Generalized Integration Operators from Weak to Strong Spaces of Vector-valued Analytic Functions

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Abstract. For a fixed nonnegative integer $m$, an analytic map $\varphi$ and an analytic function $\psi$, the generalized integration operator $I^{(m)}_{\varphi, \psi}$ is defined by

$$ I^{(m)}_{\varphi, \psi} f(z) = \int_0^z f^{(m)}(\varphi(\zeta)) \psi(\zeta) d\zeta $$

for $X$-valued analytic function $f$, where $X$ is a Banach space. Some estimates for the norm of the operator $I^{(m)}_{\varphi, \psi}: wA^p_\alpha(X) \to A^p_\alpha(X)$ are obtained. In particular, it is shown that the Volterra operator $J_b: wA^p_\alpha(X) \to A^p_\alpha(X)$ is bounded if and only if $J_b: A^2_\alpha \to A^2_\alpha$ is in the Schatten class $S_p(A^2_\alpha)$ for $2 \leq p < \infty$ and $\alpha > -1$. Some corresponding results are established for $X$-valued Hardy spaces and $X$-valued Fock spaces.

1. Introduction

Let $\Omega$ be the open unit disk $D$ or the complex plane $\mathbb{C}$, $X$ a complex Banach space and $\mathcal{H}(\Omega, X)$ the space of all $X$-valued analytic functions on $\Omega$. For $1 \leq p < \infty$ and $\alpha > -1$, the $X$-valued Bergman space $A^p_\alpha(X)$ consists of the functions $f \in \mathcal{H}(D, X)$ such that

$$ \|f\|_{A^p_\alpha(X)} = \left( \int_D \|f(z)\|^p_X dA_\alpha(z) \right)^{1/p} < \infty, $$

where $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$ and $dA$ is the Lebesgue measure on $\mathbb{C}$ normalized so that $A(D) = 1$. For $1 \leq p < \infty$, analogously, the $X$-valued Hardy space $H^p(X)$ consists of the functions $f \in \mathcal{H}(D, X)$ satisfying

$$ \|f\|_{H^p(X)} = \sup_{0 < r < 1} \left( \int_T \|f(r\zeta)\|^p_X dm(\zeta) \right)^{1/p} < \infty, $$

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where $dm$ is the normalized Lebesgue measure on $T = \partial \mathbb{D}$. For $1 \leq p < \infty$ and $\alpha > 0$, the $X$-valued Fock space $F^p_\alpha(X)$ consists of the functions $f \in \mathcal{H}(\mathbb{C}, X)$ such that
\[
\|f\|_{F^p_\alpha(X)} = \left( \frac{p\alpha}{2} \int_{\mathbb{C}} \|f(z)\|^p_X e^{-\frac{\alpha p |z|^2}{2}} dA(z) \right)^{1/p} < \infty.
\]
These spaces have been studied by many authors, see e.g. [3, 4, 7]. We also use the customary notation $H(\Omega)$, $A^p_\alpha$, $H^p$ and $F^p_\alpha$ to denote the corresponding spaces for the case $X = \mathbb{C}$. The weak versions of $X$-valued Bergman and Hardy spaces were considered by e.g. Blasco [2] and Bonet, Domaniński and Lindström [6]: the weak spaces $wA^p_\alpha(X)$ and $wH^p(X)$ consist of the functions $f \in \mathcal{H}(\mathbb{D}, X)$ for which
\[
\|f\|_{wA^p_\alpha(X)} = \sup_{x^* \in B_{X^*}} \|x^* \circ f\|_{A^p_\alpha}, \quad \|f\|_{wH^p(X)} = \sup_{x^* \in B_{X^*}} \|x^* \circ f\|_{H^p},
\]
are finite, respectively. Here and in the sequel, $X^*$ is the dual space of $X$ and $B_{X^*} = \{x^* \in X^* : \|x^*\|_{X^*} \leq 1\}$ is the closed unit ball of $X^*$. Analogously, the weak space $wF^p_\alpha(X)$ consists of $X$-valued entire functions satisfying
\[
\|f\|_{wF^p_\alpha(X)} = \sup_{x^* \in B_{X^*}} \|x^* \circ f\|_{F^p_\alpha} < \infty.
\]
It follows from [14] that $A^p_\alpha(X)$ and $wA^p_\alpha(X)$ (resp. $H^p(X)$ and $wH^p(X)$) are essential different for any infinite-dimensional Banach space $X$.

Given a fixed nonnegative integer $m$, an analytic self-map $\varphi$ of $\Omega$ and a function $\psi \in \mathcal{H}(\Omega)$, the generalized integration operator $I_{\varphi, \psi}^{(m)}$ is defined by
\[
I_{\varphi, \psi}^{(m)} f(z) = \int_0^z f^{(m)}(\varphi(\zeta))\psi(\zeta) d\zeta, \quad z \in \Omega
\]
for $f \in \mathcal{H}(\Omega, X)$. The operator $I_{\varphi, \psi}^{(m)}$ is a generalization of the Volterra type integration operator $J_b$, which is defined by
\[
J_b f(z) = \int_0^z f(\zeta)b'(\zeta) d\zeta, \quad z \in \Omega
\]
for $b \in \mathcal{H}(\Omega)$ and $f \in \mathcal{H}(\Omega, X)$. The operator $J_b$ has been studied in various $\mathbb{C}$-valued settings, see [1, 8, 12, 15, 17, 18] and the references therein. However, as far as we know, it seems that the operator $J_b$ has not been studied in the setting of spaces of vector-valued analytic functions.

Using [18, Theorem 1.3] and the following Theorem 2.1, it is easy to show that the following are equivalent for any Banach space $X$, $1 \leq p < \infty$ and $\alpha > -1$:

(a) $J_b : A^p_\alpha \to A^p_\alpha$ is bounded;
(b) $J_b: A^p_\alpha(X) \to A^p_\alpha(X)$ is bounded;
(c) $J_b: wA^p_\alpha(X) \to wA^p_\alpha(X)$ is bounded.

In the Hardy space setting, it is obvious that $J_b: wH^p(X) \to wH^p(X)$ is bounded if and only if $J_b: H^p \to H^p$ is bounded for all $1 \leq p < \infty$. Similar to the Bergman space case, using [12, Theorem 3.1] and the following Theorem 4.1, it can be proved that the following are equivalent for any Banach space $X$, $1 \leq p < \infty$ and $\alpha > -1$:

(d) $J_b: F^p_\alpha \to F^p_\alpha$ is bounded;
(e) $J_b: F^p_\alpha(X) \to F^p_\alpha(X)$ is bounded;
(f) $J_b: wF^p_\alpha(X) \to wF^p_\alpha(X)$ is bounded.

In this paper, we are interested in the boundedness of generalized integration operators on the vector-valued cases. More precisely, we give some estimates for the norms of the operators $I^{(m)}_{\varphi,\psi}$ from the weak type spaces $wA^p_\alpha(X)$, $wH^p(X)$ and $wF^p_\alpha(X)$ to the strong type spaces $A^p_\alpha(X)$, $H^p(X)$ and $F^p_\alpha(X)$. As applications, we obtain the boundedness of $J_b$ on the corresponding vector-valued cases.

Our first main result is that if $X$ is any complex infinite-dimensional Banach space, $2 \leq p < \infty$ and $\alpha > -1$, then $I^{(m)}_{\varphi,\psi}: wA^p_\alpha(X) \to A^p_\alpha(X)$ is bounded if and only if

$$\int_\mathbb{D} \frac{|\psi(z)|^p(1-|z|^2)^{\alpha+p}}{(1-|\varphi(z)|^2)^{2+\alpha+mp}} \, dA(z) < \infty.$$  

In particular, $J_b: wA^p_\alpha(X) \to A^p_\alpha(X)$ is bounded if and only if $b$ belongs to the Besov space $B_p$, which is equivalent to $J_b: A^2_\alpha \to A^2_\alpha$ is in the Schatten class $S_p(A^2_\alpha)$.

In the Hardy space setting, we need some additional conditions for the Banach space $X$. A Banach space $X$ is said $p$-uniformly PL-convex if there is a positive constant $c$ such that

$$\int_T \|x + \zeta y\|^p_X \, dm(\zeta) \geq \|x\|^p_X + c\|y\|^p_X$$

for all $x, y \in X$. For $2 \leq p < \infty$ and a complex $p$-uniformly PL-convex infinite-dimensional Banach space $X$, we obtain a lower estimate for the norm of the operator $I^{(m)}_{\varphi,\psi}: wH^p(X) \to H^p(X)$. Furthermore, if $X$ is a complex infinite-dimensional Hilbert space, we prove that $I^{(m)}_{\varphi,\psi}: wH^2(X) \to H^2(X)$ is bounded if and only if

$$\int_\mathbb{D} \frac{|\psi(z)|^2(1-|z|^2)}{(1-|\varphi(z)|^2)^{1+2m}} \, dA(z) < \infty.$$  

In particular, if $X$ is a complex infinite-dimensional Hilbert space, then $J_b: wH^2(X) \to H^2(X)$ is bounded if and only if $b$ belongs to the Dirichlet space, which is equivalent to the operator $J_b: H^2 \to H^2$ is a Hilbert-Schmidt operator.
In the Fock space case, we show that if \( X \) is any complex infinite-dimensional Banach space, \( 2 \leq p < \infty \) and \( \alpha > 0 \), then \( I_{\varphi,\psi}^{(m)} : wF^p_{\alpha}(X) \to F^p_{\alpha}(X) \) is bounded if and only if
\[
\int_{\mathbb{C}} \frac{|\psi(z)|^p (1 + |\varphi(z)|)^m}{(1 + |z|)^p} e^{\frac{ap}{2} (|z|^2 - |\varphi(z)|^2)} dA(z) < \infty.
\]
In particular, \( J_b : wF^p_{\alpha}(X) \to F^p_{\alpha}(X) \) is bounded if and only if \( b \) is a linear polynomial for \( 2 < p < \infty \), but \( J_b : wF^2_{\alpha}(X) \to F^2_{\alpha}(X) \) is bounded if and only if \( b \) is a constant. As a by-product, we obtain that the composition operator \( C_{\varphi} : wF^p_{\alpha}(X) \to F^p_{\alpha}(X) \) \( (2 \leq p < \infty) \), which is defined by \( C_{\varphi} f = f \circ \varphi \) for entire function \( \varphi \), is bounded if and only if \( \varphi(z) = az + d \) for some \( a, d \in \mathbb{C} \) with \( |a| < 1 \).

Throughout this paper, the notation \( A \lesssim B \) means that \( A \leq CB \) for some inessential constant \( C > 0 \). The converse relation \( A \gtrsim B \) is defined in an analogous manner, and if \( A \lesssim B \) and \( A \gtrsim B \) both hold, we write \( A \asymp B \).

2. Bergman space case

In this section we estimate the norm of the operator \( I_{\varphi,\psi}^{(m)} : wA^p_{\alpha}(X) \to A^p_{\alpha}(X) \). To this end, we first introduce some auxiliary results that will be used in the sequel. The first gives an equivalent norm for the space \( A^p_{\alpha}(X) \), which can be proved as that in \([4, \text{Theorem 2.5}]\).

**Theorem 2.1.** Let \( f \in \mathcal{H}(\mathbb{D}, X) \), \( n \in \mathbb{N} \), \( 1 \leq p < \infty \) and \( \alpha > -1 \). Then \( f \in A^p_{\alpha}(X) \) if and only if \( f^{(n)} \in A^{p}_{\alpha+np}(X) \).

Due to Theorem 2.1, we can define the following equivalent norm for the space \( A^p_{\alpha}(X) \):
\[
\|f\|_* = \sum_{k=0}^{n-1} \|f^{(k)}(0)\|_X + \|f^{(n)}\|_{A^{p}_{\alpha+np}(X)}.
\]

We also need the following Dvoretzky’s theorem, which can be found in \([9, \text{Chapter 19}]\).

**Theorem A.** For any \( n \in \mathbb{N} \) and \( \epsilon > 0 \) there is \( c(n, \epsilon) \in \mathbb{N} \) so that for any Banach space \( X \) of dimension at least \( c(n, \epsilon) \), there is a linear embedding \( T_n : l^n_2 \to X \) so that
\[
(1 + \epsilon)^{-1} \left( \sum_{j=1}^{n} |a_j|^2 \right)^{1/2} \leq \left\| \sum_{j=1}^{n} a_j T_n e_j \right\|_X \leq \left( \sum_{j=1}^{n} |a_j|^2 \right)^{1/2},
\]
for any \( a_1, \ldots, a_n \in \mathbb{C} \). Here \( (e_1, \ldots, e_n) \) is some fixed orthonormal basis of \( l^n_2 \).

The following lemma concerns bounded coefficient multipliers from \( A^2_{\alpha} \) to \( A^p_{\alpha} \), see for instance \([13, \text{Theorem 12.6.10}]\).

**Lemma B.** Suppose that \( 1 \leq p < \infty \) and \( \alpha > -1 \). Then the following hold.
(i) The sequence \( \{k^{(\alpha+2)/p - (\alpha+2)/2}\} \) is a bounded coefficient multiplier from \( A^2_\alpha \) to \( A^p_\alpha \) for \( 2 \leq p < \infty \).

(ii) The sequence \( \{k^{\beta}\} \) is a bounded coefficient multiplier from \( A^2_\alpha \) to \( A^p_\alpha \) for \( 1 \leq p < 2 \) and \( \beta < (\alpha + 1)/p - (\alpha + 1)/2 \).

The following well-known estimate, included here for convenience, will be used repeatedly later.

**Lemma 2.2.** For any \( \beta > -1 \) and \( 1/2 \leq t < 1 \), one has

\[
\sum_{k=1}^\infty k^\beta t^k \geq \frac{c_\beta}{(1-t)^{\beta+1}},
\]

where \( c_\beta \) is some positive constant depending only on \( \beta \).

We are now ready to estimate the norm of \( I_{\varphi, \psi}^{(m)} : wA^p_\alpha(X) \rightarrow A^p_\alpha(X) \). The first gives an upper bound of \( \|I_{\varphi, \psi}^{(m)}\|_{A^p_\alpha(X) \rightarrow A^p_\alpha(X)} \) for \( 1 \leq p < \infty \).

**Lemma 2.3.** Let \( X \) be any complex Banach space, \( 1 \leq p < \infty \) and \( \alpha > -1 \). Then

\[
\|I_{\varphi, \psi}^{(m)}\|_{wA^p_\alpha(X) \rightarrow A^p_\alpha(X)} \lesssim \left( \int_{\mathbb{D}} \frac{|\psi(z)|^p(1 - |z|^2)^{\alpha+p}}{(1 - |\varphi(z)|^2)^{2+\alpha+mp}} \, dA(z) \right)^{1/p}.
\]

**Proof.** For any \( f \in wA^p_\alpha(X) \), by the pointwise estimate of the derivative of Bergman space functions, we get

\[
\|f^{(m)}(z)\|_X^p = \sup_{x^* \in B_{X^*}} |x^* f^{(m)}(z)|^p = \sup_{x^* \in B_{X^*}} |(x^* \circ f)^{(m)}(z)|^p
\]

\[
\lesssim \sup_{x^* \in B_{X^*}} \|x^* \circ f\|_{A^p_\alpha(X)}^p \|f\|_{wA^p_\alpha(X)}^p (1 - |z|^2)^{2+\alpha+mp}.
\]

Therefore, by Theorem 2.1

\[
\|I_{\varphi, \psi}^{(m)} f\|_{A^p_\alpha(X)}^p \asymp \int_{\mathbb{D}} \|f^{(m)}(\varphi(z))\|_X^p |\psi(z)|^p (1 - |z|^2)^{\alpha+p} \, dA(z)
\]

\[
\lesssim \|f\|_{wA^p_\alpha(X)}^p \int_{\mathbb{D}} |\psi(z)|^p (1 - |\varphi(z)|^2)^{2+\alpha+mp} \, dA(z),
\]

which finishes the proof.

The following theorem is the main result of this section, which gives a norm estimate of the operator \( I_{\varphi, \psi}^{(m)} : wA^p_\alpha(X) \rightarrow A^p_\alpha(X) \) for \( 2 \leq p < \infty \).

**Theorem 2.4.** Let \( X \) be any complex infinite-dimensional Banach space, \( 2 \leq p < \infty \) and \( \alpha > -1 \). Then

\[
\|I_{\varphi, \psi}^{(m)}\|_{A^p_\alpha(X) \rightarrow A^p_\alpha(X)} \asymp \left( \int_{\mathbb{D}} |\psi(z)|^p (1 - |z|^2)^{\alpha+p} \, dA(z) \right)^{1/p}.
\]
Proof. By Lemma 2.3 we only need to proceed the lower estimate. To this end, let \( n \in \mathbb{N} \) and \( \epsilon > 0 \). According to Theorem A fix a linear embedding \( T_n : l^2_n \to X \) so that (2.1) holds. Put \( x_k^{(n)} = T_n e_k \) for \( k = 1, 2, \ldots, n \), where \((e_1, \ldots, e_n)\) is some fixed orthonormal basis of \( l^2_n \). Let \( \lambda_k = k^{(\alpha+2)/p-1/2} \), and define \( f_n : \mathbb{D} \to X \) by

\[
(2.2) \quad f_n(z) = \sum_{k=1}^{n} \lambda_k z^k x_k^{(n)} = T_n \left( \sum_{k=1}^{n} \lambda_k z^k e_k \right), \quad z \in \mathbb{D}.
\]

By Lemma B(i) and the fact that

\[
\|z^k\|_{A^\alpha_n}^2 = \frac{k! \Gamma(\alpha+2)}{\Gamma(k+\alpha+2)} \approx k^{-1-\alpha},
\]

we have

\[
\|f_n\|_{wA^{\alpha}_n(X)} = \sup_{x^* \in B_{X^*}} \|x^* \circ f_n\|_{A^\alpha_n} = \sup_{x^* \in B_{X^*}} \left\| \sum_{k=1}^{n} \lambda_k x^* (x_k^{(n)}) z^k \right\|_{A^\alpha_n} \leq \sup_{x^* \in B_{X^*}} \left( \sum_{k=1}^{n} |T_n x^*(e_k)|^2 \right)^{1/2} \leq 1.
\]

It follows from Theorem 2.1 that

\[
(2.3) \quad \|I_{\varphi, \psi}^{(m)}\|_{wA^{\alpha}_n(X) \to A^{\alpha}_n(X)} \geq \limsup_{n \to \infty} \|I_{\varphi, \psi}^{(m)} f_n\|_{A^{\alpha}_n(X)}^p \geq \limsup_{n \to \infty} \int_{\mathbb{D}} \|f_n^{(m)}(\varphi(z))\|_{X}^p |\psi(z)|^p (1 - |z|^2)^{\alpha+p} dA(z).
\]

Since \( f_n(z) = T_n \left( \sum_{k=1}^{n} \lambda_k z^k e_k \right) \), we have

\[
(2.4) \quad f_n^{(m)}(z) = T_n \left( \sum_{k=1}^{n} (k)_m \lambda_{k+m-1} z^{k-1} e_{k+m-1} \right)
\]

for \( 0 \leq m \leq n \). Here, \((k)_m = k(k+1) \cdots (k+m-1)\) for \( m \geq 1 \) and \((k)_0 = 1\), and \( \lambda_0 = 0 \). Combining (2.4) and (2.1), we establish

\[
\|f_n^{(m)}(\varphi(z))\|_{X}^p = \left\| T_n \left( \sum_{k=1}^{n} (k)_m \lambda_{k+m-1} \varphi(z)^{k-1} e_{k+m-1} \right) \right\|_{X}^p \geq \frac{1}{1 + \epsilon} \left( \sum_{k=1}^{n} (k)_m^2 \lambda_{k+m-1}^2 |\varphi(z)|^{2(k-1)} \right)^{p/2} \geq \left( \sum_{k=1}^{n} k^{2m+2(\alpha+2)/p-1} |\varphi(z)|^{2(k-1)} \right)^{p/2}.
\]
Inserting the above estimate into (2.3) and using monotone convergence theorem and Lemma 2.2 we obtain

\[ \| I_{\varphi,\psi}^{(m)} \|_{wA^p_\alpha(X) \rightarrow A^p_\alpha(X)}^p \geq \int_\mathbb{D} \left( \sum_{k=1}^{\infty} k^{2m+2(\alpha+2)/p-1} |\varphi(z)|^{2k} \right)^{p/2} |\psi(z)|^p (1 - |z|^2)^{\alpha+p} \, dA(z) \]

\[ \geq \int \left\{ \sum_{k=1}^{\infty} k^{2m+2(\alpha+2)/p-1} |\varphi(z)|^{2k} \right\}^{p/2} |\psi(z)|^p (1 - |z|^2)^{\alpha+p} \, dA(z) \]

\[ \geq c_{2m+2(\alpha+2)/p-1}^{p/2} \int \left\{ \sum_{k=1}^{\infty} k^{2m+2(\alpha+2)/p-1} |\varphi(z)|^{2k} \right\} |\psi(z)|^p (1 - |z|^2)^{\alpha+p} \, dA(z). \]

Here, \( c_{2m+2(\alpha+2)/p-1} \) is the constant defined in Lemma 2.2.

In order to obtain the desired lower estimate, we need to show

\[ \| I_{\varphi,\psi}^{(m)} \|_{wA^p_\alpha(X) \rightarrow A^p_\alpha(X)} \geq \int_{\{z \in \mathbb{D}: |\varphi(z)|^2 \leq 1/2\}} |\psi(z)|^p (1 - |z|^2)^{\alpha+p} \, dA(z). \]

Choose \( x \in X \) satisfying \( \|x\|_X = 1 \) and let

\[ g(z) = xz^m, \quad z \in \mathbb{D}. \]

Then \( g \in wA^p_\alpha(X) \) and the norm of \( g \) in \( wA^p_\alpha(X) \) only depends on \( \alpha, p \) and \( m \). Therefore, we get

\[ \| I_{\varphi,\psi}^{(m)} \|_{wA^p_\alpha(X) \rightarrow A^p_\alpha(X)} \geq \| I_{\varphi,\psi}^{(m)} g \|_{A^p_\alpha(X)}^p \geq m! \int_\mathbb{D} |\psi(z)|^p (1 - |z|^2)^{\alpha+p} \, dA(z). \]

Consequently,

\[ \int_{\{z \in \mathbb{D}: |\varphi(z)|^2 \leq 1/2\}} \frac{|\psi(z)|^p (1 - |z|^2)^{\alpha+p}}{(1 - |\varphi(z)|^2)^{\alpha+2+mp}} \, dA(z) \leq \int_\mathbb{D} |\psi(z)|^p (1 - |z|^2)^{\alpha+p} \, dA(z) \]

\[ \leq \| I_{\varphi,\psi}^{(m)} \|_{wA^p_\alpha(X) \rightarrow A^p_\alpha(X)}^p. \]

Hence (2.5) holds and the lower estimate is established. The proof is therefore complete.

\[ \square \]

For \( 1 \leq p < 2 \), using the preceding ideas we can only establish a weaker lower bound.

**Proposition 2.5.** Let \( X \) be any complex infinite-dimensional Banach space, \( 1 \leq p < 2 \) and \( \alpha > -1 \). Then

\[ \| I_{\varphi,\psi}^{(m)} \|_{wA^p_\alpha(X) \rightarrow A^p_\alpha(X)} \geq \left( \int_{\mathbb{D}} \frac{|\psi(z)|^p (1 - |z|^2)^{\alpha+p}}{(1 - |\varphi(z)|^2)^{\gamma}} \, dA(z) \right)^{1/p} \]

for \( \alpha + 1 + mp < \gamma < \alpha + 1 + p/2 + mp \).
Proof. Let $\lambda_k = k^{\beta+(1+\alpha)/2}$ with $\beta < (\alpha + 1)/p - (\alpha + 1)/2$ and define $f_n$ as (2.2). Then by Lemma B(ii) we have $\|f_n\|_{wA^p_0(X)} \lesssim 1$ for $1 \leq p < 2$. Hence Theorems 2.1 A and monotone convergence theorem yield

$$\|I^{(m)}_{\varphi,\psi}\|_{wA^p_0(X) \to A^p_0(X)}^p \geq \limsup_{n \to \infty} \|I^{(m)}_{\varphi,\psi}f_n\|^p_{A^p_0(X)}$$

$$\geq \int_{\mathbb{D}} \left( \sum_{k=1}^{\infty} (k^2)^{2(k-1)} |\varphi(z)|^{2(k-1)} \right)^{p/2} |\psi(z)|^p (1 - |z|^2)^{\alpha + p} dA(z)$$

$$\geq \int_{\mathbb{D}} \left( \sum_{k=1}^{\infty} k^{2m+2\beta+1+\alpha} |\varphi(z)|^{2k} \right)^{p/2} |\psi(z)|^p (1 - |z|^2)^{\alpha + p} dA(z)$$

for $m \geq 0$. Let $\beta > (\alpha + 1)/p - 1 - \alpha/2$, then $2m + 2\beta + 1 + \alpha > -1$ and by Lemma 2.2 we have

$$\|I^{(m)}_{\varphi,\psi}\|_{wA^p_0(X) \to A^p_0(X)}^p \geq \mathcal{C}^{p/2}_{2m+2\beta+1+\alpha} \int_{\{z \in \mathbb{D} : |\varphi(z)|^2 \geq 1/2\}} |\psi(z)|^p (1 - |\varphi(z)|^2)^{\alpha + p} (1 - |z|^2)^{\gamma} dA(z),$$

where $\gamma = (2m + 2\beta + 2 + \alpha)p/2$ satisfying

$$\alpha + 1 + mp < \gamma < \alpha + 1 + \frac{p}{2} + mp.$$ 

Similar to (2.5), we also have

$$\|I^{(m)}_{\varphi,\psi}\|_{wA^p_0(X) \to A^p_0(X)}^p \geq \int_{\{z \in \mathbb{D} : |\varphi(z)|^2 < 1/2\}} |\psi(z)|^p (1 - |\varphi(z)|^2)^{\alpha + p} (1 - |z|^2)^{\gamma} dA(z).$$

Thus the proof is finished. \qed

In particular, we have the following estimates for the norm of the Volterra type integration operator $J_b: wA^p_0(X) \to A^p_0(X)$.

**Corollary 2.6.** Let $X$ be any complex infinite-dimensional Banach space, $1 \leq p < \infty$, $\alpha > -1$ and $b \in \mathcal{H}(\mathbb{D})$.

1. If $2 \leq p < \infty$, then $J_b: wA^p_0(X) \to A^p_0(X)$ is bounded if and only if $b$ belongs to the analytic Besov space $B_p$. Moreover,

$$\|J_b\|_{wA^p_0(X) \to A^p_0(X)} \asymp \left( \int_{\mathbb{D}} |b'(z)|^p (1 - |z|^2)^{p-2} dA(z) \right)^{1/p}.$$

2. If $1 \leq p < 2$, then

$$\left( \int_{\mathbb{D}} |b'(z)|^p (1 - |z|^2)^\gamma dA(z) \right)^{1/p} \lesssim \|J_b\|_{wA^p_0(X) \to A^p_0(X)} \lesssim \left( \int_{\mathbb{D}} |b'(z)|^p (1 - |z|^2)^{p-2} dA(z) \right)^{1/p}$$

for $p/2 - 1 < \gamma < p - 1$. 
Remark 2.7. By [1] Theorem 2 [see also [18] Theorem 1.4], we know that $J_b : w A^p_\alpha(X) \to A^p_\alpha(X)$ is bounded if and only if $J_b : A^2_\alpha \to A^2_\alpha$ is in the Schatten class $S_p(A^2_\alpha)$ when $2 \leq p < \infty$.

3. Hardy space case

Let $X$ be any complex infinite-dimensional Banach space. In this section we first give a lower bound for the norm of $I_{\phi,\psi}(m) : w H^p(X) \to H^p(X)$ when $X$ is $p$-uniformly PL-convex and $2 \leq p < \infty$. To this purpose, we need the following Littlewood-Paley inequality for $H^p(X)$, which can be found in [5] Theorem 2.3.

**Theorem C.** Let $2 \leq p < \infty$ and $X$ be a Banach space. Then $X$ is $p$-uniformly PL-convex if and only if there exists $c > 0$ such that

$$\|f\|_{H^p(X)} \geq \left( \|f(0)\|_X + c \int_{\mathbb{D}} |f'(z)|^p \|X(1 - |z|^2)^{p-1} dA(z) \right)^{1/p}$$

for all $f \in H^p(X)$.

The following lemma concerns the bounded coefficient multipliers from $H^2$ to $H^p$, which is cited from [10] Theorem 1.

**Lemma D.** The sequence $\{k^{1/p-1/2}\}$ is a bounded coefficient multiplier from $H^2$ to $H^p$ for $2 \leq p < \infty$.

We now estimate the lower bound for $\|I_{\phi,\psi}(m)\|_{w H^p(X) \to H^p(X)}$.

**Proposition 3.1.** Let $2 \leq p < \infty$ and $X$ be any complex $p$-uniformly PL-convex infinite-dimensional Banach space. Then

$$\|I_{\phi,\psi}(m)\|_{w H^p(X) \to H^p(X)} \gtrsim \left( \int_{\mathbb{D}} |\psi(z)|^p (1 - |z|^2)^{p-1} dA(z) \right)^{1/p}.$$ 

**Proof.** For any given $n \in \mathbb{N}$ and $\epsilon > 0$, fix a linear embedding $T_n : l^n_2 \to X$ so that (2.1) holds. Put $x_k^{(n)} = T_n e_k$ for $k = 1, 2, \ldots, n$, where $(e_1, \ldots, e_n)$ is some fixed orthonormal basis of $l^n_2$. Consider the $X$-valued polynomials

$$f_n(z) = \sum_{k=1}^n \lambda_k z^k x_k^{(n)}, \quad z \in \mathbb{D},$$

where $\lambda_k = k^{1/2-1/2}$. Then we have

$$\|f_n\|_{w H^p(X)} = \sup_{x^* \in B_{X^*}} \left\| \sum_{k=1}^n \lambda_k z^k (x_k^{(n)})^* \right\|_{H^p} \lesssim \sup_{x^* \in B_{X^*}} \left\| \sum_{k=1}^n z^k (x_k^{(n)})^* \right\|_{H^p}$$

$$= \sup_{x^* \in B_{X^*}} \left( \sum_{k=1}^n |T_n^* x^*(e_k)|^2 \right)^{1/2} \leq 1,$$
where the inequality \( \lesssim \) follows from Lemma \([\text{D}]\). Therefore,

\[
\|I_{\varphi, \psi}^{(m)}\|_{wH^p(X) \to H^p(X)} \geq \limsup_{n \to \infty} \|I_{\varphi, \psi}^{(m)} f_n\|_{H^p(X)}.
\]

By Theorems \([\text{C}]\) \([\text{A}]\) and Lemma \([2,2]\) we obtain

\[
\|I_{\varphi, \psi}^{(m)}\|_{wH^p(X) \to H^p(X)} \geq \limsup_{n \to \infty} \|I_{\varphi, \psi}^{(m)} f_n\|_{H^p(X)}^p
\]

\[
\geq \limsup_{n \to \infty} \int_\mathbb{D} \|I_{\varphi, \psi}^{(m)} (\varphi(z))\|_{\psi(z)}^p |z|^{2p-1} dA(z)
\]

\[
\geq \int_\mathbb{D} \left( \sum_{k=1}^{\infty} k^{2m+2/p-1} |\varphi(z)|^{2k} \right)^{p/2} |\psi(z)|^{p} (1 - |z|^2)^{p-1} dA(z)
\]

\[
\geq c_2^{p/2} \int_\mathbb{D} \frac{|\psi(z)|^{p} (1 - |z|^2)^{p-1}}{(1 - |\varphi(z)|^2)^{mp+1}} dA(z)
\]

for \( m \geq 0 \). Let \( g(z) = x z^m \) for \( x \in X \) with \( \|x\|_X = 1 \), then \( \|g\|_{wH^p(X)} = 1 \). Using Theorem \([\text{C}]\) again, we have

\[
\|I_{\varphi, \psi}^{(m)}\|_{wH^p(X) \to H^p(X)} \geq \|I_{\varphi, \psi}^{(m)} g\|_{H^p(X)}^p
\]

\[
\geq \int_\mathbb{D} |\psi(z)|^{p} (1 - |z|^2)^{p-1} dA(z)
\]

\[
\geq \int_{\{|z| \geq 1/2\}} |\psi(z)|^{p} (1 - |z|^2)^{p-1} dA(z).
\]

This completes the proof.

\( \square \)

**Remark 3.2.** For the case \( 1 < p < 2 \), there are no estimates similar to the one in Theorem \([\text{C}]\). However, we can give a weaker lower bound for the norm of the operator \( I_{\varphi, \psi}^{(m)} : wH^p(X) \to H^p(X) \) via embedding Hardy spaces into Bergman spaces. If \( X \) is any complex Banach space, \( 1 < p < q < \infty \) and \( \alpha = q/p - 2 \), then \( H^p(X) \subset A^q_{\alpha}(X) \) and the inclusion is continuous. To see this, for any \( f \in H^p(X) \) and \( 0 < r < 1 \), write \( f_r(z) = f(rz) \).

By \([19]\) Corollary 4.47] and the subharmonic property of \( \|f_r\|_X \), we have

\[
\|f_r\|_{A^q_{\alpha}(X)} \leq C \|f_r\|_{H^p(X)} \leq C \|f\|_{H^p(X)}
\]

for some absolute constant \( C > 0 \). Using Fatou’s lemma, we obtain

\[
\|f\|_{A^q_{\alpha}(X)} \leq \liminf_{r \to 1} \|f_r\|_{A^q_{\alpha}(X)} \lesssim \|f\|_{H^p(X)}.
\]

Therefore, if \( X \) is any complex infinite-dimensional Banach space and \( 1 < p < 2 \), then using Theorem \([2,1]\) and the same method as in the proof of Proposition 3.1 we have

\[
\|I_{\varphi, \psi}^{(m)}\|_{wH^p(X) \to H^p(X)} \geq \left( \int_\mathbb{D} |\psi(z)|^{q} (1 - |z|^2)^{q+q/p-2} \frac{dA(z)}{(1 - |\varphi(z)|^2)^{mq+q/2}} \right)^{1/q}
\]

for \( q > p \).
If $X$ is a complex Hilbert space, we have the following Littlewood-Paley type identity for the space $H^2(X)$.

**Lemma 3.3.** Let $X$ be a complex Hilbert space, then we have

$$\|f - f(0)\|_{H^2(X)}^2 \asymp \int_D \|f'(z)\|^2_X (1 - |z|^2) dA(z)$$

for any $f \in H^2(X)$.

**Proof.** Using the Taylor expansion of $f$, this can be obtained by some elementary computations. \hfill \Box

If $X$ is a complex infinite-dimensional Hilbert space, we have the following estimate for the norm of the operator $I_{\varphi,\psi}^{(m)} : wH^2(X) \to H^2(X)$.

**Theorem 3.4.** Let $X$ be a complex infinite-dimensional Hilbert space. Then

$$\|I_{\varphi,\psi}^{(m)}\|_{wH^2(X) \to H^2(X)} \asymp \left(\int_D \left|\frac{\psi(z)}{|\varphi(z)|^{2+2m}}\right|^2 (1 - |z|^2) dA(z)\right)^{1/2}.$$ 

**Proof.** Since any Hilbert space is 2-uniformly PL-convex, the lower estimate follows from Proposition 3.1. We now consider the upper estimate. For any $f \in wH^2(X)$, by the pointwise estimate of the derivative of Hardy space functions, we have

$$\|f^{(m)}(z)\|_X^2 = \sup_{x^* \in B_{X^*}} |x^*(f^{(m)}(z))|^2 \leq \sup_{x^* \in B_{X^*}} |(x^* \circ f)^{(m)}(z)|^2 \lesssim \|f\|_{wH^2(X)}^2 (1 - |z|^2)^{1+2m}.$$ 

Therefore, by Lemma 3.3, we have

$$\|I_{\varphi,\psi}^{(m)} f\|_{H^2(X)}^2 \asymp \int_D \|f^{(m)}(\varphi(z))\|_X^2 |\psi(z)|^2 (1 - |z|^2) dA(z) \lesssim \|f\|_{wH^2(X)}^2 \int_D \frac{|\psi(z)|^2 (1 - |z|^2)}{(1 - |\varphi(z)|^2)^{1+2m}} dA(z),$$

which completes the theorem. \hfill \Box

As applications, we have the following corollaries.

**Corollary 3.5.** Let $2 \leq p < \infty$ and $X$ be any complex $p$-uniformly PL-convex infinite-dimensional Banach space. Then

$$\|J_b\|_{wH^p(X) \to H^p(X)} \gtrsim \left(\int_D |b'(z)|^p (1 - |z|^2)^{p-2} dA(z)\right)^{1/p}.$$ 

**Corollary 3.6.** Let $X$ be any complex infinite-dimensional Hilbert space. Then $J_b : wH^2(X) \to H^2(X)$ is bounded if and only if $b$ belongs to the Dirichlet space. Moreover,

$$\|J_b\|_{wH^2(X) \to H^2(X)} \asymp \left(\int_D |b'(z)|^2 dA(z)\right)^{1/2}.$$
**Remark 3.7.** Due to \cite{17} Theorem 6.7, we know that if \( 2 \leq p < \infty \) and \( X \) is a complex \( p \)-uniformly PL-convex infinite-dimensional Banach space, then the boundedness of \( J_b: \mathcal{H}^p(X) \rightarrow \mathcal{H}^p(X) \) implies \( J_b: H^2 \rightarrow H^2 \) is in the Schatten class \( S_p(H^2) \). Furthermore, if \( X \) is a complex infinite-dimensional Hilbert space, then \( J_b: wH^2(X) \rightarrow H^2(X) \) is bounded if and only if \( J_b: H^2 \rightarrow H^2 \) is a Hilbert-Schmidt operator.

4. Fock space case

In the last section, we investigate the boundedness of \( I_{\varphi,\psi}^{(m)}: wF^p_\alpha(X) \rightarrow F^p_\alpha(X) \). For this purpose, we need the following result, which characterises a \( X \)-valued Fock space function by its derivatives.

**Theorem 4.1.** Suppose \( f \in \mathcal{H}(\mathbb{C},X), 1 \leq p < \infty, \alpha > 0 \) and \( n \in \mathbb{N} \). Then

\[
\| f \|_{F^p_\alpha(X)} \lesssim \sum_{k=0}^{n-1} \| f^{(k)}(0) \|_X + \left( \int_{\mathbb{C}} \left\| \frac{f^{(n)}(z)}{(1+|z|)^n} \right\|_{X}^p e^{-\frac{\alpha p}{2} |z|^2} dA(z) \right)^{1/p}.
\]

In order to prove the above theorem, we need the following lemma.

**Lemma 4.2.** Let \( f \in \mathcal{H}(\mathbb{C},X), n \in \mathbb{N} \) and \( 1 \leq p < \infty \). Then for any \( z \in \mathbb{C} \) and \( r > 0 \), we have

\[
\| f^{(n)}(z) \|_X \lesssim \frac{1}{r^{2+np}} \int_{D(z,r)} \| f(w) \|_X^p dA(w),
\]

where \( D(z,r) = \{ w \in \mathbb{C} : |w-z| < r \} \).

**Proof.** We only need to consider the case \( z = 0 \). For any \( \rho > 0 \), Cauchy’s integral formula yields

\[
\| f^{(n)}(0) \|_X \leq \frac{n!}{2\pi} \int_{0}^{2\pi} \| f(\rho e^{i\theta}) \|_X \rho^{-n} d\theta.
\]

Multiplying by \( \rho^{n+1} \) and integrating with respect to \( \rho \) from \( r/2 \) to \( r \), we obtain

\[
\frac{r^{n+2} - (r/2)^{n+2}}{n+2} \| f^{(n)}(0) \|_X \leq \frac{n!}{2\pi} \int_{0}^{r} \int_{0}^{2\pi} \| f(\rho e^{i\theta}) \|_X d\theta d\rho.
\]

Since \( r^{n+2} - (r/2)^{n+2} \geq r^{n+2}/2 \), we arrive at

\[
\| f^{(n)}(0) \|_X \lesssim \frac{1}{r^{n+2}} \int_{D(0,r)} \| f(w) \|_X dA(w).
\]

Hölder’s inequality then gives the desired estimate. \( \square \)

**Proof of Theorem 4.1.** By Lemma 4.2, we have

\[
\| f^{(k)}(0) \|_X \lesssim \left( \int_{D(0,1)} \| f(w) \|_X^p dA(w) \right)^{1/p} \lesssim \| f \|_{F^p_\alpha(X)}.
\]
for any $0 \leq k \leq n - 1$. Using Lemma 4.2 and the estimate (8) in [12], we obtain

$$\int_{C} \left\| \frac{f^{(n)}(z)}{(1 + |z|)^{n}} \right\|_{X}^{p} e^{-\frac{\alpha \rho}{2}|z|^{2}} dA(z)$$

$$\leq \int_{C} (1 + |z|)^{2} \int_{D(z, 1/|z|)} \|f(w)\|_{X}^{p} dA(w) e^{-\frac{\alpha \rho}{2}|z|^{2}} dA(z)$$

$$\leq \int_{C} \|f(w)\|_{X}^{p} (1 + |w|)^{2} \int_{D(w, 1/2|w|)} e^{-\frac{\alpha \rho}{2}|z|^{2}} dA(z) dA(w)$$

$$\leq \int_{C} \|f(w)\|_{X}^{p} e^{-\frac{\alpha \rho}{2}|w|^{2}} dA(w),$$

where the second inequality is due to Fubini’s theorem and the facts that $w \in D(z, 1/(1 + |z|))$ implies $z \in D(w, 2/(1 + |w|))$, and $1 + |z| \lesssim 1 + |w|$ if $z \in D(w, 2/(1 + |w|))$. Combining the estimates above yields

$$\|f\|_{F_{\rho}^{p}(X)} \gtrsim \sum_{k=0}^{n-1} \|f^{(k)}(0)\|_{X} + \left( \int_{C} \left\| \frac{f^{(n)}(z)}{(1 + |z|)^{n}} \right\|_{X}^{p} e^{-\frac{\alpha \rho}{2}|z|^{2}} dA(z) \right)^{1/p}.$$  

Conversely, note that $\|f\|_{X}^{p}$ is subharmonic on $\mathbb{C}$ for any $1 \leq p < \infty$. Consequently, $M_{p}(f, r)$ is increasing with $r$, see e.g. [11, Corollary 6.6]. We claim that

$$(1) \quad \int_{C} \left\| \frac{f(z)}{(1 + |z|)^{k}} \right\|_{X}^{p} e^{-\frac{\alpha \rho}{2}|z|^{2}} dA(z) \lesssim \int_{C} \left\| \frac{f'(z)}{(1 + |z|)^{k+1}} \right\|_{X}^{p} e^{-\frac{\alpha \rho}{2}|z|^{2}} dA(z)$$

for any fixed $1 \leq p < \infty$, $k \geq 0$, and all $f \in \mathcal{H}(\mathbb{C}, X)$ with $f(0) = 0$. In fact, this can be proven by the same method as in the proof of [12, (11)]. In the case $p = 1$, for any $0 < \rho < r < \infty$, we have

$$M_{1}(f, r) - M_{1}(f, \rho) \leq \int_{T} \|f(r(\zeta)) - f(\rho(\zeta))\|_{X} dm(\zeta)$$

$$= \int_{T} \left\| \int_{\rho}^{r} f'(t(\zeta)) \partial_{t} dt \right\|_{X} dm(\zeta) \leq (r - \rho) M_{1}(f', r).$$

Therefore, (1) holds in this case. In the case $1 < p < \infty$, vector-valued version of Lemma 2.2 in [12] is needed. Carefully examining the proof of [16, Theorem 1], we see [12, Lemma 2.2] holds for vector-valued functions. Consequently, (1) also holds in this case. Then for any $f \in \mathcal{H}(\mathbb{C}, X)$, due to (1) we obtain

$$\left( \int_{C} \left\| \frac{f(z)}{(1 + |z|)^{k}} \right\|_{X}^{p} e^{-\frac{\alpha \rho}{2}|z|^{2}} dA(z) \right)^{1/p} \leq \left( \int_{C} \left\| \frac{f(z) - f(0)}{(1 + |z|)^{k}} \right\|_{X}^{p} e^{-\frac{\alpha \rho}{2}|z|^{2}} dA(z) \right)^{1/p} + \|f(0)\|_{X} \left( \int_{C} e^{-\frac{\alpha \rho}{2}|z|^{2}} (1 + |z|)^{pk} dA(z) \right)^{1/p}$$

$$\lesssim \|f(0)\|_{X} + \left( \int_{C} \left\| \frac{f'(z)}{(1 + |z|)^{k+1}} \right\|_{X}^{p} e^{-\frac{\alpha \rho}{2}|z|^{2}} dA(z) \right)^{1/p}.$$
Applying the above estimate repeatedly, we establish
\[ \|f\|_{F^p_\alpha(X)} \lesssim \sum_{k=0}^{n-1} \|f^{(k)}(0)\|_X + \left( \int_C \left\| \frac{f^{(n)}(z)}{(1+|z|)^n} \right\|^p_X \right)^{1/p} \left( e^{-\frac{\alpha p}{2}|z|^2} dA(z) \right)^{1/p}, \]
which completes the theorem.

The following lemma estimates the derivatives of Fock space functions.

**Lemma 4.3.** Let \( 0 < p < \infty \) and \( \alpha > 0 \). For any \( f \in F^p_\alpha \) and \( n \geq 0 \), the following estimate holds:
\[ |f^{(n)}(z)| \lesssim (1 + |z|^n) e^{\frac{\alpha}{2}|z|^2} \|f\|_{F^p_\alpha}. \]

**Proof.** The case \( n = 0 \) was proved in [20, Corollary 2.8]. We consider the case \( n > 0 \). For \( |z| \leq 1 \), by Cauchy’s estimate and the estimate in the case \( n = 0 \), we have
\[ |f^{(n)}(z)| \leq \frac{n!}{2\pi} \int_{|\zeta - z| = 1} \frac{|f(\zeta)|}{|z - \zeta|^{n+1}} |d\zeta| \lesssim \max_{|\zeta - z| = 1} |f(\zeta)| \lesssim \|f\|_{F^p_\alpha}. \]
For \( |z| > 1 \), arguing as above, we get
\[ |f^{(n)}(z)| \leq \frac{n!}{2\pi} \int_{|\zeta - z| = 1/|z|} \frac{|f(\zeta)|}{|z - \zeta|^{n+1}} |d\zeta| \lesssim |z|^n \max_{|\zeta - z| = 1/|z|} |f(\zeta)| \leq |z|^n e^{\frac{\alpha}{2}(1+\frac{1}{|z|})^2} \|f\|_{F^p_\alpha} \lesssim |z|^n e^{\frac{\alpha}{2}|z|^2} \|f\|_{F^p_\alpha}. \]
Combining these estimates, we obtain the desired result.

We now end this section by estimating the norm of \( I_{\varphi,\psi}^{(m)} \) on the Fock type setting.

**Theorem 4.4.** Let \( X \) be any complex infinite-dimensional Banach space, \( 2 \leq p < \infty \) and \( \alpha > 0 \). Then
\[ \|I_{\varphi,\psi}^{(m)}\|_{wF^p_\alpha(X) \to F^p_\alpha(X)} \asymp \left( \int_C \frac{|\psi(z)|^p (1 + |\varphi(z)|^m)^p}{(1 + |z|)^p} e^{-\frac{\alpha p}{2}|z|^2 - |\varphi(z)|^2} dA(z) \right)^{1/p}. \]

**Proof.** For any \( f \in wF^p_\alpha(X) \), by Theorem 4.1 and the estimate in Lemma 4.3 we get
\[ \|I_{\varphi,\psi}^{(m)} f\|_{F^p_\alpha(X)}^p \asymp \int_C \left\| \frac{f^{(m)}(\varphi(z))\psi(z)}{1 + |z|} \right\|^p_X \left( e^{-\frac{\alpha p}{2}|z|^2} dA(z) \right)^p \leq \|f\|_{wF^p_\alpha(X)} \int_C \frac{|\psi(z)|^p (1 + |\varphi(z)|^m)^p}{(1 + |z|)^p} e^{-\frac{\alpha p}{2}|z|^2 - |\varphi(z)|^2} dA(z), \]
which gives us the upper estimate.

We next consider the lower estimate. Fix \( n \in \mathbb{N} \) and \( \epsilon > 0 \). According to Theorem A, there is a linear embedding \( T_n : l^m_2 \to X \) so that (2.1) holds. Put \( x_k^{(n)} = T_n e_k \) for \( k = 1, \ldots, n \).
1, 2, \ldots, n$, where $(e_1, e_2, \ldots, e_n)$ is some fixed orthonormal basis of $l_2^n$. Define $f_n: \mathbb{C} \to X$ by

$$f_n(z) = \sum_{k=0}^{n-1} \frac{\alpha^k}{k!} z^k x_k^{(n)}, \quad z \in \mathbb{C}.$$ 

Then

$$\|f_n\|_{w F_\alpha^p(X)} = \sup_{x^* \in B_{X^*}} \left\| \sum_{k=0}^{n-1} \frac{\alpha^k}{k!} x^*(x_k^{(n)}) z^k \right\| \lesssim \sup_{x^* \in B_{X^*}} \left\| \sum_{k=0}^{n-1} \frac{\alpha^k}{k!} x^*(x_k^{(n)}) z^k \right\|_{F_\alpha^p}^{2} = \sup_{x^* \in B_{X^*}} \left( \sum_{k=0}^{n-1} |x^*(x_k^{(n)})|^2 \right)^{1/2} \leq 1,$$

where the first inequality is due to the embedding $F_\alpha^p \subset F_\alpha^q$ is bounded whenever $p \leq q$. Therefore, by Theorem 4.1 we obtain

$$\|f_n(m)\|_{w F_\alpha^p(X) \to F_\alpha^p(X)} \approx \limsup_{n \to \infty} \|f(m) f_n\|_{F_\alpha^p(X)}^p = \limsup_{n \to \infty} \int_{\mathbb{C}} \left\| \frac{f_n(m)(\varphi(z))\psi(z)}{1 + |z|} \right\|_X^p e^{-\frac{\alpha_p}{2}|z|^2} dA(z).$$

By the definition of $f_n$ and (2.1), we have

$$\|f_n(m)(\varphi(z))\|_X^p = \left\| T_n \left( \sum_{k=0}^{n-m-1} \frac{(k+1)m \alpha^{k+m}}{(k+m)!} |\varphi(z)|^{2k} \right) \right\|_X^p \approx \left( \sum_{k=0}^{n-m-1} \frac{(k+1)m \alpha^{k+m}}{(k+m)!} |\varphi(z)|^{2k} \right)^{p/2}$$

for $0 \leq m < n$. Therefore, by monotone convergence theorem, we arrive at

$$\|f_n(m)\|_{w F_\alpha^p(X) \to F_\alpha^p(X)} \gtrsim \int_{\mathbb{C}} \left( \sum_{k=0}^{\infty} \frac{(k+1)m \alpha^{k}}{k!} |\varphi(z)|^{2k} \right)^{p/2} \frac{|\psi(z)|^p e^{-\frac{\alpha_p}{2}|z|^2}}{(1 + |z|)^p} dA(z).$$

It is obvious to see

$$(1 + |\varphi(z)|^m)^p e^{\frac{\alpha_p}{2}|\varphi(z)|^2} \lesssim \left( \sum_{k=0}^{\infty} \frac{(k+1)m \alpha^{k}}{k!} |\varphi(z)|^{2k} \right)^{p/2}.$$ 

Hence we establish the lower estimate for the norm of $f_n(m)_{w F_\alpha^p(X) \to F_\alpha^p(X)}$ and the proof is complete.

**Remark 4.5.** The upper estimate for $\|f_n(m)\|_{w F_\alpha^p(X) \to F_\alpha^p(X)}$ in Theorem 4.4 is actually valid for all $1 \leq p < \infty$ and any complex Banach space $X$. \(\square\)
In particular, the boundedness of \( J_b : wF^p_\alpha(X) \to F^p_\alpha(X) \) and \( C_\varphi : wF^p_\alpha(X) \to F^p_\alpha(X) \) are characterized when \( 2 \leq p < \infty \).

**Corollary 4.6.** Let \( X \) be any complex infinite-dimensional Banach space and \( \alpha > 0 \).

1. \( J_b : wF^2_\alpha(X) \to F^2_\alpha(X) \) is bounded if and only if \( b \) is a constant.

2. If \( 2 < p < \infty \), then \( J_b : wF^p_\alpha(X) \to F^p_\alpha(X) \) is bounded if and only if \( b(z) = az + d \) for some \( a, d \in \mathbb{C} \). Moreover, \( \|J_b\|_{wF^p_\alpha(X) \to F^p_\alpha(X)} \asymp |a| \).

**Proof.** By Theorem 4.4, we have
\[
\|J_b\|_{wF^p_\alpha(X) \to F^p_\alpha(X)} \asymp \int_C \left| \frac{b'(z)}{1 + |z|} \right|^p dA(z).
\]
The subharmonicity of \( |b'|^p \) implies
\[
\left( \int_{D(w,1)} \left| \frac{b'(z)}{1 + |z|} \right|^p dA(z) \right)^{1/p} \geq \frac{|b'(w)|}{1 + |w|}.
\]
Hence the boundedness of \( J_b : wF^p_\alpha(X) \to F^p_\alpha(X) \) implies
\[
\frac{|b'(w)|}{1 + |w|} \to 0 \text{ as } |w| \to \infty,
\]
which is equivalent to \( b(z) = az + d \) for some \( a, d \in \mathbb{C} \). So it is only need to prove the necessity of Case (1), since the other case is obvious.

If \( J_b : wF^2_\alpha(X) \to F^2_\alpha(X) \) is bounded and \( b \) is not a constant, i.e., \( b(z) = az + d \) for some \( a \neq 0 \), then by the above estimate for the norm of \( J_b : wF^p_\alpha(X) \to F^p_\alpha(X) \), we have
\[
\|J_b\|_{wF^2_\alpha(X) \to F^2_\alpha(X)} \asymp |a| \left( \int_C \frac{dA(z)}{1 + |z|^2} \right)^{1/2} = \infty,
\]
which is a contradiction. \( \square \)

**Corollary 4.7.** Let \( X \) be any complex infinite-dimensional Banach space, \( 2 \leq p < \infty \) and \( \alpha > 0 \). Then \( C_\varphi : wF^p_\alpha(X) \to F^p_\alpha(X) \) is bounded if and only if \( \varphi(z) = az + d \) for some \( a, d \in \mathbb{C} \) with \( |a| < 1 \).

**Proof.** Since
\[
I^{(1)}_{\varphi,\varphi'} f(z) = \int_0^z f'(\varphi(\zeta))\varphi'(\zeta) \, d\zeta = f(\varphi(z)) - f(\varphi(0)),
\]
we obtain that \( C_\varphi : wF^p_\alpha(X) \to F^p_\alpha(X) \) is bounded if and only if \( I^{(1)}_{\varphi,\varphi'} : wF^p_\alpha(X) \to F^p_\alpha(X) \) is bounded. By Theorem 4.4, the boundedness of \( C_\varphi : wF^p_\alpha(X) \to F^p_\alpha(X) \) can be characterized by
\[
\left( \int_C \frac{\varphi'(z)^p(1 + |\varphi(z)|)^p}{(1 + |z|)^p} e^{-\frac{\alpha p}{2}(|z|^2 - |\varphi(z)|^2)} \, dA(z) < \infty. \right.
\]
If $\varphi(z) = az + d$ for some $a, d \in \mathbb{C}$ with $|a| < 1$, then it is trivial to see (4.2) holds. Conversely, the boundedness of $C_\varphi: \mathcal{F}_\alpha^p(X) \to \mathcal{F}_\alpha^p(X)$ implies that $C_\varphi: \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^p$ is bounded. Therefore, $\varphi(z) = az + d$ with $|a| < 1$ or $\varphi(z) = az$ with $|a| = 1$ (see, for instance, Exercise 4 of page 89 in [20]). The latter case obviously contradicts (4.2).

\textbf{Remark 4.8.} By Corollary 4.7 (or Corollary 4.6), we get that $\mathcal{F}_\alpha^p(X) \subsetneq \mathcal{wF}_\alpha^p(X)$ for any $2 \leq p < \infty$, $\alpha > 0$ and complex infinite-dimensional Banach space $X$. In fact, if $\mathcal{F}_\alpha^p(X) = \mathcal{wF}_\alpha^p(X)$ as linear spaces, then $\|f\|_{\mathcal{F}_\alpha^p(X)} \asymp \|f\|_{\mathcal{wF}_\alpha^p(X)}$ for any $f \in \mathcal{H}(\mathbb{C}, X)$ by open mapping theorem. Hence $C_\varphi: \mathcal{wF}_\alpha^p(X) \to \mathcal{wF}_\alpha^p(X)$ is bounded if and only if $C_\varphi: \mathcal{wF}_\alpha^p(X) \to \mathcal{F}_\alpha^p(X)$ is bounded, which in turn is equivalent to the boundedness of $C_\varphi: \mathcal{F}_\alpha^p \to \mathcal{F}_\alpha^p$. However, this is impossible by Corollary 4.7.

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\textbf{References}


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