s^*(x), q_s^*(x)\}$, $\forall x \in \mathbb{R}^N$. For $k = 1, 2, \dots$, set*

$$\alpha_k = \sup_{\substack{u \in \mathbb{Z}_k \\ \|u\|_X \leq 1}} \int_{\mathbb{R}^N} \omega(x)|u|^{r(x)} dx.$$

Then $\alpha_k \rightarrow 0$ as $k \rightarrow +\infty$.

Next, we will state the Symmetric Mountain Pass Lemma. For this purpose, we should introduce the definition of genus.

Definition 2.12. Let X be a real Banach space and A a subset of X . A is said to be symmetric if $u \in A$ implies $-u \in A$. For a closed symmetric set A which does not contain the origin, we define the genus $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If there does not exist such a k , we define $\gamma(A) = \infty$. Moreover, we set $\gamma(\emptyset) = 0$. Let Γ_k denotes the family of closed symmetric subsets A of X such that $0 \notin A$ and $\gamma(A) \geq k$.

We recall the Symmetric Mountain Pass Lemma.

Theorem 2.13. [20] *Let X be an infinite dimensional Banach space and $\mathcal{J} \in C^1(X; \mathbb{R})$ satisfy the following assertions:*

(S1) *\mathcal{J} is even, bounded from below, $\mathcal{J}(0, 0) = 0$ and $\mathcal{J}(u, v)$ satisfies the (PS) condition.*

(S2) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} \mathcal{J}(u, v) < 0$.*

Then \mathcal{J} admits a sequence of critical points $\{(u_k, v_k)\}$ such that $\mathcal{J}(u_k, v_k) < 0$, $(u_k, v_k) \neq (0, 0)$ and $\lim_{k \rightarrow \infty} (u_k, v_k) = (0, 0)$.

3. Main results

In this section we prove our main theorems, Theorems 3.4, 3.5, 3.6 and 3.10. The solutions of our problem (1.1) belong to $E = W^{s,p(x,y)}(\mathbb{R}^N) \times W^{s,q(x,y)}(\mathbb{R}^N)$, the Cartesian product of two Banach spaces, which is a reflexive Banach space endowed with the norm

$$\|(u, v)\|_E = \|u\|_{s,p(x,y)} + \|v\|_{s,q(x,y)},$$

where $\|\cdot\|_{s,p(x,y)}$ and $\|\cdot\|_{s,q(x,y)}$ are modular norms of $W^{s,p(x,y)}(\mathbb{R}^N)$ and $W^{s,q(x,y)}(\mathbb{R}^N)$ respectively.

For every (u, v) and (φ, ψ) in E , let

$$\mathcal{F}(u, v) = \int_{\mathbb{R}^N} F(x, u, v) \, dx,$$

then

$$\mathcal{F}'(u, v)(\varphi, \psi) = \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x, u, v) \varphi \, dx + \int_{\mathbb{R}^N} \frac{\partial F}{\partial v}(x, u, v) \psi \, dx.$$

Suppose F satisfies

(F₀) There exist $p_0, p_1 \in C_+(\mathbb{R}^N)$ and $q_0, q_1 \in C_+(\mathbb{R}^N)$ such that, for all $(s, t) \in \mathbb{R}^2$ and for a.e. $x \in \mathbb{R}^N$, we have

$$\begin{aligned} \left| \frac{\partial F}{\partial s}(x, s, t) \right| &\leq a_1(x) |s|^{p_0(x)-1} + a_2(x) |t|^{q_0(x)-1}, \\ \left| \frac{\partial F}{\partial t}(x, s, t) \right| &\leq b_1(x) |s|^{p_1(x)-1} + b_2(x) |t|^{q_1(x)-1}, \end{aligned}$$

where $a_1(\cdot) \in L^{\sigma_1(x)}(\mathbb{R}^N)$, $a_2(\cdot), b_1(\cdot) \in L^{\sigma_2(x)}(\mathbb{R}^N)$ and $b_2(\cdot) \in L^{\sigma_3(x)}(\mathbb{R}^N)$,

$$\sigma_1(x) = \frac{\bar{p}(x)}{\bar{p}(x) - 1}, \quad \sigma_2(x) = \frac{p_s^*(x)q_s^*(x)}{p_s^*(x)q_s^*(x) - p_s^*(x) - q_s^*(x)}, \quad \sigma_3(x) = \frac{\bar{q}(x)}{\bar{q}(x) - 1}.$$

Lemma 3.1. *Under the assumptions (F_0) , \mathcal{F} is differentiable in sense of Fréchet, i.e., for fixed $(u, v) \in E$ and given $\epsilon > 0$, there exists $\delta = \delta(\epsilon, u, v) > 0$ such that*

$$|\mathcal{F}(u + \varphi, v + \psi) - \mathcal{F}(u, v) - \mathcal{F}'(u, v)(\varphi, \psi)| \leq \epsilon(\|\varphi\|_{s,p(x,y)} + \|\psi\|_{s,q(x,y)})$$

for all $(\varphi, \psi) \in E$ with $\|\varphi\|_{s,p(x,y)} + \|\psi\|_{s,q(x,y)} < \delta$.

Proof. Let $B_r = \{x \in \mathbb{R}^N : |x| < r\}$ be the ball of radius r which centred at the origin of \mathbb{R}^N and $\widehat{B}_r = \mathbb{R}^N \setminus B_r$. Define the functional $\mathcal{F}_r : W^{s,p(x,y)}(B_r) \times W^{s,q(x,y)}(B_r) \rightarrow \mathbb{R}$ as follows:

$$\mathcal{F}_r(u, v) = \int_{B_r} F(x, u, v) dx.$$

It is easy to see that $\mathcal{F}_r \in C^1(W^{s,p(x,y)}(B_r) \times W^{s,q(x,y)}(B_r), \mathbb{R})$ and for all $(\varphi, \psi) \in W^{s,p(x,y)}(B_r) \times W^{s,q(x,y)}(B_r)$, we have

$$\mathcal{F}'_r(u, v)(\varphi, \psi) = \int_{B_r} \frac{\partial F}{\partial u}(x, u, v)\varphi dx + \int_{B_r} \frac{\partial F}{\partial v}(x, u, v)\psi dx.$$

In addition, the operator $\mathcal{F}'_r : W^{s,p(x,y)}(B_r) \times W^{s,q(x,y)}(B_r) \rightarrow (W^{s,p(x,y)}(B_r) \times W^{s,q(x,y)}(B_r))^*$ is compact. So, for all $(u, v), (\varphi, \psi) \in E$ we can write

$$\begin{aligned} & |\mathcal{F}(u + \varphi, v + \psi) - \mathcal{F}(u, v) - \mathcal{F}'(u, v)(\varphi, \psi)| \\ & \leq |\mathcal{F}_r(u + \varphi, v + \psi) - \mathcal{F}_r(u, v) - \mathcal{F}'_r(u, v)(\varphi, \psi)| \\ & \quad + \left| \int_{\widehat{B}_r} \left(F(x, u + \varphi, v + \psi) - F(x, u, v) - \frac{\partial F}{\partial u}(x, u, v)\varphi - \frac{\partial F}{\partial v}(x, u, v)\psi \right) dx \right|. \end{aligned}$$

By virtue of mean-value theorem, there exist $\mu_1, \mu_2 \in (0, 1)$ such that

$$F(x, u + \varphi, v + \psi) - F(x, u, v) = \frac{\partial F}{\partial u}(x, u + \mu_1\varphi, v)\varphi - \frac{\partial F}{\partial v}(x, u, v + \mu_2\psi)\psi.$$

Using condition (F_0) , we have

$$\begin{aligned} & \left| \int_{\widehat{B}_r} \left(F(x, u + \varphi, v + \psi) - F(x, u, v) - \frac{\partial F}{\partial u}(x, u, v)\varphi - \frac{\partial F}{\partial v}(x, u, v)\psi \right) dx \right| \\ & \leq \int_{\widehat{B}_r} \left(a_1(x)(|u + \mu_1\varphi|^{p_0(x)-1} - |u|^{p_0(x)-1})\varphi + b_2(x)(|v + \mu_2\psi|^{q_1(x)-1} - |v|^{q_1(x)-1})\psi \right) dx \\ & \leq (2^{p_0^+ - 1} - 1) \int_{\widehat{B}_r} a_1(x)|u|^{p_0(x)-1}\varphi dx + (2\mu_1)^{p_0^\pm - 1} \int_{\widehat{B}_r} a_1(x)|\varphi|^{p_0(x)} dx \\ & \quad + (2^{q_1^- - 1} - 1) \int_{\widehat{B}_r} b_2(x)|v|^{q_1(x)-1}\psi dx + (2\mu_2)^{q_1^\pm - 1} \int_{\widehat{B}_r} b_2(x)|\psi|^{q_1(x)} dx \\ & \leq (2^{p_0^+ - 1} - 1) \|a_1\|_{L^{\sigma_1(x)}(\widehat{B}_r)} \|u\|_{(p_0(x)-1)p_0^*(x)}^{\bar{p}_0} \|\varphi\|_{\frac{N}{s}} \\ & \quad + (2\mu_1)^{p_0^\pm - 1} \|a_1\|_{L^{\sigma_1(x)}(\widehat{B}_r)} (\|\varphi\|_{p_0^-(x)}^{\bar{p}_0} + \|\varphi\|_{p_0^+(x)}^{\bar{p}_0}) \end{aligned}$$

$$\begin{aligned}
 &+ (2^{q_1^+ - 1} - 1) \|b_2\|_{L^{\sigma_3(x)}(\widehat{B}_r)} \|v\|_{(q_1(x)-1)q_s^*(x)}^{\bar{q}_1} \|\psi\|_{\frac{N}{s}} \\
 &+ (2\mu_2)^{q_1^\pm - 1} \|b_2\|_{L^{\sigma_3(x)}(\widehat{B}_r)} (\|\psi\|_{q_1(x)}^{\bar{q}_1^-} + \|\psi\|_{q_1(x)}^{q_1^+}) \\
 \leq &c \|a_1\|_{L^{\sigma_1(x)}(\widehat{B}_r)} (\|u\|_{s,p(x,y)}^{\bar{p}_0} + \|\varphi\|_{s,p(x,y)}^{p_0^- - 1} + \|\varphi\|_{s,p(x,y)}^{p_0^+ - 1}) \|\varphi\|_{s,p(x,y)} \\
 &+ c \|b_2\|_{L^{\sigma_3(x)}(\widehat{B}_r)} (\|v\|_{s,q(x,y)}^{\bar{q}_1} + \|\psi\|_{s,q(x,y)}^{q_1^- - 1} + \|\psi\|_{s,q(x,y)}^{q_1^+ - 1}) \|\psi\|_{s,q(x,y)}
 \end{aligned}$$

and by the fact that

$$(3.1) \quad \|a_1\|_{L^{\sigma_1(x)}(\widehat{B}_r)} \rightarrow 0$$

and

$$(3.2) \quad \|b_2\|_{L^{\sigma_3(x)}(\widehat{B}_r)} \rightarrow 0$$

as r sufficiently large, it follows that

$$\begin{aligned}
 &\left| \int_{\widehat{B}_r} \left(F(x, u + \varphi, v + \psi) - F(x, u, v) - \frac{\partial F}{\partial u}(x, u, v)\varphi - \frac{\partial F}{\partial v}(x, u, v)\psi \right) dx \right| \\
 &\leq \epsilon (\|\varphi\|_{s,p(x,y)} + \|\psi\|_{s,q(x,y)}).
 \end{aligned}$$

It remains to show that \mathcal{F}' is continuous on E . Indeed, let $(u_n, v_n), (u, v) \in E$ such that $(u_n, v_n) \rightarrow (u, v)$, then for $(\varphi, \psi) \in E$, we have

$$\begin{aligned}
 &|\mathcal{F}'(u_n, v_n)(\varphi, \psi) - \mathcal{F}'(u, v)(\varphi, \psi)| \\
 \leq &|\mathcal{F}'_r(u_n, v_n)(\varphi, \psi) - \mathcal{F}'_r(u, v)(\varphi, \psi)| + \left| \int_{\widehat{B}_r} \frac{\partial F}{\partial u}(x, u_n, v_n)\varphi - \frac{\partial F}{\partial u}(x, u, v)\varphi dx \right| \\
 &+ \left| \int_{\widehat{B}_r} \frac{\partial F}{\partial v}(x, u_n, v_n)\psi - \frac{\partial F}{\partial v}(x, u, v)\psi dx \right|.
 \end{aligned}$$

Put

$$\begin{aligned}
 I_1 &= \left| \int_{\widehat{B}_r} \left(\frac{\partial F}{\partial u}(x, u_n, v_n)\varphi + \frac{\partial F}{\partial u}(x, u, v)\varphi \right) dx \right|, \\
 I_2 &= \left| \int_{\widehat{B}_r} \left(\frac{\partial F}{\partial v}(x, u_n, v_n)\psi + \frac{\partial F}{\partial v}(x, u, v)\psi \right) dx \right|.
 \end{aligned}$$

By (F_0) we can write

$$\begin{aligned}
 I_1 &\leq \int_{\widehat{B}_r} (a_1(x)|u_n|^{p_0(x)-1} + a_2(x)|v_n|^{q_0(x)-1})\varphi dx \\
 &+ \int_{\widehat{B}_r} (a_1(x)|u|^{p_0-1} + a_2(x)|v|^{q_0(x)-1})\varphi dx \\
 &\leq \|a_1\|_{\sigma_1(x)} \|u_n\|_{(p_0(x)-1)p_s^*(x)}^{\bar{p}_0} \|\varphi\|_{\frac{N}{s}} + \|a_2\|_{\sigma_2(x)} \|v_n\|_{(q_0(x)-1)q_s^*(x)}^{\bar{q}_0} \|\varphi\|_{p_s^*(x)} \\
 &+ \|a_1\|_{\sigma_1(x)} \|u\|_{(p_0(x)-1)p_s^*(x)}^{\bar{p}_0} \|\varphi\|_{\frac{N}{s}} + \|a_2\|_{\sigma_2(x)} \|v\|_{(q_0(x)-1)q_s^*(x)}^{\bar{q}_0} \|\varphi\|_{p_s^*(x)} \\
 &\leq (\|a_1\|_{\sigma_1(x)} \|u_n\|_{s,p(x,y)}^{\bar{p}_0} + \|a_2\|_{\sigma_2(x)} \|v_n\|_{s,q(x,y)}^{\bar{q}_0}) \|\varphi\|_{s,p(x,y)} \\
 &+ (\|a_1\|_{\sigma_1(x)} \|u\|_{s,p(x,y)}^{\bar{p}_0} + \|a_2\|_{\sigma_2(x)} \|v\|_{s,q(x,y)}^{\bar{q}_0}) \|\varphi\|_{s,p(x,y)}
 \end{aligned}$$

and

$$\begin{aligned}
 I_2 &\leq \int_{\widehat{B}_r} (b_1(x)|u_n|^{p_1-1} + b_2(x)|v_n|^{q_1(x)-1})\psi \, dx \\
 &\quad + \int_{\widehat{B}_r} (b_1(x)|u|^{p_1-1} + b_2(x)|v|^{q_1(x)-1})\psi \, dx \\
 &\leq \|b_1\|_{\sigma_2(x)} \|u_n\|_{(p_1(x)-1)p_s^*(x)}^{\bar{p}_1} \|\psi\|_{q_s^*(x)} + \|b_2\|_{\sigma_3(x)} \|v_n\|_{(q_1(x)-1)q_s^*(x)}^{\bar{q}_1} \|\psi\|_{\frac{N}{s}} \\
 &\quad + \|b_1\|_{\sigma_2(x)} \|u\|_{(p_1(x)-1)p_s^*(x)}^{\bar{p}_1} \|\psi\|_{q_s^*(x)} + \|b_2\|_{\sigma_3(x)} \|v\|_{(q_1(x)-1)q_s^*(x)}^{\bar{q}_1} \|\psi\|_{\frac{N}{s}} \\
 &\leq (\|b_1\|_{\sigma_2(x)} \|u_n\|_{s,p(x,y)}^{\bar{p}_1} + \|b_2\|_{\sigma_3(x)} \|v_n\|_{s,q(x,y)}^{\bar{q}_1}) \|\psi\|_{s,q(x,y)} \\
 &\quad + (\|b_1\|_{\sigma_2(x)} \|u\|_{s,p(x,y)}^{\bar{p}_1} + \|b_2\|_{\sigma_3(x)} \|v\|_{s,q(x,y)}^{\bar{q}_1}) \|\psi\|_{s,q(x,y)}.
 \end{aligned}$$

Since \mathcal{F}'_r is continuous on $W^{s,p(x,y)}(B_r) \times W^{s,q(x,y)}(B_r)$, we have

$$|\mathcal{F}'_r(u_n, v_n)(\varphi, \psi) - \mathcal{F}'_r(u, v)(\varphi, \psi)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Moreover, using (3.1) and (3.2), and when r is sufficiently large, I_1 and I_2 tend also to 0. Hence

$$|\mathcal{F}'(u_n, v_n)(\varphi, \psi) - \mathcal{F}'(u, v)(\varphi, \psi)| \rightarrow 0$$

as $(u_n, v_n) \rightarrow (u, v)$. This implies that \mathcal{F}' is continuous on E . □

Remark 3.2. By the same way we can get the weak-strong continuity of \mathcal{F}' , i.e., for $(u_n, v_n) \rightharpoonup (u, v)$ in E , we have $\mathcal{F}'(u_n, v_n) \rightarrow \mathcal{F}'(u, v)$ in E^* .

Definition 3.3. We say that $(u, v) \in E$ is a weak solution of (1.1) if

$$\begin{aligned}
 &M_1(I_{s,p(x,y)}(u)) \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+sp(x,y)}} \, dx dy \\
 &+ \int_{\mathbb{R}^N} |u|^{\bar{p}(x)-2} u \varphi \, dx \\
 &+ M_2(I_{s,q(x,y)}(v)) \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v(x) - v(y)|^{q(x,y)-2} (v(x) - v(y)) (\psi(x) - \psi(y))}{|x - y|^{N+sq(x,y)}} \, dx dy \\
 &+ \int_{\mathbb{R}^N} |v|^{\bar{q}(x)-2} v \psi \, dx - \mathcal{F}'(u, v)(\varphi, \psi) = 0
 \end{aligned}$$

for all $(\varphi, \psi) \in E$.

To find such a solution $(u, v) \in E$ of (1.1), we shall study the critical point of the energy functional \mathcal{J} defined as

$$\mathcal{J}(u, v) = \Psi(u, v) - \mathcal{F}(u, v),$$

where

$$\Psi(u, v) = \widehat{M}_1(I_{s,p(x,y)}(u)) + \int_{\mathbb{R}^N} \frac{1}{\bar{p}(x)} |u|^{\bar{p}(x)} \, dx + \widehat{M}_2(I_{s,q(x,y)}(v)) + \int_{\mathbb{R}^N} \frac{1}{\bar{q}(x)} |v|^{\bar{q}(x)} \, dx.$$

The same argument in [8, Lemma 3.2] combined with Lemma 3.1 implies that \mathcal{J} is well defined on E , and it is of class $C^1(E, \mathbb{R})$ and its Fréchet derivative is given by

$$\begin{aligned} & \langle \mathcal{J}'(u, v), (\varphi, \psi) \rangle \\ &= M_1(I_{s,p(x,y)}(u)) \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+sp(x,y)}} dx dy \\ & \quad + \int_{\mathbb{R}^N} |u|^{\bar{p}(x)-2} u \varphi dx \\ & \quad + M_2(I_{s,q(x,y)}(u)) \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v(x) - v(y)|^{q(x,y)-2} (v(x) - v(y)) (\psi(x) - \psi(y))}{|x - y|^{N+sq(x,y)}} dx dy \\ & \quad + \int_{\mathbb{R}^N} |v|^{\bar{q}(x)-2} v \psi dx - \mathcal{F}'(u, v)(\varphi, \psi) \end{aligned}$$

for any $(\varphi, \psi) \in E$.

3.1. The sublinear case

In this case we assume that

(F₁) $p_0^+ < p^-, q_1^+ < q^-$ and $q_0^+ < \min\{p^-, q^-\}$.

(F₂) $F(x, -s, -t) = F(x, s, t)$ for all $(x, s, t) \in \mathbb{R}^N \times \mathbb{R}^2$.

(F₃) There exist constants $R > 0, \mu < \min\{p^-, q^-\}$ and a positive function $H: \mathbb{R}^N \times \mathbb{R}^2 \rightarrow \mathbb{R}$ such that for $x \in \mathbb{R}^N, |u|, |v| > R$ and $t > 0$ sufficiently small, we have $F(x, tu, tv) \geq t^\mu H(x, u, v)$.

(F₄) There are $\delta_1, \delta_2 > 0$ such that

$$F(x, s, t) \geq h_1(x)|s|^{p_0(x)} + h_2(x)|t|^{q_1(x)} \quad \text{for } x \in \mathbb{R}^N \text{ and } 0 < s < \delta_1, 0 < t < \delta_2,$$

where $h_i \in C(\mathbb{R}^N, \mathbb{R}), h_i(x) \geq 0, i = 1, 2$ are not identical to zero.

Theorem 3.4. *Under the assumptions (M₀), (M₁), (F₀), (F₁) and (F₃), the problem (1.1) has at least one nontrivial solution.*

Proof. We will prove that \mathcal{J} is coercive. We have

$$\begin{aligned} F(x, u, v) &= \int_0^u \frac{\partial F}{\partial s}(x, s, v) dx + F(x, 0, v) \\ &= \int_0^u \frac{\partial F}{\partial s}(x, s, v) dx + \int_0^v \frac{\partial F}{\partial t}(x, 0, t) dx + F(x, 0, 0) \\ &\leq \int_0^u (a_1(x)|s|^{p_0(x)-1} + a_2(x)|v|^{q_0(x)-1}) dx + \int_0^v b_2(x)|t|^{q_1(x)-1} dx \\ &\leq C_1(a_1(x)|u|^{p_0(x)} + a_2(x)|v|^{q_0(x)-1}|u| + b_2(x)|v|^{q_1(x)}). \end{aligned}$$

Using Young’s inequality, we get

$$F(x, u, v) \leq C_2(a_1(x)|u|^{p_0(x)} + a_2(x)|v|^{q_0(x)} + a_2(x)|u|^{q_0(x)} + b_2(x)|v|^{q_1(x)}).$$

Then, from Lemma 2.8 there exist $\bar{p}_0 \in (p_0^-, p_0^+)$, $\bar{q}_0 \in (q_0^-, q_0^+)$, $\bar{q}_1 \in (q_1^-, q_1^+)$ and a constant $C_3 > 0$ such that

$$(3.3) \quad \int_{\mathbb{R}^N} F(x, u, v) dx \leq C_3 [\|u\|_{s,p(x,y)}^{\bar{p}_0} + \|v\|_{s,q(x,y)}^{\bar{q}_0} + \|u\|_{s,p(x,y)}^{\bar{q}_0} + \|v\|_{s,q(x,y)}^{\bar{q}_1}].$$

Without loss of generality, we assume that $\|v\|_{s,q(x,y)} \geq \|u\|_{s,p(x,y)}$. When $\|(u, v)\|_E > 1$, and if $\|u\|_{s,p(x,y)} > 1$, then

$$\begin{aligned} \mathcal{J}(u, v) &= \widehat{M}_1(I_{s,p(x,y)}(u)) + \int_{\mathbb{R}^N} \frac{1}{\bar{p}(x)} |u|^{\bar{p}(x)} dx + \widehat{M}_2(I_{s,q(x,y)}(v)) \\ &\quad + \int_{\mathbb{R}^N} \frac{1}{\bar{q}(x)} |v|^{\bar{q}(x)} dx - \mathcal{F}(u, v) \\ &\geq \frac{\min\{m_1, 1\}}{p^+} \|u\|_{s,p(x,y)}^{p^-} + \frac{\min\{m_2, 1\}}{q^+} \|v\|_{s,q(x,y)}^{q^-} \\ &\quad - C_3 [\|u\|_{s,p(x,y)}^{\bar{p}_0} + \|v\|_{s,q(x,y)}^{\bar{q}_0} + \|u\|_{s,p(x,y)}^{\bar{q}_0} + \|v\|_{s,q(x,y)}^{\bar{q}_1}], \end{aligned}$$

if $\|u\|_{s,p(x,y)} < 1$, then

$$\mathcal{J}(u, v) \geq \frac{\min\{m_2, 1\}}{q^+} \|v\|_{s,q(x,y)}^{q^-} - c_1 \|v\|_{s,q(x,y)}^{\bar{q}_0} + c_2 \|v\|_{s,q(x,y)}^{\bar{q}_1} - C_4.$$

Since $p_0^+ < p^-$, $q_1^+ < q^-$ and $q_0^+ < \min\{p^-, q^-\}$, we get the coercivity of \mathcal{J} . Remark 3.2 leads to conclude that \mathcal{J} is weakly lower semi-continuous. Hence, there exists $(u_1, v_1) \in E$ a critical point minimizing \mathcal{J} . This provides a solution of problem (1.1).

Now, we prove that (u_1, v_1) is nontrivial. Indeed, From (M_1) we know that the map $t \mapsto m(t) = \frac{\widehat{M}_i(t)}{t^{1/\theta}}$ is decreasing. Then, for any $t_0 > 0$ such that $t > t_0$, we have

$$m(t) \leq m(t_0) \leq C_0.$$

Hence

$$(3.4) \quad \widehat{M}_i(t) \leq \frac{\widehat{M}_i(t_0)}{t_0^{1/\theta}} t^{1/\theta} \leq C_0 t^{1/\theta}.$$

Fix $(u_0, v_0) \in E$ with $\|(u_0, v_0)\| = 1$. For $t \in (t_0, 1)$ from (3.4) and (F_3) , we have

$$\begin{aligned} \mathcal{J}(tu_0, tv_0) &= \widehat{M}_1(I_{s,p(x,y)}(tu_0)) + \int_{\mathbb{R}^N} \frac{1}{\bar{p}(x)} |tu|^{\bar{p}(x)} dx + \widehat{M}_2(I_{s,q(x,y)}(tv_0)) \\ &\quad + \int_{\mathbb{R}^N} \frac{1}{\bar{q}(x)} |tv_0|^{\bar{q}(x)} dx - \int_{\mathbb{R}^N} F(x, tu_0, tv_0) dx \end{aligned}$$

$$\begin{aligned} &\leq C_0(I_{s,p(x,y)}(tu_0))^{1/\theta} + \frac{t^{p^-}}{p^-} \int_{\mathbb{R}^N} |u_0|^{\bar{p}(x)} dx + C_0(I_{s,q(x,y)}(tv_0))^{1/\theta} \\ &\quad + \frac{t^{q^-}}{q^-} \int_{\mathbb{R}^N} |v_0|^{\bar{q}(x)} dx - \int_{\mathbb{R}^N} t^\mu H(x, u_0, v_0) dx \\ &\leq \frac{C_0 t^{\frac{p^-}{\theta}}}{(p^-)^{1/\theta}} \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_0(x) - u_0(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy \right)^{1/\theta} + \frac{t^{p^-}}{p^-} \int_{\mathbb{R}^N} |u_0|^{\bar{p}(x)} dx \\ &\quad + \frac{C_0 t^{\frac{q^-}{\theta}}}{(q^-)^{1/\theta}} \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v_0(x) - v_0(y)|^{q(x,y)}}{|x - y|^{N+sq(x,y)}} dx dy \right)^{1/\theta} + \frac{t^{q^-}}{q^-} \int_{\mathbb{R}^N} |v_0|^{\bar{q}(x)} dx \\ &\quad - t^\mu \int_{\mathbb{R}^N} H(x, u_0, v_0) dx dx. \end{aligned}$$

Since $\mu < \min\{p^-, q^-\} < \min\{\frac{p^-}{\theta}, \frac{q^-}{\theta}\}$ we can find $t > t_0 > 0$ small enough such that $\mathcal{J}(tu_0, tv_0) < 0$, i.e., $\inf_{(u,v) \in E} \mathcal{J}(u, v) < 0$. Hence, the obtained solution (u_1, v_1) of (1.1) is nontrivial. □

Theorem 3.5. *Suppose that $(M_0), (M_1), (F_0), (F_1), (F_2)$ and (F_4) are fulfilled. Then, problem (1.1) has a sequence of solutions $\{(\pm u_n, \pm v_n) : n = 1, 2, \dots\}$ such that $\mathcal{J}(\pm u_n, \pm v_n) < 0$ and $\mathcal{J}(\pm u_n, \pm v_n) \rightarrow 0$ as $n \rightarrow +\infty$.*

Proof. Choose $g \in C^\infty([0, +\infty), \mathbb{R})$ such that $0 \leq g(t) \leq 1$ for $t \in [0, +\infty)$, and for every $\epsilon > 0$, $g(t) = 1$ for $0 \leq t \leq \epsilon/2$, $g(t) = 0$ for $t \geq \epsilon$. Consider the functional

$$\begin{aligned} \mathcal{H}(u, v) &= \widehat{M}_1(I_{s,p(x,y)}(u)) + \int_{\mathbb{R}^N} \frac{1}{\bar{p}(x)} |u|^{\bar{p}(x)} dx + \widehat{M}_2(I_{s,q(x,y)}(v)) \\ &\quad + \int_{\mathbb{R}^N} \frac{1}{\bar{q}(x)} |v|^{\bar{q}(x)} dx - g(\|(u, v)\|_E) \mathcal{F}(u, v), \end{aligned}$$

we know that $\mathcal{H} \in C^1(E, \mathbb{R})$. To prove Theorem 3.5 it is sufficient to show that \mathcal{H} admits a sequence of nontrivial critical points $\{(\pm u_n, \pm v_n) : n = 1, 2, \dots\}$ such that $\mathcal{H}(\pm u_n, \pm v_n) < 0$ and $\mathcal{H}(\pm u_n, \pm v_n) \rightarrow 0$ as $n \rightarrow +\infty$.

(S1) For $\|(u, v)\|_E > \max\{1, \epsilon\}$, we have

$$\mathcal{H}(u, v) \geq \frac{\min\{m_1, 1\}}{p^+} \|u\|_{s,p(x,y)}^{p^-} + \frac{\min\{m_2, 1\}}{q^+} \|v\|_{s,q(x,y)}^{q^-},$$

which implies that $\mathcal{H}(u, v) \rightarrow +\infty$ as $\|(u, v)\|_E \rightarrow +\infty$. Hence \mathcal{H} is coercive on E and thus it is bounded from below and the (PS) sequence is bounded. From Lemma 3.1, the (PS) condition is satisfied. By (F_2) it is easy to see that $\mathcal{H}(-u, -v) = \mathcal{H}(u, v)$.

(S2) Since $h_i(x) \neq 0$ and $h_i(x) \geq 0$, $i = 1, 2$, we can find bounded sets $\Omega_1, \Omega_2 \subset \mathbb{R}^N$ such that $h_i(x) > 0$ for $x \in \Omega_i$, $i = 1, 2$. Set $\Omega = \Omega_1 \cup \Omega_2$. $W_0^{s,p(x,y)}(\Omega) \times W_0^{s,q(x,y)}(\Omega)$ is isomorphic to a subsequence of E , and so it is a subspace of E . For any k , we choose a k -dimensional linear subspaces U_k of $W_0^{s,p(x,y)}(\Omega)$ and V_k of $W_0^{s,q(x,y)}(\Omega)$ such that $U_k \times V_k \subset$

$C_0^\infty(\Omega) \times C_0^\infty(\Omega)$. As the norms on U_k (respectively V_k) are equivalent each other, there exists $0 < \rho_k < \min\{\epsilon/2, 1\}$ such that $(u, v) \in U_k \times V_k$ with $\|(u, v)\|_E \leq \rho_k$ implies $\|u\|_\infty \leq \delta_1$ and $\|v\|_\infty \leq \delta_2$. Define

$$S_{\rho_k}^{(k)} = \{(u, v) \in U_k \times V_k : \|(u, v)\|_E = \rho_k\}.$$

From the compactness of $S_{\rho_k}^{(k)}$ and condition (F_4) , we conclude that there exist constants $d_k > 0$ and $d'_k > 0$ such that

$$\int_\Omega \frac{h_1(x)}{p_0(x)} |u|^{p_0(x)} dx \geq d_k; \quad \int_\Omega \frac{h_2(x)}{q_1(x)} |v|^{q_1(x)} dx \geq d'_k, \quad \forall (u, v) \in S_{\rho_k}^{(k)}.$$

For $(u, v) \in S_{\rho_k}^{(k)}$ and $t \in (t_0, 1)$, we have

$$\begin{aligned} \mathcal{H}(tu, tv) &= \widehat{M}_1(I_{s,p(x,y)}(tu)|_{\Omega \times \Omega}) + \int_\Omega \frac{1}{\bar{p}(x)} |tu|^{\bar{p}(x)} dx + \widehat{M}_2(I_{s,q(x,y)}(tv)|_{\Omega \times \Omega}) \\ &\quad + \int_\Omega \frac{1}{\bar{q}(x)} |tv|^{\bar{q}(x)} dx - \int_\Omega F(x, tu, tv) dx \\ &\leq \frac{C_0 t^{\frac{p^-}{\theta}}}{(p^-)^{1/\theta}} \left(\int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy \right)^{1/\theta} + \frac{t^{p^-}}{p^-} \int_\Omega |u|^{\bar{p}(x)} dx \\ &\quad + \frac{C_0 t^{\frac{q^-}{\theta}}}{(q^-)^{1/\theta}} \left(\int_{\Omega \times \Omega} \frac{|v(x) - v(y)|^{q(x,y)}}{|x - y|^{N+sq(x,y)}} dx dy \right)^{1/\theta} + \frac{t^{q^-}}{q^-} \int_\Omega |v|^{\bar{q}(x)} dx \\ &\quad - t^{p_0^+} \int_\Omega \frac{h_1(x)}{p_0(x)} |u|^{p_0(x)} dx - t^{q_1^+} \int_\Omega \frac{h_2(x)}{q_1(x)} |v|^{q_1(x)} dx \\ &\leq \frac{C_0 t^{\frac{p^-}{\theta}}}{(p^-)^{1/\theta}} (\rho_k)^{1/\theta} + \frac{t^{p^-}}{p^-} \rho_k + \frac{C_0 t^{\frac{q^-}{\theta}}}{(q^-)^{1/\theta}} (\rho_k)^{1/\theta} + \frac{t^{q^-}}{q^-} \rho_k - t^{p_0^+} d_k - t^{q_1^+} d'_k. \end{aligned}$$

Since $p_0^+ < p^- < \frac{p^-}{\theta}$ and $q_1^+ < q^- < \frac{q^-}{\theta}$, we can find $t_k \in (t_0, 1)$ small enough and $\epsilon_k > 0$ such that

$$\mathcal{H}(t_k u, t_k v) \leq -\epsilon_k < 0, \quad \forall (u, v) \in S_{\rho_k}^{(k)},$$

that is,

$$\mathcal{H}(u, v) < 0, \quad \forall (u, v) \in S_{t_k \rho_k}^{(k)}.$$

Therefore, $S_{t_k \rho_k}^{(k)} \subset \{(u, v) \in E : \mathcal{H}(u, v) < 0\}$. We know that $\gamma(S_{t_k \rho_k}^{(k)}) = k + 1$, then $\gamma(\{(u, v) \in E : \mathcal{H}(u, v) < 0\}) \geq \gamma(S_{t_k \rho_k}^{(k)})$.

Let $A_k = \{(u, v) \in E : \mathcal{H}(u, v) < 0\}$, we have $A_k \in \Gamma_k$ and $\sup_{(u,v) \in A_k} \mathcal{J}(u, v) < 0$. Hence (S1) and (S2) hold. So, by Theorem 2.13 \mathcal{J} admits a sequence of critical points $\{(\pm u_k, \pm v_k)\}$ such that $\mathcal{J}(\pm u_k, \pm v_k) = c_k < 0$, $(\pm u_k, \pm v_k) \neq (0, 0)$ and $\lim_{k \rightarrow \infty} (\pm u_k, \pm v_k) = (0, 0)$.

In the following, we will prove that $c_k \rightarrow 0$ as $k \rightarrow +\infty$. By the coercivity of \mathcal{J} , there exists a constant $R > 0$ such that $\mathcal{J}(u, v) > 0$ as $\|(u, v)\|_E > R$. Taking $A \in \Gamma_k$, then

$\gamma(A) > k$. Let \mathbb{Y}_k and \mathbb{Z}_k defined in Proposition 2.9 be the subspaces of E . We have $A \cap \mathbb{Z}_k \neq 0$. Take

$$\beta_k = \sup_{\substack{(u,v) \in \mathbb{Z}_k \\ \|(u,v)\|_E \leq R}} |\mathcal{F}(u, v)|,$$

by Lemma 2.11 we have $\beta_k \rightarrow 0$ as $k \rightarrow +\infty$. When $(u, v) \in \mathbb{Z}_k$ and $\|(u, v)\|_E \leq R$, we have

$$\mathcal{J}(u, v) = \Psi(u, v) - \mathcal{F}(u, v) \geq -\mathcal{F}(u, v) \geq -\beta_k,$$

and hence $c_k \geq -\beta_k$, which concludes $c_k \rightarrow 0$ as $k \rightarrow +\infty$. This completes the proof of Theorem 3.5. □

3.2. The superlinear case

In this case we suppose that

(F₅) $p_0^- > p^+, q_1^- > q^+$ and $q_0^- > \min\{p^+, q^+\}$.

(F₆) F satisfies the Ambrosetti–Rabinowitz condition: There exist $\eta_1 > \frac{p^+}{\theta}$ and $\eta_2 > \frac{q^+}{\theta}$ such that

$$0 < F(x, s, t) \leq \frac{s}{\eta_1} \frac{\partial F}{\partial s}(x, s, t) + \frac{t}{\eta_2} \frac{\partial F}{\partial t}(x, s, t).$$

Theorem 3.6. *Let $s \in (0, 1)$. Let $p, q: \mathbb{R}^N \times \mathbb{R}^N \rightarrow (1, +\infty)$ be two continuous variable exponents with $sp^+ < N$ and $sq^+ < N$. If the hypotheses (M₀), (M₁), (F₀), (F₅) and (F₆) hold, then the problem (1.1) has at least one nontrivial weak solution.*

Lemma 3.7. *Let $s \in (0, 1)$. Let $p, q: \mathbb{R}^N \times \mathbb{R}^N \rightarrow (1, +\infty)$ be two continuous variable exponents with $sp^+ < N$ and $sq^+ < N$. Under assumptions hypotheses (M₀), (F₀) and (F₅), there exist $r > 0$ and $k > 0$ such that $\mathcal{J}(u, v) \geq k$ for every $(u, v) \in E$ satisfying $\|(u, v)\|_E = r$.*

Proof. From (3.3), we have

$$\int_{\mathbb{R}^N} F(x, u, v) dx \leq C_3 [\|u\|_{s,p(x,y)}^{\bar{p}_0} + \|v\|_{s,q(x,y)}^{\bar{q}_0} + \|u\|_{s,p(x,y)}^{\bar{q}_0} + \|v\|_{s,q(x,y)}^{\bar{q}_1}].$$

Consequently

$$\begin{aligned} \mathcal{J}(u, v) &= \widehat{M}_1(I_{s,p(x,y)}(u)) + \int_{\mathbb{R}^N} \frac{1}{\bar{p}(x)} |u|^{\bar{p}(x)} dx + \widehat{M}_2(I_{s,q(x,y)}(v)) \\ &\quad + \int_{\mathbb{R}^N} \frac{1}{\bar{q}(x)} |v|^{\bar{q}(x)} dx - \mathcal{F}(u, v) \\ &\geq \frac{\min\{m_1, 1\}}{p^+} \|u\|_{s,p(x,y)}^{p^+} + \frac{\min\{m_2, 1\}}{q^+} \|v\|_{s,q(x,y)}^{q^+} \\ &\quad - C_3 [\|u\|_{s,p(x,y)}^{\bar{p}_0} + \|v\|_{s,q(x,y)}^{\bar{q}_0} + \|u\|_{s,p(x,y)}^{\bar{q}_0} + \|v\|_{s,q(x,y)}^{\bar{q}_1}]. \end{aligned}$$

Since $p_0^- > p^+$, $q_1^- > q^+$ and $q_0^- > \min\{p^+, q^+\}$, there exist $r \in (0, 1)$ and $k > 0$ such that $\mathcal{J}(u, v) \geq k$ for every $(u, v) \in E$ satisfying $\|(u, v)\|_E = r$. \square

Lemma 3.8. *Let $s \in (0, 1)$. Let $p, q: \mathbb{R}^N \times \mathbb{R}^N \rightarrow (1, +\infty)$ be two continuous variable exponents with $sp^+ < N$ and $sq^+ < N$. Under assumptions (M_0) , (M_1) , (F_0) and (F_6) , there exists $(u_0, v_0) \in E \setminus \{0, 0\}$ such that for $\|(u_0, v_0)\|_E > r$ we have $J(u_0, v_0) < 0$.*

Proof. For any $t_0 > 0$ such that $t > t_0$, we have

$$\widehat{M}_i \leq \frac{\widehat{M}_i(t_0)}{t_0^{1/\theta}} t^{1/\theta} \leq C_0 t^{1/\theta}.$$

On the other hand, we claim that the assumption (F_6) implies the following inequality [15]

$$F(x, s, t) \geq C_4(|s|^{\eta_1} + |t|^{\eta_2}) - 1, \quad \forall (x, s, t) \in \mathbb{R}^N \times \mathbb{R}^2.$$

Choose $(u_0, v_0) \in E$, $u_0, v_0 > 0$ and $\|(u_0, v_0)\|_E > r$. It follows that if $t > t_0 > 0$ is large enough

$$\begin{aligned} \mathcal{J}(tu_0, tv_0) &= \widehat{M}_1(I_{s,p(x,y)}(tu_0)) + \int_{\mathbb{R}^N} \frac{1}{\widehat{p}(x)} |tu_0|^{\widehat{p}(x)} dx + \widehat{M}_2(I_{s,q(x,y)}(tv_0)) \\ &\quad + \int_{\mathbb{R}^N} \frac{1}{\widehat{q}(x)} |tv_0|^{\widehat{q}(x)} dx - \int_{\mathbb{R}^N} F(x, u_0, v_0) dx \\ &\leq C_0(I_{s,p(x,y)}(tu_0))^{1/\theta} + \frac{t^{p^+}}{p^-} \int_{\mathbb{R}^N} |u_0|^{\widehat{p}(x)} dx + C_0(I_{s,q(x,y)}(tv_0))^{1/\theta} \\ &\quad + \frac{t^{q^+}}{q^-} \int_{\mathbb{R}^N} |v_0|^{\widehat{q}(x)} dx - C_4 \int_{\mathbb{R}^N} (|tu_0|^{\eta_1} + |tv_0|^{\eta_2}) dx - C'_4 \\ &\leq \frac{C_0 t^{\frac{p^+}{\theta}}}{(p^-)^{1/\theta}} \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_0(x) - u_0(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy \right)^{1/\theta} + \frac{t^{p^+}}{p^-} \int_{\mathbb{R}^N} |u_0|^{\widehat{p}(x)} dx \\ &\quad + \frac{C_0 t^{\frac{q^+}{\theta}}}{(q^-)^{1/\theta}} \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v_0(x) - v_0(y)|^{q(x,y)}}{|x - y|^{N+sq(x,y)}} dx dy \right)^{1/\theta} + \frac{t^{q^+}}{q^-} \int_{\mathbb{R}^N} |v_0|^{\widehat{q}(x)} dx \\ &\quad - C_4 t^{\eta_1} \int_{\mathbb{R}^N} |u_0|^{\eta_1} - C_4 t^{\eta_2} \int_{\mathbb{R}^N} |v_0|^{\eta_2} dx - C'_4. \end{aligned}$$

Since $\eta_1 > \frac{p^+}{\theta} > p^+$ and $\eta_2 > \frac{q^+}{\theta} > q^+$, we conclude that $\mathcal{J}(tu_0, tv_0) < 0$ and $\mathcal{J}(tu_0, tv_0) \rightarrow -\infty$ as $t \rightarrow \infty$. \square

Lemma 3.9. *The functional \mathcal{J} satisfies the Palais–Smale condition $(PS)_c$ for any $c \in \mathbb{R}$.*

Proof. Let $(u_n, v_n) \subset E$ be a sequence satisfying

$$\mathcal{J}(u_n, v_n) \leq c \text{ for some } c > 0, \quad \text{and} \quad \mathcal{J}'(u_n, v_n) \rightarrow 0_{E^*} \text{ as } n \rightarrow \infty.$$

We will show that (u_n, v_n) is a bounded sequence. We have

$$\begin{aligned}
 c &\geq \mathcal{J}(u_n, v_n) = \widehat{M}_1(I_{s,p(x,y)}(u_n)) + \int_{\mathbb{R}^N} \frac{1}{\bar{p}(x)} |u_n|^{\bar{p}(x)} dx + \widehat{M}_2(I_{s,q(x,y)}(v_n)) \\
 &\quad + \int_{\mathbb{R}^N} \frac{1}{\bar{q}(x)} |v_n|^{\bar{q}(x)} dx - \int_{\mathbb{R}^N} F(x, u_n, v_n) dx \\
 &\geq \theta M_1(I_{s,p(x,y)}(u_n)) I_{s,p(x,y)}(u_n) + \frac{1}{p^+} \int_{\mathbb{R}^N} |u_n|^{\bar{p}(x)} dx \\
 &\quad + \theta M_2(I_{s,q(x,y)}(v_n)) I_{s,q(x,y)}(v_n) + \frac{1}{q^+} \int_{\mathbb{R}^N} |v_n|^{\bar{q}(x)} dx \\
 &\quad - \int_{\mathbb{R}^N} F(x, u_n, v_n) dx \\
 &\geq \frac{\theta}{p^+} m_1 \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy + \frac{1}{p^+} \int_{\mathbb{R}^N} |u_n|^{\bar{p}(x)} dx \\
 &\quad + \frac{\theta}{q^+} m_2 \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^{q(x,y)}}{|x - y|^{N+sq(x,y)}} dx dy \\
 &\quad + \frac{1}{q^+} \int_{\mathbb{R}^N} |v_n|^{\bar{q}(x)} dx - \int_{\mathbb{R}^N} F(x, u_n, v_n) dx
 \end{aligned}$$

On the other hand, we have

$$\langle \mathcal{J}'(u_n, v_n), (u_n, v_n) \rangle \leq \varepsilon_n \|u_n, v_n\|_E \xrightarrow{n \rightarrow +\infty} 0.$$

So

$$\begin{aligned}
 &\varepsilon_n (\|u_n\|_{s,p(x,y)} + \|v_n\|_{s,q(x,y)}) \\
 &\geq M_1(I_{s,p(x,y)}(u_n)) \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy + \int_{\mathbb{R}^N} |u_n|^{\bar{p}(x)} dx \\
 &\quad + M_2(I_{s,q(x,y)}(v_n)) \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^{q(x,y)}}{|x - y|^{N+sq(x,y)}} dx dy + \int_{\mathbb{R}^N} |v_n|^{\bar{q}(x)} dx \\
 &\quad - \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x, u_n, v_n) u_n dx - \int_{\mathbb{R}^N} \frac{\partial F}{\partial v}(x, u_n, v_n) v_n dx.
 \end{aligned}$$

Thus, by condition (F_5) , we get

$$\begin{aligned}
 c + \varepsilon &\geq \left(\frac{\theta}{p^+} m_1 - \frac{m_1}{\eta_1} \right) \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy + \left(\frac{1}{p^+} - \frac{1}{\eta_1} \right) \int_{\mathbb{R}^N} |u_n|^{\bar{p}(x)} dx \\
 &\quad + \left(\frac{\theta}{q^+} m_2 - \frac{m_2}{\eta_2} \right) \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v_n(x) - v_n(y)|^{q(x,y)}}{|x - y|^{N+sq(x,y)}} dx dy + \left(\frac{1}{q^+} - \frac{1}{\eta_2} \right) \int_{\mathbb{R}^N} |v_n|^{\bar{q}(x)} dx \\
 &\quad + \int_{\mathbb{R}^N} \left(\frac{1}{\eta_1} \frac{\partial F}{\partial u}(x, u_n, v_n) u_n + \frac{1}{\eta_2} \frac{\partial F}{\partial v}(x, u_n, v_n) v_n - F(x, u_n, v_n) \right) dx \\
 &\geq \min \left\{ \left(\frac{\theta}{p^+} - \frac{1}{\eta_1} \right) m_1, \left(\frac{1}{p^+} - \frac{1}{\eta_1} \right) \right\} \rho_{s,p(x,y)}(u_n)
 \end{aligned}$$

$$\begin{aligned} & + \min \left\{ \left(\frac{\theta}{q^+} - \frac{1}{\eta_2} \right) m_2, \left(\frac{1}{q^+} - \frac{1}{\eta_2} \right) \right\} \rho_{s,q(x,y)}(v_n) \\ \geq & \min \left\{ \left(\frac{\theta}{p^+} - \frac{1}{\eta_1} \right) m_1, \left(\frac{1}{p^+} - \frac{1}{\eta_1} \right) \right\} \|u_n\|_{s,p(x,y)}^{p^\pm} \\ & + \min \left\{ \left(\frac{\theta}{q^+} - \frac{1}{\eta_2} \right) m_2, \left(\frac{1}{q^+} - \frac{1}{\eta_2} \right) \right\} \|v_n\|_{s,q(x,y)}^{q^\pm}, \end{aligned}$$

where $\pm = +$ if $\|\cdot\|_{s,\cdot(x,y)} \leq 1$ and $\pm = -$ if $\|\cdot\|_{s,\cdot(x,y)} \geq 1$. Since $\eta_1 > \frac{p^+}{\theta} > p^+$ and $\eta_2 > \frac{q^+}{\theta} > q^+$, we conclude that (u_n, v_n) is bounded in E . Hence, there exists a subsequence denoted by (u_n, v_n) that converges weakly to $(u, v) \in E$. We will prove that (u_n, v_n) converges strongly to (u, v) . Indeed, we have

$$\begin{aligned} & \langle \mathcal{J}'(u_n, v_n), (u_n - u, 0) \rangle \\ = & M_1(I_{s,p(x,y)}(u_n)) \\ & \times \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x,y)-2} (u_n(x) - u_n(y))(u_n(x) - u(x) + u_n(y) - u(y))}{|x - y|^{N+sp(x,y)}} dx dy \\ & + \int_{\mathbb{R}^N} |u_n|^{\bar{p}(x)-2} u_n (u_n - u) dx - \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x, u_n, v_n) (u_n - u) dx. \end{aligned}$$

By (F_0) we have

$$\begin{aligned} & \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x, u_n, v_n) (u_n - u) dx \\ \leq & \int_{\mathbb{R}^N} a_1(x) |u_n|^{p_0(x)-1} (u_n - u) dx + \int_{\mathbb{R}^N} a_2(x) |v_n|^{q_0(x)-1} (u_n - u) dx \\ \leq & 2^{p_0^+-1} \left(\int_{\mathbb{R}^N} a_1(x) |u_n - u|^{p_0(x)} dx + \int_{\mathbb{R}^N} a_1(x) |u|^{p_0(x)-1} (u_n - u) dx \right) \\ & + 2^{q_0^+-1} \left(\int_{\mathbb{R}^N} a_2(x) |v_n - v|^{q_0(x)-1} (u_n - u) dx + \int_{\mathbb{R}^N} a_2(x) |v|^{q_0(x)-1} (u_n - u) dx \right) \\ \leq & 2^{p_0^+-1} \left(\int_{\mathbb{R}^N} a_1(x) |u_n - u|^{p_0(x)} dx + \int_{\mathbb{R}^N} a_1(x) |u|^{p_0(x)-1} (u_n - u) dx \right) \\ & + 2^{q_0^+-1} \left(\int_{\mathbb{R}^N} a_2(x) \frac{q_0(x) - 1}{q_0(x)} |v_n - v|^{q_0(x)} dx + \int_{\mathbb{R}^N} a_2(x) \frac{1}{q_0(x)} |u_n - u|^{q_0(x)} dx \right. \\ & \quad \left. + \int_{\mathbb{R}^N} a_2(x) |v|^{q_0(x)-1} (u_n - u) dx \right) \\ \leq & 2^{p_0^+-1} \left(\int_{\mathbb{R}^N} a_1(x) |u_n - u|^{p_0(x)} dx + \int_{\mathbb{R}^N} a_1(x) |u|^{p_0(x)-1} (u_n - u) dx \right) \\ & + 2^{q_0^+-1} \left(\frac{q_0^+ - 1}{q_0^-} \int_{\mathbb{R}^N} a_2(x) |v_n - v|^{q_0(x)} dx + \frac{1}{q_0^-} \int_{\mathbb{R}^N} a_2(x) |u_n - u|^{q_0(x)} dx \right. \\ & \quad \left. + \int_{\mathbb{R}^N} a_2(x) |v|^{q_0(x)-1} (u_n - u) dx \right). \end{aligned}$$

Since $u_n \rightharpoonup u$ in $W^{s,p(x,y)}(\mathbb{R}^N)$, we have $\int_{\mathbb{R}^N} a_1(x)|u|^{p_0(x)-1}(u_n - u) dx \xrightarrow{n \rightarrow +\infty} 0$ and $\int_{\mathbb{R}^N} a_2(x)|v|^{q_0(x)-1}(u_n - u) dx \xrightarrow{n \rightarrow +\infty} 0$. Since $p_0(x) < p_s^*(x)$ and $q_0(x) < \min\{p_s^*(x), q_s^*(x)\}$ a.e. $x \in \mathbb{R}^N$. Combining Lemmas 2.6 and 2.7, we conclude that (u_n) converges strongly to u in $L^{p_0(x)}_{a_1(x)}(\mathbb{R}^N)$ and in $L^{q_0(x)}_{a_2(x)}(\mathbb{R}^N)$ and (v_n) converges strongly to v in $L^{q_0(x)}_{a_2(x)}(\mathbb{R}^N)$ and so

$$\int_{\mathbb{R}^N} a_1(x)|u_n - u|^{p_0(x)} dx \xrightarrow{n \rightarrow +\infty} 0, \quad \int_{\mathbb{R}^N} a_2(x)|u_n - u|^{q_0(x)} dx \xrightarrow{n \rightarrow +\infty} 0,$$

$$\int_{\mathbb{R}^N} a_2(x)|v_n - v|^{q_0(x)} dx \xrightarrow{n \rightarrow +\infty} 0.$$

Consequently

$$(3.5) \quad \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x, u_n, v_n)(u_n - u) dx \xrightarrow{n \rightarrow +\infty} 0.$$

Similarly,

$$\int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x, u, v)(u_n - u) dx \xrightarrow{n \rightarrow +\infty} 0.$$

From (3.5) and using the fact that $\|\mathcal{J}'(u_n, v_n)\|_{E^*} \rightarrow 0$ as $n \rightarrow \infty$, it is easy to see that

$$M_1(I_{s,p(x,y)}(u_n))$$

$$\times \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y))(u_n(x) - u(x) + u_n(y) - u(y))}{|x - y|^{N+sp(x,y)}} dx dy$$

$$+ \int_{\mathbb{R}^N} |u_n|^{\bar{p}(x)-2} u_n (u_n - u) dx \xrightarrow{n \rightarrow +\infty} 0.$$

On the other hand, since u_n converges weakly to u in $W^{s,p(x,y)}(\Omega)$, we have $\langle \mathcal{J}'(u, v), (u_n - u, 0) \rangle \rightarrow 0$ as $n \rightarrow \infty$ or

$$M_1(I_{s,p(x,y)}(u))$$

$$\times \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(u_n(x) - u(x) + u_n(y) - u(y))}{|x - y|^{N+sp(x,y)}} dx dy$$

$$+ \int_{\mathbb{R}^N} |u|^{\bar{p}(x)-2} u (u_n - u) dx - \int_{\mathbb{R}^N} \frac{\partial F}{\partial u}(x, u, v)(u_n - u) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which implies that

$$M_1(I_{s,p(x,y)}(u))$$

$$\times \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(u_n(x) - u(x) + u_n(y) - u(y))}{|x - y|^{N+sp(x,y)}} dx dy$$

$$+ \int_{\mathbb{R}^N} |u|^{\bar{p}(x)-2} u (u_n - u) dx \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence,

$$\begin{aligned} & M_1(I_{s,p(x,y)}(u_n)) \\ & \times \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y)) - |u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))}{|x - y|^{N+sp(x,y)}} \\ & \quad \times (u_n(x) - u(x) + u_n(y) - u(y)) \, dx dy \\ & + [M_1(I_{s,p(x,y)}(u_n)) - M_1(I_{s,p(x,y)}(u))] \\ & \times \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(u_n(x) - u(x) + u_n(y) - u(y))}{|x - y|^{N+sp(x,y)}} \, dx dy \\ & + \int_{\mathbb{R}^N} (|u_n|^{\bar{p}(x)-2}u_n - |u|^{\bar{p}(x)-2}u)(u_n - u) \, dx \xrightarrow{n \rightarrow +\infty} 0. \end{aligned}$$

Using (M_1) , we can easily obtain that

$$\widehat{M}_1(t) \leq \tilde{c}_1 t^{1/\theta}.$$

Hence, from (M_0) , it follows that

$$m_1 \leq M_1(t) \leq \widehat{M}_1(t) \leq \frac{\tilde{c}_1}{\theta_1} t^{\frac{1-\theta_1}{\theta_1}}.$$

Since $\{u_n\} \subset W^{s,p(x,y)}(\mathbb{R}^N)$ and $u \in W^{s,p(x,y)}(\mathbb{R}^N)$, by Proposition 2.9, we deduce that $M_1(I_{s,p(x,y)}(u_n))$ and $M_1(I_{s,p(x,y)}(u))$ are bounded.

For the continuity of Φ_u defined by

$$\Phi_u(\varphi) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp(x,y)}} \, dx dy,$$

we obtain it as the same in [7] where the authors have shown that the operator \mathcal{L} associated to the $(-\Delta_{p(x)})^s$ defined as

$$\begin{aligned} \mathcal{L}: W_0^{s,p(x,y)} & \longrightarrow (W_0^{s,p(x,y)})^* \\ u & \longmapsto \mathcal{L}(u): W_0^{s,p(x,y)} \longrightarrow \mathbb{R} \\ & \quad \varphi \longmapsto \langle \mathcal{L}(u), \varphi \rangle \end{aligned}$$

such that

$$\langle \mathcal{L}(u), \varphi \rangle = \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp(x,y)}} \, dx dy$$

is an homeomorphism and in particular is continuous.

Thus, we have

$$\begin{aligned} & [M_1(I_{s,p(x,y)}(u_n)) - M_1(I_{s,p(x,y)}(u))] \\ & \times \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(u_n(x) - u(x) + u_n(y) - u(y))}{|x - y|^{N+sp(x,y)}} \, dx dy \xrightarrow{n \rightarrow +\infty} 0, \end{aligned}$$

so that

$$\begin{aligned} & \min\{1, m_1\} \|u_n - u\|_{s,p(x,y)}^{p^+} \\ & \leq M_1(I_{s,p(x,y)}(u_n)) \\ & \quad \times \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y)) - |u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))}{|x - y|^{N+sp(x,y)}} \\ & \quad \quad \times (u_n(x) - u(x) + u_n(y) - u(y)) \, dx dy \\ & \quad + \int_{\mathbb{R}^N} (|u_n|^{\bar{p}(x)-2}u_n - |u|^{\bar{p}(x)-2}u)(u_n - u) \, dx \xrightarrow{n \rightarrow +\infty} 0. \end{aligned}$$

Hence, $u_n \rightarrow u$ in $W^{s,p(x,y)}(\mathbb{R}^N)$. In the same way we show that $v_n \rightarrow v$ in $W^{s,q(x,y)}(\mathbb{R}^N)$. Thus, \mathcal{J} satisfies the $(PS)_c$ condition. □

Proof of Theorem 3.6. The proof of Theorem 3.6 can be deduced from the following arguments:

- (1) We have that J is of class $C^1(E, \mathbb{R})$ and J' is continuous.
- (2) In Lemma 3.7, under the hypothesis of Theorem 3.6, we have proved that there exist $r > 0$ and $k > 0$ such that $J(u, v) \geq k$ for every $(u, v) \in E$ satisfying $\|(u, v)\|_E = r$.
- (3) In Lemma 3.8, under the hypothesis of Theorem 3.6, we have proved that there exists $(u, v) \in E \setminus \{0, 0\}$ such that for $\|(u, v)\|_E > r$ we have $J(u, v) \leq 0$.
- (4) In Lemma 3.9, under the hypothesis of Theorem 3.6, we have proved that J satisfies the Palais–Smale conditions on E .

Therefore, by the Mountain Pass Theorem, we conclude that the functional J has at least one nontrivial critical point (u, v) in E , which is a weak solution of our problem (1.1). This achieves the proof. □

Theorem 3.10. *Let $s \in (0, 1)$. Let $p, q: \mathbb{R}^N \times \mathbb{R}^N \rightarrow (1, +\infty)$ be two continuous variable exponents with $sp^+ < N$ and $sq^+ < N$. Assume that the assumptions (M_0) , (M_1) , (F_0) , (F_5) and (F_6) hold. Then problem (1.1) has a sequence of nontrivial solutions $\{(u_n, v_n)\}$ such that*

$$\mathcal{J}(u_n, v_n) \rightarrow +\infty \quad \text{as } n \rightarrow +\infty.$$

Proof. It suffices to show that \mathcal{J} has an unbounded sequence of critical points. We verify that \mathcal{J} satisfies the conditions of Fountain Theorem. Indeed, for $n = 1, 2, 3, \dots$ set

$$\begin{aligned} \alpha_n &= \sup_{\substack{(u,v) \in Z_n \\ \|(u,v)\|_E \leq 1}} \int_{\mathbb{R}^N} a_1(x)|u|^{p_0(x)} \, dx, & \beta_n &= \sup_{\substack{(u,v) \in Z_n \\ \|(u,v)\|_E \leq 1}} \int_{\mathbb{R}^N} a_2(x)|v|^{q_0(x)-1}|u| \, dx, \\ \gamma_n &= \sup_{\substack{(u,v) \in Z_n \\ \|(u,v)\|_E \leq 1}} \int_{\mathbb{R}^N} b_2(x)|v|^{q_1(x)} \, dx, & \xi_n &= \alpha_n + \beta_n + \gamma_n. \end{aligned}$$

Then $\alpha_n, \beta_n, \gamma_n > 0$ and by Lemma 2.11 we have

$$\alpha_n \rightarrow 0, \quad \beta_n \rightarrow 0, \quad \gamma_n \rightarrow 0 \quad \text{and} \quad \xi_n \rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty.$$

Let $(u, v) \in \mathbb{Z}_n$ with $\|(u, v)\|_E > 1$, then

$$\begin{aligned} \mathcal{J}(u, v) &\geq m_1 I_{s,p(x,y)}(u) + \int_{\mathbb{R}^N} \frac{1}{\bar{p}(x)} |u|^{\bar{p}(x)} dx + m_2 I_{s,q(x,y)}(v) + \int_{\mathbb{R}^N} \frac{1}{\bar{q}(x)} |v|^{\bar{q}(x)} dx \\ &\quad - \int_{\mathbb{R}^N} (a_1(x)|u|^{p_0(x)} + a_2(x)|v|^{q_0(x)-1}|u| + b_2(x)|v|^{q_1(x)}) dx \\ &\geq \frac{m_1}{p^+} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy + \frac{1}{p^+} \int_{\mathbb{R}^N} |u|^{\bar{p}(x)} dx \\ &\quad + \frac{m_2}{q^+} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v(x) - v(y)|^{q(x,y)}}{|x - y|^{N+sq(x,y)}} dx dy + \frac{1}{q^+} \int_{\mathbb{R}^N} |v_n|^{\bar{q}(x)} dx \\ &\quad - \alpha_n \|u\|_{s,p(x,y)}^{\bar{p}_0} - \gamma_n \|v\|_{s,q(x,y)}^{\bar{q}_1} - \beta_n \|u\|_{s,p(x,y)} \|v\|_{s,q(x,y)}^{\bar{q}_0} \\ &\geq \min \left\{ \frac{\min\{m_1, 1\}}{p^+}, \frac{\min\{m_2, 1\}}{q^+} \right\} \|(u, v)\|_E^{\min\{p^-, q^-\}} \\ &\quad - C_5 \alpha_n \|(u, v)\|_E^{\bar{p}_0} - C_6 \gamma_n \|(u, v)\|_E^{\bar{q}_1} - C_7 \beta_n \|(u, v)\|_E^{\bar{q}_0+1} \\ &\geq \min \left\{ \frac{\min\{m_1, 1\}}{p^+}, \frac{\min\{m_2, 1\}}{q^+} \right\} \|(u, v)\|_E^{\min\{p^-, q^-\}} - C_8 \xi_n \|(u, v)\|_E^{\bar{\omega}}, \end{aligned}$$

where $\bar{\omega} = \max\{\bar{p}_0, \bar{q}_0, \bar{q}_1 + 1\}$. Put $C = \min \left\{ \frac{\min\{m_1, 1\}}{p^+}, \frac{\min\{m_2, 1\}}{q^+} \right\}$. At this stage we fix

$$r_n = \left(\frac{C \min\{p^-, q^-\}}{C_8 \bar{\omega} \xi_n} \right)^{\frac{1}{\bar{\omega} - \min\{p^-, q^-\}}} \rightarrow +\infty \quad \text{as} \quad n \rightarrow +\infty.$$

Consequently, when $(u, v) \in \mathbb{Z}_n$ and $\|(u, v)\|_E = r_n$, we have

$$\mathcal{J}(u, v) \geq \frac{\bar{\omega} - \min\{p^-, q^-\}}{\min\{p^-, q^-\}} \left(\frac{C \min\{p^-, q^-\}}{\bar{\omega}} \right)^{\frac{\bar{\omega}}{\bar{\omega} - \min\{p^-, q^-\}}} \left(\frac{1}{\xi_n} \right)^{\frac{\min\{p^-, q^-\}}{\bar{\omega} - \min\{p^-, q^-\}}}.$$

Since $\xi_n \rightarrow 0$ and $\bar{\omega} > \max\{p^-, q^-\}$, we get

$$\inf_{\substack{(u,v) \in \mathbb{Z}_n \\ \|(u,v)\|_E = r_n}} \mathcal{J}(u, v) \rightarrow +\infty \quad \text{as} \quad n \rightarrow \infty.$$

Hence (A1) is satisfied.

On the other hand, for any $(u, v) \in \mathbb{Y}_n$ with $\|(u, v)\|_E = 1$ and $t > 1$, we have

$$\begin{aligned} \mathcal{J}(tu_0, tv_0) &= \widehat{M}_1(I_{s,p(x,y)}(tu_0)) + \int_{\mathbb{R}^N} \frac{1}{\bar{p}(x)} |tu|^{\bar{p}(x)} dx + \widehat{M}_2(I_{s,q(x,y)}(tv_0)) \\ &\quad + \int_{\mathbb{R}^N} \frac{1}{\bar{q}(x)} |tv_0|^{\bar{q}(x)} dx - \int_{\mathbb{R}^N} F(x, u_0, v_0) dx \end{aligned}$$

$$\begin{aligned}
&\leq C_0(I_{s,p(x,y)}(tu_0))^{1/\theta} + \frac{t^{p^+}}{p^-} \int_{\mathbb{R}^N} |u_0|^{\bar{p}(x)} dx + C_0(I_{s,q(x,y)}(tv_0))^{1/\theta} \\
&\quad + \frac{t^{q^+}}{q^-} \int_{\mathbb{R}^N} |v_0|^{\bar{q}(x)} dx - C_4 \int_{\mathbb{R}^N} (|tu_0|^{\eta_1} + |tv_0|^{\eta_2}) dx - C'_4 \\
&\leq \frac{C_0 t^{\frac{p^+}{\theta}}}{(p^-)^{1/\theta}} \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_0(x) - u_0(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dx dy \right)^{1/\theta} + \frac{t^{p^+}}{p^-} \int_{\mathbb{R}^N} |u_0|^{\bar{p}(x)} dx \\
&\quad + \frac{C_0 t^{\frac{q^+}{\theta}}}{(q^-)^{1/\theta}} \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v_0(x) - v_0(y)|^{q(x,y)}}{|x - y|^{N+sq(x,y)}} dx dy \right)^{1/\theta} + \frac{t^{q^+}}{q^-} \int_{\mathbb{R}^N} |v_0|^{\bar{q}(x)} dx \\
&\quad - C_4 t^{\eta_1} \int_{\mathbb{R}^N} |u_0|^{\eta_1} dx + C_4 t^{\eta_2} \int_{\mathbb{R}^N} |v_0|^{\eta_2} dx - C'_4.
\end{aligned}$$

Since $\eta_1 > \frac{p^+}{\theta} > p^+$ and $\eta_2 > \frac{q^+}{\theta} > q^+$, we conclude that $\mathcal{J}(tu, tv) \rightarrow -\infty$ as $t \rightarrow +\infty$. Thus (A2) is satisfied. This ends the proof. \square

Acknowledgments

The authors would like to thank the anonymous referee who provided useful and detailed comments on this manuscript.

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