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Traveling Waves for a Discrete Diffusion Epidemic Model with Delay

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Abstract. This paper is concerned with traveling wave solutions in a discrete diffusion epidemic model with delayed transmission. Employing the way of contradictory discussions and the bilateral Laplace transform, we obtain the nonexistence of nontrivial positive bounded traveling wave solutions. Utilizing the super-/sub-solutions method and the fixed point theory, we derive the existence of nontrivial positive traveling wave solutions with both super-critical and critical speeds. Our results indicate that the critical speed is the minimal speed.

1. Introduction

Considering the environment which individuals live in can be divided into countably discrete niches and the influence of latent period of the disease, we investigate a discrete diffusion epidemic model with delay

$$(1.1)\qquad \begin{cases} \frac{dS_n(t)}{dt} = d_s[S_{n+1}(t) + S_{n-1}(t) - 2S_n(t)] - \frac{\beta S_n(t)I_n(t-\tau)}{S_n(t)+I_n(t-\tau)+R_n(t)}, \\ \frac{dI_n(t)}{dt} = d_i[I_{n+1}(t) + I_{n-1}(t) - 2I_n(t)] + \frac{\beta S_n(t)I_n(t-\tau)}{S_n(t)+I_n(t-\tau)+R_n(t)} - (\gamma+\delta)I_n(t), \\ \frac{dR_n(t)}{dt} = d_r[R_{n+1}(t) + R_{n-1}(t) - 2R_n(t)] + \gamma I_n(t), \quad n \in \mathbb{Z}, \end{cases}$$

where $S_n(t)$, $I_n(t)$ and $R_n(t)$ refer to the densities of susceptible, infected and recovered individuals in time t and niches n, respectively. The coefficients $d_s, d_i, d_r > 0$ denote the diffusion rates of each class, $\beta > 0$ stands for the transmission rate, $\gamma > 0$ represents the recovery rate, $\delta \geq 0$ is the disease-induced death rate and $\tau \geq 0$ is the latent period. Model (1.1) with standard incidence $\beta SI/(S+I+R)$ describes that individuals can move freely in a patchy habitat and a part of infected individuals will be removed from the community due to disease-induced death, while other recovered individuals will return into the population, which capture the dynamical behavior of disease propagation.

In mathematical biology, traveling wave solutions can describe the phase transition that an epidemic transmits geographically with a constant speed from the initial state to

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the final state. The main purpose in this paper is to explore the existence and the minimal speed of traveling wave solutions for (1.1). By a traveling wave solution with a constant speed c of (1.1), we mean that it is in the form of

$$(S_n, I_n, R_n)(t) = (S, I, R)(\xi), \quad \xi = n + ct,$$

which satisfies the following ordinary differential system

$$\begin{cases} cS'(\xi) = d_s[S(\xi+1) + S(\xi-1) - 2S(\xi)] - \frac{\beta S(\xi) I(\xi - c\tau)}{S(\xi)+I(\xi-c\tau)+R(\xi)}, \\ cI'(\xi) = d_i[I(\xi+1) + I(\xi-1) - 2I(\xi)] + \frac{\beta S(\xi) I(\xi - c\tau)}{S(\xi)+I(\xi-c\tau)+R(\xi)} - (\gamma+\delta)I(\xi), \\ cR'(\xi) = d_r[R(\xi+1) + R(\xi-1) - 2R(\xi)] + \gamma I(\xi), \quad \xi \in \mathbb{R}, \end{cases} \tag{1.2}$$

with the asymptotic boundary conditions

$$(S, I, R)(-\infty) = (S_1, 0, 0) \quad \text{and} \quad (S, I, R)(+\infty) = (S_2, 0, \gamma(S_1 - S_2)/(\gamma+\delta)),$$

where $S_1 > 0$ is a given constant and the constant $S_2 \in [0, S_1)$ will be proved to exist.

In the last several decades, many theoretical issues concerning local-diffusion (or nonlocal-diffusion) epidemic models with (or without) time delay have been attracted considerable attention. Particularly, the existence and nonexistence of traveling waves for these models have been well-studied because these information can forecast whether or not an epidemic transmit in the crowd and how fast the epidemic invades geographically. In recent years, Wang et al. [20] studied a local-diffusion epidemic model

$$\begin{cases} \partial_t S = d_1 \partial_{xx} S - \beta SI/(S+I+R), \\ \partial_t I = d_2 \partial_{xx} I + \beta SI/(S+I+R) - (\gamma+\delta)I, \\ \partial_t R = d_3 \partial_{xx} R + \gamma I, \end{cases} \tag{1.3}$$

where $S(x,t)$, $I(x,t)$ and $R(x,t)$ are the densities of susceptible, infected and recovered individuals in location x and time t, respectively. For the biological interpretation of model (1.3) and its coefficients, one can refer to [20]. They proved that if $R_0 = \beta/(\gamma+\delta) > 1$, $c > c^* = 2\sqrt{d_2(\beta-\gamma-\delta)}$ and $d_3 < 2d_2$, then (1.3) has a nontrivial nonnegative traveling wave solution satisfying $S(-\infty) := S_{-\infty} > S(+\infty) := S_\infty \geq 0$, $I(\pm\infty) = 0$, $R(-\infty) = 0$ and $R(+\infty) = \gamma(S_{-\infty} - S_\infty)/(\gamma+\delta)$; if $0 < R_0 \leq 1$ or $c < c^*$, then (1.3) admits no nontrivial nonnegative traveling waves. In reality, latent period of many diseases seems to be inevitable. Removing the unnatural condition $d_3 < 2d_2$, He and Tsai [12] obtained the existence of nontrivial nonnegative traveling wave solutions with both super-critical and critical speeds for a discrete delayed version of (1.3). For the investigation of other local-diffusion epidemic systems, we suggest the readers to see [1, 6–8, 10, 13, 14, 22, 23, 25, 28, 32, 37–40, 43]. To describe a long range process in a spatially continuous environment, Yang et al. [34] explored the nonlocal version of system (1.3)

$$
\begin{cases}
\partial_t S = d_1(J * S - S) - \beta SI/(S + I + R), \\
\partial_t I = d_2(J * I - I) + \beta SI/(S + I + R) - (\gamma + \delta)I, \\
\partial_t R = d_3(J * R - R) + \gamma I,
\end{cases}
\tag{1.4}
$$

where "$*$" is the standard convolution with respect to spatial variable, $J \in C^1(\mathbb{R})$, $J(x) = J(-x) \geq 0$, $\int_{\mathbb{R}} J(y)\,dy = 1$ and J is compactly supported. They showed that when $R_0 = \beta/(\gamma + \delta) > 1$, there is a constant $c^* > 0$ such that for every $c > c^*$, system (1.4) admits a nontrivial nonnegative traveling wave solution with $S(-\infty) := S_{-\infty} > S(+\infty) := S_\infty \geq 0$, $I(\pm\infty) = 0$ and $R(-\infty) = 0$. Moreover, $R(+\infty) = \gamma(S_{-\infty} - S_\infty)/(\gamma + \delta)$ if $\limsup_{\xi \to +\infty} R(\xi) < +\infty$. When $0 < R_0 \leq 1$ or $0 < c < c^*$, system (1.4) has no nontrivial nonnegative traveling waves. Very recently, Wei et al. [25] investigated a nonlocal delayed version of model (1.4) and derived the existence of nontrivial positive traveling waves with super-critical and critical speeds. For more study of nonlocal diffusion epidemic systems, we refer to [2, 3, 9, 15–19, 21, 33, 35, 41, 44]. To study the nonlocal process in a spatially discrete environment, the present authors [26] proposed a two-component discrete diffusion epidemic model with delay

$$
\begin{cases}
\frac{dS_n(t)}{dt} = d_1[S_{n+1}(t) + S_{n-1}(t) - 2S_n(t)] - \frac{\beta S_n(t) I_n(t-\tau)}{S_n(t) + I_n(t-\tau)}, \\
\frac{dI_n(t)}{dt} = d_2[I_{n+1}(t) + I_{n-1}(t) - 2I_n(t)] + \frac{\beta S_n(t) I_n(t-\tau)}{S_n(t) + I_n(t-\tau)} - \gamma I_n(t), \quad n \in \mathbb{Z},
\end{cases}
\tag{1.5}
$$

and established the existence and nonexistence of traveling wave solutions for this system. Let us recall the proof strategy in [26]. Firstly, we constructed a pair of super-/sub-solutions on the real line and defined an invariant cone of a functional space with weighted norm by this pair of super-/sub-solutions. Secondly, we applied Schauder's fixed point theorem to prove that (1.5) has a traveling wave solution with super-critical speed. Thirdly, by analysis method we obtained the asymptotic boundary, positiveness and other properties of the traveling wave solution. Fourthly, utilizing the similar way as for the super-critical traveling wave solution, we still derived the critical traveling wave solution via another pair of super-/sub-solutions. Finally, for the nonexistence theorems, we mainly used the way of contradictory arguments and the bilateral Laplace transform to achieve the goal. The results in [26] are summarised as follows. If $\beta > \gamma$, then there is some constant $c^* > 0$ such that for each $c \geq c^*$, model (1.5) admits a nontrivial positive bounded traveling wave solution. If $\beta \leq \gamma$ or $c < c^*$, then (1.5) has no nontrivial positive bounded traveling wave solutions. For other progress of discrete diffusion epidemic models, see [5, 11, 24, 30, 36, 42].

We should point out that the difference-differential epidemic models in the existing references [5, 11, 24, 26, 30, 36, 42] are two-component systems, while (1.1) is indeed a three-component system and we need to overcome some difficulties. Due to the deficiency of monotonicity for (1.1), it is hard to obtain the exact boundaries of S-component and R-component at plus infinity. However, by analysis technique, we still derive the existence of the limits for S-component and R-component at plus infinity under the condition $\limsup_{\xi\to+\infty} R(\xi) < +\infty$. Because of the appearance of second order difference operators, it seems difficult to deduce a priori estimate of R-component, which is a key estimate for using the method of the bilateral Laplace transform to prove the nonexistence results. Herein, we construct a nonnegative bounded smooth cut-off function and make full use the structure of system (1.2) to obtain this a priori estimate.

Now we sketch our ideas and organization as follows. Section 2 is devoted to stating some preliminaries. In Section 3, we apply the reduction to absurdity together with the bilateral Laplace transform to establish the nonexistence of nontrivial positive bounded traveling wave solutions in (1.1). In Section 4, to explore the existence of a super-critical traveling wave solution in (1.1), we first construct a pair of super-/sub-solutions for (1.2); secondly, we introduce a convex cone Ω_X of initial functions defined in a large bounded closed interval $[-X, X]$, whose elements sandwich between super-solution and sub-solution; thirdly, we define a nonlinear operator $\mathcal{O}$ on Ω_X and present that $\mathcal{O}\colon \Omega_X \mapsto \Omega_X$ is completely continuous with respect to the supremum norm in $C([-X, X], \mathbb{R}^3)$; fourthly, we use Schauder's fixed point theorem on this cone to obtain the existence of a fixed point for $\mathcal{O}$, which guarantees that the existence of a solution for (1.2) on $[-X, X]$; fifthly, by a limiting method we extend the existence of the solution on $[-X, X]$ to the unbounded spatial domain $\mathbb{R}$; finally, by delicate analysis we show the positiveness and asymptotic boundaries of the traveling wave solutions. In Section 5, to investigate the existence of a critical traveling wave solution in (1.1), we construct another pair of super-/sub-solutions for (1.2) and utilize the analogous manner as for the super-critical traveling wave solution to reach our goal. Then we further deduce some properties concerning the traveling wave solutions.

2. Preliminaries

Let us start with the definition of the super-/sub-solutions for (1.2). In the sequel,

$$D[u](\xi) := u(\xi + 1) + u(\xi - 1) - 2u(\xi).$$

Definition 2.1. The nonnegative continuous function pairs $(S_+, I_+, R_+)(\xi)$ and $(S_-, I_-,$

$R_-)(\xi)$ are named as a pair of super-/sub-solutions for (1.2) if they satisfy

$$\begin{aligned} &d_sD[S_+](\xi) - cS'_+(\xi) - \frac{\beta S_+(\xi)I_-(\xi - c\tau)}{S_+(\xi) + I_-(\xi - c\tau) + R_+(\xi)} \le 0,\\ &d_iD[I_+](\xi) - cI'_+(\xi) + \frac{\beta S_+(\xi)I_+(\xi - c\tau)}{S_+(\xi) + I_+(\xi - c\tau) + R_-(\xi)} - (\gamma+\delta)I_+(\xi) \le 0,\\ &d_rD[R_+](\xi) - cR'_+(\xi) + \gamma I_+(\xi) \le 0,\\ &d_sD[S_-](\xi) - cS'_-(\xi) - \frac{\beta S_-(\xi)I_+(\xi - c\tau)}{S_-(\xi) + I_+(\xi - c\tau) + R_-(\xi)} \ge 0,\\ &d_iD[I_-](\xi) - cI'_-(\xi) + \frac{\beta S_-(\xi)I_-(\xi - c\tau)}{S_-(\xi) + I_-(\xi - c\tau) + R_+(\xi)} - (\gamma+\delta)I_-(\xi) \ge 0,\\ &d_rD[R_-](\xi) - cR'_-(\xi) + \gamma I_-(\xi) \ge 0, \end{aligned}$$

except for finitely many points on the whole real line.

Now we establish a couple of lemmas which will be utilized to prove our main results.

Lemma 2.2. *Let* $R_0 := \beta/(\gamma+\delta) > 1$ *and*

$$F(\rho, c) := d_i(e^{\rho} + e^{-\rho} - 2) - c\rho + \beta e^{-\rho c\tau} - \gamma - \delta.$$

Then there exists a pair of positive real numbers (ρ^*, c^*) *such that*

$$F(\rho^*, c^*) = d_i(e^{\rho^*} + e^{-\rho^*} - 2) - c^*\rho^* + \beta e^{-\rho^* c^* \tau} - \gamma - \delta = 0 \tag{2.1}$$

and

$$F_\rho(\rho^*, c^*) = d_i(e^{\rho^*} - e^{-\rho^*}) - c^* - \beta c^* \tau e^{-\rho^* c^* \tau} = 0. \tag{2.2}$$

Moreover, the following statements are valid.

(i) *If* $c \in (0, c^*)$*, then* $F(\rho, c) > 0$ *for* $\rho \in [0, +\infty)$*.*

(ii) *If* $c \in (c^*, +\infty)$*, then the equation* $F(\rho, c) = 0$ *admits two positive roots* $\rho_1(c) := \rho_1$ *and* $\rho_2(c) := \rho_2$ *with* $\rho^* \in (\rho_1, \rho_2)$ *such that* $F(\rho, c) > 0$ *for* $\rho \in [0, +\infty) \setminus [\rho_1, \rho_2]$ *and* $F(\rho, c) < 0$ *for* $\rho \in (\rho_1, \rho_2)$*.*

Proof. It follows that $F(+\infty, c) = +\infty$ for each $c > 0$ and $F(\rho, +\infty) = -\infty$ for each $\rho > 0$. Since $R_0 > 1$, we compute that

$$\begin{gathered} F(0, c) = \beta - \gamma - \delta > 0, \quad F_c(\rho, c) = -\rho - \beta\rho\tau e^{-\rho c\tau} < 0, \quad \forall\, \rho > 0,\\ F(\rho, 0) = d_i(e^{\rho} + e^{-\rho} - 2) + \beta - \gamma - \delta \ge \beta - \gamma - \delta > 0,\\ F_\rho(0, c) = \left(d_i e^{\rho} - d_i e^{-\rho} - c - \beta c\tau e^{-\rho c\tau}\right)\Big|_{\rho=0} = -c - \beta c\tau < 0, \quad \forall\, c > 0 \end{gathered}$$

and

$$F_{\rho\rho}(\rho, c) = d_i e^{\rho} + d_i e^{-\rho} + \beta c^2 \tau^2 e^{-\rho c \tau} > 0.$$

By these calculations, we show the rough graphs of function $F(\rho, c)$ for each $c > 0$ in Figure 2.1.

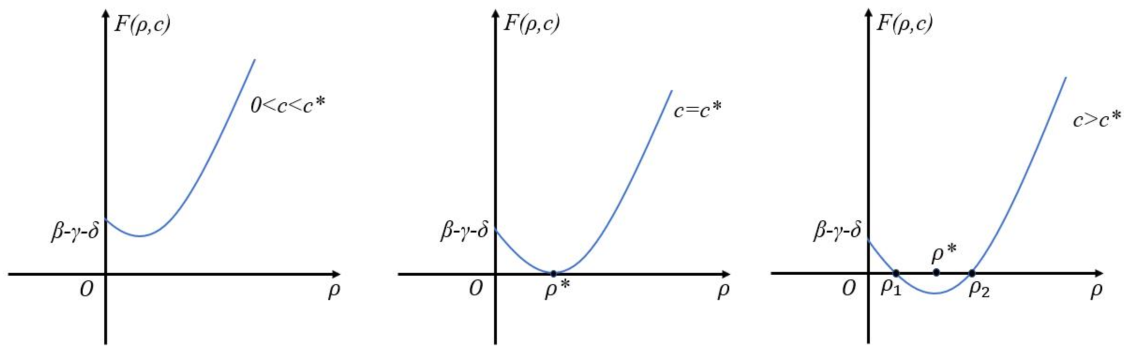

Figure 2.1: $F(\rho, c)$ when $0 < c < c^*$, $c = c^*$ and $c > c^*$, respectively.

In view of this figure, we obtain the desired results of this lemma. □

Lemma 2.3. *Let*

$$G(\rho, c) := c\rho - d_r(e^{\rho} + e^{-\rho} - 2).$$

Then for each $c > 0$, there exists a constant $\rho_3 > 0$ such that $G(\rho, c) > 0$ with $\rho \in (0, \rho_3)$.

Proof. Elementary computations give that $G(0, c) = 0$, $G_{\rho}(0, c) = c > 0$, $G_{\rho\rho}(\rho, c) = -d_r(e^{\rho} + e^{-\rho}) < 0$ and $G(+\infty, c) = -\infty$ for each $c > 0$. With the aid of the above computations, we present the rough graph of the function $G(\rho, c)$ for each $c > 0$ in Figure 2.2.

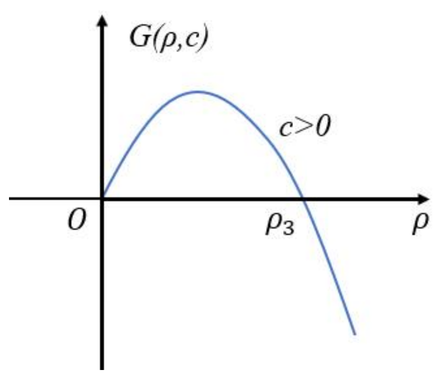

Figure 2.2: $G(\rho, c)$ when $c > 0$.

By the virtue of this figure, we end the proof. □

Lemma 2.4. *Suppose that* $(S, I, R)(\xi) \in C^1(\mathbb{R}, \mathbb{R}^3)$ *is a nontrivial positive solution of* (1.2) *with the wave speed* $c > 0$ *and satisfies*

$$S(-\infty) = S_1, \quad \sup_{\xi\in\mathbb{R}} S(\xi) \le S_1, \quad I(\pm\infty) = 0, \quad R(-\infty) = 0, \quad \sup_{\xi\in\mathbb{R}} R(\xi) < +\infty,$$

where $S_1 > 0$ *is a given constant. Then*

$$\int_{\mathbb{R}} \frac{S(\xi)I(\xi - c\tau)}{S(\xi) + I(\xi - c\tau) + R(\xi)}\, d\xi < +\infty \tag{2.3}$$

and

$$\int_{\mathbb{R}} I(\xi)\, d\xi < +\infty. \tag{2.4}$$

Proof. Integrating the first equation in (1.2) over $[x, y]$ gives

$$\begin{aligned}
&\int_x^y \frac{\beta S(\xi)I(\xi - c\tau)}{S(\xi) + I(\xi - c\tau) + R(\xi)}\, d\xi \\
&= d_s \int_x^y D[S](\xi) - c\int_x^y S'(\xi)\, d\xi \\
&= d_s \int_x^y \int_0^1 S'(\xi + \theta)\, d\theta d\xi - d_s \int_x^y \int_0^1 S'(\xi - \theta)\, d\theta d\xi - cS(y) + cS(x) \\
&= d_s \int_0^1 [S(y + \theta) - S(x + \theta)]\, d\theta + d_s \int_0^1 [S(x - \theta) - S(y - \theta)]\, d\theta - cS(y) + cS(x) \\
&\le (2d_s + c)S_1, \quad (\text{since } \sup_{\xi\in\mathbb{R}} S(\xi) \le S_1 \text{ and } S(\xi) > 0 \text{ on } \mathbb{R}),
\end{aligned}$$

which together with the positiveness of $S(\xi)$, $I(\xi)$ and $R(\xi)$ on $\mathbb{R}$ implies that (2.3) holds. Since $I(\xi) \in C^1(\mathbb{R})$ is nontrivial, positive and $I(\pm\infty) = 0$, there is some constant $C_0 > 0$ such that $I(\xi) \le C_0$ on $\mathbb{R}$. Then an integration of the second equation in (1.2) over $[\eta, \zeta]$ yields

$$\begin{aligned}
&(\gamma + \delta) \int_\eta^\zeta I(\xi)\, d\xi \\
&= d_i \int_\eta^\zeta D[I](\xi) - c\int_\eta^\zeta I'(\xi)\, d\xi + \int_\eta^\zeta \frac{\beta S(\xi)I(\xi - c\tau)}{S(\xi) + I(\xi - c\tau) + R(\xi)}\, d\xi \\
&\le d_i \int_\eta^\zeta \int_0^1 I'(\xi + \theta)\, d\theta d\xi - d_i \int_\eta^\zeta \int_0^1 I'(\xi - \theta)\, d\theta d\xi - cI(\zeta) + cI(\eta) + (2d_s + c)S_1 \\
&= d_i \int_0^1 [I(\zeta + \theta) - I(\eta + \theta)]\, d\theta + d_i \int_0^1 [I(\eta - \theta) - I(\zeta - \theta)]\, d\theta \\
&\quad - cI(\zeta) + cI(\eta) + (2d_s + c)S_1 \\
&\le (2d_i + c)C_0 + (2d_s + c)S_1.
\end{aligned}$$

This combined with the positiveness of $I(\xi)$ on $\mathbb{R}$ ensures that (2.4) holds. The proof is finished. □

3. Nonexistence of traveling waves

This section is to establish the nonexistence of nontrivial positive bounded traveling wave solutions to (1.1).

Theorem 3.1. *For a given constant $S_1 > 0$, if $(R_0, c) \in (0,1] \times \mathbb{R} \cup (1, +\infty) \times (-\infty, c^*)$, then system* (1.1) *has no nontrivial positive traveling wave solutions* $(S, I, R)(\xi)$ *satisfying*

$$S(-\infty) = S_1, \quad \sup_{\xi\in\mathbb{R}} S(\xi) \leq S_1, \quad I(\pm\infty) = 0, \quad R(-\infty) = 0, \quad \sup_{\xi\in\mathbb{R}} R(\xi) < +\infty. \tag{3.1}$$

Proof. By the reduction to absurdity, we assume that $(S, I, R)(\xi) \in C^1(\mathbb{R}, \mathbb{R}^3)$ is a nontrivial positive traveling wave solution of (1.1) satisfying (3.1). Then we divide the proof into three cases.

First case: $R_0 \leq 1$ and $c \in \mathbb{R}$. An integration of the second equation in (1.2) over $\mathbb{R}$ yields that

$$\begin{aligned}(\gamma+\delta)\int_{\mathbb{R}} I(\xi)\,d\xi &= d_i\int_{\mathbb{R}} D[I](\xi)\,d\xi - c\int_{\mathbb{R}} I'(\xi)\,d\xi + \int_{\mathbb{R}} \frac{\beta S(\xi)I(\xi-c\tau)}{S(\xi)+I(\xi-c\tau)+R(\xi)}\,d\xi \\ &= \int_{\mathbb{R}} \frac{\beta S(\xi)I(\xi-c\tau)}{S(\xi)+I(\xi-c\tau)+R(\xi)}\,d\xi \quad (\text{by (2.4) and } I(\pm\infty)=0) \\ &< \beta\int_{\mathbb{R}} I(\xi-c\tau)\,d\xi \quad (\text{by the positiveness of } S,\ I,\ R \text{ on } \mathbb{R}) \\ &\leq (\gamma+\delta)\int_{\mathbb{R}} I(\xi)\,d\xi \quad (\text{since } R_0 \leq 1).\end{aligned}$$

A contradiction appears.

Second case: $R_0 > 1$ and $0 < c < c^*$. It follows from (3.1) that

$$\frac{\beta S(\xi)}{S(\xi)+I(\xi-c\tau)+R(\xi)} \to \beta \quad \text{as } \xi \to -\infty. \tag{3.2}$$

By (3.2) and $R_0 > 1$, we have that there is a constant $\xi^* \ll 0$ such that

$$\frac{\beta S(\xi)}{S(\xi)+I(\xi-c\tau)+R(\xi)} > \frac{\beta+\gamma+\delta}{2} \quad \text{for } \xi < \xi^*. \tag{3.3}$$

Then from (3.3) and the second equation in (1.2), we deduce that

$$cI'(\xi) \geq d_i D[I](\xi) + \frac{\beta+\gamma+\delta}{2}[I(\xi-c\tau)-I(\xi)] + \frac{\beta-\gamma-\delta}{2}I(\xi) \quad \text{for } \xi < \xi^*. \tag{3.4}$$

Using (2.4) we define the improper integral

$$H(\xi) := \int_{-\infty}^{\xi} I(\eta)\,d\eta \quad \text{for } \xi \in \mathbb{R}.$$

Integrating (3.4) over $(-\infty,\xi]$ gives

$$\frac{\beta-\gamma-\delta}{2}H(\xi)\le cI(\xi)-d_iD[H](\xi)-\frac{\beta+\gamma+\delta}{2}[H(\xi-c\tau)-H(\xi)] \quad \text{for } \xi<\xi^*, \tag{3.5}$$

where we have used $I(-\infty)=0$. By dominated convergence theorem and $H(-\infty)=0$, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\xi} D[H](\xi)\,d\eta \\
&=\lim_{z\to-\infty}\int_z^{\xi}[H(\eta+1)-H(\eta)]\,d\eta+\lim_{z\to-\infty}\int_z^{\xi}[H(\eta-1)-H(\eta)]\,d\eta \\
&=\lim_{z\to-\infty}\int_z^{\xi}\int_0^1 H'(\eta+\theta)\,d\theta d\eta-\lim_{z\to-\infty}\int_z^{\xi}\int_0^1 H'(\eta-\theta)\,d\theta d\eta \\
&=\lim_{z\to-\infty}\int_0^1[H(\xi+\theta)-H(z+\theta)]\,d\theta-\lim_{z\to-\infty}\int_0^1[H(\xi-\theta)-H(z-\theta)]\,d\theta \\
&=\int_0^1[H(\xi+\theta)-H(\xi-\theta)]\,d\theta
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
\int_{-\infty}^{\xi}[H(\eta-c\tau)-H(\eta)]\,d\eta&=\lim_{z\to-\infty}\int_z^{\xi}[H(\eta-c\tau)-H(\eta)]\,d\eta \\
&=-c\tau\lim_{z\to-\infty}\int_z^{\xi}\int_0^1 H'(\eta-c\tau\theta)\,d\theta d\eta \\
&=-c\tau\lim_{z\to-\infty}\int_0^1[H(\xi-c\tau\theta)-H(z-c\tau\theta)]\,d\theta \\
&=-c\tau\int_0^1 H(\xi-c\tau\theta)\,d\theta,
\end{aligned} \tag{3.7}$$

which guarantee that $H(\xi+1)+H(\xi-1)-2H(\xi)$ and $H(\xi-c\tau)-H(\xi)$ are integrable on $\mathbb{R}$. Thus integrating (3.5) on $(-\infty,\xi]$ and utilizing (3.6) and (3.7) yield that

$$\begin{aligned}
&\frac{\beta-\gamma-\delta}{2}\int_{-\infty}^{\xi}H(\eta)\,d\eta+d_i\int_0^1[H(\xi+\theta)-H(\xi-\theta)]\,d\theta \\
&\le cH(\xi)+\frac{c\tau(\beta+\gamma+\delta)}{2}\int_0^1 H(\xi-c\tau\theta)\,d\theta \quad \text{for } \xi<\xi^*.
\end{aligned} \tag{3.8}$$

Note that $H(\xi)$ is strictly increasing on $\mathbb{R}$ due to the positiveness of $I(\xi)$ on $\mathbb{R}$. Then it follows from (3.8) that

$$\frac{\beta-\gamma-\delta}{2}\int_{-\infty}^{\xi}H(\eta)\,d\eta\le\left[c+\frac{c\tau(\beta+\gamma+\delta)}{2}\right]H(\xi) \quad \text{for } \xi<\xi^*. \tag{3.9}$$

From (3.9) and the monotonicity of $H(\xi)$, we obtain that there is a large enough constant $\eta_0>0$ such that

$$\frac{\eta_0(\beta-\gamma-\delta)}{2}H(\xi-\eta_0)\le\left[c+\frac{c\tau(\beta+\gamma+\delta)}{2}\right]H(\xi) \quad \text{for } \xi<\xi^*,$$

which implies that

$$H(\xi-\eta_0)\le \sigma H(\xi)\quad \text{for } \xi<\xi^*, \tag{3.10}$$

where $\sigma\in(0,1)$ is a small enough constant. Denote

$$\mu_0 := \frac{1}{\eta_0}\ln\frac{1}{\sigma}\quad \text{and}\quad J(\xi) := H(\xi)e^{-\mu_0\xi}. \tag{3.11}$$

We infer from (3.10) and (3.11) that

$$J(\xi-\eta_0)=H(\xi-\eta_0)e^{-\mu_0(\xi-\eta_0)}\le \sigma H(\xi)e^{-\mu_0(\xi-\eta_0)}=J(\xi)\quad \text{for } \xi<\xi^*,$$

which coupled with $J(\xi)>0$ ensures that $J(-\infty)$ exists. Also it is easy to see that $J(+\infty)=0$. Hence there exists a constant $J_0>0$ such that

$$J(\xi)\le J_0\quad \text{for } \xi\in\mathbb{R}. \tag{3.12}$$

Obviously, one can have from the second equation in (1.2) that

$$cI'(\xi)\le d_iD[I](\xi)+\beta I(\xi-c\tau)-(\gamma+\delta)I(\xi). \tag{3.13}$$

Integrating (3.13) over $(-\infty,\xi]$ leads to

$$cI(\xi)\le d_iD[H](\xi)+\beta H(\xi-c\tau)-(\gamma+\delta)H(\xi). \tag{3.14}$$

Then it follows from (3.12)–(3.14) that

$$\sup_{\xi\in\mathbb{R}}\left\{I(\xi)e^{-\mu_0\xi}\right\}<+\infty\quad \text{and}\quad \sup_{\xi\in\mathbb{R}}\left\{I'(\xi)e^{-\mu_0\xi}\right\}<+\infty. \tag{3.15}$$

Let $\upsilon(\xi)\in C^\infty(\mathbb{R},[0,1])$ be a nondecreasing function such that $\upsilon_N(\xi)=\upsilon(\xi/N)$ for $N\in\mathbb{N}$ and

$$\upsilon(\xi)=\begin{cases}0 & \text{if } \xi\in(-\infty,-2],\\ 1 & \text{if } \xi\in[-1,+\infty).\end{cases}$$

Multiplying the third equation in (1.2) by $e^{-\nu\xi}\upsilon_N(\xi)$ and integrating the resultant equation over $\mathbb{R}$, we obtain

$$c\int_{\mathbb{R}}R'(\xi)e^{-\nu\xi}\upsilon_N(\xi)\,d\xi=d_r\int_{\mathbb{R}}D[R](\xi)e^{-\nu\xi}\upsilon_N(\xi)\,d\xi+\gamma\int_{\mathbb{R}}I(\xi)e^{-\nu\xi}\upsilon_N(\xi)\,d\xi. \tag{3.16}$$

By direct calculations, we deduce that

$$\int_{\mathbb{R}}R'(\xi)e^{-\nu\xi}\upsilon_N(\xi)\,d\xi=\nu\int_{\mathbb{R}}R(\xi)e^{-\nu\xi}\upsilon_N(\xi)\,d\xi-\int_{\mathbb{R}}R(\xi)e^{-\nu\xi}\upsilon_N'(\xi)\,d\xi \tag{3.17}$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}} [R(\xi+1)+R(\xi-1)]e^{-\nu\xi}\upsilon_N(\xi)\,d\xi \\
&= \int_{\mathbb{R}} R(\xi+1)e^{-\nu\xi}\upsilon_N(\xi)\,d\xi + \int_{\mathbb{R}} R(\xi-1)e^{-\nu\xi}\upsilon_N(\xi)\,d\xi \\
&= e^{\nu}\int_{\mathbb{R}} R(\xi)e^{-\nu\xi}\upsilon_N(\xi-1)\,d\xi + e^{-\nu}\int_{\mathbb{R}} R(\xi)e^{-\nu\xi}\upsilon_N(\xi+1)\,d\xi \\
&\le (e^{\nu}+e^{-\nu})\int_{\mathbb{R}} R(\xi)e^{-\nu\xi}\,d\xi,
\end{aligned}
\tag{3.18}
$$

since $\upsilon_N(\xi-1)\le 1$ and $\upsilon_N(\xi+1)\le 1$ for $\xi\in\mathbb{R}$. Plugging (3.17) and (3.18) into (3.16) yields

(3.19)

$$
\begin{aligned}
&(c\nu+2d_r)\int_{\mathbb{R}} R(\xi)e^{-\nu\xi}\upsilon_N(\xi)\,d\xi - d_r(e^{\nu}+e^{-\nu})\int_{\mathbb{R}} R(\xi)e^{-\nu\xi}\,d\xi - c\int_{\mathbb{R}} R(\xi)e^{-\nu\xi}\upsilon_N'(\xi)\,d\xi \\
&\le \gamma\int_{\mathbb{R}} I(\xi)e^{-\nu\xi}\upsilon_N(\xi)\,d\xi.
\end{aligned}
$$

Recall that $G(\nu,c)=c\nu+2d_r-d_r(e^{\nu}+e^{-\nu})>0$ for each $\nu\in(0,\rho_3)$ (see Lemma 2.3). Then passing to the limits in (3.19) as $N\to\infty$ gives

$$
\int_{\mathbb{R}} R(\xi)e^{-\nu\xi}\,d\xi \le \frac{\gamma}{G(\nu,c)}\int_{\mathbb{R}} I(\xi)e^{-\nu\xi}\,d\xi \quad \text{for } \nu\in(0,\rho_3).
$$

Hence by (3.15) we obtain that

$$
\int_{\mathbb{R}} R(\xi)e^{-\nu\xi}\,d\xi < +\infty \quad \text{for } \nu\in(0,\underline{\mu}) \text{ with } \underline{\mu} := \min\{\mu_0,\rho_3\}.
\tag{3.20}
$$

It follows from (3.15) and (3.20) that

$$
\int_{\mathbb{R}} e^{-\rho\xi}\frac{I(\xi-c\tau)[I(\xi-c\tau)+R(\xi)]}{S(\xi)+I(\xi-c\tau)+R(\xi)}\,d\xi < +\infty \quad \text{for } \rho\in(0,\mu_0+\underline{\mu}).
\tag{3.21}
$$

By (3.15) and the boundedness of $I(\xi)$ on $\mathbb{R}$, we define the two-sided Laplace transform of $I(\xi)$ by

$$
L(\rho) := \int_{\mathbb{R}} I(\xi)e^{-\rho\xi}\,d\xi \quad \text{with } 0<\operatorname{Re}\rho<\mu_0.
$$

Note that the second equation in (1.2) can be rewritten as

$$
d_i D[I](\xi) - cI'(\xi) + \beta I(\xi-c\tau) - (\gamma+\delta)I(\xi) = \frac{\beta I(\xi-c\tau)[I(\xi-c\tau)+R(\xi)]}{S(\xi)+I(\xi-c\tau)+R(\xi)}.
\tag{3.22}
$$

By taking the two-sided Laplace transform on (3.22), we have

$$
L(\rho)F(\rho,c) = \int_{\mathbb{R}} e^{-\rho\xi}\frac{\beta I(\xi-c\tau)[I(\xi-c\tau)+R(\xi)]}{S(\xi)+I(\xi-c\tau)+R(\xi)}\,d\xi.
\tag{3.23}
$$

One can see that $L(\rho)$ on the left-hand side in (3.23) is well-defined for $\rho \in (0, \mu_0)$, while

$$\int_{\mathbb{R}} e^{-\rho\xi} \frac{\beta I(\xi - c\tau)[I(\xi - c\tau) + R(\xi)]}{S(\xi) + I(\xi - c\tau) + R(\xi)} \, d\xi$$

on the right-hand side is well-defined for $\rho \in (0, \mu_0 + \underline{\mu})$ (see (3.21)). According to the property of Laplace transform [29], we obtain that these two integrals are analytical on the whole right half plane; see the analogously discussions in [12,20,22,23,26,27,40,41,44]. Due to $F(\rho, c) \to +\infty$ as $\rho \to +\infty$ (see the proof in Lemma 2.2), *a contradiction occurs* in (3.23).

Third case: $R_0 > 1$ and $c \leq 0$. From (3.8), we get

$$\begin{aligned} &\frac{\beta - \gamma - \delta}{2} \int_{-\infty}^{\xi} H(\eta) \, d\eta + d_i \int_0^1 [H(\xi + \theta) - H(\xi - \theta)] \, d\theta \\ &\leq cH(\xi) + \frac{c\tau(\beta + \gamma + \delta)}{2} \int_0^1 H(\xi - c\tau\theta) \, d\theta \quad \text{for } \xi < \xi^*. \end{aligned} \tag{3.24}$$

With aid of the positiveness and monotonicity of $H(\xi)$ on $\mathbb{R}$, we conclude that inequality (3.24) *does not hold* under the conditions $R_0 > 1$ and $c \leq 0$, since the left-hand side in (3.24) is greater than zero, while the right-hand side in (3.24) is less than or equal to zero. *A contradiction appears.* Combining the above three cases, we finish the proof. □

4. Existence of super-critical traveling waves

In this section, we shall prove the existence result under the conditions $R_0 > 1$ and $c > c^*$. For this purpose, recalling the definition of ρ_1 in Lemma 2.2, we select a constant $\rho_4 \in (0, \rho_1)$ to be small enough such that

$$M_1 := S_1 + \frac{\beta e^{-\rho_1 c\tau}}{d_s(2 - e^{\rho_4} - e^{-\rho_4}) + c\rho_4} > S_1, \tag{4.1}$$

where $S_1 > 0$ is a given constant. Then choose a constant $\sigma_1 > 0$ such that the following algebraic equation

$$S_1 - M_1 e^{\rho_4 \xi} = \sigma_1 e^{-\frac{\beta}{c}\xi}$$

admits two negative roots and we denote the bigger one by ξ_1. Set

$$\xi_2 := \frac{1}{\rho_1} \ln[(R_0 - 1)S_1] \quad \text{and} \quad \xi_3 := \frac{1}{\epsilon_1} \ln \frac{1}{M_2},$$

where the constants $\epsilon_1 \in (0, \min\{\rho_2 - \rho_1, \rho_3, \rho_4\})$ is chosen to be sufficiently small and $M_2 \gg 1$ such that

$$\xi_3 < \xi_2, \quad \xi_3 < \xi_1 \quad \text{and} \quad S_1 - M_1 e^{\rho_4 \xi_3} \geq S_1/2. \tag{4.2}$$

Also we pick the constant

$$M_3 \geq \max\left\{\frac{\gamma e^{(\rho_1-\epsilon_1)\xi_2}}{G(\epsilon_1, c)}, \frac{\gamma(R_0-1)S_1 e^{-\epsilon_1\xi_2}}{G(\epsilon_1, c)}\right\}, \tag{4.3}$$

where $G(\cdot,\cdot)$ is defined in Lemma 2.3. By the choices of above parameters, we construct the following nonnegative continuous functions on $\mathbb{R}$.

$$S_+(\xi) := S_1, \qquad S_-(\xi) := \begin{cases} S_1 - M_1 e^{\rho_4 \xi}, & \xi < \xi_1, \\ \sigma_1 e^{-\frac{\beta}{c}\xi}, & \xi \geq \xi_1, \end{cases}$$

$$I_+(\xi) := \begin{cases} e^{\rho_1 \xi}, & \xi < \xi_2, \\ (R_0-1)S_1, & \xi \geq \xi_2, \end{cases} \qquad I_-(\xi) := \begin{cases} e^{\rho_1\xi} - M_2 e^{(\rho_1+\epsilon_1)\xi}, & \xi < \xi_3, \\ 0, & \xi \geq \xi_3, \end{cases}$$

$$R_+(\xi) := M_3 e^{\epsilon_1 \xi}, \qquad R_-(\xi) := 0.$$

In order to describe that the parameters are admissible, we show Figures 4.1, 4.2 and 4.3.

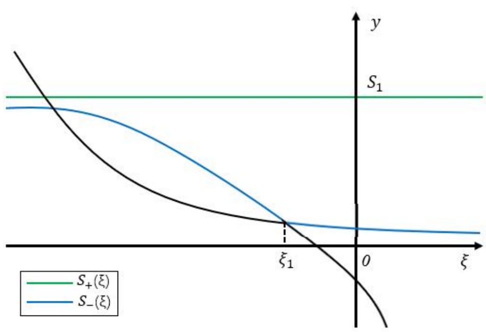

Figure 4.1: $S_+(\xi)$ and $S_-(\xi)$ when $R_0 > 1$ and $c > c^*$.

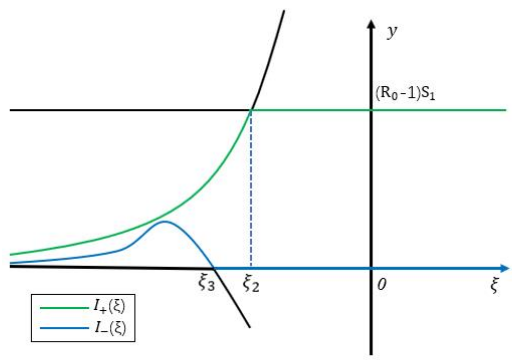

Figure 4.2: $I_+(\xi)$ and $I_-(\xi)$ when $R_0 > 1$ and $c > c^*$.

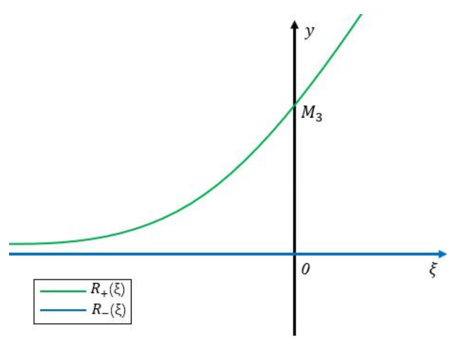

Figure 4.3: $R_+(\xi)$ and $R_-(\xi)$ when $R_0 > 1$ and $c > c^*$.

Lemma 4.1. *The functions* $S_\pm(\xi)$, $I_\pm(\xi)$ *and* $R_\pm(\xi)$ *satisfy*

$$d_sD[S_+](\xi) - cS'_+(\xi) - \frac{\beta S_+(\xi)I_-(\xi - c\tau)}{S_+(\xi) + I_-(\xi - c\tau) + R_+(\xi)} \le 0, \quad \xi \in \mathbb{R}, \tag{4.4}$$

$$d_iD[I_+](\xi) - cI'_+(\xi) + \frac{\beta S_+(\xi)I_+(\xi - c\tau)}{S_+(\xi) + I_+(\xi - c\tau) + R_-(\xi)} - (\gamma+\delta)I_+(\xi) \le 0, \quad \xi \ne \xi_2, \tag{4.5}$$

$$d_rD[R_+](\xi) - cR'_+(\xi) + \gamma I_+(\xi) \le 0, \quad \xi \in \mathbb{R}, \tag{4.6}$$

$$d_sD[S_-](\xi) - cS'_-(\xi) - \frac{\beta S_-(\xi)I_+(\xi - c\tau)}{S_-(\xi) + I_+(\xi - c\tau) + R_-(\xi)} \ge 0, \quad \xi \ne \xi_1, \tag{4.7}$$

$$d_iD[I_-](\xi) - cI'_-(\xi) + \frac{\beta S_-(\xi)I_-(\xi - c\tau)}{S_-(\xi) + I_-(\xi - c\tau) + R_+(\xi)} - (\gamma+\delta)I_-(\xi) \ge 0, \quad \xi \ne \xi_3, \tag{4.8}$$

$$d_rD[R_-](\xi) - cR'_-(\xi) + \gamma I_-(\xi) \ge 0, \quad \xi \in \mathbb{R}. \tag{4.9}$$

Proof. From the definitions of $S_+(\xi)$, $I_-(\xi)$ and $R_\pm(\xi)$ on $\mathbb{R}$, we obtain that inequalities (4.4) and (4.9) hold naturally. Subsequently, we give the detailed proofs of inequalities (4.5)–(4.8).

Proof of (4.5). If $\xi < \xi_2$, then $I_+(\xi) = e^{\rho_1\xi}$, $I_+(\xi-1) = e^{\rho_1(\xi-1)}$, $I_+(\xi - c\tau) = e^{\rho_1(\xi - c\tau)}$ and $I_+(\xi+1) \le e^{\rho_1(\xi+1)}$. It follows from Lemma 2.2 that

$$\begin{aligned}
&d_iD[I_+](\xi) - cI'_+(\xi) + \frac{\beta S_+(\xi)I_+(\xi - c\tau)}{S_+(\xi) + I_+(\xi - c\tau) + R_-(\xi)} - (\gamma+\delta)I_+(\xi)\\
&\le d_iD[I_+](\xi) - cI'_+(\xi) + \beta I_+(\xi - c\tau) - (\gamma+\delta)I_+(\xi)\\
&\le e^{\rho_1\xi}\big[d_i(e^{\rho_1} + e^{-\rho_1} - 2) - c\rho_1 + \beta e^{-\rho_1 c\tau} - \gamma - \delta\big]\\
&= e^{\rho_1\xi}F(\rho_1, c) = 0.
\end{aligned}$$

If $\xi > \xi_2$, then $I_+(\xi) = I_+(\xi+1) = (R_0-1)S_1$, $I_+(\xi-1) \le (R_0-1)S_1$, $I_+(\xi - c\tau) \le (R_0-1)S_1$, $S_+(\xi) = S_1$ and $R_-(\xi) = 0$. A direct computation leads to

$$\begin{aligned}
&d_iD[I_+](\xi) - cI'_+(\xi) + \frac{\beta S_+(\xi)I_+(\xi - c\tau)}{S_+(\xi) + I_+(\xi - c\tau) + R_-(\xi)} - (\gamma+\delta)I_+(\xi)\\
&\le \frac{\beta S_1(R_0-1)S_1}{S_1 + (R_0-1)S_1} - (\gamma+\delta)(R_0-1)S_1 = 0.
\end{aligned}$$

Proof of (4.6). By the definitions of $I_+(\xi)$ and $R_+(\xi)$ on $\mathbb{R}$, we derive from Lemma 2.3 and (4.3) that

$$\begin{aligned}
&d_rD[R_+](\xi) - cR'_+(\xi) + \gamma I_+(\xi)\\
&= d_r\big[M_3e^{\epsilon_1(\xi+1)} + M_3e^{\epsilon_1(\xi-1)} - 2M_3e^{\epsilon_1\xi}\big] - cM_3\epsilon_1e^{\epsilon_1\xi} + \gamma e^{\rho_1\xi}\\
&= M_3e^{\epsilon_1\xi}\left[d_r(e^{\epsilon_1} + e^{-\epsilon_1} - 2) - c\epsilon_1 + \frac{\gamma e^{(\rho_1-\epsilon_1)\xi}}{M_3}\right]\\
&\le M_3e^{\epsilon_1\xi}\left[\frac{\gamma e^{(\rho_1-\epsilon_1)\xi_2}}{M_3} - G(\epsilon_1, c)\right] \le 0 \quad \text{for } \xi < \xi_2
\end{aligned}$$

and

$$
\begin{aligned}
&d_r D[R_+](\xi) - cR_+'(\xi) + \gamma I_+(\xi)\\
&= d_r\big[M_3 e^{\epsilon_1(\xi+1)} + M_3 e^{\epsilon_1(\xi-1)} - 2M_3 e^{\epsilon_1\xi}\big] - cM_3\epsilon_1 e^{\epsilon_1\xi} + \gamma(R_0-1)S_1\\
&= M_3 e^{\epsilon_1\xi}\left[d_r(e^{\epsilon_1}+e^{-\epsilon_1}-2) - c\epsilon_1 + \frac{\gamma(R_0-1)S_1 e^{-\epsilon_1\xi}}{M_3}\right]\\
&\le M_3 e^{\epsilon_1\xi}\left[\frac{\gamma(R_0-1)S_1 e^{-\epsilon_1\xi_2}}{M_3} - G(\epsilon_1,c)\right] \le 0 \quad \text{for } \xi \ge \xi_2.
\end{aligned}
$$

Proof of (4.7). If $\xi < \xi_1$, then $S_-(\xi) = S_1 - M_1 e^{\rho_4\xi}$, $S_-(\xi-1) = S_1 - M_1 e^{\rho_4(\xi-1)}$, $S_-(\xi+1) \ge S_1 - M_1 e^{\rho_4(\xi+1)}$, $I_+(\xi - c\tau) \le e^{\rho_1(\xi - c\tau)}$ and $R_-(\xi) = 0$. By (4.1) we compute that

$$
\begin{aligned}
&d_s D[S_-](\xi) - cS_-'(\xi) - \frac{\beta S_-(\xi) I_+(\xi-c\tau)}{S_-(\xi) + I_+(\xi-c\tau) + R_-(\xi)}\\
&\ge d_s D[S_-](\xi) - cS_-'(\xi) - \beta I_+(\xi - c\tau)\\
&\ge d_s\big[2M_1 e^{\rho_4\xi} - M_1 e^{\rho_4(\xi+1)} - M_1 e^{\rho_4(\xi-1)}\big] + cM_1\rho_4 e^{\rho_4\xi} - \beta e^{\rho_1(\xi-c\tau)}\\
&= e^{\rho_4\xi}\big[M_1 d_s(2 - e^{\rho_4} - e^{-\rho_4}) + cM_1\rho_4 - \beta e^{(\rho_1-\rho_4)\xi - \rho_1 c\tau}\big]\\
&\ge e^{\rho_4\xi}\big\{M_1\big[d_s(2 - e^{\rho_4} - e^{-\rho_4}) + c\rho_4\big] - \beta e^{-\rho_1 c\tau}\big\}\\
&\ge 0 \quad \text{for } \xi < \xi_1,
\end{aligned}
$$

where we have used the fact that $e^{(\rho_1-\rho_4)\xi} < 1$ for $\xi < \xi_1 < 0$. If $\xi > \xi_1$, then $S_-(\xi) = \sigma_1 e^{-\frac{\beta}{c}\xi}$, $S_-(\xi+1) = \sigma_1 e^{-\frac{\beta}{c}(\xi+1)}$ and $S_-(\xi-1) \ge \sigma_1 e^{-\frac{\beta}{c}(\xi-1)}$. We obtain that

$$
\begin{aligned}
&d_s D[S_-](\xi) - cS_-'(\xi) - \frac{\beta S_-(\xi) I_+(\xi-c\tau)}{S_-(\xi) + I_+(\xi-c\tau) + R_-(\xi)}\\
&\ge d_s D[S_-](\xi) - cS_-'(\xi) - \beta S_-(\xi)\\
&\ge d_s\big[\sigma_1 e^{-\frac{\beta}{c}(\xi+1)} + \sigma_1 e^{-\frac{\beta}{c}(\xi-1)} - 2\sigma_1 e^{-\frac{\beta}{c}\xi}\big] + \beta\sigma_1 e^{-\frac{\beta}{c}\xi} - \beta\sigma_1 e^{-\frac{\beta}{c}\xi}\\
&= d_s\sigma_1 e^{-\frac{\beta}{c}\xi}\big(e^{-\beta/c} + e^{\beta/c} - 2\big)\\
&\ge 0 \quad \text{for } \xi > \xi_1.
\end{aligned}
$$

Proof of (4.8). By (4.2) we get for $\xi < \xi_3$ that

$$
I_-(\xi) = e^{\rho_1\xi} - M_2 e^{(\rho_1+\epsilon_1)\xi}, \quad S_-(\xi) = S_1 - M_1 e^{\rho_4\xi} \ge S_1/2 \quad \text{and} \quad R_+(\xi) = M_3 e^{\epsilon_1\xi}. \tag{4.10}
$$

For $\xi < \xi_3$, we deduce from (4.10) that

$$
\begin{aligned}
&-\beta I_-(\xi - c\tau) + \frac{\beta S_-(\xi) I_-(\xi-c\tau)}{S_-(\xi) + I_-(\xi-c\tau) + R_+(\xi)}\\
&= -\frac{\beta I_-^2(\xi-c\tau) + \beta I_-(\xi-c\tau)R_+(\xi)}{S_-(\xi) + I_-(\xi-c\tau) + R_+(\xi)} \ge -\frac{\beta I_-^2(\xi-c\tau) + \beta I_-(\xi-c\tau)R_+(\xi)}{S_-(\xi)} \qquad (4.11)\\
&\ge -\frac{2\beta}{S_1}\big[e^{2\rho_1\xi} + M_3 e^{(\rho_1+\epsilon_1)\xi}\big],
\end{aligned}
$$

since $I_-(\xi - c\tau) \leq I_+(\xi) = e^{\rho_1 \xi}$ for $\xi < \xi_3$. Recalling that $\epsilon_1 \in (0, \min\{\rho_2 - \rho_1, \rho_3, \rho_4\})$, we have for $\xi < \xi_3 < 0$ that

$$e^{(\rho_1 - \epsilon_1)\xi} < 1, \quad F(\rho_1, c) = 0 \quad \text{and} \quad F(\rho_1 + \epsilon_1, c) < 0. \tag{4.12}$$

Noting that $M_2 \gg 1$ and using (4.11) and (4.12), we derive for $\xi < \xi_3$ that

$$\begin{aligned}
& d_i D[I_-](\xi) - cI_-'(\xi) + \frac{\beta S_-(\xi) I_-(\xi - c\tau)}{S_-(\xi) + I_-(\xi - c\tau) + R_+(\xi)} - (\gamma + \delta) I_-(\xi) \\
&= d_i D[I_-](\xi) - cI_-'(\xi) + \beta I_-(\xi - c\tau) - (\gamma + \delta) I_-(\xi) - \beta I_-(\xi - c\tau) \\
&\quad + \frac{\beta S_-(\xi) I_-(\xi - c\tau)}{S_-(\xi) + I_-(\xi - c\tau) + R_+(\xi)} \\
&\geq e^{\rho_1 \xi} F(\rho_1, c) - M_2 e^{(\rho_1 + \epsilon_1)\xi} F(\rho_1 + \epsilon_1, c) - \frac{2\beta}{S_1} \left[e^{2\rho_1 \xi} + M_3 e^{(\rho_1 + \epsilon_1)\xi} \right] \\
&= -e^{(\rho_1 + \epsilon_1)\xi} F(\rho_1 + \epsilon_1, c) \left[M_2 - \frac{2\beta e^{(\rho_1 - \epsilon_1)\xi} + 2\beta M_3}{-S_1 F(\rho_1 + \epsilon_1, c)} \right] \\
&\geq -e^{(\rho_1 + \epsilon_1)\xi} F(\rho_1 + \epsilon_1, c) \left[M_2 - \frac{2\beta(1 + 2M_3)}{-S_1 F(\rho_1 + \epsilon_1, c)} \right] \geq 0.
\end{aligned}$$

If $\xi > \xi_3$, then $I_-(\xi) = 0$ and inequality (4.8) holds immediately. The proof of this lemma is finished. □

Now we introduce a non-empty, bounded, closed and convex subset of $C([-X, X], \mathbb{R}^3)$

$$\begin{aligned}
\Omega_X := \Big\{ (\phi, \varphi, \chi)(\xi) \in C([-X, X], \mathbb{R}^3) \mid\ & \phi(-X) = S_-(-X), \varphi(-X) = I_-(-X), \\
& \chi(-X) = R_-(-X), S_-(\xi) \leq \phi(\xi) \leq S_+(\xi), \\
& I_-(\xi) \leq \varphi(\xi) \leq I_+(\xi), R_-(\xi) \leq \chi(\xi) \leq R_+(\xi) \\
& \text{for any } \xi \in [-X, X] \Big\},
\end{aligned}$$

which is endowed with the usual supremum norm, where $X \gg l := \max\{|\xi_3|, c\tau, 1\}$. On a closed interval $[-X - l, X + l]$, we construct the following nonnegative continuous functions

$$\widehat{\phi}(\xi) := \begin{cases} S_-(\xi), & \xi \in I_1, \\ \phi(\xi), & \xi \in I_2, \\ \phi(X), & \xi \in I_3, \end{cases} \quad \widehat{\varphi}(\xi) := \begin{cases} I_-(\xi), & \xi \in I_1, \\ \varphi(\xi), & \xi \in I_2, \\ \varphi(X), & \xi \in I_3, \end{cases} \quad \widehat{\chi}(\xi) := \begin{cases} R_-(\xi), & \xi \in I_1, \\ \chi(\xi), & \xi \in I_2, \\ \chi(X), & \xi \in I_3, \end{cases}$$

where $I_1 = [-X - l, -X]$, $I_2 = (-X, X)$, $I_3 = [X, X + l]$ and $(\phi, \varphi, \chi)(\xi) \in \Omega_X$. For any $\xi \in [-X - l, X + l]$, one can check that

$$S_-(\xi) \leq \widehat{\phi}(\xi) \leq S_+(\xi), \quad I_-(\xi) \leq \widehat{\varphi}(\xi) \leq I_+(\xi) \quad \text{and} \quad R_-(\xi) \leq \widehat{\chi}(\xi) \leq R_+(\xi). \tag{4.13}$$

Consider an initial value problem

(4.14)
$$\begin{cases} cS'(\xi) = d_s\big[\widehat{\phi}(\xi+1) + \widehat{\phi}(\xi-1) - 2S(\xi)\big] - \alpha S(\xi) + \alpha\phi(\xi) - \frac{\beta\phi(\xi)\widehat{\varphi}(\xi-c\tau)}{\phi(\xi)+\widehat{\varphi}(\xi-c\tau)+\chi(\xi)}, \\ cI'(\xi) = d_i\big[\widehat{\varphi}(\xi+1) + \widehat{\varphi}(\xi-1) - 2I(\xi)\big] + \frac{\beta\phi(\xi)\widehat{\varphi}(\xi-c\tau)}{\phi(\xi)+\widehat{\varphi}(\xi-c\tau)+\chi(\xi)} - (\gamma+\delta)I(\xi), \\ cR'(\xi) = d_r\big[\widehat{\chi}(\xi+1) + \widehat{\chi}(\xi-1) - 2R(\xi)\big] + \gamma\varphi(\xi), \\ S(-X) = S_-(-X), \quad I(-X) = I_-(-X), \quad R(-X) = R_-(-X), \end{cases}$$

where $\xi \in [-X, X]$ and the constant $\alpha > \beta$. The general theory of ordinary differential equations guarantees that initial problem (4.14) has a unique solution $(S_X, I_X, R_X)(\xi) \in C^1([-X,X],\mathbb{R}^3)$. Also the solution of (4.14) can be written by the following integral form

$$\begin{cases} S_X(\xi) = e^{-\frac{2d_s+\alpha}{c}(\xi+X)}S_-(-X) + \frac{1}{c}\int_{-X}^{\xi} e^{\frac{2d_s+\alpha}{c}(\eta-\xi)}H_1(\phi,\varphi,\chi)(\eta)\,d\eta, \\ I_X(\xi) = e^{-\frac{2d_i+\gamma+\delta}{c}(\xi+X)}I_-(-X) + \frac{1}{c}\int_{-X}^{\xi} e^{\frac{2d_i+\gamma+\delta}{c}(\eta-\xi)}H_2(\phi,\varphi,\chi)(\eta)\,d\eta, \\ R_X(\xi) = \frac{1}{c}\int_{-X}^{\xi} e^{\frac{2d_r}{c}(\eta-\xi)}H_3(\phi,\varphi,\chi)(\eta)\,d\eta, \end{cases} \tag{4.15}$$

where

$$H_1(\phi,\varphi,\chi)(\eta) = d_s\widehat{\phi}(\eta+1) + d_s\widehat{\phi}(\eta-1) + \alpha\phi(\eta) - \frac{\beta\phi(\eta)\widehat{\varphi}(\eta-c\tau)}{\phi(\eta)+\widehat{\varphi}(\eta-c\tau)+\chi(\eta)},$$
$$H_2(\phi,\varphi,\chi)(\eta) = d_i\widehat{\varphi}(\eta+1) + d_i\widehat{\varphi}(\eta-1) + \frac{\beta\phi(\eta)\widehat{\varphi}(\eta-c\tau)}{\phi(\eta)+\widehat{\varphi}(\eta-c\tau)+\chi(\eta)},$$
$$H_3(\phi,\varphi,\chi)(\eta) = d_r\widehat{\chi}(\eta+1) + d_r\widehat{\chi}(\eta-1) + \gamma\varphi(\eta).$$

Note that since $\alpha > \beta$, $H_1(\phi,\varphi,\chi)$ is decreasing with respect to φ and is increasing in both ϕ and χ; $H_2(\phi,\varphi,\chi)$ is decreasing with respect to χ and increasing in both ϕ and φ; $H_3(\phi,\varphi,\chi)$ is increasing in both φ and χ.

By (4.15) we define a nonlinear operator $\mathcal{O} = (\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3)\colon \Omega_X \mapsto C^1([-X,X],\mathbb{R}^3)$ as follows.

$$\mathcal{O}_1(\phi,\varphi,\chi)(\xi) := S_X(\xi), \quad \mathcal{O}_2(\phi,\varphi,\chi)(\xi) := I_X(\xi) \quad \text{and} \quad \mathcal{O}_3(\phi,\varphi,\chi)(\xi) := R_X(\xi).$$

In the next two lemmas, we will prove that the operator $\mathcal{O}$ is completely continuous which maps Ω_X into itself.

Lemma 4.2. *The operator $\mathcal{O}$ satisfies $\mathcal{O}(\Omega_X) \subseteq \Omega_X$.*

Proof. For any $(\phi,\varphi,\chi)(\xi) \in \Omega_X$, we only need to show that

$$S_-(\xi) \le \mathcal{O}_1(\phi,\varphi,\chi)(\xi) \le S_+(\xi), \quad I_-(\xi) \le \mathcal{O}_2(\phi,\varphi,\chi)(\xi) \le I_+(\xi)$$

and

$$R_-(\xi) \le \mathcal{O}_3(\phi,\varphi,\chi)(\xi) \le R_+(\xi) \quad \text{for } \xi \in [-X,X].$$

By the monotonicity of $H_1(\phi,\varphi,\chi)$, we obtain from (4.4), (4.7) and (4.13) that

$$\begin{aligned}
&d_s\big[\widehat{\phi}(\xi+1)+\widehat{\phi}(\xi-1)-2S_+(\xi)\big]-cS_+'(\xi)-\alpha S_+(\xi)+\alpha\phi(\xi)\\
&\quad-\frac{\beta\phi(\xi)\widehat{\varphi}(\xi-c\tau)}{\phi(\xi)+\widehat{\varphi}(\xi-c\tau)+\chi(\xi)}\\
&\le d_sD[S_+](\xi)-cS_+'(\xi)-\alpha S_+(\xi)+\alpha S_+(\xi)-\frac{\beta S_+(\xi)I_-(\xi-c\tau)}{S_+(\xi)+I_-(\xi-c\tau)+R_+(\xi)}\\
&\le 0 \quad \text{for } \xi\in[-X,X]
\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}
&d_s\big[\widehat{\phi}(\xi+1)+\widehat{\phi}(\xi-1)-2S_-(\xi)\big]-cS_-'(\xi)-\alpha S_-(\xi)+\alpha\phi(\xi)\\
&\quad-\frac{\beta\phi(\xi)\widehat{\varphi}(\xi-c\tau)}{\phi(\xi)+\widehat{\varphi}(\xi-c\tau)+\chi(\xi)}\\
&\ge d_sD[S_-](\xi)-cS_-'(\xi)-\alpha S_-(\xi)+\alpha S_-(\xi)-\frac{\beta S_-(\xi)I_+(\xi-c\tau)}{S_-(\xi)+I_+(\xi-c\tau)+R_-(\xi)}\\
&\ge 0 \quad \text{for } \xi\in[-X,\xi_1)\cup(\xi_1,X].
\end{aligned} \tag{4.17}$$

Inequalities (4.16) and (4.17) are equivalent to

$$cS_+'(\xi)+(2d_s+\alpha)S_+(\xi)\ge H_1(\phi,\varphi,\chi)(\xi) \quad \text{for } \xi\in[-X,X]$$

and

$$cS_-'(\xi)+(2d_s+\alpha)S_-(\xi)\le H_1(\phi,\varphi,\chi)(\xi) \quad \text{for } \xi\in[-X,\xi_1)\cup(\xi_1,X],$$

which imply that

$$\begin{aligned}
S_+(\xi)&\ge e^{-\frac{2d_s+\alpha}{c}(\xi+X)}S_+(-X)+\frac{1}{c}\int_{-X}^{\xi}e^{\frac{2d_s+\alpha}{c}(\eta-\xi)}H_1(\phi,\varphi,\chi)(\eta)\,d\eta\\
&\ge e^{-\frac{2d_s+\alpha}{c}(\xi+X)}S_-(-X)+\frac{1}{c}\int_{-X}^{\xi}e^{\frac{2d_s+\alpha}{c}(\eta-\xi)}H_1(\phi,\varphi,\chi)(\eta)\,d\eta\\
&=\mathcal{O}_1(\phi,\varphi,\chi)(\xi) \quad \text{for } \xi\in[-X,X]
\end{aligned}$$

and

$$\begin{aligned}
S_-(\xi)&\le e^{-\frac{2d_s+\alpha}{c}(\xi+X)}S_-(-X)+\frac{1}{c}\int_{-X}^{\xi}e^{\frac{2d_s+\alpha}{c}(\eta-\xi)}H_1(\phi,\varphi,\chi)(\eta)\,d\eta\\
&=\mathcal{O}_1(\phi,\varphi,\chi)(\xi) \quad \text{for } \xi\in[-X,\xi_1)\cup(\xi_1,X].
\end{aligned}$$

In analogous manners, one can have that

$$\begin{aligned}
I_+(\xi)&\ge\mathcal{O}_2(\phi,\varphi,\chi)(\xi) \quad \text{for } \xi\in[-X,\xi_2)\cup(\xi_2,X],\\
I_-(\xi)&\le\mathcal{O}_2(\phi,\varphi,\chi)(\xi) \quad \text{for } \xi\in[-X,\xi_3)\cup(\xi_3,X]
\end{aligned}$$

and

$$R_-(\xi) \le \mathcal{O}_3(\phi,\varphi,\chi)(\xi) \le R_+(\xi) \quad \text{for } \xi \in [-X,X].$$

By the continuity of $(S_\pm, I_\pm, R_\pm)(\xi)$ and the operator $\mathcal{O}$ on $[-X,X]$, we complete the proof. □

Lemma 4.3. *The operator $\mathcal{O}$ is completely continuous with respect to the supremum norm in $C([-X,X],\mathbb{R}^3)$.*

Proof. Since $(S_X, I_X, R_X)(\xi) \in C^1([-X,X],\mathbb{R}^3)$ satisfies (4.15), for $(\phi,\varphi,\chi)(\xi) \in \Omega_X$, we get that $S'_X(\xi)$, $I'_X(\xi)$ and $R'_X(\xi)$ are bounded on $[-X,X]$. Then applying Arzela-Ascoli theorem gives that the operator $\mathcal{O}$ is compact.

For any $\Phi_i(\xi) := (\phi_i,\varphi_i,\chi_i)(\xi) \in \Omega_X$, $i=1,2$, we deduce that

$$\begin{aligned}
&|H_1(\phi_1,\varphi_1,\chi_1)(\xi) - H_1(\phi_2,\varphi_2,\chi_2)(\xi)| \\
&\le d_s|\widehat{\phi}_1(\xi+1) - \widehat{\phi}_2(\xi+1) + \widehat{\phi}_1(\xi-1) - \widehat{\phi}_2(\xi-1)| + \alpha|\phi_1(\xi) - \phi_2(\xi)| \\
&\quad + \left| \frac{\beta\phi_1(\xi)\widehat{\varphi}_1(\xi-c\tau)}{\phi_1(\xi)+\widehat{\varphi}_1(\xi-c\tau)+\chi_1(\xi)} - \frac{\beta\phi_2(\xi)\widehat{\varphi}_2(\xi-c\tau)}{\phi_2(\xi)+\widehat{\varphi}_2(\xi-c\tau)+\chi_2(\xi)} \right| \\
&\le 2d_s \sup_{\xi\in[-X,X]} |\phi_1(\xi)-\phi_2(\xi)| + \alpha \sup_{\xi\in[-X,X]} |\phi_1(\xi)-\phi_2(\xi)| + 2\beta \sup_{\xi\in[-X,X]} |\phi_1(\xi)-\phi_2(\xi)| \\
&\quad + 2\beta \sup_{\xi\in[-X,X]} |\varphi_1(\xi)-\varphi_2(\xi)| + \beta \sup_{\xi\in[-X,X]} |\chi_1(\xi)-\chi_2(\xi)| \\
&\le (2d_s+\alpha+5\beta) \sup_{\xi\in[-X,X]} |\Phi_1(\xi)-\Phi_2(\xi)|.
\end{aligned}$$

Then we derive that

$$\begin{aligned}
&|\mathcal{O}_1(\phi_1,\varphi_1,\chi_1)(\xi) - \mathcal{O}_1(\phi_2,\varphi_2,\chi_2)(\xi)| \\
&\le \frac{1}{c}\int_{-X}^{\xi} e^{\frac{2d_s+\alpha}{c}(\eta-\xi)} |H_1(\phi_1,\varphi_1,\chi_1)(\eta) - H_1(\phi_2,\varphi_2,\chi_2)(\eta)|\, d\eta \\
&\le \frac{2d_s+\alpha+5\beta}{c} \sup_{\xi\in[-X,X]} |\Phi_1(\xi)-\Phi_2(\xi)| \int_{-X}^{\xi} e^{\frac{2d_s+\alpha}{c}(\eta-\xi)}\, d\eta \\
&\le \left(1+\frac{5\beta}{2d_s+\alpha}\right) \sup_{\xi\in[-X,X]} |\Phi_1(\xi)-\Phi_2(\xi)|.
\end{aligned}$$

Similarly, one can obtain that

$$|\mathcal{O}_j(\phi_1,\varphi_1,\chi_1)(\xi) - \mathcal{O}_j(\phi_2,\varphi_2,\chi_2)(\xi)| \le C \sup_{\xi\in[-X,X]} |\Phi_1(\xi)-\Phi_2(\xi)| \quad \text{for } j=2,3,$$

where the positive constant C depends on d_i, d_r, β, γ and δ. This ends the proof. □

Applying Lemmas 4.2, 4.3 and Schauder's fixed point theorem yields that the operator $\mathcal{O}$ has a fixed point, which is a solution of the system

$$\begin{cases} cS'_X(\xi) = d_s D[S_X](\xi) - \frac{\beta S_X(\xi) I_X(\xi - c\tau)}{S_X(\xi) + I_X(\xi - c\tau) + R_X(\xi)}, \\ cI'_X(\xi) = d_i D[I_X](\xi) + \frac{\beta S_X(\xi) I_X(\xi - c\tau)}{S_X(\xi) + I_X(\xi - c\tau) + R_X(\xi)} - (\gamma + \delta) I_X(\xi), \\ cR'_X(\xi) = d_r D[R_X](\xi) + \gamma I_X(\xi), \end{cases} \tag{4.18}$$

such that

$$S_-(\xi) \le S_X(\xi) \le S_+(\xi), \quad I_-(\xi) \le I_X(\xi) \le I_+(\xi)$$

and

$$R_-(\xi) \le R_X(\xi) \le R_+(\xi) \quad \text{for } \xi \in [-X + l, X - l].$$

Lemma 4.4. *If $R_0 > 1$ and $c > c^*$, then system* (1.2) *admits a solution $(S, I, R)(\xi)$ on $\mathbb{R}$ such that*

$$S_-(\xi) \le S(\xi) \le S_+(\xi), \quad I_-(\xi) \le I(\xi) \le I_+(\xi) \quad \textit{and} \quad R_-(\xi) \le R(\xi) \le R_+(\xi). \tag{4.19}$$

Proof. Select an increasing sequence $\{X_n\}_{n \in \mathbb{N}}$ satisfying $X_n \gg l$ for each n and $X_n \to +\infty$ as $n \to \infty$. Let $(S_{X_n}, I_{X_n}, R_{X_n})(\xi)$, $n \in \mathbb{N}$, be the solution of (4.18) with $X = X_n$. For any fixed $N \in \mathbb{N}$, since $R_+(\xi)$ is bounded on $[-X_N, X_N]$, we have that the sequences

$$\{S_{X_n}(\xi)\}_{n \ge N}, \ \{I_{X_n}(\xi)\}_{n \ge N}, \ \{R_{X_n}(\xi)\}_{n \ge N} \text{ and } \left\{ \frac{S_{X_n}(\xi) I_{X_n}(\xi - c\tau)}{S_{X_n}(\xi) + I_{X_n}(\xi - c\tau) + R_{X_n}(\xi)} \right\}_{n \ge N}$$

are uniformly bounded on $[-X_N, X_N]$. By (4.18) with $X = X_n$, we get that the sequences

$$\{S'_{X_n}(\xi)\}_{n \ge N}, \quad \{I'_{X_n}(\xi)\}_{n \ge N} \quad \text{and} \quad \{R'_{X_n}(\xi)\}_{n \ge N}$$

are uniformly bounded on $[-X_N + l, X_N - l]$. Differentiating (4.18) with $X = X_n$, we obtain that the sequences

$$\{S''_{X_n}(\xi)\}_{n \ge N}, \quad \{I''_{X_n}(\xi)\}_{n \ge N} \quad \text{and} \quad \{R''_{X_n}(\xi)\}_{n \ge N}$$

are uniformly bounded on $[-X_N + 2l, X_N - 2l]$. Utilizing Arzela-Ascoli theorem and a standard diagonal extraction argument, we deduce that there is a subsequence which is still denoted by $(S_{X_n}, I_{X_n}, R_{X_n})(\xi)$ such that

$$S_{X_n}(\xi) \to S(\xi), \quad I_{X_n}(\xi) \to I(\xi), \quad R_{X_n}(\xi) \to R(\xi) \quad \text{in } C^1_{\text{loc}}(\mathbb{R}) \text{ as } n \to \infty.$$

Moreover, there holds

$$S_-(\xi) \le S(\xi) \le S_+(\xi), \ \ I_-(\xi) \le I(\xi) \le I_+(\xi) \ \text{ and } \ R_-(\xi) \le R(\xi) \le R_+(\xi) \ \text{ for } \xi \in \mathbb{R}.$$

The proof of this lemma is finished. □

Based on Lemma 4.4, we will proving the following result.

Theorem 4.5. *For a given constant* $S_1 > 0$*, if* $R_0 > 1$ *and* $c > c^*$*, then system* (1.1) *has a nontrivial positive traveling wave solution* $(S, I, R)(\xi)$ *satisfying*

(i) $0 < S(\xi) < S_1$, $0 < I(\xi) < (R_0 - 1)S_1$ *and* $R(\xi) > 0$ *for* $\xi \in \mathbb{R}$.

(ii) $(S, I, R)(-\infty) = (S_1, 0, 0)$. *If* $\xi \to -\infty$*, then* $I(\xi) = O(e^{\rho_1 \xi})$.

(iii) $I(+\infty) = 0$, $S(+\infty) := S_2$ *exists and* $S_2 < S_1$.

(iv) $(\gamma + \delta) \int_{\mathbb{R}} I(\xi)\, d\xi = \beta \int_{\mathbb{R}} \frac{S(\xi)I(\xi - c\tau)}{S(\xi)+I(\xi-c\tau)+R(\xi)}\, d\xi = c(S_1 - S_2)$.

(v) *If* $\limsup_{\xi\to+\infty} R(\xi) < +\infty$*, then* $R(+\infty) = \gamma(S_1 - S_2)/(\gamma+\delta)$ *and* $S'(\xi), I'(\xi), R'(\xi) \to 0$ *as* $\xi \to \pm\infty$.

Proof. (i) By (4.19), we have that $S(\xi) > 0$ on $\mathbb{R}$. Suppose that $I(\widetilde{\xi}_1) = 0$ for some $\widetilde{\xi}_1 \in \mathbb{R}$, then $I'(\widetilde{\xi}_1) = 0$. It follows from the second equation in (1.2) that

$$I(\widetilde{\xi}_1 + 1) = I(\widetilde{\xi}_1 - 1) = 0.$$

By induction we obtain that

$$I(\widetilde{\xi}_1 - n) = 0 \quad \text{for } n \in \mathbb{Z},$$

which contradicts that $I(\xi) \geq I_-(\xi) > 0$ for $\xi \in (-\infty, \xi_3)$. Hence $I(\xi) > 0$ on $\mathbb{R}$. Assume that $R(\widetilde{\xi}_2) = 0$ for some $\widetilde{\xi}_2 \in \mathbb{R}$, then $R'(\widetilde{\xi}_2) = 0$. From the third equation in (1.2), we deduce that $I(\widetilde{\xi}_2) = 0$, a contradiction appears. Thus $R(\xi) > 0$ on $\mathbb{R}$. Suppose that $S(\widetilde{\xi}_3) = S_1$ for some $\widetilde{\xi}_3 \in \mathbb{R}$. Then $S'(\widetilde{\xi}_3) = 0$. By the first equation in (1.2), we have that

$$\begin{aligned} 0 &= d_s D[S](\widetilde{\xi}_3) - cS'(\widetilde{\xi}_3) - \frac{\beta S(\widetilde{\xi}_3) I(\widetilde{\xi}_3 - c\tau)}{S(\widetilde{\xi}_3) + I(\widetilde{\xi}_3 - c\tau) + R(\widetilde{\xi}_3)} \\ &\leq -\frac{\beta S_1 I(\widetilde{\xi}_3 - c\tau)}{S_1 + I(\widetilde{\xi}_3 - c\tau) + R(\widetilde{\xi}_3)} < 0, \end{aligned}$$

which yields a contradiction. So $S(\xi) < S_1$ on $\mathbb{R}$. Assume that $I(\widetilde{\xi}_4) = (R_0 - 1)S_1$ for some $\widetilde{\xi}_4 \in \mathbb{R}$. Then $I'(\widetilde{\xi}_4) = 0$. Using the second equation in (1.2), we get that

$$\begin{aligned} 0 &= d_i D[I](\widetilde{\xi}_4) - cI'(\widetilde{\xi}_4) + \frac{\beta S(\widetilde{\xi}_4) I(\widetilde{\xi}_4 - c\tau)}{S(\widetilde{\xi}_4) + I(\widetilde{\xi}_4 - c\tau) + R(\widetilde{\xi}_4)} - (\gamma + \delta) I(\widetilde{\xi}_4) \\ &< \frac{\beta S_1 (R_0 - 1) S_1}{S_1 + (R_0 - 1) S_1} - (\gamma + \delta)(R_0 - 1) S_1 = 0, \end{aligned}$$

which leads to a contradiction. Therefore, $I(\xi) < (R_0 - 1)S_1$ on $\mathbb{R}$.

(ii) Applying squeeze rule in (4.19) yields that

$$(S,I,R)(-\infty)=(S_1,0,0) \quad \text{and} \quad I(\xi)=O(e^{\rho_1\xi}) \quad \text{as } \xi\to-\infty.$$

(iii) We claim that $I(\xi)$ is integrable on $\mathbb{R}$. Integrating the first equation in (1.2) over $[x,y]$ yields

$$\begin{aligned}
&\int_x^y \frac{\beta S(\xi)I(\xi-c\tau)}{S(\xi)+I(\xi-c\tau)+R(\xi)}\,d\xi\\
&= d_s\int_x^y D[S](\xi)-c\int_x^y S'(\xi)\,d\xi\\
&= d_s\int_x^y\int_0^1 S'(\xi+\theta)\,d\theta d\xi - d_s\int_x^y\int_0^1 S'(\xi-\theta)\,d\theta d\xi - cS(y)+cS(x)\\
&= d_s\int_0^1 [S(y+\theta)-S(x+\theta)]\,d\theta + d_s\int_0^1 [S(x-\theta)-S(y-\theta)]\,d\theta - cS(y)+cS(x)\\
&< (2d_s+c)S_1 \quad (\text{since } 0<S(\xi)<S_1 \text{ on } \mathbb{R}),
\end{aligned}$$

which together with the positiveness of $S(\xi)$, $I(\xi)$ and $R(\xi)$ on $\mathbb{R}$ implies

$$\int_{\mathbb{R}} \frac{S(\xi)I(\xi-c\tau)}{S(\xi)+I(\xi-c\tau)+R(\xi)}\,d\xi<+\infty.$$

Note that $0<I(\xi)<(R_0-1)S_0$ for $\xi\in\mathbb{R}$. Then an integration of the second equation in (1.2) over $[\eta,\zeta]$ gives

$$\begin{aligned}
&(\gamma+\delta)\int_\eta^\zeta I(\xi)\,d\xi\\
&= d_i\int_\eta^\zeta D[I](\xi)-c\int_\eta^\zeta I'(\xi)\,d\xi+\int_\eta^\zeta \frac{\beta S(\xi)I(\xi-c\tau)}{S(\xi)+I(\xi-c\tau)+R(\xi)}\,d\xi\\
&< d_i\int_\eta^\zeta\int_0^1 I'(\xi+\theta)\,d\theta d\xi - d_i\int_\eta^\zeta\int_0^1 I'(\xi-\theta)\,d\theta d\xi - cI(\zeta)+cI(\eta)+(2d_s+c)S_1\\
&= d_i\int_0^1 [I(\zeta+\theta)-I(\eta+\theta)]\,d\theta + d_i\int_0^1 [I(\eta-\theta)-I(\zeta-\theta)]\,d\theta\\
&\quad - cI(\zeta)+cI(\eta)+(2d_s+c)S_1\\
&< (2d_i+c)(R_0-1)S_0+(2d_s+c)S_1.
\end{aligned}$$

This together with the positiveness of $I(\xi)$ on $\mathbb{R}$ ensures that

$$\int_{-\infty}^{\infty} I(\xi)\,d\xi<+\infty.$$

Recall that $0<S(\xi)<S_1$, $0<I(\xi)<(R_0-1)S_1$ and $R(\xi)>0$ for $\xi\in\mathbb{R}$. Then it follows from the second equation in (1.2) that $I'(\xi)$ is uniformly bounded on $\mathbb{R}$. Hence

$I(+\infty) = 0$. Now we investigate the existence of $S(+\infty)$. Assume for the contrary that $\limsup_{\xi\to+\infty} S(\xi) > \liminf_{\xi\to+\infty} S(\xi)$. Then applying Fluctuation Lemma [31, Lemma 2.2] yields that there are two sequences $\{\xi_n\}_{n\in\mathbb{N}}$ and $\{\eta_n\}_{n\in\mathbb{N}}$ satisfying $\xi_n, \eta_n \to +\infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} S(\xi_n) = \limsup_{\xi\to+\infty} S(\xi) := m_1 \quad \text{and} \quad \lim_{n\to\infty} S(\eta_n) = \liminf_{\xi\to+\infty} S(\xi) := m_2 < m_1. \tag{4.20}$$

Denote

$$S_n(y) := S(\eta_n + y), \quad I_n(y) := I(\eta_n + y) \quad \text{and} \quad R_n(y) := R(\eta_n + y), \quad y \in \mathbb{R}.$$

We infer from $I(+\infty) = 0$ that $I_n(y) \to 0$ in $C_{\text{loc}}(\mathbb{R})$ as $n \to \infty$. By (4.19) and the first equation in (1.2), we deduce that $S(\xi)$, $S'(\xi)$ and $S''(\xi)$ are uniformly bounded on $\mathbb{R}$. Then there exists a subsequence $\{n_k\}$ by a diagonal extraction process, which is still denoted by $\{n\}$, such that $S_n(y) \to S_\infty(y)$ in $C^1_{\text{loc}}(\mathbb{R})$ as $n \to \infty$. Note that $S_\infty(0) = m_2$. From (4.20) and the first equation in (1.2), we have that

$$cS_n'(y) = d_s D[S_n](y) - \frac{\beta S_n(y) I_n(y - c\tau)}{S_n(y) + I_n(y - c\tau) + R_n(y)}, \quad y \in \mathbb{R}. \tag{4.21}$$

Passing to the limits in (4.21) as $n \to \infty$ gives that

$$cS_\infty'(y) = d_s D[S_\infty](y), \quad y \in \mathbb{R}. \tag{4.22}$$

With the aid of [4, Theorem 3.1 and Remark 3.1], we get from (4.22) that

$$S_\infty(y) = a_1 + a_2 e^{\nu y}, \quad y \in \mathbb{R}, \tag{4.23}$$

where a_1, a_2 are constants and ν is the unique positive root of $d_s(e^\nu + e^{-\nu} - 2) - c\nu = 0$. Then by the boundedness of $S_\infty(y)$ on $\mathbb{R}$ and (4.23), we derive that $a_2 = 0$. Thus $S_\infty(y) = a_1 = S_\infty(0) = m_2$, which ensures that

$$\lim_{n\to\infty} S(\eta_n + y) = m_2 \quad \text{in } C^1_{\text{loc}}(\mathbb{R}). \tag{4.24}$$

Analogously,

$$\lim_{n\to\infty} S(\xi_n + y) = m_1 \quad \text{in } C^1_{\text{loc}}(\mathbb{R}). \tag{4.25}$$

By the similar arguments as that in Lemma 2.4, one can deduce that

$$\int_{\mathbb{R}} \frac{S(\xi) I(\xi - c\tau)}{S(\xi) + I(\xi - c\tau) + R(\xi)} \, d\xi < +\infty,$$

which leads to

$$\lim_{n\to\infty} \int_{\eta_n}^{\xi_n} \frac{S(\xi) I(\xi - c\tau)}{S(\xi) + I(\xi - c\tau) + R(\xi)} \, d\xi = 0. \tag{4.26}$$

Integrating the first equation in (1.2) over $[\eta_n, \xi_n]$ and using (4.24)–(4.26) and dominated convergence theorem, we obtain that

$$\begin{aligned}
0 &< c(m_1 - m_2) \\
&= c \lim_{n\to\infty} [S(\xi_n) - S(\eta_n)] \\
&= d_s \lim_{n\to\infty} \int_{\eta_n}^{\xi_n} [S(\xi+1) - S(\xi)]\, d\xi + d_s \lim_{n\to\infty} \int_{\eta_n}^{\xi_n} [S(\xi-1) - S(\xi)]\, d\xi \\
&\quad - \lim_{n\to\infty} \int_{\eta_n}^{\xi_n} \frac{\beta S(\xi) I(\xi - c\tau)}{S(\xi) + I(\xi - c\tau) + R(\xi)}\, d\xi \\
&= d_s \lim_{n\to\infty} \int_{\eta_n}^{\xi_n} \int_0^1 S'(\xi+\theta)\, d\theta d\xi - d_s \lim_{n\to\infty} \int_{\eta_n}^{\xi_n} \int_0^1 S'(\xi-\theta)\, d\theta d\xi \\
&= d_s \lim_{n\to\infty} \int_0^1 [S(\xi_n+\theta) - S(\eta_n+\theta)]\, d\theta - d_s \lim_{n\to\infty} \int_0^1 [S(\xi_n-\theta) - S(\eta_n-\theta)]\, d\theta \\
&= 0.
\end{aligned}$$

A contradiction appears. Hence $S(+\infty)$ exists and we denote it by S_2. Then we present $S_2 < S_1$. Since $S(\xi) < S_1$ on $\mathbb{R}$, we have $S_2 \le S_1$. Assume that $S_2 = S_1$. An integration of the first equation in (1.2) over $\mathbb{R}$ yields

$$\int_{\mathbb{R}} \frac{S(\xi) I(\xi - c\tau)}{S(\xi) + I(\xi - c\tau) + R(\xi)}\, d\xi = 0,$$

which contradicts that fact that

$$\int_{\mathbb{R}} \frac{S(\xi) I(\xi - c\tau)}{S(\xi) + I(\xi - c\tau) + R(\xi)}\, d\xi > 0.$$

Therefore, $S_2 < S_1$.

(iv) Integrating the first two equations in (1.2) over $\mathbb{R}$, respectively, and using the asymptotic boundaries of $S(\xi)$ and $I(\xi)$, we have

$$(\gamma + \delta) \int_{\mathbb{R}} I(\xi)\, d\xi = \beta \int_{\mathbb{R}} \frac{S(\xi) I(\xi - c\tau)}{S(\xi) + I(\xi - c\tau) + R(\xi)}\, d\xi = c(S_1 - S_2). \tag{4.27}$$

Suppose that $\limsup_{\xi\to+\infty} R(\xi) > \liminf_{\xi\to+\infty} R(\xi)$. Then there exist two sequences $\{\zeta_n\}$ and $\{\vartheta_n\}$ satisfying $\zeta_n, \vartheta_n \to +\infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} R(\zeta_n) = \limsup_{\xi\to+\infty} R(\xi) := m_3 \quad \text{and} \quad \lim_{n\to\infty} R(\vartheta_n) = \liminf_{\xi\to+\infty} R(\xi) := m_4 < m_3.$$

By the analogous arguments as that in (iii), one can obtain that

$$\lim_{n\to\infty} R(\zeta_n + y) = m_3 \quad \text{and} \quad \lim_{n\to\infty} R(\vartheta_n + y) = m_4 \quad \text{in } C^1_{\text{loc}}(\mathbb{R}).$$

Hence integrating the third equation in (1.2) over $[\vartheta_n, \zeta_n]$ and taking $n \to \infty$, we get

$$\begin{aligned}
0 &< c(m_3 - m_4) \\
&= c \lim_{n\to\infty} [R(\zeta_n) - R(\vartheta_n)] \\
&= d_r \lim_{n\to\infty} \int_{\vartheta_n}^{\zeta_n} [R(\xi+1) - R(\xi)]\, d\xi + d_r \lim_{n\to\infty} \int_{\vartheta_n}^{\zeta_n} [R(\xi-1) - R(\xi)]\, d\xi \\
&\quad + \lim_{n\to\infty} \int_{\vartheta_n}^{\zeta_n} \gamma I(\xi)\, d\xi \\
&= d_r \lim_{n\to\infty} \int_{\vartheta_n}^{\zeta_n} \int_0^1 R'(\xi+\theta)\, d\theta d\xi - d_r \lim_{n\to\infty} \int_{\vartheta_n}^{\zeta_n} \int_0^1 R'(\xi-\theta)\, d\theta d\xi \\
&= d_r \lim_{n\to\infty} \int_0^1 [R(\zeta_n+\theta) - R(\vartheta_n+\theta)]\, d\theta - d_r \lim_{n\to\infty} \int_0^1 [R(\zeta_n-\theta) - R(\vartheta_n-\theta)]\, d\theta \\
&= 0.
\end{aligned}$$

This contradiction guarantees that the existence of $R(+\infty)$. Integrating the third equation in (1.2) and utilizing (4.27), we obtain that

$$R(+\infty) = \frac{\gamma(S_1 - S_2)}{\gamma + \delta}.$$

Passing to the limits in (1.2) as $\xi \to \pm\infty$, respectively, and employing the asymptotic boundary of $(S, I, R)(\xi)$, we get

$$S'(\xi), I'(\xi), R'(\xi) \to 0 \quad \text{as } \xi \to \pm\infty.$$

The proof is completed. □

Remark 4.6. In Theorem 4.5(v), we make use of a condition $\limsup_{\xi\to+\infty} R(\xi) < +\infty$ to prove the existence of $R(+\infty)$. From the view of mathematical biology, this condition may fit reality. However, we do not obtain it by rigorous analysis. We leave it for future investigation.

5. Existence of critical traveling waves

In this section, we will prove the existence result under the conditions $R_0 > 1$ and $c = c^*$. For this, we set

$$M_4 := \frac{(R_0 - 1) S_1 \rho^* e^{\rho^*+1}}{\rho^* + 1} \quad \text{and} \quad \xi_5 := -\frac{1}{\rho^*} - 1,$$

where $\rho^* > 0$ is defined in Lemma 2.2 and $S_1 > 0$ is a given constant. Choose a sufficient small constant $\rho_5 \in (0, \rho^*)$ and a suitable constant $\sigma_2 > 0$ such that $\rho_5^{-1} \gg S_1$ and

$$S_1 - \rho_5^{-1} e^{\rho_5 \xi} = \sigma_2 e^{-\frac{\beta}{c^*}\xi}$$

has two negative roots and we select the bigger one as ξ_4. By the choice of ρ_5, one can deduce that

(5.1)
$$\xi_4 < \xi_5 \quad \text{and} \quad \rho_5^{-1} d_s(2 - e^{\rho_5} - e^{-\rho_5}) + c^* + \beta M_4(\xi - c^*\tau)e^{(\rho^*-\rho_5)\xi - \rho^* c^* \tau} \geq 0 \quad \text{for } \xi < \xi_4.$$

We choose suitable constants $M_6 > 0$ and $\epsilon_2 \in (0, \min\{\rho_3, \rho^*\})$ such that

(5.2)
$$-G(\epsilon_2, c^*) + \frac{\gamma(R_0 - 1)S_1 e^{-\epsilon_2 \xi_5}}{M_6} \leq 0 \quad \text{and} \quad -G(\epsilon_2, c^*) - \frac{\gamma M_4}{M_6}\xi e^{(\rho^* - \epsilon_2)\xi} \leq 0 \quad \text{for } \xi < \xi_5.$$

Notice the fact that

$$2M_4^2(-\xi)^{3/2}(\xi - c^*\tau)^2 e^{\rho^*(\xi - c^*\tau)} - 2M_4 M_6(-\xi)^{3/2}(\xi - c^*\tau)e^{\epsilon_2 \xi} \to 0 \quad \text{as } \xi \to -\infty,$$

then there exists a sufficiently large constant $|\xi_*|$ with $\xi_* < 0$ such that

(5.3)
$$2M_4^2(-\xi)^{3/2}(\xi - c^*\tau)^2 e^{\rho^*(\xi - c^*\tau)} - 2M_4 M_6(-\xi)^{3/2}(\xi - c^*\tau)e^{\epsilon_2 \xi} < \frac{(c^*\tau)^2}{16} S_1 \text{ and } 1 + \frac{c^*\tau}{\xi} > 0$$

for $\xi < \xi_*$. Define $\xi_6 := -M_5^2/M_4^2$ and take the constant $M_5 \gg 1$ such that

$$\xi_6 < \xi_4, \quad \xi_6 < \xi_* \quad \text{and} \quad S_1 - \rho_5^{-1} e^{\rho_5 \xi_6} \geq S_1/2. \tag{5.4}$$

With the choices of above parameters, we introduce the following nonnegative continuous functions on $\mathbb{R}$.

$$S_+^*(\xi) := S_1, \qquad S_-^*(\xi) := \begin{cases} S_1 - \rho_5^{-1} e^{\rho_5 \xi}, & \xi < \xi_4, \\ \sigma_2 e^{-\frac{\beta}{c^*}\xi}, & \xi \geq \xi_4, \end{cases}$$

$$I_+^*(\xi) := \begin{cases} -M_4 \xi e^{\rho^* \xi}, & \xi < \xi_5, \\ (R_0 - 1)S_1, & \xi \geq \xi_5, \end{cases} \qquad I_-^*(\xi) := \begin{cases} \left[-M_4\xi - M_5(-\xi)^{1/2}\right] e^{\rho^* \xi}, & \xi < \xi_6, \\ 0, & \xi \geq \xi_6, \end{cases}$$

$$R_+^*(\xi) := M_6 e^{\epsilon_2 \xi}, \qquad R_-^*(\xi) := 0.$$

To illustrate that the parameters are admissible, we give Figures 5.1, 5.2 and 5.3.

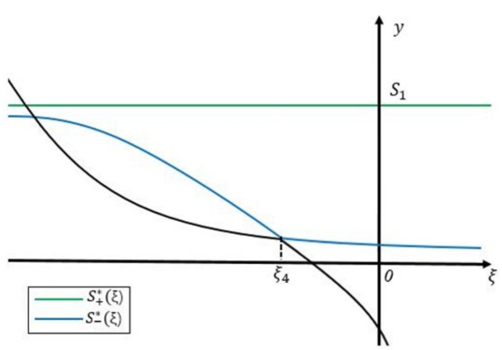

Figure 5.1: $S_+^*(\xi)$ and $S_-^*(\xi)$ when $R_0 > 1$ and $c = c^*$.

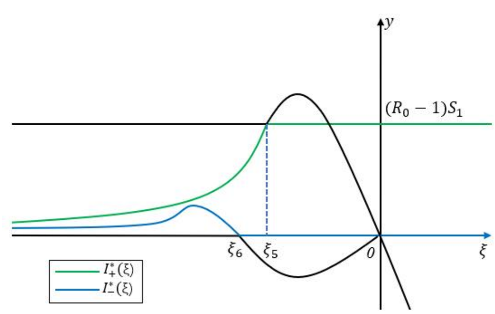

Figure 5.2: $I_+^*(\xi)$ and $I_-^*(\xi)$ when $R_0 > 1$ and $c = c^*$.

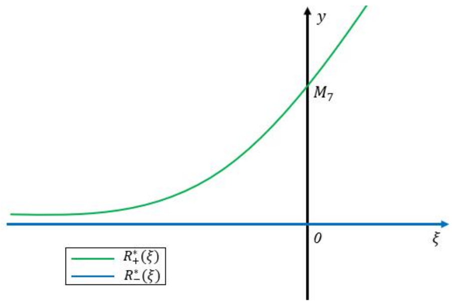

Figure 5.3: $R_+^*(\xi)$ and $R_-^*(\xi)$ when $R_0 > 1$ and $c = c^*$.

Lemma 5.1. *The functions* $S_\pm^*(\xi)$, $I_\pm^*(\xi)$ *and* $R_\pm^*(\xi)$ *satisfy*

$$d_s D[S_+^*](\xi) - c^*(S_+^*)'(\xi) - \frac{\beta S_+^*(\xi) I_-^*(\xi - c^*\tau)}{S_+^*(\xi) + I_-^*(\xi - c^*\tau) + R_+^*(\xi)} \le 0, \quad \xi \in \mathbb{R}, \tag{5.5}$$

$$d_i D[I_+^*](\xi) - c^*(I_+^*)'(\xi) + \frac{\beta S_+^*(\xi) I_+^*(\xi - c^*\tau)}{S_+^*(\xi) + I_+^*(\xi - c^*\tau) + R_-^*(\xi)} - (\gamma + \delta) I_+^*(\xi) \le 0, \quad \xi \ne \xi_5, \tag{5.6}$$

$$d_r D[R_+^*](\xi) - c^*(R_+^*)'(\xi) + \gamma I_+^*(\xi) \le 0, \quad \xi \in \mathbb{R}, \tag{5.7}$$

$$d_s D[S_-^*](\xi) - c^*(S_-^*)'(\xi) - \frac{\beta S_-^*(\xi) I_+^*(\xi - c^*\tau)}{S_-^*(\xi) + I_+^*(\xi - c^*\tau) + R_-^*(\xi)} \ge 0, \quad \xi \ne \xi_4, \tag{5.8}$$

$$d_i D[I_-^*](\xi) - c^*(I_-^*)'(\xi) + \frac{\beta S_-^*(\xi) I_-^*(\xi - c^*\tau)}{S_-^*(\xi) + I_-^*(\xi - c^*\tau) + R_+^*(\xi)} - (\gamma + \delta) I_-^*(\xi) \ge 0, \quad \xi \ne \xi_6, \tag{5.9}$$

$$d_r D[R_-^*](\xi) - c^*(R_-^*)'(\xi) + \gamma I_-^*(\xi) \ge 0, \quad \xi \in \mathbb{R}. \tag{5.10}$$

Proof. It is easy to verify that (5.5) and (5.10) hold under the definitions of $S_+^*(\xi)$, $I_-^*(\xi)$ and $R_\pm^*(\xi)$ on $\mathbb{R}$. The rest of the proof is devoted to proving inequalities (5.6)–(5.9).

Proof of (5.6). When $\xi < \xi_5$, we get that $I_+^*(\xi) = -M_4 \xi e^{\rho^* \xi}$, $I_+^*(\xi - c^*\tau) = -M_4(\xi - c^*\tau) e^{\rho^*(\xi - c^*\tau)}$, $I_+^*(\xi - 1) = -M_4(\xi - 1) e^{\rho^*(\xi - 1)}$, $I_+^*(\xi + 1) \le -M_4(\xi + 1) e^{\rho^*(\xi+1)}$ and $(I_+^*)'(\xi) = -M_4 e^{\rho^* \xi}(1 + \rho^* \xi)$. Then by (2.1) and (2.2), we compute that

$$\begin{aligned}
& d_i D[I_+^*](\xi) - c^*(I_+^*)'(\xi) + \frac{\beta S_+^*(\xi) I_+^*(\xi - c^*\tau)}{S_+^*(\xi) + I_+^*(\xi - c^*\tau) + R_-^*(\xi)} - (\gamma + \delta) I_+^*(\xi) \\
\le\ & d_i D[I_+^*](\xi) - c^*(I_+^*)'(\xi) + \beta I_+^*(\xi - c^*\tau) - (\gamma + \delta) I_+^*(\xi) \\
\le\ & d_i \big[- M_4(\xi + 1) e^{\rho^*(\xi+1)} - M_4(\xi - 1) e^{\rho^*(\xi-1)} + 2 M_4 \xi e^{\rho^* \xi} \big] \\
& + c^* M_4 e^{\rho^* \xi}(1 + \rho^* \xi) - \beta M_4(\xi - c^*\tau) e^{\rho^*(\xi - c^*\tau)} + (\gamma + \delta) M_4 \xi e^{\rho^* \xi} \\
=\ & -M_4 \xi e^{\rho^* \xi} F(\rho^*, c^*) - M_4 e^{\rho^* \xi} F_\rho(\rho^*, c^*) = 0.
\end{aligned}$$

When $\xi > \xi_5$, we have that $I_+^*(\xi) = I_+^*(\xi + 1) = (R_0 - 1) S_1$, $I_+^*(\xi - c^*\tau) \le (R_0 - 1) S_1$,

$I_+^*(\xi-1)\le (R_0-1)S_1$, $S_+^*(\xi)=S_1$ and $R_-^*(\xi)=0$. Then we infer that

$$\begin{aligned}&d_iD[I_+^*](\xi)-c^*(I_+^*)'(\xi)+\frac{\beta S_+^*(\xi)I_+^*(\xi-c^*\tau)}{S_+^*(\xi)+I_+^*(\xi-c^*\tau)+R_-^*(\xi)}-(\gamma+\delta)I_+^*(\xi)\\ \le&\frac{\beta S_1(R_0-1)S_1}{S_1+(R_0-1)S_1}-(\gamma+\delta)(R_0-1)S_1=0.\end{aligned}$$

Proof of (5.7). By the expression of $I_+^*(\xi)$ and $R_+^*(\xi)$ on $\mathbb{R}$, we obtain from Lemma 2.3 and (5.2) that

$$\begin{aligned}&d_rD[R_+^*](\xi)-c^*(R_+^*)'(\xi)+\gamma I_+^*(\xi)\\ =&d_r\left[M_6e^{\epsilon_2(\xi+1)}+M_6e^{\epsilon_2(\xi-1)}-2M_6e^{\epsilon_2\xi}\right]-c^*M_6\epsilon_2e^{\epsilon_2\xi}-\gamma M_4\xi e^{\rho^*\xi}\\ =&M_6e^{\epsilon_2\xi}\left[d_r(e^{\epsilon_2}+e^{-\epsilon_2}-2)-c^*\epsilon_2-\frac{\gamma M_4\xi e^{(\rho^*-\epsilon_2)\xi}}{M_6}\right]\\ =&M_6e^{\epsilon_2\xi}\left[-G(\epsilon_2,c^*)-\frac{\gamma M_4\xi e^{(\rho^*-\epsilon_2)\xi}}{M_6}\right]\le 0\quad\text{for } \xi<\xi_5\end{aligned}$$

and

$$\begin{aligned}&d_rD[R_+^*](\xi)-c^*(R_+^*)'(\xi)+\gamma I_+^*(\xi)\\ =&d_r\left[M_6e^{\epsilon_2(\xi+1)}+M_6e^{\epsilon_2(\xi-1)}-2M_6e^{\epsilon_2\xi}\right]-c^*M_6\epsilon_2e^{\epsilon_2\xi}+\gamma(R_0-1)S_1\\ =&M_6e^{\epsilon_2\xi}\left[d_r(e^{\epsilon_2}+e^{-\epsilon_2}-2)-c^*\epsilon_2+\frac{\gamma(R_0-1)S_1e^{-\epsilon_2\xi}}{M_6}\right]\\ =&M_6e^{\epsilon_2\xi}\left[-G(\epsilon_2,c^*)+\frac{\gamma(R_0-1)S_1e^{-\epsilon_2\xi_5}}{M_6}\right]\le 0\quad\text{for } \xi\ge\xi_5.\end{aligned}$$

Proof of (5.8). When $\xi<\xi_4$, $S_-^*(\xi)=S_1-\rho_5^{-1}e^{\rho_5\xi}$, $S_-^*(\xi-1)=S_1-\rho_5^{-1}e^{\rho_5(\xi-1)}$, $S_-^*(\xi+1)\ge S_1-\rho_5^{-1}e^{\rho_5(\xi+1)}$, $I_+^*(\xi-c^*\tau)=-M_4(\xi-c^*\tau)e^{\rho^*(\xi-c^*\tau)}$ and $R_-^*(\xi)=0$. Then it follows from (5.1) that

$$\begin{aligned}&d_sD[S_-^*](\xi)-c^*(S_-^*)'(\xi)-\frac{\beta S_-^*(\xi)I_+^*(\xi-c^*\tau)}{S_-^*(\xi)+I_+^*(\xi-c^*\tau)+R_-^*(\xi)}\\ \ge&d_sD[S_-^*](\xi)-c^*(S_-^*)'(\xi)-\beta I_+^*(\xi-c^*\tau)\\ \ge&d_s\left[2\rho_5^{-1}e^{\rho_5\xi}-\rho_5^{-1}e^{\rho_5(\xi+1)}-\rho_5^{-1}e^{\rho_5(\xi-1)}\right]+c^*e^{\rho_5\xi}+\beta M_4(\xi-c^*\tau)e^{\rho^*(\xi-c^*\tau)}\\ =&e^{\rho_5\xi}\left[\rho_5^{-1}d_s(2-e^{\rho_5}-e^{-\rho_5})+c^*+\beta M_4(\xi-c^*\tau)e^{(\rho^*-\rho_5)\xi-\rho^*c^*\tau}\right]\\ \ge&0\quad\text{for } \xi<\xi_4.\end{aligned}$$

When $\xi>\xi_4$, $S_-^*(\xi)=\sigma_2e^{-\frac{\beta}{c^*}\xi}$, $S_-^*(\xi+1)=\sigma_2e^{-\frac{\beta}{c^*}(\xi+1)}$ and $S_-^*(\xi-1)\ge\sigma_2e^{-\frac{\beta}{c^*}(\xi-1)}$. Then we have that

$$d_sD[S_-^*](\xi)-c^*(S_-^*)'(\xi)-\frac{\beta S_-^*(\xi)I_+^*(\xi-c^*\tau)}{S_-^*(\xi)+I_+^*(\xi-c^*\tau)+R_-^*(\xi)}$$

$$
\begin{aligned}
&\geq d_s D[S_-^*](\xi) - c^*(S_-^*)'(\xi) - \beta S_-^*(\xi) \\
&\geq d_s \left[\sigma_2 e^{-\frac{\beta}{c^*}(\xi+1)} + \sigma_2 e^{-\frac{\beta}{c^*}(\xi-1)} - 2\sigma_2 e^{-\frac{\beta}{c^*}\xi}\right] + \beta\sigma_2 e^{-\frac{\beta}{c^*}\xi} - \beta\sigma_2 e^{-\frac{\beta}{c^*}\xi} \\
&= d_s \sigma_2 e^{-\frac{\beta}{c^*}\xi}\big(e^{-\beta/c^*} + e^{\beta/c^*} - 2\big) \\
&\geq 0 \quad \text{for } \xi > \xi_4.
\end{aligned}
$$

Proof of (5.9). By (5.4) we have for $\xi < \xi_6$ that

$$
\begin{aligned}
S_-^*(\xi) &= S_1 - \rho_5^{-1} e^{\rho_5 \xi} \geq S_1/2, \\
I_-^*(\xi) &= \big[-M_4\xi - M_5(-\xi)^{1/2}\big]e^{\rho^*\xi}, \\
(I_-^*)'(\xi) &= \left[-M_4 + \frac{1}{2}M_5(-\xi)^{-1/2}\right]e^{\rho^*\xi} + \big[-M_4\rho^*\xi - M_5\rho^*(-\xi)^{1/2}\big]e^{\rho^*\xi}, \\
I_-^*(\xi-1) &= \big[-M_4(\xi-1) - M_5(-\xi+1)^{1/2}\big]e^{\rho^*(\xi-1)}, \\
I_-^*(\xi-c^*\tau) &= \big[-M_4(\xi-c^*\tau) - M_5(-\xi+c^*\tau)^{1/2}\big]e^{\rho^*(\xi-c^*\tau)}, \\
I_-^*(\xi+1) &\geq \big[-M_4(\xi+1) - M_5(-\xi-1)^{1/2}\big]e^{\rho^*(\xi+1)}.
\end{aligned}
\tag{5.11}
$$

Applying Taylor's formula, we deduce for $\xi < \xi_6$ that

$$
\begin{aligned}
(-\xi+1)^{1/2} &\leq (-\xi)^{1/2} + \frac{1}{2}(-\xi)^{-1/2}, \\
(-\xi-1)^{1/2} &\leq (-\xi)^{1/2} - \frac{1}{2}(-\xi)^{-1/2}, \\
(-\xi+c^*\tau)^{1/2} &\leq (-\xi)^{1/2} + \frac{c^*\tau}{2}(-\xi)^{-1/2} - \frac{(c^*\tau)^2}{8}(-\xi)^{-3/2} + \frac{(c^*\tau)^3}{16}(-\xi)^{-5/2}.
\end{aligned}
\tag{5.12}
$$

Noting that $I_-^*(\xi - c^*\tau) \leq I_+^*(\xi - c^*\tau) = -M_4(\xi - c^*\tau)e^{\rho^*(\xi-c^*\tau)}$ for $\xi < \xi_6$, we get from (5.11) that

$$
\begin{aligned}
&-\beta I_-^*(\xi-c^*\tau) + \frac{\beta S_-^*(\xi) I_-(\xi-c^*\tau)}{S_-^*(\xi) + I_-^*(\xi-c^*\tau) + R_+^*(\xi)} \\
&= -\frac{\beta (I_-^*)^2(\xi-c^*\tau) + \beta I_-^*(\xi-c^*\tau)R_+^*(\xi)}{S_-^*(\xi) + I_-^*(\xi-c^*\tau) + R_+^*(\xi)} \geq -\frac{\beta (I_-^*)^2(\xi-c^*\tau) + \beta I_-^*(\xi-c^*\tau)R_+^*(\xi)}{S_-^*(\xi)} \\
&\geq -\frac{2\beta}{S_1}\big[M_4^2(\xi-c^*\tau)^2 e^{2\rho^*(\xi-c^*\tau)} - M_4M_6(\xi-c^*\tau)e^{(\rho^*+\epsilon_2)\xi-\rho^*c^*\tau}\big] \quad \text{for } \xi < \xi_6.
\end{aligned}
\tag{5.13}
$$

Using (5.3), (5.4), (5.11)–(5.13), (2.1) and (2.2), we obtain for $\xi < \xi_6$ that

$$
\begin{aligned}
&d_i D[I_-^*](\xi) - c^*(I_-^*)'(\xi) + \frac{\beta S_-^*(\xi) I_-^*(\xi-c^*\tau)}{S_-^*(\xi) + I_-^*(\xi-c^*\tau) + R_+^*(\xi)} - (\gamma+\delta)I_-^*(\xi) \\
&= d_i D[I_-^*](\xi) - c^*(I_-^*)'(\xi) + \beta I_-^*(\xi-c^*\tau) - (\gamma+\delta)I_-^*(\xi) - \beta I_-^*(\xi-c^*\tau) \\
&\quad + \frac{\beta S_-^*(\xi) I_-^*(\xi-c^*\tau)}{S_-^*(\xi) + I_-^*(\xi-c^*\tau) + R_+^*(\xi)}
\end{aligned}
$$

$$
\begin{aligned}
&\geq d_i\Big\{\big[-M_4(\xi+1)-M_5(-\xi-1)^{1/2}\big]e^{\rho^*(\xi+1)}+\big[-M_4(\xi-1)-M_5(-\xi+1)^{1/2}\big]e^{\rho^*(\xi-1)}\\
&\quad-2\big[-M_4\xi-M_5(-\xi)^{1/2}\big]e^{\rho^*\xi}\Big\}\\
&\quad-c^*\left\{\left[-M_4+\frac{1}{2}M_5(-\xi)^{-1/2}\right]e^{\rho^*\xi}+\big[-M_4\rho^*\xi-M_5\rho^*(-\xi)^{1/2}\big]e^{\rho^*\xi}\right\}\\
&\quad+\beta\big[-M_4(\xi-c^*\tau)-M_5(-\xi+c^*\tau)^{1/2}\big]e^{\rho^*(\xi-c^*\tau)}-(\gamma+\delta)\big[-M_4\xi-M_5(-\xi)^{1/2}\big]e^{\rho^*\xi}\\
&\quad-\frac{2\beta}{S_1}\big[M_4^2(\xi-c^*\tau)^2e^{2\rho^*(\xi-c^*\tau)}-M_4M_6(\xi-c^*\tau)e^{(\rho^*+\epsilon_2)\xi-\rho^*c^*\tau}\big]\\
&\geq d_i\left\{\left[-M_4(\xi+1)-M_5\left((-\xi)^{1/2}-\frac{1}{2}(-\xi)^{-1/2}\right)\right]e^{\rho^*(\xi+1)}\right.\\
&\quad\left.+\left[-M_4(\xi-1)-M_5\left((-\xi)^{1/2}+\frac{1}{2}(-\xi)^{-1/2}\right)\right]e^{\rho^*(\xi-1)}-2\big[-M_4\xi-M_5(-\xi)^{1/2}\big]e^{\rho^*\xi}\right\}\\
&\quad-c^*\left[-M_4+\frac{1}{2}M_5(-\xi)^{-1/2}\right]e^{\rho^*\xi}-c^*\rho^*\big[-M_4\xi-M_5(-\xi)^{1/2}\big]e^{\rho^*\xi}-\beta M_4(\xi-c^*\tau)e^{\rho^*(\xi-c^*\tau)}\\
&\quad-\beta M_5\left[(-\xi)^{1/2}+\frac{c^*\tau}{2}(-\xi)^{-1/2}-\frac{(c^*\tau)^2}{8}(-\xi)^{-3/2}+\frac{(c^*\tau)^3}{16}(-\xi)^{-5/2}\right]e^{\rho^*(\xi-c^*\tau)}\\
&\quad-(\gamma+\delta)\big[-M_4\xi-M_5(-\xi)^{1/2}\big]e^{\rho^*\xi}\\
&\quad-\frac{2\beta}{S_1}\big[M_4^2(\xi-c^*\tau)^2e^{2\rho^*(\xi-c^*\tau)}-M_4M_6(\xi-c^*\tau)e^{(\rho^*+\epsilon_2)\xi-\rho^*c^*\tau}\big]\\
&=\big[-M_4\xi-M_5(-\xi)^{1/2}\big]e^{\rho^*\xi}F(\rho^*,c^*)+\left[-M_4+\frac{1}{2}M_5(-\xi)^{-1/2}\right]e^{\rho^*\xi}F_\rho(\rho^*,c^*)\\
&\quad+\beta M_5\left[\frac{(c^*\tau)^2}{8}(-\xi)^{-3/2}-\frac{(c^*\tau)^3}{16}(-\xi)^{-5/2}\right]e^{\rho^*(\xi-c^*\tau)}\\
&\quad-\frac{2\beta}{S_1}\big[M_4^2(\xi-c^*\tau)^2e^{2\rho^*(\xi-c^*\tau)}-M_4M_6(\xi-c^*\tau)e^{(\rho^*+\epsilon_2)\xi-\rho^*c^*\tau}\big]\\
&=\beta M_5(-\xi)^{-3/2}e^{\rho^*(\xi-c^*\tau)}\frac{(c^*\tau)^2}{16}\left(1+\frac{c^*\tau}{\xi}\right)\\
&\quad+\frac{\beta}{S_1}(-\xi)^{-3/2}e^{\rho^*(\xi-c^*\tau)}\\
&\quad\times\left[\frac{(c^*\tau)^2}{16}M_5S_1-2M_4^2(-\xi)^{3/2}(\xi-c^*\tau)^2e^{\rho^*(\xi-c^*\tau)}+2M_4M_6(-\xi)^{3/2}(\xi-c^*\tau)e^{\epsilon_2\xi}\right]\\
&\geq 0.
\end{aligned}
$$

When $\xi>\xi_6$, $I_-^*(\xi)=0$ and inequality (5.9) follows trivially. The claim of this lemma is shown. □

Now we state the existence result of critical traveling wave solution for (1.1).

Theorem 5.2. *For a given constant $S_1>0$, if $R_0>1$ and $c=c^*$, then system* (1.1) *has a traveling wave solution $(S,I,R)(\xi)$ satisfying the following assertions.*

(i) $0<S(\xi)<S_1$, $0<I(\xi)<(R_0-1)S_1$ *and* $R(\xi)>0$ *for* $\xi\in\mathbb{R}$.

(ii) $(S,I,R)(-\infty)=(S_1,0,0)$. *If* $\xi\to-\infty$, *then* $I(\xi)=O(-\xi e^{\rho^*\xi})$.

(iii) $I(+\infty) = 0$, $S(+\infty) := S_2$ *exists and* $S_2 < S_1$.

(iv) $(\gamma + \delta) \int_{\mathbb{R}} I(\xi)\, d\xi = \beta \int_{\mathbb{R}} \frac{S(\xi) I(\xi - c^*\tau)}{S(\xi)+I(\xi-c^*\tau)+R(\xi)}\, d\xi = c^*(S_1 - S_2)$.

(v) *If* $\limsup_{\xi\to+\infty} R(\xi) < +\infty$, *then* $R(+\infty) = \gamma(S_1 - S_2)/(\gamma+\delta)$ *and* $S'(\xi), I'(\xi), R'(\xi) \to 0$ *as* $\xi \to \pm\infty$.

(vi) $\frac{\partial c^*}{\partial d_i} > 0$, $\frac{\partial c^*}{\partial \beta} > 0$ *and* $\frac{\partial c^*}{\partial \tau} < 0$.

Proof. Applying the functions $S^*_\pm(\xi)$, $I^*_\pm(\xi)$ and $R^*_\pm(\xi)$ defined at the beginning of this section and the analogous discussions in Section 4, we can obtain system (1.1) admits a critical traveling wave solution $(S, I, R)(\xi)$ satisfying (i)–(v). In the following, we shall prove (vi) of this theorem. From Lemma 2.2, we can compute that

$$F_{c^*}(\rho^*, c^*) = -\rho^* - \beta\rho^*\tau e^{-\rho^* c^* \tau} < 0, \quad F_{d_i}(\rho^*, c^*) = e^{\rho^*} + e^{-\rho^*} - 2 > 0,$$
$$F_\beta(\rho^*, c^*) = e^{-\rho^* c^* \tau} > 0 \quad \text{and} \quad F_\tau(\rho^*, c^*) = -\beta\rho^* c^* e^{-\rho^* c^* \tau} < 0,$$

which together with derivative rule for implicit functions implies that

$$\frac{\partial c^*}{\partial d_i} > 0, \quad \frac{\partial c^*}{\partial \beta} > 0 \quad \text{and} \quad \frac{\partial c^*}{\partial \tau} < 0.$$

The proof of this theorem is completed. □

Remark 5.3. Theorems 3.1, 4.5 and 5.2 mainly reveal the sufficient conditions of existence and nonexistence of traveling wave solutions for (1.1), and the characterization of their minimal speed. One can observe that the obtained traveling waves include the pulse-type (I-component) and front-type (S-component and R-component) traveling waves. Notice that $I(-\infty) = 0$ for both $c > c^*$ and $c = c^*$, while the exact decay rates of $I(-\infty) = 0$ are distinct for these cases. The limit value $I(+\infty) = 0$ implies that the infected individuals will disappear after a long time.

Remark 5.4. Inequalities in Theorem 5.2(vi) reflect that the geographical movement of the infected individuals and the interaction between infected individuals and susceptible individuals can accelerate the speed of propagation of the epidemic, while the time delay can slow down the speed of transmission of the epidemic.

By Theorems 4.5 and 5.2, we can further obtain some properties concerning the functions $S(\xi)$, $I(\xi)$ and $R(\xi)$.

Proposition 5.5. *Let* $(S, I, R)(\xi)$ *be a nontrivial positive traveling wave solutions of* (1.1) *for each* $c \geq c^*$. *Then for* $j = 1, 2, 3$, *the functions* $e^{k_j \xi} U_j(\xi)$ *are strictly increasing and* $U_j(\xi \pm 1)/U_j(\xi)$ *are uniformly bounded on* $\mathbb{R}$, *where* $k_1 = (2d_s + \beta)/c$, $k_2 = (2d_i + \gamma + \delta)/c$, $k_3 = 2d_r/c$, $U_1(\xi) = S(\xi)$, $U_2(\xi) = I(\xi)$ *and* $U_3(\xi) = R(\xi)$.

Proof. Here we only prove the corresponding results for $R(\xi)$ since one can use the similar arguments to deduce the remainder results. By the third equation in (1.2), we have

$$\begin{aligned}\big[e^{\frac{2d_r}{c}\xi}R(\xi)\big]' &= e^{\frac{2d_r}{c}\xi}R'(\xi)+\frac{2d_r}{c}e^{\frac{2d_r}{c}\xi}R(\xi)\\ &= \frac{d_r}{c}e^{\frac{2d_r}{c}\xi}[R(\xi+1)+R(\xi-1)]+\frac{\gamma}{c}e^{\frac{2d_r}{c}\xi}I(\xi)>0\end{aligned}\tag{5.14}$$

for $\xi\in\mathbb{R}$, which implies that the function $e^{\frac{2d_r}{c}\xi}R(\xi)$ is strictly increasing on $\mathbb{R}$. Thus we obtain that

$$e^{\frac{2d_r}{c}\xi}R(\xi)>e^{\frac{2d_r}{c}(\xi-1)}R(\xi-1)\quad \text{for } \xi\in\mathbb{R},$$

which is equivalent to

$$\frac{R(\xi-1)}{R(\xi)}<e^{2d_r/c}\quad \text{for } \xi\in\mathbb{R}.$$

Observing (5.14) gives

$$\big[e^{\frac{2d_r}{c}\xi}R(\xi)\big]'>\frac{d_r}{c}e^{\frac{2d_r}{c}\xi}R(\xi+1)\quad \text{for } \xi\in\mathbb{R}.\tag{5.15}$$

Integrating (5.15) over $[\xi,\xi+1]$ and using the monotonicity of $e^{\frac{2d_r}{c}\xi}R(\xi)$, we obtain

$$\begin{aligned}e^{\frac{2d_r}{c}(\xi+1)}R(\xi+1)-e^{\frac{2d_r}{c}\xi}R(\xi) &\ge \frac{d_r}{c}\int_{\xi}^{\xi+1}e^{\frac{2d_r}{c}\eta}R(\eta+1)\,d\eta\\ &> \frac{d_r}{c}e^{\frac{2d_r}{c}\xi}R(\xi+1)\quad \text{for } \xi\in\mathbb{R},\end{aligned}$$

that is,

$$R(\xi+1)>\left[R(\xi)+\frac{d_r}{c}R(\xi+1)\right]e^{-2d_r/c}\quad \text{for } \xi\in\mathbb{R}.\tag{5.16}$$

Inserting (5.16) into (5.15) yields

$$\begin{aligned}\big[e^{\frac{2d_r}{c}\xi}R(\xi)\big]' &> \frac{d_r}{c}e^{\frac{2d_r}{c}\xi}\left[R(\xi)+\frac{d_r}{c}R(\xi+1)\right]e^{-2d_r/c}\\ &> \frac{d_r^2}{c^2}e^{-4d_r/c}e^{\frac{2d_r}{c}(\xi+1)}R(\xi+1)\quad \text{for } \xi\in\mathbb{R}.\end{aligned}\tag{5.17}$$

Integrating (5.17) over $[\xi-1/2,\xi]$ gives

$$\begin{aligned}e^{\frac{2d_r}{c}\xi}R(\xi) &\ge e^{\frac{2d_r}{c}(\xi-1/2)}R(\xi-1/2)+\frac{d_r^2}{c^2}e^{-4d_r/c}\int_{\xi-1/2}^{\xi}e^{\frac{2d_r}{c}(\eta+1)}R(\eta+1)\,d\eta\\ &> \frac{d_r^2}{2c^2}e^{-4d_r/c}e^{\frac{2d_r}{c}(\xi+1/2)}R(\xi+1/2)\quad \text{for } \xi\in\mathbb{R},\end{aligned}\tag{5.18}$$

where we have used the monotonicity of $e^{\frac{2d_r}{c}\xi}R(\xi)$. Then it follows from (5.18) that

$$\frac{R(\xi+1/2)}{R(\xi)}<\frac{2c^2}{d_r^2}e^{3d_r/c}\quad \text{for } \xi\in\mathbb{R},$$

which implies that

$$\frac{R(\xi+1)}{R(\xi)} = \frac{R(\xi+1)}{R(\xi+1/2)} \cdot \frac{R(\xi+1/2)}{R(\xi)} < \frac{4c^4}{d_r^4} e^{6d_r/c} \quad \text{for } \xi \in \mathbb{R}.$$

The proof is completed. □

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