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A Gradient Estimate Related Fractional Maximal Operators for a p-Laplace Problem in Morrey Spaces

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Abstract. In the present paper, we deal with the global regularity estimates for the p-Laplace equations with data in divergence form

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div}(|F|^{p-2}F) \quad \text{in } \Omega,$$

in Morrey spaces with natural data $F \in L^p(\Omega;\mathbb{R}^n)$ and nonhomogeneous boundary data belongs to $W^{1,p}(\Omega)$. Motivated by the work of [M.-P. Tran, T.-N. Nguyen, *New gradient estimates for solutions to quasilinear divergence form elliptic equations with general Dirichlet boundary data*, J. Differential Equations **268** (2020), no. 4, 1427–1462], this paper extends that of global Lorentz–Morrey gradient estimates in which the 'good-λ' technique was undertaken for a class of more general equations, and further, global regularity of weak solutions will be given in terms of fractional maximal operators.

1. Introduction and statements of main results

During the last few decades, most of the work done so far dealt with p-Laplace equations, which have their relevance in mathematical and physical applications. It is worth mentioning that the study of regularity estimates for p-Laplace problems turned out to be challenging and attracted a great attention by a number of researchers from many scientific fields through the years. In several recent contributions to p-Laplace equations and their generalizations such as the systems or double phase ones, etc, the validity of Calderón–Zygmund estimates, regularity theory (higher integrability or differentiability of the gradients of solutions, etc) of nonlinear problems, etc have been established and by now rather developed. We herein recommend [4, 5, 15, 16, 19] for rich literature and many references so far.

Let us consider the following p-Laplace problem with divergence form data

$$-\Delta_p u = -\operatorname{div}(f) \ \text{ in } \Omega, \quad u = g \ \text{ on } \partial\Omega, \tag{1.1}$$

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where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator with $p > 1$. It is remarked here that the divergence term $\operatorname{div}(f)$ in (1.1) can be replaced by the other form $\operatorname{div}(|F|^{p-2}F)$ by changing of the vector field as follows

$$f = |F|^{p-2}F \quad \Longleftrightarrow \quad F = |f|^{\frac{2-p}{p-1}} f,$$

which ensures that

$$f \in L^{\frac{p}{p-1}}(\Omega) \quad \Longleftrightarrow \quad F \in L^p(\Omega).$$

The present work is mainly devoted to the regularity property of p-Laplace equations with nonhomogeneous Dirichlet boundary data (1.1) in the framework of Morrey spaces. Furthermore, the results shown in this paper cover that of a larger class of quasilinear elliptic equations as below

$$\begin{cases} -\operatorname{div}(A(x,\nabla u)) = -\operatorname{div}(|F|^{p-2}F) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here, the nonlinear operator $A\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ in this problem is a Carathéodory mapping, i.e., $A(\,\cdot\,,\xi)$ is measurable on Ω for every $\xi \in \mathbb{R}^n$, and $A(x,\cdot\,)$ is continuous on $\mathbb{R}^n$ for almost every $x \in \Omega$. Moreover, as in series of works on this quasi-linear elliptic equation, we also consider A satisfying the standard growth and monotone conditions. More precisely, we assume that there exist constants $1 < p \le n$ and $\Lambda \ge 1$ such that

$$|A(x,\nu)| \le \Lambda |\nu|^{p-1},$$
$$\langle A(x,\nu) - A(x,\mu), \nu - \mu \rangle \ge \Lambda^{-1}(|\nu|^2 + |\mu|^2)^{\frac{p-2}{2}} |\nu - \mu|^2$$

for a.e. x in Ω and every $(\nu,\mu) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$. It is easy to see that the p-Laplace problem (1.1) is a special case of (1.2) when $A(x,\nu) = |\nu|^{p-2}\nu$. The regularity property of weak solutions to (1.2) can be studied under the natural data $F \in L^p(\Omega)$ and the boundary condition $g \in W^{1,p}(\Omega)$. The leitmotif of our investigations is the aim at assumptions on domain Ω, that has its complement satisfying p-capacity uniform thickness. To our knowledge, this condition is the *minimal regularity requirement* for the regularity estimates up to boundary. However, a huge number of interesting regularity results have been investigated for weak solutions to (1.2) under various assumptions of domain, we address the reader to the papers can be found in [6–11, 29, 34, 40] or similar works done on the same topic.

A deep discussion of various developments and generalizations of this equation is made. For instance, for $p \ge 2$, Iwaniec in a very first work [23] proved the L^q-estimates for the gradient of solutions

$$F \in L^q \quad \Longrightarrow \quad \nabla u \in L^q \quad \text{for all } p \le q \le \infty,$$

and later by DiBenedetto–Manfredi [17] extended the results with a broader range of p: $1 < p < \infty$, to the case of elliptic systems and in local BMO spaces. Since then, the nonlinear Calderón–Zygmund theory has a number of contributions, we also refer to [13] by Caffaralli and Peral, some research papers [6, 10] by Byun, Wang and Ryu, or [29] by Mengesha and Phuc, [27] and more recent works on elliptic equations with non-standard growth [1, 14], etc. Recently, the global Lorentz gradient estimates were established by Tran and Nguyen in [32, 33, 38, 40, 41] using the good-λ technique or level-set inequality, where regularity results were preserved in fractional maximal operators $\mathbf{M}_\alpha$. More precisely, in [40] the authors proved the following result

$$\mathbf{M}_\alpha(|F|^p + |\nabla g|^p) \in L^{q,s}(\Omega) \quad \Longrightarrow \quad \mathbf{M}_\alpha(|\nabla u|^p) \in L^{q,s}(\Omega).$$

It is worth mentioning that the fractional maximal operators $\mathbf{M}_\alpha$ has a relation to the Riesz potential $\mathbf{I}_\alpha$ and the fractional derivatives (see [3, 28, 30, 33]). Inspired by these aforementioned studies and the recent works dealt with global Lorentz gradient estimates, we are going to establish the result that is particularly given as follows

$$|F|^p + |\nabla g|^p \in L^{q,s;\kappa}(\Omega) \quad \Longrightarrow \quad \mathbf{M}_\alpha(|\nabla u|^p) \in L^{\frac{\kappa q}{\kappa - \alpha q}, \frac{\kappa s}{\kappa - \alpha q};\kappa}(\Omega).$$

It should be noted that in the general context, result is obtained in Lorentz–Morrey spaces whose definition will be given in the next section.

Let us briefly discuss the technique behind our work. We mention here the very effective approach first proposed by Acerbi and Mingione in [1, 2], that allows us to give harmonic analysis and interpolation free proof of Calderón–Zygmund estimates. Later, under the different viewpoint, the use of so-called 'good-λ' approach is devoted to regularity results (at least for most types of quasi-linear elliptic equations), see [31, 35, 37, 39]. Needless to say, we can follow this approach to obtain both regularity in interior domain as well as on the boundary of domain. To better specify our results, one considers the problem under an additional assumption of domain Ω that has p-capacity uniform thickness complement corresponding to two constants $c_0, r_0 > 0$ (see Section 2 below).

We are now ready to state the main results hereafter.

Theorem 1.1. *Let $F \in L^p(\Omega)$, $g \in W^{1,p}(\Omega)$ with $1 < p \leq n$ and Ω be an open bounded domain in $\mathbb{R}^n$. Assume that u is a weak solution to equation* (1.2) *and Ω has p-capacity uniform thickness complement corresponding to two constants $c_0, r_0 > 0$. Let $\xi_0 \in \Omega$ and $0 < r < \operatorname{diam}(\Omega)$ fixed. Then there exists a constant $\Theta = \Theta(n, p, \Lambda) > p$ such that for $0 \leq \alpha < np/\Theta$ and $\varepsilon \in (0, \varepsilon_0)$ one can find $b_\varepsilon > 0$ satisfying the following inequality*

$$\begin{aligned} &\mathcal{L}^n(\{\mathbf{M}_\alpha(\chi_{B_{10r}(\xi_0)}|\nabla u|^p) > \varepsilon^{-a}\lambda, \mathbf{M}_\alpha(\chi_{B_{10r}(\xi_0)}(|F|^p + |\nabla g|^p)) \leq b_\varepsilon\lambda\} \cap B_r(\xi_0)) \\ &\leq C\varepsilon\mathcal{L}^n(\{\mathbf{M}_\alpha(\chi_{B_{10r}(\xi_0)}|\nabla u|^p) > \lambda\} \cap B_r(\xi_0)) \end{aligned} \tag{1.3}$$

for any $\lambda > \lambda_0$*, where* λ_0 *is given by*

$$\lambda_0 := \varepsilon^{\frac{p}{\Theta}-1} r^{\alpha-n} \|\nabla u\|^p_{L^p(B_{10r}(\xi_0)\cap\Omega)}. \tag{1.4}$$

Here $a := \frac{p}{\Theta} - \frac{\alpha}{n}$*,* $\varepsilon_0 = \varepsilon_0(n,a,\alpha) \in (0,1)$ *and the constant* C *depends on the given data* $(n,p,\Lambda,\alpha,c_0,r_0,\mathrm{diam}(\Omega))$.

We stress here that we are going to use the common notation $\mathcal{L}^n(E)$ for the Lebesgue measure of $E \subset \mathbb{R}^n$. Moreover, for the sake of brevity, in the main theorems and in what follows, the set $\{y \in \Omega : |h(y)| > t\}$ is simply denoted by $\{|h| > t\}$. On the other hand, we tacitly extend u, F and g by zero to $\mathbb{R}^n \setminus \Omega$ in all terms of (1.3).

Theorem 1.2. *Let* $F \in L^p(\Omega)$*,* $g \in W^{1,p}(\Omega)$ *with* $1 < p \leq n$ *and* Ω *be an open bounded domain in* $\mathbb{R}^n$*. Assume that* u *is a weak solution to equation* (1.2) *and* Ω *has p-capacity uniform thickness complement corresponding to two constants* $c_0, r_0 > 0$*. Then there exist constants* $\Theta = \Theta(n,p,\Lambda) > p$ *and* $\beta_0 \in (0,1/2]$ *such that for every*

$$0 \leq \alpha < \frac{pn}{\Theta}, \quad 1 < q < \frac{\Theta}{p}, \quad 0 < s \leq \infty$$

and

$$\max\left\{pq(1-\beta_0); \frac{\alpha}{\frac{1}{q} - \frac{p}{\Theta} + \frac{\alpha}{n}}\right\} < \kappa \leq n,$$

there holds

$$\|\mathbf{M}_\alpha(|\nabla u|^p)\|_{L^{\frac{\kappa q}{\kappa-\alpha q}, \frac{\kappa s}{\kappa - \alpha q};\kappa}(\Omega)} \leq C \| |F|^p + |\nabla g|^p \|_{L^{q,s;\kappa}(\Omega)}. \tag{1.5}$$

Here the constant C *depends only on* n*,* p*,* Λ*,* α*,* c_0*,* r_0*,* $\mathrm{diam}(\Omega)$*,* q*,* s *and* κ*.*

The rest of this paper is organized in the following way. Section 2 is dedicated to the fundamental notation, definitions and state the main assumptions which will be considered throughout the paper. Section 3 contains various comparison estimates with a reference homogeneous problem, that are very useful in our main proofs later. And finally, in Section 4, we exhibit the proofs of our main results in this paper.

2. Notation and fundamental definitions

Throughout the paper, the domain Ω is an open bounded set in $\mathbb{R}^n$. The diameter of Ω will be denoted by $\mathrm{diam}(\Omega)$. In addition, we will denote by $B_r(\xi)$ the open ball in $\mathbb{R}^n$ of radius $r > 0$ and centered at $\xi \in \mathbb{R}^n$. The integral average of a function $f \in L^1(\mathcal{U})$ over the measurable subset $\mathcal{U}$ of $\mathbb{R}^n$ will be written as follows

$$\fint_{\mathcal{U}} f(x)\,dx = \frac{1}{\mathcal{L}^n(\mathcal{U})} \int_{\mathcal{U}} f(x)\,dx,$$

where notation $\mathcal{L}^n(\mathcal{U})$ stands for the Lebesgue measure of $\mathcal{U} \subset \mathbb{R}^n$. Along the paper, we denote $\mathcal{U}^c$ for the complement of $\mathcal{U}$ in $\mathbb{R}^n$.

Moreover, from now on, the letter C stands for the universal constant that may change from line to line, the dependencies on prescribed parameters will be emphasized in parentheses, if needed.

Definition 2.1 (Distributional solution)**.** A function $u \in W^{1,p}(\Omega)$ is called weak (or distributional) solution to (1.2) if the following variational formula

$$\int_\Omega \langle A(x,\nabla u), \nabla\varphi\rangle \, dx = \int_\Omega \langle |F|^{p-2}F, \nabla\varphi\rangle \, dx$$

holds for all test function $\varphi \in W_0^{1,p}(\Omega)$.

Definition 2.2 (The p-capacity)**.** Let $Q \subset \Omega$, the p-capacity of Q, namely $\operatorname{cap}_p(Q,\Omega)$, will be defined as follows

$$\operatorname{cap}_p(Q,\Omega) = \begin{cases} \inf_{\substack{\psi\in C_c^\infty(\Omega)\\ \chi_Q\psi\geq 1}} \int_\Omega |\nabla\psi|^p \, dx & \text{if } Q \text{ is compact},\\ \sup_{\substack{Q'\subseteq Q\\ Q' \text{compact}}} \operatorname{cap}_p(Q',\Omega) & \text{if } Q \text{ is open},\\ \inf_{\substack{Q'\subseteq Q\\ Q' \text{open}}} \operatorname{cap}_p(Q',\Omega) & \text{otherwise}. \end{cases}$$

Definition 2.3 (Domains with uniformly p-capacity thick complement)**.** The complement of Ω is said to satisfy a p-capacity uniform thickness (p-CUT) condition if there exist c_0 and $r_0 > 0$ such that

$$\operatorname{cap}_p(\Omega^c \cap \overline{B}_\varrho(y), B_{2\varrho}(y)) \geq c_0 \operatorname{cap}_p(\overline{B}_\varrho(y), B_{2\varrho}(y)) \tag{2.1}$$

for every $y \in \Omega^c$ and $0 < \varrho \leq r_0$.

For convenience of the reader, let us recall here some well-known remarks related to the p-capacity uniform thickness condition.

Remark 2.4. Every $\Omega^c \neq \emptyset$ satisfies a p-CUT condition if $p > n$, and hence this condition is nontrivial only if $p \leq n$. Moreover, if Ω^c satisfies a p-CUT condition, then it satisfies a q-CUT for all $q \geq p$.

Remark 2.5. We note that the p-CUT condition implies that every points on $y \in \partial\Omega$ is regular, that means

$$\int_0^1 \left(\frac{\operatorname{cap}_p(\Omega^c \cap \overline{B}_\varrho(y), B_{2\varrho}(y))}{\operatorname{cap}_p(\overline{B}_\varrho(y), B_{2\varrho}(y))} \right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} = \infty$$

for the p-Laplace equation, where the thickness of Ω^c near $\partial\Omega$ can be measured by capacity densities. This condition is called the Wiener criterion which is important in regularity of boundary points.

Remark 2.6. Such assumption (2.1) is very mild and essential for higher integrability results. Domains whose complement satisfy p-CUT condition include domains with Lipschitz continuous boundaries.

We now recall the definition of Lorentz space $L^{q,s}(\Omega)$ and Lorentz–Morrey space $L^{q,s;\kappa}(\Omega)$ which is studied in many literature such as [22].

Definition 2.7 (Lorentz spaces)**.** Let $0 < q < \infty$ and $0 < s \leq \infty$, the Lorentz space $L^{q,s}(\Omega)$ contains all of Lebesgue measurable maps f on Ω such that the quasi-norm $\|f\|_{L^{q,s}(\Omega)}$ is finite, where

$$\|f\|_{L^{q,s}(\Omega)} := \left[q \int_0^\infty \lambda^{s-1} \mathcal{L}^n(\{|f| > \lambda\})^{\frac{s}{q}} \, d\lambda \right]^{\frac{1}{s}}$$

as $s \neq \infty$ and otherwise

$$\|f\|_{L^{q,\infty}(\Omega)} := \sup_{\lambda > 0} \lambda \mathcal{L}^n(\{|f| > \lambda\})^{\frac{1}{q}}.$$

The space $L^{q,\infty}(\Omega)$ is well-known as the Marcinkiewicz space or the usual weak Lebesgue space. In particular, the Lorentz space $L^{q,q}(\Omega)$ is exactly the Lebesgue space $L^q(\Omega)$.

Definition 2.8 (Lorentz–Morrey spaces)**.** Let $0 < q < \infty$ and $0 < s \leq \infty$, we say that a measurable map $f \in L^{q,s}(\Omega)$ belongs to the Lorentz–Morrey spaces $L^{q,s;\kappa}(\Omega)$ for $0 \leq \kappa \leq n$ if $\|f\|_{L^{q,s;\kappa}(\Omega)}$ is finite, where

$$\|f\|_{L^{q,s;\kappa}(\Omega)} := \sup_{\substack{0 < \varrho < \mathrm{diam}(\Omega) \\ \xi \in \Omega}} \varrho^{\frac{\kappa - n}{q}} \|f\|_{L^{q,s}(B_\varrho(\xi) \cap \Omega)}.$$

In a special case when $\kappa = n$, the Lorentz–Morrey space $L^{q,s;\kappa}(\Omega)$ is not different from the Lorentz space $L^{q,s}(\Omega)$.

Regarding [25, 26], let us reproduce the definition of *fractional maximal operators* as follows.

Definition 2.9 (Fractional maximal operators)**.** Let $0 \leq \alpha \leq n$ and a locally integrable function $f \colon \mathbb{R}^n \to \mathbb{R}^+$, we call $\mathbf{M}_\alpha f$ the fractional maximal operator of f given by

$$\mathbf{M}_\alpha f(x) = \sup_{\varrho > 0} \varrho^\alpha \fint_{B_\varrho(x)} |f(y)| \, dy, \quad x \in \mathbb{R}^n.$$

We have known that the Hardy–Littlewood maximal function $\mathbf{M}$ of locally integrable maps f in $\mathbb{R}^n$ is defined by

$$\mathbf{M} f(x) = \sup_{\varrho > 0} \fint_{B_\varrho(x)} |f(y)| \, dy, \quad x \in \mathbb{R}^n.$$

This maximal function is obviously a typical case of fractional maximal function $\mathbf{M}_\alpha$ when $\alpha = 0$.

Properties of Hardy–Littlewood maximal function and its fractional operators play a key role in gradient estimates of the weak solution to our problem. The maximal function has been successfully used in studying the regularity theory of partial differential equations. Duzaar and Mingione in their contributions [18, 20] introduced the gradient estimates employing fractional maximal functions and nonlinear potentials. With further important technical developments, it allows us to give many research papers here [7,11,31,35,39,41], etc. The following lemma will recall some useful properties of maximal and fractional maximal operators, see [41] for the detailed proof of this lemma.

Lemma 2.10. (see [41]) *The operator $\mathbf{M}_\alpha$ is bounded from Lebesgue space $L^s(\mathbb{R}^n)$ to Marcinkiewicz space $L^{\frac{ns}{n-\alpha s},\infty}(\mathbb{R}^n)$ for $s \geq 1$ and $0 \leq \alpha s < n$. Precisely, there is a positive constant $C = C(n,s,\alpha)$ such that*

$$\lambda^{\frac{ns}{n-\alpha s}} \mathcal{L}^n(\{x \in \mathbb{R}^n : \mathbf{M}_\alpha f(x) > \lambda\}) \leq C \|f\|_{L^s(\mathbb{R}^n)}^{\frac{ns}{n-\alpha s}}$$

for all $f \in L^s(\mathbb{R}^n)$ and $\lambda > 0$.

3. Comparisons with homogeneous problem

In this section, we always assume that Ω satisfies the p-capacity uniformly thickness condition with two given constants c_0, $r_0 > 0$. We consider $u \in W^{1,p}(\Omega)$ as a weak solution to (1.2) with given data $F \in L^p(\Omega)$ and $g \in W^{1,p}(\Omega)$. The following lemma is standard, see also [40] for the proof.

Lemma 3.1. *There exists a positive constant C depending on n, p and Λ such that*

$$\int_\Omega |\nabla u|^p \, dx \leq C \int_\Omega (|F|^p + |\nabla g|^p) \, dx.$$

Let us take $x_0 \in \overline{\Omega}$, $R \in (0, r_0/2]$ and denote $\Omega_{2R} = B_{2R}(x_0) \cap \Omega$. Assume that $v \in W^{1,p}(\Omega_{2R})$ is the unique solution to the following equation which can be considered as the homogeneous type of equation (1.2) in Ω_{2R}:

$$\begin{cases} -\operatorname{div} A(x, \nabla v) = 0 & \text{in } \Omega_{2R}, \\ v = u - g & \text{on } \partial\Omega_{2R}. \end{cases} \tag{3.1}$$

The next lemma gives the reverse Hölder inequality of ∇v in Ω_{2R} and the comparison between the integral average of ∇v on two different balls. That is a type of Gehring's lemma is applied to obtain the higher integrability of weak solutions v for the reference homogeneous problem (3.1). We refer to [21, Theorem 6.7] and [24] for further reading.

Lemma 3.2. *There exists a constant $\Theta = \Theta(n,p,\Lambda) > p$ such that*

$$\left(\fint_{\Omega_R} |\nabla v|^{\Theta}\, dx\right)^{\frac{1}{\Theta}} \le C \left(\fint_{\Omega_{2R}} |\nabla v|^{p}\, dx\right)^{\frac{1}{p}}. \tag{3.2}$$

Moreover there exists a constant $\beta_0 = \beta_0(n,p,\Lambda) \in (0,1/2]$ such that for any $s \in (0,p]$ the following inequality

$$\left(\fint_{B_\varrho(y)} |\nabla v|^{s}\, dx\right)^{\frac{1}{s}} \le C \left(\frac{\varrho}{r}\right)^{\beta_0 - 1} \left(\fint_{B_r(y)} |\nabla v|^{s}\, dx\right)^{\frac{1}{s}}$$

holds for all $B_\varrho(y) \subset B_r(y) \subset \Omega_{2R}$. Here the constant C depends on n, p and Λ.

Let us now present a local comparison estimate between ∇v and ∇u in the following lemma.

Lemma 3.3. *For every $\varepsilon \in (0,1)$, one can find a positive constant $c_\varepsilon = C(n,p,\varepsilon) > 0$ such that*

$$\fint_{\Omega_{2R}} |\nabla u - \nabla v|^p\, dx \le \varepsilon \fint_{\Omega_{2R}} |\nabla u|^p\, dx + c_\varepsilon \fint_{\Omega_{2R}} |F|^p + |\nabla g|^p\, dx. \tag{3.3}$$

Proof. We also refer to [40] for the proof of the following inequality

$$\begin{aligned}\fint_{\Omega_{2R}} |\nabla u - \nabla v|^p\, dx \le C &\fint_{\Omega_{2R}} |F|^p + |\nabla g|^p\, dx \\ &+ \left(\fint_{\Omega_{2R}} |\nabla u|^p\, dx\right)^{\frac{p-1}{p}} \left(\fint_{\Omega_{2R}} |F|^p + |\nabla g|^p\, dx\right)^{\frac{1}{p}},\end{aligned}$$

which implies to (3.3) by applying Young's inequality. □

We next prove a technical Lemma 3.5 which is useful for the proof of our main theorem. This lemma can be observed by combining the comparison estimate (3.3) and the well-known result stated in many literatures, see [21] for instance.

Lemma 3.4. *Given a, b, $D > 0$, $k_0 \in (0,1)$ and $0 \le s < t$. Let $\varphi\colon [0,D] \to \mathbb{R}^+$ be a non-decreasing function satisfying*

$$\varphi(\varrho) \le a\left[\left(\frac{\varrho}{r}\right)^t + \varepsilon\right]\varphi(r) + br^s$$

for any $0 < \varrho \le k_0 r < D$ and $\varepsilon > 0$. One can find a constant $\varepsilon_0 = \varepsilon_0(a,t,s,k_0) > 0$ such that if $\varepsilon \in (0,\varepsilon_0)$ then

$$\varphi(\varrho) \le C\left[\left(\frac{\varrho}{r}\right)^{\vartheta}\varphi(r) + b\varrho^s\right]$$

for all $\vartheta \in [s,t]$ and $0 < \varrho \le r \le D$, where $C = C(a,t,\vartheta) > 0$.

Lemma 3.5. *Let $\beta_0 \in (0,1/2]$ given in Lemma 3.2 and $\beta \in (p(1-\beta_0), n]$. There holds*

$$\int_{B_\varrho(y)\cap\Omega} |\nabla u|^p \, dx \leq C\varrho^{n-\beta}\mathbf{M}^D_\beta(|F|^p + |\nabla g|^p)(y) \tag{3.4}$$

for all $y \in \Omega$ and $0 < \varrho < D = \mathrm{diam}(\Omega)$.

Proof. Let $y \in \Omega$ and $0 < \varrho < D$ where $D = \mathrm{diam}(\Omega)$, we introduce a non-decreasing function φ as follows

$$\varphi(\varrho) = \int_{B_\varrho(y)\cap\Omega} |\nabla u|^p \, dx, \quad \varrho \in (0, D].$$

In order to apply Lemma 3.4, we need to check the validation of hypotheses in this lemma. Firstly, it is easy to see that

$$\varphi(\varrho) \leq C\int_{B_\varrho(y)\cap\Omega} |\nabla u - \nabla v|^p \, dx + C\int_{B_\varrho(y)\cap\Omega} |\nabla v|^p \, dx. \tag{3.5}$$

Moreover, thanks to Lemma 3.2 with $s = p$, $\varrho < 2r/3$ and by performing a simple computation, one obtains

$$\begin{aligned}\int_{B_\varrho(y)\cap\Omega} |\nabla v|^p \, dx &\leq C\left(\frac{\varrho}{r}\right)^{n+p(\beta_0-1)}\int_{B_{2r/3}(y)\cap\Omega} |\nabla v|^p \, dx \\ &\leq \left(\frac{\varrho}{r}\right)^{n+p(\beta_0-1)}\int_{B_r(y)\cap\Omega} |\nabla u|^p \, dx.\end{aligned} \tag{3.6}$$

On the other hand, by applying the comparison estimate (3.3) in Lemma 3.3, we have the following estimate

$$\begin{aligned}\int_{B_\varrho(y)\cap\Omega} |\nabla u - \nabla v|^p \, dx &\leq C\int_{B_r(y)\cap\Omega} |\nabla u - \nabla v|^p \, dx \\ &\leq C\left(\varepsilon\int_{B_r(y)\cap\Omega} |\nabla u|^p \, dx + c_\varepsilon\int_{B_r(y)\cap\Omega} (|F|^p + |\nabla g|^p) \, dx\right)\end{aligned} \tag{3.7}$$

for all $\varepsilon \in (0,1)$. Combining the above approximations in (3.5), (3.6) and (3.7), one gets that

$$\begin{aligned}\varphi(\varrho) &\leq C\left[\left(\frac{\varrho}{r}\right)^{n+p(\beta_0-1)} + \varepsilon\right]\int_{B_r(y)\cap\Omega} |\nabla u|^p \, dx + c_\varepsilon\int_{B_r(y)\cap\Omega} (|F|^p + |\nabla g|^p) \, dx \\ &\leq C\left[\left(\frac{\varrho}{r}\right)^{n+p(\beta_0-1)} + \varepsilon\right]\varphi(r) + c_\varepsilon r^{n-\beta}\mathbf{M}^D_\beta(|F|^p + |\nabla g|^p)(y).\end{aligned}$$

Applying Lemma 3.4, one can find $\varepsilon_0 > 0$ such that with $s = n - \beta < n + p(\beta_0 - 1) = t$ and $\varepsilon \in (0, \varepsilon_0)$ there holds

$$\varphi(\varrho) \leq C\left[\left(\frac{\varrho}{r}\right)^{n-\beta}\varphi(r) + \varrho^{n-\beta}\mathbf{M}^D_\beta(|F|^p + |\nabla g|^p)(y)\right].$$

In particular, this inequality holds for $r = D$, i.e.,

$$\int_{B_\varrho(y)\cap\Omega} |\nabla u|^p \, dx \le C\varrho^{n-\beta} \left[D^{\beta-n} \int_\Omega |\nabla u|^p \, dx + \mathbf{M}_\beta^D (|F|^p + |\nabla g|^p)(y) \right]. \tag{3.8}$$

Furthermore, based on Lemma 3.1 we can check that

$$D^{\beta-n} \int_\Omega |\nabla u|^p \, dx \le C\mathbf{M}_\beta^D (|F|^p + |\nabla g|^p)(y),$$

which ensures us to conclude (3.4) from (3.8). □

4. Proofs of the main theorems

Our proofs in this section rely on the following important ingredient: the substitution of Calderón–Zygmund–Krylov–Safonov decomposition, which allows us to apply with balls instead of cubes.

Lemma 4.1 (Covering Lemma)**.** *Let Ω be an open bound domain in $\mathbb{R}^n$ such that its complement satisfies the p-capacity uniform thickness condition with c_0, $r_0 > 0$. Consider two measurable subsets $\mathcal{V} \subset \mathcal{W} \subset \Omega$ satisfying two following hypotheses for some constants $\varepsilon \in (0,1)$ and $r \in (0, r_0]$:*

(i) $\mathcal{L}^n(\mathcal{V}) \le \varepsilon \mathcal{L}^n(B_r(0))$;

(ii) $\forall\, x \in \Omega$ *and* $\varrho \in (0, r]$, *if* $\mathcal{L}^n(\mathcal{V} \cap B_\varrho(x)) > \varepsilon \mathcal{L}^n(B_\varrho(x))$ *then* $\Omega \cap B_\varrho(x) \subset \mathcal{W}$.

Then there exists a constant $C = C(n) > 0$ such that $\mathcal{L}^n(\mathcal{V}) \le C\varepsilon \mathcal{L}^n(\mathcal{W})$.

To our knowledge, this lemma is a classical result in measure theory, and it plays an essential role to prove 'good-λ' Theorem 1.1. We suggest for the interested reader the references [12, Lemma 4.2] or [42].

Proof of Theorem 1.1. For every $\alpha \in [0, np/\Theta)$, $\xi_0 \in \Omega$ and $0 < r < \mathrm{diam}(\Omega)$, we denote $Q_1 = B_r(\xi_0)$ and $Q_2 = B_{10r}(\xi_0)$. The constant Θ will be determined later. Let us consider two measurable subsets of Q_1 as follows

$$\mathcal{G}_\lambda(t) = \{\mathbf{M}_\alpha(\chi_{Q_2} |\nabla u|^p) > t\lambda\} \cap Q_1$$

and

$$\mathcal{H}_\lambda(t) = \{\mathbf{M}_\alpha(\chi_{Q_2}(|F|^p + |\nabla g|^p)) \le t\lambda\} \cap Q_1$$

for $\lambda, t \ge 0$. Inequality (1.3) can be rewritten as

$$\mathcal{L}^n(\mathcal{G}_\lambda(\varepsilon^{-a}) \cap \mathcal{H}_\lambda(b_\varepsilon)) \le C\varepsilon \mathcal{L}^n(\mathcal{G}_\lambda(1)).$$

The main idea of this proof is to apply Lemma 4.1 with $\mathcal{V} = \mathcal{G}_\lambda(\varepsilon^{-a}) \cap \mathcal{H}_\lambda(b_\varepsilon)$ and $\mathcal{W} = \mathcal{G}_\lambda(1)$. It is sufficient to show that two hypotheses (i) and (ii) in Lemma 4.1 hold.

Let us prove the first one. Thanks to Lemma 2.10, we can estimate as follows

$$\mathcal{L}^n(\mathcal{V}) \le \mathcal{L}^n(\mathcal{G}_\lambda(\varepsilon^{-a})) \le C\left(\frac{1}{\varepsilon^{-a}\lambda}\int_\Omega \chi_{Q_2}|\nabla u|^p\,dx\right)^{\frac{n}{n-\alpha}}. \tag{4.1}$$

For $\lambda > \lambda_0$ with λ_0 is given by (1.4), one obtains from (4.1) that

$$\mathcal{L}^n(\mathcal{V}) \le C\left(\frac{1}{\varepsilon^{-a}\varepsilon^{\frac{p}{\Theta}-1}r^{\alpha-n}\|\nabla u\|^p_{L^p(Q_2\cap\Omega)}}\int_\Omega \chi_{Q_2}|\nabla u|^p\,dx\right)^{\frac{n}{n-\alpha}},$$

which is equivalent to the following inequality

$$\mathcal{L}^n(\mathcal{V}) \le C\varepsilon^{\frac{n\left(a+1-\frac{p}{\Theta}\right)}{n-\alpha}}\mathcal{L}^n(B_r(0)) \le C\varepsilon\mathcal{L}^n(B_r(0)). \tag{4.2}$$

In the last estimate of (4.2), we emphasize that the constraint

$$\frac{n\left(a+1-\frac{p}{\Theta}\right)}{n-\alpha} \ge 1 \quad\Longleftrightarrow\quad a \ge \frac{p}{\Theta} - \frac{\alpha}{n},$$

which will be valid with the choice of the constant a at the end of the proof.

Next, we will prove (ii) in Lemma 4.1 by contradiction. Assume that one can find $x \in \Omega$ and $\varrho \in (0,r]$ such that $\Omega \cap B_\varrho(x) \not\subset \mathcal{W}$. To obtain a contradiction, we need to show that

$$\mathcal{L}^n(\mathcal{V} \cap B_\varrho(x)) \le \varepsilon\mathcal{L}^n(B_\varrho(x)). \tag{4.3}$$

Without loss of generality we may assume that

$$\mathcal{V} \cap B_\varrho(x) \neq \emptyset \quad\text{and}\quad \Omega \cap B_\varrho(x) \cap (\mathbb{R}^n \setminus \mathcal{W}) \neq \emptyset.$$

Thus there exist $x_1 \in \mathcal{V} \cap B_\varrho(x)$ and $x_2 \in \Omega \cap B_\varrho(x)$ such that

$$\mathbf{M}_\alpha(\chi_{Q_2}(|F|^p + |\nabla g|^p))(x_1) \le b_\varepsilon\lambda \quad\text{and}\quad \mathbf{M}_\alpha(\chi_{Q_2}|\nabla u|^p)(x_2) \le \lambda. \tag{4.4}$$

For any $y \in B_\varrho(x)$, one can separate as follows

$$\mathbf{M}_\alpha(\chi_{Q_2}|\nabla u|^p)(y) = \max\left\{\sup_{0<\rho<\varrho}\rho^\alpha \fint_{B_\rho(y)}\chi_{Q_2}|\nabla u|^p\,dx; \sup_{\rho\ge\varrho}\rho^\alpha\fint_{B_\rho(y)}\chi_{Q_2}|\nabla u|^p\,dx\right\}. \tag{4.5}$$

The first term can be estimated from the fact that $B_\rho(y) \subset B_{2\varrho}(x)$ for all $\rho \in (0,\varrho)$. This deduces that

$$\begin{aligned}\sup_{0<\rho<\varrho}\rho^\alpha\fint_{B_\rho(y)}\chi_{Q_2}|\nabla u|^p\,dx &\le \sup_{0<\rho<\varrho}\rho^\alpha\fint_{B_\rho(y)}\chi_{Q_2\cap B_{2\varrho}(x)}|\nabla u|^p\,dx \\ &\le \mathbf{M}_\alpha(\chi_{Q_2\cap B_{2\varrho}(x)}|\nabla u|^p)(y).\end{aligned} \tag{4.6}$$

Next, we estimate the second one from (4.4) as follows

$$\begin{aligned}\sup_{\rho\geq\varrho}\rho^{\alpha}\fint_{B_\rho(y)}\chi_{Q_2}|\nabla u|^p\,dx &\leq \sup_{\rho\geq\varrho}\rho^{\alpha}\frac{\mathcal{L}^n(B_{3\rho}(x_2))}{\mathcal{L}^n(B_\rho(y))}\fint_{B_{3\rho}(x_2)}\chi_{Q_2}|\nabla u|^p\,dx \\ &\leq 3^{n-\alpha}\mathbf{M}_\alpha(\chi_{Q_2}|\nabla u|^p)(x_2) \\ &\leq 3^{n-\alpha}\lambda.\end{aligned} \tag{4.7}$$

Here we remark that the first inequality in (4.7) yields from the following relation

$$B_\rho(y)\subset B_{\rho+\varrho}(x)\subset B_{\rho+2\varrho}(x_2)\subset B_{3\rho}(x_2),\quad \forall\,\rho\geq\varrho.$$

Collecting estimates in (4.5), (4.6) and (4.7), one concludes that

$$\mathcal{G}_\lambda(\varepsilon^{-a})\cap B_\varrho(x)=\{\mathbf{M}_\alpha(\chi_{Q_2\cap B_{2\varrho}(x)}|\nabla u|^p)>\varepsilon^{-a}\lambda\}\cap B_\varrho(x), \tag{4.8}$$

if provided $\varepsilon^{-a}>3^{n-\alpha}$.

If $B_{2\varrho}(x)\subset\Omega$ then we choose

$$x_0=x\quad\text{and}\quad R=2\varrho. \tag{4.9}$$

Otherwise, if $B_{2\varrho}(x)\cap\partial\Omega\neq\emptyset$ we choose $x_0\in\partial\Omega$ such that

$$|x-x_0|=\operatorname{dist}(x,\partial\Omega)\leq 2\varrho\quad\text{and}\quad R=4\varrho. \tag{4.10}$$

It is clear to see that $B_{2\varrho}(x)\subset B_R(x_0)$ which follows from (4.8) that for all $\varepsilon\in(0,\varepsilon_0)$ with $\varepsilon_0=3^{-\frac{n-\alpha}{a}}$, there holds

$$\mathcal{G}_\lambda(\varepsilon^{-a})\cap B_\varrho(x)=\{\mathbf{M}_\alpha(\chi_{Q_2\cap B_R(x_0)}|\nabla u|^p)>\varepsilon^{-a}\lambda\}\cap B_\varrho(x). \tag{4.11}$$

Let v be the unique solution to (3.1) in $\Omega_{2R}=B_{2R}(x_0)\cap\Omega$. Lemma 3.3 ensures that the reverse Hölder inequality (3.2) and the comparison estimate (3.3) hold. Using the following fundamental inequality

$$|\nabla u|^p\leq 2^p(|\nabla v|^p+|\nabla u-\nabla v|^p),$$

it implies from (4.11) that

$$\begin{aligned}\mathcal{L}^n(\mathcal{G}_\lambda(\varepsilon^{-a})\cap B_\varrho(x)) &\leq \mathcal{L}^n(\{\mathbf{M}_\alpha(\chi_{Q_2\cap B_R(x_0)}|\nabla v|^p)>2^{-p}\varepsilon^{-a}\lambda\}\cap B_\varrho(x)) \\ &\quad+\mathcal{L}^n(\{\mathbf{M}_\alpha(\chi_{Q_2\cap B_R(x_0)}|\nabla u-\nabla v|^p)>2^{-p}\varepsilon^{-a}\lambda\}\cap B_\varrho(x)).\end{aligned} \tag{4.12}$$

Applying Lemma 2.10 with different values of s, $s=\Theta/p$ and $s=1$ corresponding to two terms on the right-hand side of (4.12) respectively, we obtain that

$$
\begin{aligned}
\mathcal{L}^n(\mathcal{G}_\lambda(\varepsilon^{-a}) \cap B_\varrho(x)) &\le C\left(\frac{1}{(2^{-p}\varepsilon^{-a}\lambda)^{\frac{\Theta}{p}}}\int_{Q_2\cap B_R(x_0)} |\nabla v|^\Theta\,dx\right)^{\frac{n}{n-\alpha\frac{\Theta}{p}}} \\
&\quad + C\left(\frac{1}{2^{-p}\varepsilon^{-a}\lambda}\int_{Q_2\cap B_R(x_0)} |\nabla u - \nabla v|^p\,dx\right)^{\frac{n}{n-\alpha}} \\
&\le C\left(\frac{\varrho^n}{(\varepsilon^{-a}\lambda)^{\frac{\Theta}{p}}}\fint_{B_R(x_0)} \chi_{Q_2}|\nabla v|^\Theta\,dx\right)^{\frac{n}{n-\alpha\frac{\Theta}{p}}} \\
&\quad + C\left(\frac{\varrho^n}{\varepsilon^{-a}\lambda}\fint_{B_R(x_0)} \chi_{Q_2}|\nabla u - \nabla v|^p\,dx\right)^{\frac{n}{n-\alpha}}.
\end{aligned}
\tag{4.13}
$$

The choice of R and x_0 in (4.9) or (4.10) makes sure that

$$
B_{2R}(x_0) \subset B_{8\varrho}(x_0) \subset B_{12\varrho}(x) \subset B_{13\varrho}(x_1) \cap B_{13\varrho}(x_2),
$$

which guarantees from (4.4) that

$$
\begin{aligned}
\fint_{B_{2R}(x_0)} \chi_{Q_2}|\nabla u|^p\,dx &\le \frac{\mathcal{L}^n(B_{13\varrho}(x_2))}{\mathcal{L}^n(B_{2R}(x_0))}\fint_{B_{13\varrho}(x_2)} \chi_{Q_2}|\nabla u|^p\,dx \\
&\le C\varrho^{-\alpha}\mathbf{M}_\alpha(\chi_{Q_2}|\nabla u|^p)(x_2) \le C\varrho^{-\alpha}\lambda,
\end{aligned}
$$

and similarly

$$
\fint_{B_{2R}(x_0)} \chi_{Q_2}(|F|^p + |\nabla g|^p)\,dx \le C\varrho^{-\alpha}\mathbf{M}_\alpha(\chi_{Q_2}(|F|^p + |\nabla g|^p))(x_1) \le C\varrho^{-\alpha}b_\varepsilon\lambda.
$$

Applying two above estimates into (3.3) and (3.2), it deduces that

$$
\fint_{B_{2R}(x_0)} \chi_{Q_2}|\nabla u - \nabla v|^p\,dx \le C(\delta + c_\delta b_\varepsilon)\varrho^{-\alpha}\lambda \tag{4.14}
$$

for all $\delta > 0$, and

$$
\begin{aligned}
\fint_{B_R(x_0)} \chi_{Q_2}|\nabla v|^\Theta\,dx &\le C\left(\fint_{B_{2R}(x_0)} \chi_{Q_2}|\nabla v|^p\,dx\right)^{\frac{\Theta}{p}} \\
&\le C\left(\fint_{B_{2R}(x_0)} \chi_{Q_2}|\nabla u|^p\,dx + \fint_{B_{2R}(x_0)} \chi_{Q_2}|\nabla u - \nabla v|^p\,dx\right)^{\frac{\Theta}{p}} \\
&\le C\big[(1 + \delta + c_\delta b_\varepsilon)\varrho^{-\alpha}\lambda\big]^{\frac{\Theta}{p}}.
\end{aligned}
\tag{4.15}
$$

Substituting (4.14) and (4.15) into (4.13), one gets that

$$
\mathcal{L}^n(\mathcal{G}_\lambda(\varepsilon^{-a}) \cap B_\varrho(x)) \le C\left(\frac{\varrho^n}{(\varepsilon^{-a}\lambda)^{\frac{\Theta}{p}}}\big[(1 + \delta + c_\delta b_\varepsilon)\varrho^{-\alpha}\lambda\big]^{\frac{\Theta}{p}}\right)^{\frac{n}{n-\alpha\frac{\Theta}{p}}}
$$

$$
\begin{aligned}
&+ C\left(\frac{\varrho^n}{\varepsilon^{-a}\lambda}(\delta + c_\delta b_\varepsilon)\varrho^{-\alpha}\lambda\right)^{\frac{n}{n-\alpha}} \\
&\le C\varrho^n\left(\varepsilon^{\frac{a\Theta n}{np-\alpha\Theta}} + (\varepsilon^a\delta)^{\frac{n}{n-\alpha}}\right),
\end{aligned}
$$

where $b_\varepsilon = \delta c_\delta^{-1}$. Let us balance the exponent of ε in this inequality by taking $\delta = \varepsilon^{\frac{an(\Theta-p)}{np-\alpha\Theta}} \in (0,1)$, we may conclude that

$$
\mathcal{L}^n(\mathcal{G}_\lambda(\varepsilon^{-a}) \cap B_\varrho(x)) \le C\varepsilon^{\frac{a\Theta n}{np-\alpha\Theta}}\varrho^n,
$$

which leads to (4.3) by choosing $a = \frac{p}{\Theta} - \frac{\alpha}{n}$. This finishes the proof. □

Lemma 4.2. *Let $f \in L^{q,s;\kappa}(\Omega)$ for $1 < q < \infty$, $0 < s \le \infty$ and $0 < \kappa \le n$. For $0 \le \tau < \alpha \le \kappa/q$, there exist constants $C = C(n,q,\kappa,\alpha) > 0$ and $\upsilon = \frac{q(\alpha-\tau)}{\kappa-\tau q} \in (0,1]$ such that*

$$
\mathbf{M}_\alpha(f)(\xi) \le C[\mathbf{M}_\tau(f)(\xi)]^{1-\upsilon}(\|f\|_{L^{q,s;\kappa}(\Omega)})^\upsilon \tag{4.16}
$$

for all $\xi \in \Omega$. In particular, there holds

$$
\|\mathbf{M}^D_{\frac{\kappa}{q}}(f)\|_{L^\infty(\Omega)} \le C\|f\|_{L^{q,s;\kappa}(\Omega)}, \tag{4.17}
$$

where $D = \mathrm{diam}(\Omega)$.

Proof. Let $\xi \in \Omega$, $\varrho > 0$ and $0 < \upsilon \le 1$, we may decompose as follows

$$
\varrho^{\alpha-n}\int_{B_\varrho(\xi)} f(x)\,dx = \left(\varrho^{\tau-n}\int_{B_\varrho(\xi)} f(x)\,dx\right)^{1-\upsilon}\left(\varrho^{\gamma-n}\int_{B_\varrho(\xi)} f(x)\,dx\right)^{\upsilon},
$$

where $\gamma = \tau + \frac{\alpha-\tau}{\upsilon}$. For $q > 1$, thanks to [36, Lemma 2.1] or [22, Exercise 1.1.11, p. 13], one has the following estimate

$$
\begin{aligned}
\varrho^{\alpha-n}\int_{B_\varrho(\xi)} f(x)\,dx &\le C[\mathbf{M}_\tau(f)(\xi)]^{1-\upsilon}\left(\varrho^{\gamma-n}\varrho^{n-\frac{n}{q}}\|f\|_{L^{q,\infty}(B_\varrho(\xi))}\right)^{\upsilon} \\
&\le C[\mathbf{M}_\tau(f)(\xi)]^{1-\upsilon}\left(\varrho^{\gamma-\frac{n}{q}}\|f\|_{L^{q,s}(B_\varrho(\xi))}\right)^{\upsilon},
\end{aligned}
$$

which yields that

$$
\varrho^{\alpha-n}\int_{B_\varrho(\xi)} f(x)\,dx \le C[\mathbf{M}_\tau(f)(\xi)]^{1-\upsilon}(\|f\|_{L^{q,s;\gamma q}(\Omega)})^{\upsilon}. \tag{4.18}
$$

We may choose $\gamma q = \kappa \Leftrightarrow \upsilon = \frac{q(\alpha-\tau)}{\kappa-\tau q}$ to obtain (4.16). In particular, inequality (4.17) is valid by taking the supremum both sides of (4.18) for all $0 < \varrho < D$ and $\xi \in \Omega$ with $\upsilon = 1 \Leftrightarrow \alpha = \kappa/q$. □

Lemma 4.3. *Let $f \in L^{q,s;\kappa}(\Omega)$ for $1 < q < \infty$, $0 < s \leq \infty$ and $0 < \kappa \leq n$. For $0 < \alpha \leq \kappa/q$, there exists a constant $C = C(n,q,\kappa,\alpha) > 0$ such that*

$$\sup_{\substack{0<r<D\\ \xi\in\Omega}} r^{-\alpha+\frac{\kappa}{q}-\frac{n}{\sigma}} \|\mathbf{M}_\alpha f\|_{L^{\sigma,\vartheta}(B_r(\xi))} \leq C\|f\|_{L^{q,s;\kappa}(\Omega)}, \tag{4.19}$$

where $\sigma = \frac{\kappa q}{\kappa-\alpha q}$ and $\vartheta = \frac{\kappa s}{\kappa-\alpha q}$.

Proof. We first remark that $q > 1$ implies to $\sigma(1-\upsilon) > 1$ with $\upsilon = \alpha q/\kappa$. For this reason, applying Lemma 4.2 and boundedness of maximal operator, there holds

$$\begin{aligned}
\|\mathbf{M}_\alpha f\|_{L^{\sigma,\vartheta}(B_r(\xi))} &\leq C\|(\mathbf{M}f)^{1-\upsilon}\|_{L^{\sigma,\vartheta}(B_r(\xi))}\|f\|^{\upsilon}_{L^{q,s;\kappa}(\Omega)}\\
&= C\|\mathbf{M}f\|^{1-\upsilon}_{L^{\sigma(1-\upsilon),\vartheta(1-\upsilon)}(B_r(\xi))}\|f\|^{\upsilon}_{L^{q,s;\kappa}(\Omega)}\\
&\leq C\|f\|^{1-\upsilon}_{L^{\sigma(1-\upsilon),\vartheta(1-\upsilon)}(B_r(\xi))}\|f\|^{\upsilon}_{L^{q,s;\kappa}(\Omega)}.
\end{aligned}$$

From this reason, the left-hand side of (4.19) can be estimated as follows

$$\begin{aligned}
&\sup_{\substack{0<r<D\\ \xi\in\Omega}} r^{-\alpha+\frac{\kappa}{q}-\frac{n}{\sigma}} \|\mathbf{M}_\alpha f\|_{L^{\sigma,\vartheta}(B_r(\xi))}\\
&\leq C \sup_{\substack{0<r<D\\ \xi\in\Omega}} \left(r^{\frac{\sigma\left(\frac{\kappa}{q}-\alpha\right)-n}{\sigma(1-\upsilon)}} \|f\|_{L^{\sigma(1-\upsilon),\vartheta(1-\upsilon)}(B_r(\xi))}\right)^{1-\upsilon} \|f\|^{\upsilon}_{L^{q,s;\kappa}(\Omega)}\\
&\leq C\|f\|^{1-\upsilon}_{L^{\sigma(1-\upsilon),\vartheta(1-\upsilon);\sigma\left(\frac{\kappa}{q}-\alpha\right)}(\Omega)}\|f\|^{\upsilon}_{L^{q,s;\kappa}(\Omega)}.
\end{aligned} \tag{4.20}$$

It notes here the determination of σ and ϑ in the statement of this lemma ensures that

$$\sigma(1-\upsilon) = q, \quad \vartheta(1-\upsilon) = s, \quad \sigma\left(\frac{\kappa}{q}-\alpha\right) = \kappa,$$

which deduces to (4.19) from (4.20). The proof is complete. $\square$

Henceforth, Theorem 1.2 is proved directly using the specific Theorem 1.1 and definition of Lorentz–Morrey spaces in general.

Proof of Theorem 1.2. Let $\xi \in \Omega$ and $0 < r < D = \mathrm{diam}(\Omega)$. For simplicity of notation, let us denote as follows

$$\mathbb{U} = \chi_{B_{10r}(\xi)}|\nabla u|^p \quad \text{and} \quad \mathbb{G} = \chi_{B_{10r}(\xi)}(|F|^p + |\nabla g|^p).$$

According to Theorem 1.1, there exists a constant $\Theta = \Theta(n,p,\Lambda) > p$ such that the following inequality

$$\mathcal{L}^n\big(\{\mathbf{M}_\alpha\mathbb{U} > \varepsilon^{\frac{\alpha}{n}-\frac{p}{\Theta}}\lambda, \mathbf{M}_\alpha\mathbb{G} \leq b_\varepsilon\lambda\} \cap B_r(\xi)\big) \leq C\varepsilon\mathcal{L}^n(\{\mathbf{M}_\alpha\mathbb{U} > \lambda\} \cap B_r(\xi)) \tag{4.21}$$

holds for any $\lambda > \lambda_0 := \varepsilon^{\frac{p}{\Theta}-1} r^{\alpha-n} \|\mathbb{U}\|_{L^1(\Omega)}$ and ε small enough, where the positive constant C depends on n, p, Λ, α, c_0, r_0 and $\mathrm{diam}(\Omega)$. We deduce from (4.21) that

$$\begin{aligned}(4.22)\qquad &\mathcal{L}^n\big(\big\{\mathbf{M}_\alpha \mathbb{U} > \varepsilon^{\frac{\alpha}{n}-\frac{p}{\Theta}}\lambda\big\} \cap B_r(\xi)\big) \\ &\le \mathcal{L}^n(\{\mathbf{M}_\alpha \mathbb{G} > b_\varepsilon \lambda\} \cap B_r(\xi)) + C\varepsilon \mathcal{L}^n(\{\mathbf{M}_\alpha \mathbb{U} > \lambda\} \cap B_r(\xi)), \quad \forall\, \lambda > \lambda_0.\end{aligned}$$

On the other hand, let us introduce two parameters defined as

$$(4.23)\qquad \sigma = \frac{\kappa q}{\kappa - \alpha q} \quad \text{and} \quad \vartheta = \frac{\kappa s}{\kappa - \alpha q},$$

then the norm of $\mathbf{M}_\alpha \mathbb{U}$ in Lorentz space $L^{\sigma,\vartheta}(B_r(\xi))$ can be rewritten as below

$$\|\mathbf{M}_\alpha \mathbb{U}\|^{\vartheta}_{L^{\sigma,\vartheta}(B_r(\xi))} = \varepsilon^{\vartheta\left(\frac{\alpha}{n}-\frac{p}{\Theta}\right)} \sigma \int_0^\infty \lambda^\vartheta \mathcal{L}^n\big(\big\{\mathbf{M}_\alpha \mathbb{U} > \varepsilon^{\frac{\alpha}{n}-\frac{p}{\Theta}}\lambda\big\} \cap B_r(\xi)\big)^{\frac{\vartheta}{\sigma}} \frac{d\lambda}{\lambda}.$$

We first separate this integral over $(0,\infty)$ by two integrals over $(0,\lambda_0)$ and (λ_0,∞) respectively, then we apply (4.22) for the term over (λ_0,∞), there holds

$$\begin{aligned}\|\mathbf{M}_\alpha \mathbb{U}\|^{\vartheta}_{L^{\sigma,\vartheta}(B_r(\xi))} \le \varepsilon^{\vartheta\left(\frac{\alpha}{n}-\frac{p}{\Theta}\right)} \Bigg[&\sigma \lambda_0^\vartheta \mathcal{L}^n(B_r(\xi))^{\frac{\vartheta}{\sigma}} \\ &+ C\sigma \int_{\lambda_0}^\infty \lambda^\vartheta \mathcal{L}^n(\{\mathbf{M}_\alpha \mathbb{G} > b_\varepsilon \lambda\} \cap B_r(\xi))^{\frac{\vartheta}{\sigma}} \frac{d\lambda}{\lambda} \\ &+ C\varepsilon^{\frac{\vartheta}{\sigma}} \sigma \int_{\lambda_0}^\infty \lambda^\vartheta \mathcal{L}^n(\{\mathbf{M}_\alpha \mathbb{U} > \lambda\} \cap B_r(\xi))^{\frac{\vartheta}{\sigma}} \frac{d\lambda}{\lambda}\Bigg].\end{aligned}$$

By replacing $\lambda_0 = \varepsilon^{\frac{p}{\Theta}-1} r^{\alpha-n} \|\mathbb{U}\|_{L^1(\Omega)}$ which is defined from the beginning of the proof, one gets that

$$\begin{aligned}&\|\mathbf{M}_\alpha \mathbb{U}\|^{\vartheta}_{L^{\sigma,\vartheta}(B_r(\xi))} \\ &\le C\varepsilon^{\vartheta\left(\frac{\alpha}{n}-\frac{p}{\Theta}\right)} \left[\left(\varepsilon^{\frac{p}{\Theta}-1} r^{\alpha-n} \|\mathbb{U}\|_{L^1(\Omega)}\right)^\vartheta r^{\frac{n\vartheta}{\sigma}} + b_\varepsilon^{-\vartheta} \|\mathbf{M}_\alpha \mathbb{G}\|^\vartheta_{L^{\sigma,\vartheta}(B_r(\xi))} + \varepsilon^{\frac{\vartheta}{\sigma}} \|\mathbf{M}_\alpha \mathbb{U}\|^\vartheta_{L^{\sigma,\vartheta}(B_r(\xi))}\right].\end{aligned}$$

This inequality is equivalent to the following inequality

$$\begin{aligned}(4.24)\qquad \|\mathbf{M}_\alpha \mathbb{U}\|_{L^{\sigma,\vartheta}(B_r(\xi))} &\le C\varepsilon^{\frac{\alpha}{n}-1} r^{\alpha-n+\frac{n}{\sigma}} \|\mathbb{U}\|_{L^1(\Omega)} + Cb_\varepsilon^{-1} \varepsilon^{\frac{\alpha}{n}-\frac{p}{\Theta}} \|\mathbf{M}_\alpha \mathbb{G}\|_{L^{\sigma,\vartheta}(B_r(\xi))} \\ &\quad + C\varepsilon^{\frac{\alpha}{n}-\frac{p}{\Theta}+\frac{1}{\sigma}} \|\mathbf{M}_\alpha \mathbb{U}\|_{L^{\sigma,\vartheta}(B_r(\xi))}.\end{aligned}$$

For every $1 < q < \Theta/p$, $0 < s \le \infty$ and $\frac{\alpha}{\frac{1}{q}-\frac{p}{\Theta}+\frac{\alpha}{n}} < \kappa \le n$, it is a simple matter to check from the setting (4.23) that $\frac{\alpha}{n} - \frac{p}{\Theta} + \frac{1}{\sigma} > 0$. For this reason, one may choose ε in (4.24) small enough such that $C\varepsilon^{\frac{\alpha}{n}-\frac{p}{\Theta}+\frac{1}{\sigma}} < 1/2$, it follows that

$$(4.25)\qquad \|\mathbf{M}_\alpha \mathbb{U}\|_{L^{\sigma,\vartheta}(B_r(\xi))} \le Cr^{\alpha-n+\frac{n}{\sigma}} \|\mathbb{U}\|_{L^1(\Omega)} + C\|\mathbf{M}_\alpha \mathbb{G}\|_{L^{\sigma,\vartheta}(B_r(\xi))}.$$

Thanks to (3.4) in Lemma 3.5, it gives us the existence of a constant $\beta_0 \in (0,1/2]$ such that under assumption $\kappa > pq(1-\beta_0)$, one has

$$r^{\frac{\kappa}{q}-n} \|\mathbb{U}\|_{L^1(\Omega)} \le C\|\mathbf{M}^D_{\frac{\kappa}{q}}(|F|^p + |\nabla g|^p)\|_{L^\infty(\Omega)}.$$

Combining the above inequality with (4.17) in Lemma 4.2, it follows that

$$r^{\frac{\kappa}{q}-n}\|\mathbb{U}\|_{L^1(\Omega)} \leq C\||F|^p + |\nabla g|^p\|_{L^{q,s;\kappa}(\Omega)}. \tag{4.26}$$

Collecting between (4.25) and (4.26) and taking supremum both sides of the inequality for $0 < \varrho < D$ and $\xi \in \Omega$, there holds

$$\begin{aligned}
&\sup_{\substack{0<\varrho<D \\ \xi\in\Omega}} r^{-\alpha+\frac{\kappa}{q}-\frac{n}{\sigma}}\|\mathbf{M}_\alpha(\chi_{B_{10r}(\xi)}(|\nabla u|^p))\|_{L^{\sigma,\vartheta}(B_r(\xi))} \\
&\leq C\||F|^p + |\nabla g|^p\|_{L^{q,s;\kappa}(\Omega)} + \sup_{\substack{0<\varrho<D \\ \xi\in\Omega}} r^{-\alpha+\frac{\kappa}{q}-\frac{n}{\sigma}}\|\mathbf{M}_\alpha\mathbb{G}\|_{L^{\sigma,\vartheta}(B_r(\xi))}.
\end{aligned} \tag{4.27}$$

We deduce from (4.27) that

$$\|\mathbf{M}_\alpha(|\nabla u|^p)\|_{L^{\sigma,\vartheta;\sigma\left(\frac{\kappa}{q}-\alpha\right)}(\Omega)} \leq C\||F|^p + |\nabla g|^p\|_{L^{q,s;\kappa}(\Omega)} + \mathcal{K},$$

where the second term $\mathcal{K}$ is given by

$$\mathcal{K} = \sup_{\substack{0<\varrho<D \\ \xi\in\Omega}} r^{-\alpha+\frac{\kappa}{q}-\frac{n}{\sigma}}\|\mathbf{M}_\alpha(\chi_{B_{10r}(\xi)}(|F|^p + |\nabla g|^p))\|_{L^{\sigma,\vartheta}(B_r(\xi))}.$$

An easy computation from (4.23) shows that $\sigma(\kappa/q - \alpha) = \kappa$ which leads to

$$\|\mathbf{M}_\alpha(|\nabla u|^p)\|_{L^{\sigma,\vartheta;\kappa}(\Omega)} \leq C\||F|^p + |\nabla g|^p\|_{L^{q,s;\kappa}(\Omega)} + \mathcal{K}. \tag{4.28}$$

Finally, the statement in (1.5) will be proved from (4.28) once we have the following estimate

$$\mathcal{K} \leq C\||F|^p + |\nabla g|^p\|_{L^{q,s;\kappa}(\Omega)}.$$

This inequality is immediately observed by Lemma 4.3 and hence we complete the proof. $\square$

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