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Int-amplified Endomorphisms on Normal Projective Surfaces

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Abstract. We investigate int-amplified endomorphisms on normal projective surfaces. We prove that the output of the equivariant MMP is either a Q-abelian surface, a (equivariant) quasi-étale quotient of a smooth projective surface, a Mori dream space, or a projective cone of an elliptic curve.

1. Introduction

In this paper, we work over an algebraically closed field k of characteristic zero. A self-morphism $f\colon X \to X$ on a projective variety X is called int-amplified if there exists an ample Cartier divisor H on X such that $f^*H - H$ is ample. Int-amplified endomorphisms are compatible with minimal model program (MMP), as shown in [11, 12]. Also, existence of such endomorphisms imposes strong constraint to the singularities of the varieties. Therefore, it seems possible to classify all int-amplified endomorphisms or varieties admitting an int-amplified endomorphism. In this paper, we investigate int-amplified endomorphisms on normal projective surfaces.

To state our main theorem, we fix the terminology.

Definition 1.1. (1) A morphism $h\colon Y \to X$ between varieties is called quasi-étale if h is étale at every codimension one point on Y.

(2) A variety X is called Q-abelian if there exists a finite surjective quasi-étale morphism $A \to X$ from an abelian variety A.

The linear equivalence and $\mathbb{Q}$-linear equivalence of divisors on normal projective varieties are denoted by $\sim$ and $\sim_{\mathbb{Q}}$ respectively. The Iitaka dimension of a $\mathbb{Q}$-Cartier divisor D on a normal projective variety is denoted by $\kappa(D)$.

The following is the main theorem of this paper.

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Theorem 1.2. *Let X be a normal projective surface over k. Let $f\colon X \to X$ be an int-amplified endomorphism. Then X is $\mathbb{Q}$-Gorenstein log canonical (lc) and we have the following sequence of morphisms:*

$$X = X_1 \to \cdots \to X_r \to C$$

where

- *$X_i \to X_{i+1}$ is the divisorial contraction of a K_{X_i}-negative extremal ray for $i = 1, \ldots, r-1$;*
- *$X_r \to C$ is a Fano contraction of a K_{X_r}-negative extremal ray if K_X is not pseudo-effective;*
- *we ignore "$\to C$" if K_X is pseudo-effective;*
- *there exists a positive integer n such that f^n induces endomorphisms on X_i and C (in such case, we call the sequence f^n-equivariant MMP).*

Moreover, one of the following holds:

(1) *$K_{X_1} \sim_{\mathbb{Q}} 0$. In this case, $r = 1$ and X is a Q-abelian variety;*

(2) *C is an elliptic curve, $r = 1$ and X_1 is smooth;*

(3) *$C \simeq \mathbb{P}^1$, $\kappa(-K_{X_r}) = 0$. In this case, X is klt, $r = 1$, there exists a quasi-étale finite surjection $h\colon Y \to X$ of degree 2 from a smooth projective surface Y, which is a minimal ruled surface over an elliptic curve, and an endomorphism $f_Y\colon Y \to Y$ such that*

$$\begin{tikzcd} Y \arrow[r, "f_Y"] \arrow[d, "h"'] & Y \arrow[d, "h"] \\ X \arrow[r, "f^n"'] & X \end{tikzcd}$$

is commutative;

(4) *$C \simeq \mathbb{P}^1$, $\kappa(-K_{X_r}) = 1$. In this case, X is klt, $r = 1$, there exists a quasi-étale finite surjection $h\colon Y \to X$ from a smooth projective surface Y, which is a minimal ruled surface over an elliptic curve, and an endomorphism $f_Y\colon Y \to Y$ such that*

$$\begin{tikzcd} Y \arrow[r, "f_Y"] \arrow[d, "h"'] & Y \arrow[d, "h"] \\ X \arrow[r, "f^n"'] & X \end{tikzcd}$$

is commutative;

(5) $C \simeq \mathbb{P}^1$, $\kappa(-K_{X_r}) = 2$. *In this case, X is klt and a Mori dream space.*

(6) *C is a point, the Picard number of X_r is one and $-K_{X_r}$ is ample. In this case, X is a projective cone of an elliptic curve or a Mori dream space.*

Remark 1.3. The structure of X in Theorem 1.2(1) and (6) are already known (cf. [3, 11, 14]). The essential result of this paper is the construction of quasi-étale covers in the cases (3) and (4).

Remark 1.4. We refer [8, Definiton 1.10] for the definition of Mori dream spaces.

Remark 1.5. All the cases (1)–(6) in Theorem 1.2 actually happen. There are trivial examples for (1), (2), (5), (6):

(1) X is an abelian surface and f is the multiplication by n map for some $n > 1$.

(2) X is the product of $\mathbb{P}^1$ and an elliptic curve and f is the product of non-isomorphic surjective endomorphisms on each factor.

(5) $X = \mathbb{P}^1 \times \mathbb{P}^1$ and f is the product of non-isomorphic surjective endomorphisms on each factor.

(6) $X = \mathbb{P}^2$ and f is a non-isomorphic surjective endomorphism.

For more examples, see for instance [6, 13]. We give examples for (3), (4) in Section 7.

Remark 1.6. Notation as in Theorem 1.2. Let $g\colon X \to X$ be any surjective endomorphism. Then, by [12, Theorem 4.6], g^m induces endomorphisms on X_r and C for some $m > 0$. Moreover, in case (3), the induced endomorphism g_r on X_r lifts to an endomorphism on Y. Indeed, by the proof of Lemma 4.2, the curve "C" in Lemma 4.2 and Proposition 4.3 is totally invariant under g_r. Therefore, by Proposition 4.3, g_r lifts to the quasi-étale cover.

2. Notation and terminology

Throughout this paper, the ground field k is an algebraically closed field of characteristic zero. A variety is an irreducible reduced separated scheme of finite type over k. A subvariety means an irreducible reduced closed subscheme. Divisor on a normal projective variety means Weil divisor.

For a self-morphism $f\colon X \to X$ of a variety X, a subset $S \subset X$ is called totally invariant under f if $f^{-1}(S) = S$ as sets.

- The pseudo-effective cone of a projective variety X is denoted by $\overline{\mathrm{Eff}}(X)$.
- The ramification divisor of a finite surjective morphism $f\colon X \to Y$ between normal projective varieties is denoted by R_f.

- Let D, E be two $\mathbb{Q}$-Weil divisors on a normal projective variety. We write $D \geq E$ if the divisor $D - E$ is effective.

3. Preliminaries

3.1.

Let X be a normal variety, and let $\mu\colon X' \to X$ be a proper birational morphism from a normal variety X'. If $\Delta \subset X$ is a $\mathbb{Q}$-divisor, we denote by $\mu_*^{-1}(\Delta)$ its strict transform.

A log pair is a tuple (X, Δ) where X is a normal variety and $\Delta = \sum_i d_i \Delta_i$ is a $\mathbb{Q}$-divisor on X with $d_i \leq 1$ for all i. We say that the pair (X, Δ) is log canonical (lc) (resp. purely log terminal (plt), resp. Kawamata log terminal (klt)) if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and for every proper birational morphism $\mu\colon X' \to X$ from a normal variety X' we can write

$$K_{X'} + \mu_*^{-1}(\Delta) = \mu^*(K_X + \Delta) + \sum_j a(E_j, X, \Delta) E_j,$$

where the divisor E_j are μ-exceptional and $a(E_j, X, \Delta) \geq -1$ (resp. $a(E_j, X, \Delta) > -1$, resp. $a(E_j, X, \Delta) > -1$ and $d_i < 1$ for all i) for all j. If the pair (X, Δ) is lc, we say that a subvariety $Z \subset X$ is an lc center if there exists a morphism $\mu\colon X' \to X$ as above and a μ-exceptional divisor E such that $Z = \mu(E)$ and $a(E, X, \Delta) = -1$.

A variety X is called lc, (resp. klt) if so is the pair $(X, 0)$. A variety X is called $\mathbb{Q}$-Gorenstein if the canonical divisor K_X is $\mathbb{Q}$-Cartier and $\mathbb{Q}$-factorial if every Weil divisor on X is $\mathbb{Q}$-Cartier. If a variety is lc, then it is $\mathbb{Q}$-Gorenstein by definition. A surface is $\mathbb{Q}$-factorial if it has rational singularities and it has rational singularities if it is klt (see [10, Theorem 5.22] and [1, Theorem 4.6]).

3.2.

We gather several facts on endomorphisms that we use later. The first two lemmas are about the relationship between endomorphisms and singularities.

Lemma 3.1. (see [17, Proposition 7.7], cf. [4, Lemma 2.10, Theorem 1.4]) *Let X be a normal projective surface and $f\colon X \to X$ a surjective endomorphism with $\deg f > 1$. Let $C \subset X$ be a reduced effective divisor such that $f^{-1}(C) = C$. Then (X, C) is an lc $\mathbb{Q}$-Gorenstein pair and any lc center of (X, C) is not contained in $\operatorname{Supp} R_f$ and totally invariant if we replace f by a suitable power f^n*

By setting $C = 0$, we get the following.

Lemma 3.2. *Let X be normal projective surface and $f\colon X \to X$ a surjective endomorphism with $\deg f > 1$. Then X is $\mathbb{Q}$-Gorenstein lc and any lc center of X is not contained in $\operatorname{Supp} R_f$ and totally invariant if we replace f by a suitable power f^n.*

We recall basic properties and fundamental theorems on int-amplified endomorphisms.

Lemma 3.3. (1) *Let X be a normal projective variety, $f\colon X \to X$ a surjective morphism, and $n > 0$ a positive integer. Then f is int-amplified if and only if so is f^n.*

(2) *Let $\pi\colon X \to Y$ be a surjective morphism between normal projective varieties. Let $f\colon X \to X$, $g\colon Y \to Y$ be surjective endomorphisms such that $\pi \circ f = g \circ \pi$. If f is int-amplified, then so is g.*

(3) *Let $\pi\colon X \dashrightarrow Y$ be a dominant rational map between normal projective varieties of same dimension. Let $f\colon X \to X$, $g\colon Y \to Y$ be surjective endomorphisms such that $\pi \circ f = g \circ \pi$. Then f is int-amplified if and only if so is g.*

(4) *f is int-amplified if and only if all the eigenvalues of $f^*\colon N^1(X) \to N^1(X)$ have modulus greater than one. Here $N^1(X)$ is the group of Cartier divisors on X modulo numerical equivalence.*

Proof. See [11, Theorem 1.1 and Lemmas 3.3, 3,5, 3.6]. □

Lemma 3.4. [11, Theorem 1.5] *Let X be a normal $\mathbb{Q}$-Gorenstein projective variety and $f\colon X \to X$ an int-amplified endomorphism. Then $-K_X$ is numerically equivalent to an effective $\mathbb{Q}$-Cartier divisor.*

Proposition 3.5. [11, Theorem 5.2] *Let X be a normal $\mathbb{Q}$-Gorenstein projective variety and $f\colon X \to X$ an int-amplified endomorphism. If K_X is pseudo-effective, then $K_X \sim_{\mathbb{Q}} 0$. If, moreover, X is klt, then X is a Q-abelian variety, there exists a quasi-étale finite morphism $A \to X$ from an abelian variety A and some power f^n of f lifts to a self-morphism of A.*

The following easy lemma makes MMP equivariant under certain endomorphisms.

Lemma 3.6. *Let X be an lc projective variety and $f\colon X \to X$ a surjective endomorphism. Let $R \subset \overline{NE}(X)$ be a K_X-negative extremal ray and $\pi\colon X \to Y$ the contraction of R. Suppose $f_*R = R$. Then there exists a surjective endomorphism $Y \to Y$ such that*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Y \arrow[r] & Y. \end{tikzcd}$$

Proof. This is true because the contraction is determined by the ray. □

We will use the following lemma to prove kltness.

Lemma 3.7. *Consider the following commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ C \arrow[r, "g"'] & C \end{tikzcd}$$

where X is a normal projective surface, C is a smooth projective curve, f is an int-amplified endomorphism, g is an endomorphism and π is a surjective morphism with connected fibers. Then X is klt.

Proof. By Lemma 3.2, X is $\mathbb{Q}$-Gorenstein lc and we may assume an lc center P of X is totally invariant under f. Then $\pi(P)$ is totally invariant and the fibre F of P is also totally invariant. In particular, since $F_{\text{red}} \leq R_f$, we have $P \in \operatorname{Supp}(R_f)$, but this contradicts to Lemma 3.2. $\square$

4. Int-amplified endomorphisms on two dimensional Mori fiber spaces

Proposition 4.1. *Consider the following commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ C \arrow[r, "g"'] & C \end{tikzcd}$$

where X is a $\mathbb{Q}$-Gorenstein lc projective surface, f is an int-amplified endomorphism, π is a K_X-negative extremal ray contraction to a smooth projective curve C and g is an endomorphism. Then

(1) *C is isomorphic to $\mathbb{P}^1$ or an elliptic curve;*

(2) *If C is an elliptic curve, then f does not have non-empty totally invariant finite set and X is smooth;*

(3) *If $C \simeq \mathbb{P}^1$ and $-K_X$ is not big, then f does not have non-empty totally invariant finite set.*

Proof. (1) Since f is int-amplified, so is g and that means $\deg g > 1$. This implies C is isomorphic to $\mathbb{P}^1$ or an elliptic curve.

(2) Suppose C is an elliptic curve. Then g is an étale non-isomorphic morphism, and therefore g and its iterates have no totally invariant points. Thus f also does not have non-empty totally invariant finite set. By Lemma 3.7, X is klt and $\mathbb{Q}$-factorial.

If π has a singular fiber, it is not generically reduced. Indeed, if a fiber F of π is generically reduced, it is integral. (It is irreducible because π is a Mori fiber space over a curve and X is $\mathbb{Q}$-factorial. Every fiber of π is Cohen-Macaulay and thus it is reduced if generically reduced.) Since π is flat and general fibers are $\mathbb{P}^1$, the arithmetic genus of F is zero and this implies $F \simeq \mathbb{P}^1$. This is a contradiction.

Assume π has a singular fiber $F = \pi^* P$. Since g is étale, $(g^n)^* P$ is a reduced divisor but every coefficient of $\pi^*(g^n)^* P = (f^n)^* F$ is greater than one for any n. This implies there are infinitely many singular fibers of π, but this is absurd. Thus all fibers of π are regular and therefore X is smooth.

(3) Note that by Lemma 3.7, X is klt and $\mathbb{Q}$-factorial. Assume f admits a totally invariant finite set. Replacing f by its iterate, we may assume f has a totally invariant point. Since $-K_X$ is not big and the Picard number of X is two, $-K_X$ generates an extremal ray of the pseudo-effective cone $\overline{\mathrm{Eff}}(X)$. Another ray is generated by the fiber class F. Since F is preserved under f^*, $-K_X$ is also preserved and we write $f^*(-K_X) \equiv q(-K_X)$ where q is an integer greater than one (cf. Lemma 3.3(4)). Then $R_f \equiv K_X - f^* K_X \equiv (q-1)(-K_X)$, i.e., R_f generates the extremal ray different than the one generated by F. Now the reduced fiber containing the totally invariant point is contained in the support of R_f. This is a contradiction. $\square$

Lemma 4.2. *Consider the following commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "g"'] & \mathbb{P}^1 \end{tikzcd}$$

where X is a klt projective surface, f is an int-amplified endomorphism, π is a K_X-negative extremal ray contraction and g is an endomorphism. Let R_f be the ramification divisor of f. If $\kappa(-K_X) = 0$, then $f^*(-K_X) \sim_{\mathbb{Q}} q(-K_X)$ for some integer $q > 1$, $(R_f)_{\mathrm{red}} =: C$ is a smooth irreducible curve and the following holds:

- $C \sim_{\mathbb{Q}} -K_X$;
- $f^{-1}(C) = C$ *as sets;*
- $R_f = (q-1)C$ *as Weil divisors.*

Proof. Note that X is $\mathbb{Q}$-factorial since it is a klt surface. Moreover, $\mathrm{Pic}(X)_{\mathbb{Q}} \simeq N^1(X)_{\mathbb{Q}}$ since X is rational. Let $\overline{\mathrm{Eff}}(X) = \overline{NE}(X) = \mathbb{R}_{\geq 0} F + \mathbb{R}_{\geq 0} v$ where F is the fiber class. Note that we have

$$f^* F = \deg g F, \quad f^* v = qv$$

for some $q \in \mathbb{R}_{>1}$. By Lemma 3.4, we can write $-K_X = aF + bv$ in $N^1(X)_{\mathbb{R}}$ for some $a, b \geq 0$. Since π is a K_X-negative contraction, we have $0 < (-K_X \cdot F) = b(v \cdot F)$. This implies $b > 0$ and $(v \cdot F) > 0$. Therefore, $a = 0$. Indeed, if $a > 0$, $-K_X$ is contained in the interior of $\overline{\mathrm{Eff}}(X)$ and it means $-K_X$ is big, which contradicts to our assumption. Thus $-K_X$ generates an extremal ray, q is an integer, and $f^*(-K_X) \sim_{\mathbb{Q}} q(-K_X)$.

Now, since $R_f \sim K_X - f^*K_X \sim_{\mathbb{Q}} (q-1)(-K_X)$, $\kappa(R_f) = 0$ and R_f generate the extremal ray of $\overline{\mathrm{Eff}}(X)$. This implies R_f is irreducible. Set $C = (R_f)_{\mathrm{red}}$. Since $f^*R_f \sim_{\mathbb{Q}} qR_f$ and $\kappa(R_f) = 0$, $f^{-1}(R_f) = R_f$ (in other words, $f^{-1}(C) = C$) as sets. Thus, by the definition of the ramification divisor, $R_f = (q-1)C$. From this, we get $-K_X \sim_{\mathbb{Q}} C$.

Now we apply Lemma 3.1. Since f does not ramify along fibers, there is no totally invariant finite set. Thus, by Lemma 3.1, (X, C) has no lc center. Then (X, C) is plt, in particular, C is normal (cf. [10, Proposition 5.51]). $\square$

Proposition 4.3. *Consider the following commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "g"'] & \mathbb{P}^1 \end{tikzcd}$$

where X is a klt projective surface, f is an int-amplified endomorphism, π is a K_X-negative extremal ray contraction and g is an endomorphism. Let R_f be the ramification divisor of f. If $\kappa(-K_X) = 0$, then $(R_f)_{\mathrm{red}} =: C$ is an elliptic curve. Moreover, let $X' = X \times_{\mathbb{P}^1} C$ and $\widetilde{X}$ be the normalization of X'_{red}. Then

- *$\widetilde{X}$ is smooth;*
- *the projection $\widetilde{\pi}\colon \widetilde{X} \to C$ is a Fano contraction of a $K_{\widetilde{X}}$ negative extremal ray (i.e., $\widetilde{X}$ is a minimal ruled surface over C);*
- *the finite morphism $h\colon \widetilde{X} \to X$ is quasi-étale of degree 2;*
- *f induces an int-amplified endomorphism on $\widetilde{X}$:*

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[rr, bend left, "h"] \arrow[dr, "\widetilde{\pi}"'] & X' \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ & C \arrow[r] & \mathbb{P}^1. \end{tikzcd}$$

Proof. We use the notation in Lemma 4.2. The restriction of f on C has degree larger than one, so C is isomorphic to $\mathbb{P}^1$ or an elliptic curve. Note that $\pi|_C\colon C \to \mathbb{P}^1$ is a double cover. To see this, let F be a general fiber of π. Then by Lemma 4.2 and the adjunction formula, we have $(F \cdot C) = -(F \cdot K_X) = (F^2) - (2p_a(F) - 2) = 2$. Here $p_a(F) = 0$ since the generic fiber of π is the projective line.

Step 1. We assume $C \simeq \mathbb{P}^1$ and deduce contradiction. Form the following commutative diagram:

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[dr, "\widetilde{\pi}"'] \arrow[rr, bend left, "h"] & X' \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ & C \arrow[r, "\pi|_C"'] & \mathbb{P}^1 \end{tikzcd}$$

where $X' = X \times_C \mathbb{P}^1$ and $\widetilde{X}$ is the normalization of $(X')_{\mathrm{red}}$. Since f induces an endomorphism of C, it induces an endomorphism $\widetilde{f}$ on $\widetilde{X}$ which is int-amplified. By Lemma 3.7, $\widetilde{X}$ is klt. By Lemma 3.4, $-K_{\widetilde{X}}$ is $\mathbb{Q}$-linearly effective (note that $\widetilde{X}$ is rational since general fibers of $\widetilde{\pi}$ is rational). Let R_h be the ramification divisor of h. Then, by pushing the ramification formula by h, we get

$$(\deg h)(-K_X) \sim h_*(-K_{\widetilde{X}}) + h_* R_h.$$

Since $\kappa(-K_X) = 0$ and $-K_X \sim_{\mathbb{Q}} C$, we get $\operatorname{Supp} R_h \subset h^{-1}(C)$. Note that h is not ramified along horizontal divisors (i.e., divisors whose image by $\widetilde{\pi}$ is equal to C) since h is the base change of generically étale morphism $\pi|_C$ over an open subset of $\mathbb{P}^1$. Thus $R_h = 0$ and h is quasi-étale. Then we get $R_{\widetilde{f}} = h^* R_f$ where $R_{\widetilde{f}}$ is the ramification divisor of $\widetilde{f}$. In particular, $\widetilde{f}$ does not ramify along curves contracted by $\widetilde{\pi}$. This implies $\widetilde{f}$ does not have totally invariant finite set. Indeed, if there is a totally invariant finite set $S \subset \widetilde{X}$, then $\widetilde{\pi}(S)$ is totally invariant under $f|_C$. Since $f|_C$ is not an isomorphism, $f|_C$ is branched over $\widetilde{\pi}(S)$ and thus $\widetilde{f}$ is ramified along fibers over $\widetilde{\pi}(S)$, which we just show does not happen. Moreover, any curve which is contracted by $\widetilde{\pi}$ is $K_{\widetilde{X}}$-negative since $K_{\widetilde{X}} \sim h^* K_X$ and h is finite. If the contraction of one of such curves is a divisorial contraction, then the contraction is equivariant with respect to some iterate of $\widetilde{f}$ by [12, Theorem 4.6]. Then the contracted curve must be totally invariant under (some iterate of) $\widetilde{f}$ and the image of it by $\widetilde{\pi}$ is a totally invariant point of some iterate of $f|_C$. This is absurd because $R_{\widetilde{f}}$ is horizontal. Therefore, $\widetilde{\pi}$ is a Fano contraction (Note $K_{\widetilde{X}} \sim h^* K_X$ is not nef over C). Since $h_*(-K_{\widetilde{X}}) \sim \deg h(-K_X)$ and $\kappa(-K_X) = 0$, $\kappa(-K_{\widetilde{X}}) = 0$. Now, we can apply Lemma 4.2 to $\widetilde{X}$ and $\widetilde{f}$, and it says $R_{\widetilde{f}}$ is irreducible. But $\operatorname{Supp} R_h = h^{-1}(C)$ is not irreducible since $\pi|_C \colon C \to \mathbb{P}^1$ has degree two. This is a contradiction.

Step 2. Now we assume C is an elliptic curve. Form the following commutative diagram as in Step 1:

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[rr, bend left, "h"] \arrow[dr, "\widetilde{\pi}"'] & X' \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ & C \arrow[r, "\pi|_C"'] & \mathbb{P}^1. \end{tikzcd}$$

Since $\pi|_C$ is a double cover, h has degree 2. As in Step 1, f induces an int-amplified endomorphism $\widetilde{f}$ on $\widetilde{X}$ and $\widetilde{X}$ is $\mathbb{Q}$-Gorenstein lc. Consider the following equations:

$$R_h + h^*R_f = R_{\widetilde{f}} + \widetilde{f}^*R_h, \tag{4.1}$$

$$h^*R_f = (q-1)h^*C, \tag{4.2}$$

$$\widetilde{f}^*h^*C = h^*f^*C = qh^*C. \tag{4.3}$$

By construction, h^*C has two components and each coefficient is 1. By (4.3), $\widetilde{f}$ is ramified along each component of h^*C with ramification index q. Thus we have $R_{\widetilde{f}}-(q-1)h^*C \geq 0$. By (4.2), $R_{\widetilde{f}}-h^*R_f \geq 0$. By (4.1), $R_h - \widetilde{f}^*R_h \geq 0$, and this implies R_h is totally invariant under $\widetilde{f}$ as a set. Since h is not ramified along horizontal curves by construction, every component of R_h is contracted by $\widetilde{\pi}$. If $R_h \neq 0$, $f|_C$ has a non-empty totally invariant set. This is absurd because $f|_C$ is étale and not isomorphic. Therefore, we get $R_h = 0$, i.e., h is quasi-étale. Moreover, if there is a $K_{\widetilde{X}}$-negative extremal divisorial contraction, it is equivariant with respect to some iterate of $\widetilde{f}$ by [12, Theorem 4.6]. This implies there is a totally invariant point of some iterate of $f|_C$, but this is absurd. Thus $\widetilde{\pi}$ is a Fano contraction. By Proposition 4.1(2), $\widetilde{X}$ is smooth. □

Proposition 4.4. *Consider the following commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "g"'] & \mathbb{P}^1 \end{tikzcd}$$

where X is a klt projective surface with $\kappa(-K_X) = 1$, f is an int-amplified endomorphism, π is a K_X-negative extremal ray contraction and g is an endomorphism. Then there exists a positive integer n and an elliptic curve E on X such that $f^n(E) = E$ satisfying the following properties. Let $X' = X \times_{\mathbb{P}^1} E$ and $\widetilde{X}$ be the normalization of X'_{red}. Then

- *$\widetilde{X}$ is smooth;*
- *the projection $\widetilde{\pi}\colon \widetilde{X} \to E$ is a Fano contraction of a $K_{\widetilde{X}}$ negative extremal ray (i.e., $\widetilde{X}$ is a minimal ruled surface over E);*

- *the finite morphism* $h\colon \widetilde{X} \to X$ *is quasi-étale;*
- f^n *induces an int-amplified endomorphism on* $\widetilde{X}$*:*

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[rr, bend left, "h"] \arrow[dr, "\widetilde{\pi}"'] & X' \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ & E \arrow[r, "\pi|_E"'] & \mathbb{P}^1. \end{tikzcd}$$

Proof. Since $-K_X$ is not big, $-K_X$ generates the extremal ray of $\overline{\mathrm{Eff}}(X)$ other than the one generated by the fiber class of π (cf. the proof of Lemma 4.2). Therefore, we can show $(-K_X)^2 \geq 0$. Since the other extremal ray is K_X-negative, we get $(-K_X)^2 = 0$ (otherwise, $-K_X$ is ample, but $\kappa(-K_X) = 1$). Moreover, $-K_X$ is semi-ample because it is $\mathbb{Q}$-linearly equivalent to at least two irreducible effective divisors and has self-intersection 0. Let $\mu\colon X \to \mathbb{P}^1$ be the morphism defined by $-mK_X$ for sufficiently divisible m. Since f preserves the ray $\mathbb{R}_{\geq 0}(-K_X)$, it induces a non-invertible endomorphism $g'\colon \mathbb{P}^1 \to \mathbb{P}^1$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mu"'] & X \arrow[d, "\mu"] \\ \mathbb{P}^1 \arrow[r, "g'"'] & \mathbb{P}^1 \end{tikzcd}$$

is commutative.

Since g' is non-isomorphic, it has infinitely many periodic points (cf. [5]). General fibers of μ are elliptic curves because $(K_X)^2 = 0$. Thus, if we replace f by a suitable power, we may assume there exists a point $P \in \mathbb{P}^1$ such that $g'(P) = P$ and $\mu^{-1}(P) =: E$ is an elliptic curve.

Consider the following diagram:

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[rr, bend left, "h"] \arrow[dr, "\widetilde{\pi}"'] & X' \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ & E \arrow[r, "\pi|_E"'] & \mathbb{P}^1. \end{tikzcd}$$

where $X' = X \times_{\mathbb{P}^1} E$ and $\widetilde{X}$ is the normalization of $(X')_{\mathrm{red}}$. Since E is preserved by f, it induces an int-amplified endomorphism $\widetilde{f}$ on $\widetilde{X}$. Therefore, $\widetilde{X}$ is $\mathbb{Q}$-Gorenstein klt by Lemma 3.7.

First, we prove h is quasi-étale. Let R_h be the ramification divisor and fix a canonical divisor $K_{\widetilde{X}}$ of $\widetilde{X}$ so that $-h^*K_X = -K_{\widetilde{X}} + R_h$. By Lemma 3.4, there exists an effective

$\mathbb{Q}$-Cartier divisor D on $\widetilde{X}$ such that $D \equiv -K_{\widetilde{X}}$. Then we get $-(\deg h)K_X \equiv h_*R_h + h_*D$. For any fiber E' of μ, we have $0 = (-(\deg h)K_X \cdot E') = (h_*R_h \cdot E') + (h_*D \cdot E')$. Since E' is nef and h_*R_h, h_*D are effective, we get $(h_*R_h \cdot E') = 0$. Therefore, h_*R_h has no irreducible component that is contained in a fiber of π. Since h is finite, R_h also has no irreducible component that is contained in a fiber of $\widetilde{\pi}$. By the construction of h, R_h has no $\widetilde{\pi}$-horizontal component, and hence we get $R_h = 0$.

By the same argument as in the last part of the proof of Proposition 4.3, $\widetilde{\pi}$ is a Fano contraction and $\widetilde{X}$ is smooth. □

5. Int-amplified endomorphisms on surfaces with big anti-canonical divisor

Lemma 5.1. (cf. [3, Theorem 5.5]) *Let X be a normal $\mathbb{Q}$-factorial rational projective surface with $-K_X$ is big. Then X is a Mori dream space.*

Proof. Take the minimal resolution $\nu\colon Y \to X$. Then, by negativity lemma, we have $-K_Y = -\nu^*K_X + E$ where E is a ν-exceptional effective divisor. In particular, $-K_Y$ is also big. Since Y is rational, Y is a Mori dream space by [16, Theorem 1]. By [15, Theorem 1.1], X is also a Mori dream space. □

Lemma 5.2. *Let X be a normal projective surface. Let $f\colon X \to X$ be an int-amplified endomorphism. Suppose we have the following f-equivariant MMP:*

$$X = X_1 \to \cdots \to X_r$$

where $X_i \to X_{i+1}$ is the divisorial contraction of a K_{X_i}-negative extremal ray for $i = 1, \ldots, r-1$. If $-K_{X_r}$ is big and X_r is $\mathbb{Q}$-factorial, then $-K_X$ is also big.

Proof. Let $\nu\colon X \to X_r$ be the composite of the divisorial contractions, then the all exceptional divisors $E_1, \ldots, E_{r-1}$ of ν are totally invariant and $E_i \leq R_f$ for all i since f is int-amplified (cf. [11, Lemma 3.11]). Write $-K_X \sim_{\mathbb{Q}} \nu^*(-K_{X_r}) + E$ where $E = \sum_{i=1}^{r-1} a_i E_i$. By the ramification formula, we get $(f^n)^*(-K_X) \sim -K_X + (f^{n-1})^* R_f + \cdots + R_f$ for $n > 0$. Since E_i are components of R_f and totally invariant under f, $E + (f^{n-1})^* R_f + \cdots + R_f$ is effective for large n. Therefore, the divisor

$$(f^n)^*(-K_X) \sim_{\mathbb{Q}} \nu^*(-K_{X_r}) + E + (f^{n-1})^* R_f + \cdots + R_f$$

is big and hence so is $-K_X$. □

Proposition 5.3. (cf. [3, Theorem 5.1]) *Let X be a normal projective surface. Let $f\colon X \to X$ be an int-amplified endomorphism. Suppose we can run f-equivariant MMP:*

$$X = X_1 \to \cdots \to X_r \to C$$

where

- *$X_i \to X_{i+1}$ is the divisorial contraction of a K_{X_i}-negative extremal ray for $i = 1, \dots, r-1$;*
- *$X_r \to C$ is the Fano contraction of a K_{X_r}-negative extremal ray;*
- *C is a projective line or a point.*

Suppose $-K_{X_r}$ is big. If C is a projective line, then X is a Mori dream space. If C is a point, then X is a Mori dream space or a projective cone of an elliptic curve.

Proof. If C is a projective line, then X is klt by Lemma 3.7. In particular X is $\mathbb{Q}$-factorial, and X is a Mori dream space by Lemmas 5.1 and 5.2.

If C is a point, then X is a projective cone of an elliptic curve or rational surface with rational singularities by the last part in the proof of [3, Theorem 5.1]. If X has rational singularities, then X is $\mathbb{Q}$-factorial by [1, Theorem 4.6] and a Mori dream space by Lemmas 5.1 and 5.2. □

6. Proof of the main theorem

Proof of Theorem 1.2. Let $f: X \to X$ be an int-amplified endomorphism of normal projective surface. By Lemma 3.2, X is $\mathbb{Q}$-Gorenstein lc. By [9, Theorem 2.3.6], we can run a MMP for X. By [12, Theorem 4.6] and Lemma 3.6, if we replace f by a suitable power, every K_X-negative extremal ray contraction is f-equivariant and the induced morphism on the target is also int-amplified (Lemma 3.3). Therefore, we can repeat this process and get

$$X = X_1 \to \cdots \to X_r \to C$$

where

- $p_i: X_i \to X_{i+1}$ is the divisorial contraction of a K_{X_i}-negative extremal ray for $i = 1, \dots, r-1$;
- K_{X_r} is nef and ignore "$\to C$" for this case, or $X_r \to C$ is the Fano contraction of a K_{X_r}-negative extremal ray.

By replacing f by its iterate, we assume f induces int-amplified endomorphisms f_i on X_i.

(1) When K_X is pseudo-effective, then by Lemma 3.4, $K_X \equiv 0$. By [7, Theorem 1.2], $K_X \sim_{\mathbb{Q}} 0$ and in particular, $r = 1$. By [14, Theorem A], X is a Q-abelian variety.

(2) When K_X is not pseudo-effective, then the out put of MMP must be a Fano contraction (cf. [2, Corollary 1.1.7]). Note that $p_i(\mathrm{Exc}(p_i))$ is a non-empty finite set totally invariant under f_{i+1}.

(a) If C is an elliptic curve, by Proposition 4.1(2), f_r admits no totally invariant finite set. Therefore, $r = 1$ and by Proposition 4.1(2) again, $X = X_1$ is smooth.

(b) If $C \simeq \mathbb{P}^1$ and $\kappa(-K_{X_r}) = 0$, then $r = 1$ by Lemma 4.1(3) and X is klt by Lemma 3.7. By Proposition 4.3, we get a desired quasi-étale cover as in the statement.

(c) If $C \simeq \mathbb{P}^1$ and $\kappa(-K_{X_r}) = 1$, then $r = 1$ by Lemma 4.1(3) and $X = X_1$ is klt by Lemma 3.7. By Proposition 4.4, we get a desired quasi-étale cover as in the statement.

(d) If $C \simeq \mathbb{P}^1$ and $\kappa(-K_{X_r}) = 2$, then X is klt by Lemma 3.7 and hence X is a Mori dream space by Proposition 5.3.

(e) If C is a point, then X_r has Picard number one and $-K_{X_r}$ is ample. By Proposition 5.3, X is a Mori dream space or a projective cone of an elliptic curve. □

7. Examples

Proposition 7.1. *The cases* (3) *and* (4) *in Theorem* 1.2 *occur.*

Let E be an elliptic curve. We write $[m]\colon E \to E$ the multiplication by m map for every integer m. Take an invertible $\mathcal{O}_E$-module $\mathcal{L}$ with $\deg \mathcal{L} = 0$. Consider the projective bundle $p\colon Y = \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}) \to E$.

Lemma 7.2. (1) *For any isomorphism $\varphi\colon [-1]^*\mathcal{L} \to \mathcal{L}^{-1}$, we have*

$$\begin{tikzcd}
{[-1]^*([-1]^*\mathcal{L})} \arrow[r] \arrow[d, "{[-1]^*\varphi}"'] & {([-1]\circ[-1])^*\mathcal{L}} \arrow[rr] & & \mathcal{L} \\
{[-1]^*(\mathcal{L}^{-1})} \arrow[r] & {([-1]^*\mathcal{L})^{-1}} \arrow[rr, "\varphi^\vee"'] & & (\mathcal{L}^{-1})^{-1} \arrow[u]
\end{tikzcd}$$

commutative, where unlabeled arrows are canonical isomorphisms.

(2) *Let $n > 1$ an integer. For every isomorphism $\varphi\colon [-1]^*\mathcal{L} \to \mathcal{L}^{-1}$, there exists an isomorphism $\psi\colon [n]^*\mathcal{L} \to \mathcal{L}^n$ such that the following diagram is commutative:*

$$\begin{tikzcd}
{[-1]^*([n]^*\mathcal{L})} \arrow[r, "{[-1]^*\psi}"] \arrow[d] & {[-1]^*(\mathcal{L}^n)} \arrow[r] & {([-1]^*\mathcal{L})^n} \arrow[r, "\varphi^{\otimes n}"] & (\mathcal{L}^{-1})^n \arrow[d] \\
{[n]^*([-1]^*\mathcal{L})} \arrow[r, "{[n]^*\varphi}"'] & {[n]^*(\mathcal{L}^{-1})} \arrow[r] & {([n]^*\mathcal{L})^{-1}} \arrow[r, "\psi^\vee"'] & (\mathcal{L}^n)^{-1}
\end{tikzcd}$$

where unlabeled arrows are canonical isomorphisms.

Proof. We may assume $\mathcal{L} = \mathcal{O}_E(x - 0)$ where $0 \in E$ is the identity and $x \in E$ is a closed point. Take any non-zero rational functions f, g on E so that

$$\begin{aligned} [-1]^*(x-0) &= -(x-0) + \operatorname{div} f, \\ [n]^*(x-0) &= n(x-0) + \operatorname{div} g. \end{aligned}$$

(1) We can reduce to prove that $([-1]^*f)/f = 1$. This function is constant by definition. Take a two torsion point $z \in E \setminus \{0, x, y\}$, where $y \in E$ is the inverse element of x. Then $(([-1]^*f)/f)(z) = f(z)/f(z) = 1$.

(2) We can reduce to find g such that

$$\frac{[n]^*f}{f^n g[-1]^*g} = 1.$$

The left hand side is a constant, say a, by the definition of f and g. Replace g by $\sqrt{a}g$. Then f and $\sqrt{a}g$ satisfy the desired formula. $\square$

Let $n > 1$ be an integer. Fix two isomorphisms $\varphi\colon [-1]^*\mathcal{L} \to \mathcal{L}^{-1}$, $\psi\colon [n]^*\mathcal{L} \to \mathcal{L}^n$ as in Lemma 7.2(2).

Consider the following diagram:

$$\begin{tikzcd}
Y \arrow[r, "\alpha"'] \arrow[dr, "p"'] \arrow[rrr, bend left, "F"] & \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^n) \arrow[r, "\Psi"] \arrow[d] & \mathbb{P}(\mathcal{O}_E \oplus [n]^*\mathcal{L}) \arrow[r, "\beta"'] \arrow[dl, "{[n]^*p}"] & Y \arrow[d, "p"] \\
 & E \arrow[rr, "{[n]}"'] & & E
\end{tikzcd}$$

where $[n]^*p$ is the base change of p by $[n]$, β is the projection, Ψ is the isomorphism over E induced by ψ, and α is the morphism over E defined by the canonical inclusion $\mathcal{O}_E \oplus \mathcal{L}^n \to \mathrm{Sym}^n(\mathcal{O}_E \oplus \mathcal{L})$. Define $F\colon Y \to Y$ to be the composite $F = \beta \circ \Psi \circ \alpha$. Note that F is an int-amplified endomorphism.

Similarly, consider the following diagram:

$$\begin{tikzcd}
Y \arrow[r, "\iota"'] \arrow[dr, "p"'] \arrow[rrr, bend left, "\sigma"] & \mathbb{P}(\mathcal{O}_E \oplus \mathcal{L}^{-1}) \arrow[r, "\Phi"] \arrow[d] & \mathbb{P}(\mathcal{O}_E \oplus [-1]^*\mathcal{L}) \arrow[r, "\gamma"'] \arrow[dl, "{[-1]^*p}"] & Y \arrow[d, "p"] \\
 & E \arrow[rr, "{[-1]}"'] & & E
\end{tikzcd}$$

where $[-1]^*p$ is the base change of p by $[-1]$, γ is the projection, Φ is the isomorphism over E induced by φ, and ι is the isomorphism over E induced by $\mathcal{O}_E \oplus \mathcal{L} \simeq \mathcal{L} \oplus \mathcal{O}_E \simeq (\mathcal{O}_E \oplus \mathcal{L}^{-1}) \otimes \mathcal{L}$. Define $\sigma\colon Y \to Y$ to be the composite $\sigma = \gamma \circ \Phi \circ \iota$. Then, by Lemma 7.2(1), we get $\sigma \circ \sigma = \mathrm{id}$. By Lemma 7.2(2), we get $F \circ \sigma = \sigma \circ F$. (By taking base changes, reduce to equations of morphisms between projective bundles over a common base and use Lemma 7.2.)

Let $X := Y/\langle\sigma\rangle$ be the quotient of Y by the involution σ. Then X is a projective klt surface and we get the following commutative diagram:

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "p"'] & X \arrow[d, "\pi"] \\ E \arrow[r] & E/\langle[-1]\rangle \simeq \mathbb{P}^1 \end{tikzcd}$$

where the horizontal arrows are quotient morphisms and π is the induced morphism by p. Note that h is quasi-étale since the set of fixed points of σ is finite. Since $F \circ \sigma = \sigma \circ F$, F descends to an int-amplified endomorphism $f\colon X \to X$. Also, $[n]\colon E \to E$ induces an endomorphism $g\colon \mathbb{P}^1 \to \mathbb{P}^1$ and the above diagram is equivariant under these endomorphisms.

We have $h^*K_X \sim K_Y$ because h is quasi-étale. Therefore, π is a K_X-negative extremal ray contraction and $\kappa(-K_X) = \kappa(-K_Y)$. Moreover, $\kappa(-K_Y) = 0$ if $\mathcal{L}$ is non-torsion in $\mathrm{Pic}^0(E)$ and $\kappa(-K_Y) = 1$ if $\mathcal{L}$ is torsion. The morphism $f\colon X \to X$ is an example of the case Theorem 1.2(3) or (4) depending on whether $\mathcal{L}$ is non-torsion or not.

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