Reconstruction of the Sturm–Liouville Operator on a Graph from Subinterval Nodes

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Abstract. We consider inverse subinterval-nodal problems for the Sturm–Liouville operator on a star graph with mixed boundary conditions in pendant vertices and the standard matching conditions at the interior vertex. Can subinterval nodes recognize star-shaped quantum graphs? In this paper we give a positive answer. It is shown that the data of subinterval-nodes near the interior vertex can uniquely determine the potential on a graph and boundary conditions.

1. Introduction

We consider a compact star graph G with the vertex set $V = \{v_0, v_1, v_2, \ldots, v_m\}$ and edge set $E = \{e_1, \ldots, e_m\}$, where $m \geq 3$, v_0 is the internal vertex, $v_1, \ldots, v_m$ are the boundary vertices and each edge is of equal length 1. For convenience, we parameterize each e_j by $x \in [0, 1]$ such that $x = 0$ corresponds to the boundary vertex v_j and $x = 1$ corresponds to the internal vertex v_0. Consider the boundary value problem $L := L(q, h)$ on the graph G generated by the following Sturm–Liouville equations

$$(1.1) \qquad -y_j'' + q_j(x)y_j = \lambda y_j, \quad x \in (0, 1), \ j = 1, 2, \ldots, m,$$

together with the Robin and/or Dirichlet conditions on the boundary vertices

$$y_j'(0, \lambda) - h_j y_j(0, \lambda) = 0, \quad j = 1, 2, \ldots, p,$$
$$y_j(0, \lambda) = 0, \quad j = p+1, p+2, \ldots, m,$$

and the standard matching conditions at the internal vertex

$$y_1(1, \lambda) = y_j(1, \lambda), \ j = 2, \ldots, m, \quad \sum_{j=1}^{m} y_j'(1, \lambda) = 0,$$

where λ is the spectral parameter, $1 \leq p < m$, $h = \{h_j\}_{j=1,2,\ldots,p}$ are real constants, the potential $q = \{q_j\}_{j=1,2,\ldots,m}$ is real-valued function in $\bigoplus_{j=1}^{m} L_2[0,1]$.

In this work, we consider the inverse nodal problems on a star graph with mixed boundary conditions by using subinterval nodal data, which amounts to subinterval nodes (zeros) of eigenfunctions. These problems are related to some questions in mechanics and mathematical physics (see, for example, [14]). Inverse nodal problems for Sturm–Liouville operators on an interval have been studied fairly completely in [5, 6, 9, 11, 14, 19] and other papers. Differential operators on graphs (networks, trees) often appear in natural sciences and engineering (see [8, 10, 15–17] and the references therein). On inverse nodal problems for differential operators on graphs there are only a few findings. For example, the works [2, 18, 20] prove that the set of nodal points, which is dense in the whole interval considered, uniquely determines the parameters of the boundary conditions and the potential functions on a graph. As well as in the work [3] the uniqueness of the potential for given nodal data is proved and authors give a construction of the potential as a limit of a sequence of functions whose nth term is dependent only on the nth eigenvalue and its associated nodal data, which is dense in the whole interval considered.

When solving the inverse problem, to avoid over-determinedness of the inverse nodal problem one hopes to get the expected result with the least/optimal input data. In the inverse node problem of a finite interval, the node data of subinterval is enough to determine the uniqueness [5, 6, 19]. Inspired by it, together with the well-known inverse spectral analysis with partial information on the potential and eigenvalues [4, 7], can the node data on the subinterval determine the quantum graph? This paper gives a positive answer.

This work establishes uniqueness theorems of the inverse subinterval nodal problems for the Sturm–Liouville operator on a star-type graph. We prove that the data of subinterval-nodes near the interior vertex can uniquely determine the potential on a graph and boundary conditions.

An outline of this paper is as follows. In Section 2, some preliminaries are provided. Section 3 deals with the inverse nodal problem on the whole interval. Section 4 is devoted to the statements of the inverse subinterval nodal problems.

2. Preliminaries

In this section, we review some facts about the Sturm–Liouville operator L on the graph G and the growth of entire functions. Let $C_i(x,\lambda)$, $i = 1,2,\ldots,p$ and $S_i(x,\lambda)$, $i = p+1, p+2, \ldots, m$ be the solutions of equations (1.1) under the initial conditions

$$C_i(0,\lambda) = 1, \quad C_i'(0,\lambda) = h_i, \quad S_i(0,\lambda) = 0, \quad S_i'(0,\lambda) = 1.$$

Then for large $|\lambda| \to \infty$, the solutions $C_i(x, \lambda)$ and $S_i(x, \lambda)$ have the asymptotic formulas, uniformly in $x \in [0, 1]$,

$$
\begin{aligned}
C_i(x, \lambda) &= \cos \rho x + (h_i + [q_i]_x) \frac{\sin \rho x}{\rho} + o\left(\frac{e^{\tau x}}{\rho}\right), \\
S_i(x, \lambda) &= \frac{\sin \rho x}{\rho} - [q_i]_x \frac{\cos \rho x}{\rho^2} + o\left(\frac{e^{\tau x}}{\rho^2}\right),
\end{aligned}
$$

(2.1)

where $\lambda = \rho^2$, $\tau = |\operatorname{Im} \rho|$, and $[q_i]_x := \frac{1}{2} \int_0^x q_i(t)\, dt$.

As is shown in [1], L has a countable set of real eigenvalues, which can be enumerated as $\{\lambda_{nk}\}_{n=1}^\infty$, $k = 1, 2, \ldots, m$ (counting with their multiplicities), and satisfy the following asymptotic formulae

$$
\begin{aligned}
\rho_{n1} &= \rho_{n1}^0 + \frac{\omega}{n\pi} + o(n^{-1}), \\
\rho_{n2} &= \rho_{n2}^0 + \frac{\omega}{n\pi} + o(n^{-1}), \\
\rho_{nk} &= \left(n - \frac{1}{2}\right)\pi + O(n^{-1}), \qquad k \in J_3, \\
\rho_{nk} &= n\pi + O(n^{-1}), \qquad\qquad k \in J_4,
\end{aligned}
$$

(2.2)

where $\lambda_{nk} = \rho_{nk}^2$, $\rho_{n1}^0 = (n-1)\pi + \arccos\sqrt{\frac{p}{m}}$, $\rho_{n2}^0 = n\pi - \arccos\sqrt{\frac{p}{m}}$,

$$
\omega = \frac{1}{p}\sum_{j=1}^p \omega_j + \frac{1}{m-p}\sum_{j=p+1}^m \omega_j, \qquad \omega_j = \begin{cases} h_j + [q_j]_1 & \text{if } j = 1, 2, \ldots, p, \\ [q_j]_1 & \text{if } j = p+1, p+2, \ldots, m, \end{cases}
$$

J_3 and J_4 are some fixed sets of indices such that $J_3 \cup J_4 = \{3, 4, \ldots, m\}$, $J_3 \cap J_4 = \emptyset$, $|J_3| = p - 1$, $|J_4| = m - p - 1$. For definiteness, we assume that $3 \in J_3$ and $4 \in J_4$ if these sets are nonempty.

Let $0 \le a \le 1$. Denote by L_j the Sturm–Liouville problem on each edge e_j for $j = 1, 2, \ldots, m$ defined by

$$
\begin{aligned}
-y_j'' + q_j(x)y_j &= \lambda y_j, & x &\in (0, a), \\
y_j'(0, \lambda) - h_j y_j(0, \lambda) = y_j(a, \lambda) &= 0, & j &= 1, 2, \ldots, p
\end{aligned}
$$

and

$$
\begin{aligned}
-y_j'' + q_j(x)y_j &= \lambda y_j, & x &\in (0, a), \\
y_j(0, \lambda) = y_j(a, \lambda) &= 0, & j &= p+1, p+2, \ldots, m.
\end{aligned}
$$

The function $M_j(x, \lambda)$ defined by

$$
M_j(x, \lambda) = \begin{cases} -\dfrac{C_j'(x, \lambda)}{C_j(x, \lambda)} & \text{if } j = 1, 2, \ldots, p, \\[2mm] -\dfrac{S_j'(x, \lambda)}{S_j(x, \lambda)} & \text{if } j = p+1, p+2, \ldots, m \end{cases}
$$

is called the Weyl function of L_j. According to [4], the following asymptotic formulae hold

$$(2.3) \qquad\qquad M_j(x, \lambda) = i\rho + o(1)$$

uniformly in $x \in [\delta, a]$ for $|\lambda| \to \infty$ in any sector $\varepsilon < \arg(\lambda) < \pi - \varepsilon$ for $\varepsilon > 0$, where $\delta \in (0, a]$. In addition, we need a classical estimate of Levinson in [13] and a Phragmén–Lindelöf-type result in [12].

Lemma 2.1. *Let $\{z_n\}_{n \geq 1}$ be a sequence of complex numbers so that*

$$\lim_{n \to \infty} \frac{n}{z_n} = d$$

holds for some $d \in \mathbb{R}$ and there exists a constant $c > 0$ such that

$$|z_n - z_m| \geq c|n - m|.$$

Define

$$f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{z_n^2}\right).$$

If $z_n = 0$, we substitute $1 - z^2/z_n^2$ by z^2. Then for some $C > 0$ there hold

$$|f(z)| \leq C e^{\pi d |\operatorname{Im} z| + \varepsilon |z|}, \quad \forall \varepsilon > 0 \ \text{ as } |z| \to \infty$$

and

$$\left|\frac{1}{f(z)}\right| \leq C e^{-\pi d |\operatorname{Im} z| + \varepsilon |z|} \quad \text{if } |z - z_n| \geq \frac{1}{8}c.$$

Lemma 2.2. *Let $f(z)$ be an entire function of zero exponential type, i.e.,*

$$\limsup_{r \to \infty} \frac{\ln M(r)}{r} \leq 0, \quad \text{where } M(r) = \max_{\varphi} \left\{\left|f(re^{i\varphi})\right|\right\}.$$

If $f(z)$ is bounded along a line, then $f(z)$ is a constant. In particular, $f(z) \to 0$ when $|z| \to \infty$ along a line, then $f(z) \equiv 0$.

3. Inverse nodal problem on the whole interval

Put

$$Y(x, \lambda) = \{y_i(x, \lambda)\}_{i=1,2,\ldots,m}, \quad y_i(x, \lambda) = \begin{cases} A_i(\lambda) C_i(x, \lambda) & \text{if } i = 1, 2, \ldots, p, \\ A_i(\lambda) S_i(x, \lambda) & \text{if } i = p+1, p+2, \ldots, m. \end{cases}$$

Then the function $Y(x, \lambda)$ satisfies equations and the boundary conditions. If λ^* is an eigenvalue of the problem then the function $Y(x, \lambda_0)$ is an eigenfunction.

Using the asymptotic expression (2.2) of the eigenvalue $\lambda_{n1} := \rho_{n1}^2$, when $n \to \infty$, we obtain the asymptotics for the components of the eigenfunction $Y(x, \lambda_{n1})$, uniformly in $x \in [0, 1]$:

$$C_i(x, \lambda_{n1}) = \cos \rho_{n1}^0 x + \frac{\beta_i(x)}{2\pi(n-1)} \sin \rho_{n1}^0 x + o\left(\frac{1}{n}\right), \quad i = 1, 2, \ldots, p,$$

where $\beta_i(x) = 2[q_i]_x + 2h_i - \omega x$, and for $i = p+1, p+2, \ldots, m-1$,

$$S_i(x, \lambda_{n1}) = \frac{\sin \rho_{n1}^0 x}{\rho_{n1}^0} + \frac{\gamma_i(x)}{2\pi^2(n-1)^2} \cos \rho_{n1}^0 x + o\left(\frac{1}{n^2}\right),$$

where $\gamma_i(x) = -2[q_i]_x + \omega x$.

Similarly, taking $\lambda_{n2} := \rho_{n2}^2$ in (2.2), for large $n \to \infty$, we also get the asymptotics for the components of the eigenfunction $Y(x, \lambda_{n2})$, uniformly in $x \in [0, 1]$:

$$S_m(x, \lambda_{n2}) = \frac{\sin \rho_{n2}^0 x}{\rho_{n2}^0} + \frac{\gamma_m(x)}{2\pi^2 n^2} \cos \rho_{n2}^0 x + o\left(\frac{1}{n^2}\right).$$

Fix $i = 1, 2, \ldots, m$. There exists N_0 such that for $n \geq N_0$ the function $C_i(x, \lambda_{n1})$ (or $S_i(x, \lambda_{n1})$, $S_m(x, \lambda_{n2})$) has exactly $n-1$ (simple) zeros inside the interval $(0, 1)$, that is, $0 < x_{ni}^1 < \cdots < x_{ni}^{n-1} < 1$. The point sets $X_i := \{x_{ni}^j\}_{n \geq N_0}$ ($i = 1, 2, \ldots, m-1$, $j = 1, 2, \ldots, n-1$) are called the nodes on the edge e_i with respect to the eigenvalues λ_{n1}, $X_m := \{x_{nm}^j\}_{n \geq N_0}$ with respect to the eigenvalues λ_{n2}.

Denote

$$\alpha_n^j := \frac{(j - \frac{1}{2})\pi}{\rho_{n1}^0}, \quad \beta_n^j := \frac{j\pi}{\rho_{n1}^0}, \quad \gamma_n^j := \frac{j\pi}{\rho_{n2}^0},$$

where $j = 1, 2, \ldots, n-1$. Taking asymptotic formula (2.1) for eigenfunctions into account, for large n, the following asymptotic formulae for the nodes hold uniformly in j:

$$x_{ni}^j = \begin{cases} \alpha_n^j + \frac{\beta_i(x)}{2\pi^2(n-1)^2} + o\left(\frac{1}{n^2}\right) & \text{if } i = 1, 2, \ldots, p, \\ \beta_n^j - \frac{\gamma_i(x)}{2\pi^2(n-1)^2} + o\left(\frac{1}{n^2}\right) & \text{if } i = p+1, p+2, \ldots, m-1, \\ \gamma_n^j - \frac{\gamma_m(x)}{2\pi^2(n-1)^2} + o\left(\frac{1}{n^2}\right) & \text{if } i = m. \end{cases}$$

Note that for the fixed $i = 1, 2, \ldots, m$ the nodal set X_i is dense in $(0, 1)$. Without loss of generality one may assume that $\omega = 0$, which can be achieved by shifting a constant ω. Analyzing the asymptotic expressions x_{ni}^j we have the following statements.

Fixed $i = 1, 2, \ldots, m$ and $x \in [0, 1]$. Suppose that $X_i^0 \subset X_i$ is dense on $(0, 1)$ and choose $\{x_{ni}^{j_{ni}}\} \subset X_i^0$ such that $\lim_{n \to \infty} x_{ni}^{j_{ni}} = x$. Then the following finite limits hold:

(a) $\lim_{n \to \infty} 2\pi^2(n-1)^2(x_{ni}^{j_{ni}} - \alpha_n^j) = g_i(x)$, $i = 1, 2, \ldots, p$, where $g_i(x) = 2[q_i]_x + 2h_i$;

(b) $\lim_{n \to \infty} 2\pi^2(n-1)^2(x_{ni}^{j_{ni}} - \beta_n^j) = h_i(x)$, $i = p+1, p+2, \ldots, m-1$, where $h_i(x) = 2[q_i]_x$;

(c) $\lim_{n\to\infty} 2\pi^2(n-1)^2(x_{nm}^{j_{ni}} - \gamma_n^j) = f_m(x)$, where $f_m(x) = 2[q_m]_x$.

Now we can provide a uniqueness theorem and constructive algorithm for the inverse nodal problem on the whole interval $(0,1)$.

Theorem 3.1. *Fix $i = 1, 2, \ldots, m$. Suppose that a set of nodes $X_i^0 \subset X_i$ is dense on $(0,1)$. Then the data X_i^0 can uniquely determine $q_i(x)$ a.e. on $(0,1)$ and h_i. And the constructive algorithm is as follows:*

(a) $q_i(x) = g_i'(x)$, $h_i = g_i(0)/2$, $i = 1, 2, \ldots, p$;

(b) $q_i(x) = h_i'(x)$, $i = p+1, p+2, \ldots, m-1$;

(c) $q_m(x) = f_m'(x)$.

4. Inverse nodal problems on the subinterval

Let $0 < a < 1$. Denote the node $X_{i,a}^0 \subset X_i \cap (a,1)$ on a subinterval $(a,1)$ of the edge e_i. From Theorem 3.1 we get the following statement.

Theorem 4.1. *Fix $i = 1, 2, \ldots, m$ and $a \in (0,1)$. Suppose that a set of nodes $X_{i,a}^0$ is dense on $(a,1)$. Then the data $X_{i,a}^0$ can uniquely determine $q_i(x)$ a.e. on $(a,1)$.*

For the node set $X_{i,a}^0$, set $X_a^0 = \bigcup_{i=1}^{m-1} X_{i,a}^0$, $S_{X_a^0} = \{n : \exists j, i, x_{ni}^j \in X_{i,a}^0\}$, $S_{X_{m,a}^0} = \{n : \exists j, x_{nm}^j \in X_{m,a}^0\}$, and

$$\Lambda_1 := \{\lambda_{n1}\}_{n \in S_{X_a^0}}, \quad \Lambda_2 := \{\lambda_{n2}\}_{n \in S_{X_{m,a}^0}}.$$

Introduce the counting function

$$N_k(t) := \#\{n \mid \lambda_{nk} < t^2, \lambda_{nk} \in \Lambda_k\}, \quad t \in \mathbb{R}^+, \ k = 1, 2,$$

and assume that

$$(4.1) \qquad\qquad \Lambda_1 \cap \Lambda_2 = \emptyset, \quad \lim_{t\to\infty} \frac{N_k(t)}{t} =: \frac{\alpha_k}{\pi}, \quad 0 \le \alpha_k \le 1.$$

Finally, the result that the partial nodes on the subinterval $[a,1]$ of each edge e_i uniquely determine $q_i(x)$ on the whole $(0,1)$ and h_i is described as follows.

Theorem 4.2. *Fix $a \in (0,1)$. Suppose that a set of nodes $X_{i,a}^0$ is twin-dense in $(a,1)$, $i = 1, 2, \ldots, m$. If there exist $\beta \in [0,1]$ and $\delta > 0$ such that for sufficiently large t*

$$(4.2) \qquad \sum_{k=1}^{2} N_k(t) \ge 2a \left\{ \beta \left[\frac{t}{\pi} + \frac{1}{2} \right] + (1-\beta) \left(\left[\frac{t}{\pi} \right] + \frac{1}{2} \right) + O(t^{-\delta}) \right\}$$

holds, where $[x]$ denotes the largest integer not greater than x, then $\bigcup_{i=1}^{m} X_{i,a}^0$ uniquely determines $q_i(x)$ $(i = 1, 2, \ldots, m)$ a.e. on $[0,1]$ and h_i $(i = 1, 2, \ldots, p)$.
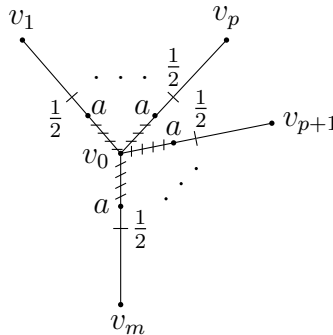

Figure 4.1: Graph with m edges.

Take the node set $X_{i,a}^0$ $(i = 2, 3, \ldots, m)$ such that the eigenvalue sets responding to $S_{X_{i,a}^0}$ satisfy $\Lambda_1 = \{\lambda_{n1}\}_{n \geq N_0}$ and $\Lambda_2 = \{\lambda_{n2}\}_{n \geq N_0}$. Select $\Lambda_3 \subset \{\lambda_{n3}\}_{n \geq 1}$ and $\#\Lambda_3 = 2N_0$. Assume that $\Lambda_1 \cap \Lambda_2 = \emptyset$ and $\prod_{j=1}^{p} C_j(1, \lambda_{nk}) \prod_{j=p+1}^{m} S_j(1, \lambda_{nk}) \neq 0$ for all $\lambda_{nk} \in \Lambda_1 \cup \Lambda_2$. From Theorem 4.2, together with Theorem 3.2 in [1], we have the following corollary.

Corollary 4.3. *Fix $a \in (0,1)$. Suppose that a set of nodes $X_{i,a}^0$ is twin-dense in $(a,1)$, $i = 2, 3, \ldots, m$. Then $\bigcup_{i=2}^{m} X_{i,a}^0$ and Λ_3 uniquely determine $q_i(x)$ $(i = 1, 2, \ldots, m)$ a.e. on $[0,1]$ and h_i $(i = 1, 2, \ldots, p)$.*

Remark 4.4. (1) In Corollary 4.3 one has $\lim_{t \to \infty} \frac{N_k(t)}{t} =: \frac{1}{\pi}$ so the condition (4.2) is true.

(2) In Theorem 4.2 one needs an assumption that a set of nodes $X_{i,a}^0$ is "twin-dense" in $(a,1)$. In fact, one may use "paired dense" defined in Definition 1.1 in [6] instead of "twin-dense". The conclusion of the theorem still holds.

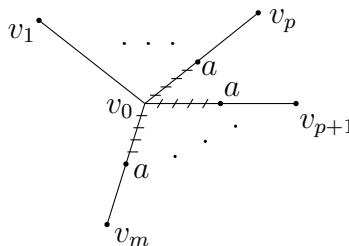Figure 4.2: Graph with m edges.

Proof of Theorem 4.2. For proving the theorem, together with L we consider a boundary value problem $\widetilde{L} = L(\widetilde{q}, \widetilde{h})$ of the same form but with a different potential q and the

parameter h. We agree that if a certain symbol α denotes an object related to L, then $\widetilde{\alpha}$ will denote an analogous object related to $\widetilde{L}$. Consider two boundary value problems L and $\widetilde{L}$ with the assumption that $X_{i,a}^0 = \widetilde{X}_{i,a}^0$.

Step 1: To deduce that $M_j(a, \lambda_{nk}) = \widetilde{M}_j(a, \lambda_{nk})$ for all $\lambda_{nk} \in \Lambda_1 \cup \Lambda_2$. Under the assumption of Theorem 4.2, from Theorem 4.1 and Lemma 1 in [20] we know

$$q_i(x) = \widetilde{q}_i(x) \text{ for } x \in [a, 1] \quad \text{and} \quad \lambda_{nk} = \widetilde{\lambda}_{nk} \text{ for all } \lambda_{nk} \in \Lambda_1 \cup \Lambda_2.$$

From this the following equalities hold

$$(C_i \widetilde{C}_i' - \widetilde{C}_i C_i')(a, \lambda_{nk}) - (C_i \widetilde{C}_i' - \widetilde{C}_i C_i')(x_{nk}^{j_{nk}}, \lambda_{nk}) = \int_{x_{nk}^{j_{nk}}}^{a} (\widetilde{q}_i - q_i)(t)(C\widetilde{C})(t, \lambda_{nk}) \, dt \equiv 0$$

for $i = 1, 2, \ldots, p$ and

$$(S_i \widetilde{S}_i' - \widetilde{S}_i S_i')(a, \lambda_{nk}) - (S_i \widetilde{S}_i' - \widetilde{S}_i S_i')(x_{nk}^{j_{nk}}, \lambda_{nk}) = \int_{x_{nk}^{j_{nk}}}^{a} (\widetilde{q}_i - q_i)(t)(S\widetilde{S})(t, \lambda_{nk}) \, dt \equiv 0$$

for $i = p + 1, p + 2, \ldots, m$, where $x_{nk}^{j_{nk}} \in [a, 1]$. Therefore we have

$$M_i(a, \lambda_{nk}) = \widetilde{M}_i(a, \lambda_{nk}) \quad \text{for all } \lambda_{nk} \in \Lambda_1 \cup \Lambda_2.$$

Step 2: To deduce that $M_j(a, \lambda) = \widetilde{M}_j(a, \lambda)$ for all $\lambda \in \mathbb{C}$. For $i = 1, 2, \ldots, p$ let us introduce

$$(4.3) \qquad H_i(a, \lambda) = C_i(a, \lambda)\widetilde{C}_i'(a, \lambda) - C_i'(a, \lambda)\widetilde{C}_i(a, \lambda)$$

$$(4.4) \qquad = C_i(a, \lambda)\widetilde{C}_i(a, \lambda)(M_i(a, \lambda) - \widetilde{M}_i(a, \lambda)).$$

Together with Step 1, (4.4) shows that

$$(4.5) \qquad H_i(a, \lambda_{nk}) = 0$$

for all $\lambda_{nk} \in \Lambda_1 \cup \Lambda_2$. From (2.1) and (4.3) we get

$$(4.6) \qquad |H_i(a, \lambda)| = O(e^{2a\tau})$$

for $|\lambda| \to \infty$. On the other hand, by virtue of (2.1), (2.3) and (4.4), this yields

$$(4.7) \qquad |H_i(a, \lambda)| = o(e^{2a\tau})$$

for $|\lambda| \to \infty$ in any sector $\varepsilon < \arg \lambda < \pi - \varepsilon$ for some $\varepsilon > 0$. Define

$$(4.8) \qquad F(\lambda) := \frac{H_i(a, \lambda)}{\prod_{k=1}^{2} W_k(\lambda)},$$

where

$$W_k(\lambda) := \prod_{\lambda_{nk} \in \Lambda_1 \cup \Lambda_2} \left(1 - \frac{\lambda}{\lambda_{nk}}\right), \quad k = 1, 2.$$

If $\lambda_{nk} = 0$, we substitute $1 - \lambda/\lambda_{nk}$ by λ. By virtue of (4.5), we see that $F(\lambda)$ is an entire function in λ.

Next we will prove $F(\lambda) \equiv 0$. Firstly, since $N_k(\rho_{nk}) = n - 1$ as $n \to \infty$, it follows from (4.1) that

$$(4.9) \qquad \lim_{n \to \infty} \frac{n}{\rho_{nk}} = \frac{\alpha_k}{\pi}, \quad k = 1, 2.$$

Then Lemma 2.1, together with (4.9), shows that there exist constants $c > 0$ and $C > 0$ such that

$$\frac{1}{|W_k(\lambda)|} \leq C e^{-\alpha_k \tau + \varepsilon r}, \quad \forall \lambda \in G_c,$$

where $\varepsilon > 0$ is arbitrary, and

$$G_c := \left\{\lambda : |\rho - \rho_{nk}| \geq \frac{1}{8}c, \lambda_{nk} \in \Lambda_k, k = 1, 2\right\}.$$

Therefore we obtain

$$(4.10) \qquad \frac{1}{|W_1(\lambda)W_2(\lambda)|} \leq C e^{-(\alpha_1 + \alpha_2)\tau + 2\varepsilon r}, \quad \forall \lambda \in G_c$$

for sufficiently large λ. Moreover, from (4.2) we deduce that $\alpha_1 + \alpha_2 \geq 2a$. Thus (4.6), (4.10) and (4.8) imply

$$|F(\lambda)| = O(e^{2\varepsilon r}), \quad \forall \lambda \in G_c$$

for sufficiently large λ. Consequently the maximum modulus principle shows that

$$(4.11) \qquad |F(\lambda)| \leq C e^{2\varepsilon|\lambda|}, \quad \lambda \in \mathbb{C}.$$

We see that $F(\lambda)$ is of zero exponential type by the arbitrariness of ε. On the other hand, by the known method in [7], we get that for large $|y|$,

$$(4.12) \qquad \ln|W_1(iy)W_2(iy)| = \int_1^\infty \frac{\sum_{k=1}^2 N_k(\sqrt{t})}{t} \frac{y^2}{y^2 + t^2} \, dt + O(1).$$

Combining these with (4.2) and (4.12) yields

$$\ln|W_1(iy)W_2(iy)| \geq 2a\sqrt{\frac{|y|}{2}} + O(1),$$

which follows

$$|W_1(iy)W_2(iy)| \geq C e^{2a\sqrt{\frac{|y|}{2}}}, \quad C > 0 \text{ is some constant.}$$

By virtue of (4.7), we get

$$|H_i(a, iy)| = o\left(e^{2a\sqrt{\frac{|y|}{2}}}\right).$$

Hence

$$|F(iy)| = o(1),$$

which follows

$$F(\lambda) \equiv 0, \quad \lambda \in \mathbb{C}$$

by Lemma 2.2 and (4.11). Consequently, for $i = 1, 2, \ldots, p$,

$$H_i(a, \lambda) \equiv 0 \quad \text{and} \quad M_i(a, \lambda) \equiv \widetilde{M}_i(a, \lambda), \quad \lambda \in \mathbb{C},$$

due to (4.4). Similarly, we also get that for $i = p+1, p+2, \ldots, m$,

$$M_i(a, \lambda) \equiv \widetilde{M}_i(a, \lambda), \quad \lambda \in \mathbb{C}.$$

Step 3: To deduce that $q_i(x) = \widetilde{q}_i(x)$ $(i = 1, 2, \ldots, m)$ a.e. on $[0,1]$ and $h_i = \widetilde{h}_i$ $(i = 1, 2, \ldots, p)$. From the fact that for $i = 1, 2, \ldots, m$,

$$M_i(a, \lambda) \equiv \widetilde{M}_i(a, \lambda), \quad \lambda \in \mathbb{C}.$$

By Borg–Marchenko uniqueness theorem [4], we obtain

$$q_i(x) = \widetilde{q}_i(x), \; i = 1, 2, \ldots, m, \text{ a.e. on } [0,1] \quad \text{and} \quad h_i = \widetilde{h}_i, \; i = 1, 2, \ldots, p.$$

The proof is finished. $\square$

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