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Generalized Integration Operators from Weak to Strong Spaces of Vector-valued Analytic Functions

Jiale Chen* and Maofa Wang

Abstract. For a fixed nonnegative integer m, an analytic map φ and an analytic function ψ, the generalized integration operator $I_{\varphi,\psi}^{(m)}$ is defined by

$$I_{\varphi,\psi}^{(m)} f(z) = \int_0^z f^{(m)}(\varphi(\zeta))\psi(\zeta)\, d\zeta$$

for X-valued analytic function f, where X is a Banach space. Some estimates for the norm of the operator $I_{\varphi,\psi}^{(m)} \colon wA_\alpha^p(X) \to A_\alpha^p(X)$ are obtained. In particular, it is shown that the Volterra operator $J_b \colon wA_\alpha^p(X) \to A_\alpha^p(X)$ is bounded if and only if $J_b \colon A_\alpha^2 \to A_\alpha^2$ is in the Schatten class $S_p(A_\alpha^2)$ for $2 \leq p < \infty$ and $\alpha > -1$. Some corresponding results are established for X-valued Hardy spaces and X-valued Fock spaces.

1. Introduction

Let Ω be the open unit disk $\mathbb{D}$ or the complex plane $\mathbb{C}$, X a complex Banach space and $\mathcal{H}(\Omega, X)$ the space of all X-valued analytic functions on Ω. For $1 \leq p < \infty$ and $\alpha > -1$, the X-valued Bergman space $A_\alpha^p(X)$ consists of the functions $f \in \mathcal{H}(\mathbb{D}, X)$ such that

$$\|f\|_{A_\alpha^p(X)} = \left(\int_{\mathbb{D}} \|f(z)\|_X^p \, dA_\alpha(z) \right)^{1/p} < \infty,$$

where $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \, dA(z)$ and dA is the Lebesgue measure on $\mathbb{C}$ normalized so that $A(\mathbb{D}) = 1$. For $1 \leq p < \infty$, analogously, the X-valued Hardy space $H^p(X)$ consists of the functions $f \in \mathcal{H}(\mathbb{D}, X)$ satisfying

$$\|f\|_{H^p(X)} = \sup_{0<r<1} \left(\int_{\mathbb{T}} \|f(r\zeta)\|_X^p \, dm(\zeta) \right)^{1/p} < \infty,$$

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*Corresponding author.

where dm is the normalized Lebesgue measure on $\mathbb{T} = \partial\mathbb{D}$. For $1 \le p < \infty$ and $\alpha > 0$, the X-valued Fock space $F^p_\alpha(X)$ consists of the functions $f \in \mathcal{H}(\mathbb{C}, X)$ such that

$$\|f\|_{F^p_\alpha(X)} = \left(\frac{p\alpha}{2}\int_{\mathbb{C}} \|f(z)\|^p_X e^{-\frac{\alpha p}{2}|z|^2}\, dA(z)\right)^{1/p} < \infty.$$

These spaces have been studied by many authors, see e.g. [3, 4, 7]. We also use the customary notation $\mathcal{H}(\Omega)$, A^p_α, H^p and F^p_α to denote the corresponding spaces for the case $X = \mathbb{C}$. The weak versions of X-valued Bergman and Hardy spaces were considered by e.g. Blasco [2] and Bonet, Domański and Lindström [6]: the weak spaces $wA^p_\alpha(X)$ and $wH^p(X)$ consist of the functions $f \in \mathcal{H}(\mathbb{D}, X)$ for which

$$\|f\|_{wA^p_\alpha(X)} = \sup_{x^* \in B_{X^*}} \|x^* \circ f\|_{A^p_\alpha}, \quad \|f\|_{wH^p(X)} = \sup_{x^* \in B_{X^*}} \|x^* \circ f\|_{H^p},$$

are finite, respectively. Here and in the sequel, X^* is the dual space of X and $B_{X^*} = \{x^* \in X^* : \|x^*\|_{X^*} \le 1\}$ is the closed unit ball of X^*. Analogously, the weak space $wF^p_\alpha(X)$ consists of X-valued entire functions satisfying

$$\|f\|_{wF^p_\alpha(X)} = \sup_{x^* \in B_{X^*}} \|x^* \circ f\|_{F^p_\alpha} < \infty.$$

It follows from [14] that $A^p_\alpha(X)$ and $wA^p_\alpha(X)$ (resp. $H^p(X)$ and $wH^p(X)$) are essential different for any infinite-dimensional Banach space X.

Given a fixed nonnegative integer m, an analytic self-map φ of Ω and a function $\psi \in \mathcal{H}(\Omega)$, the generalized integration operator $I^{(m)}_{\varphi,\psi}$ is defined by

$$I^{(m)}_{\varphi,\psi} f(z) = \int_0^z f^{(m)}(\varphi(\zeta))\psi(\zeta)\, d\zeta, \quad z \in \Omega$$

for $f \in \mathcal{H}(\Omega, X)$. The operator $I^{(m)}_{\varphi,\psi}$ is a generalization of the Volterra type integration operator J_b, which is defined by

$$J_b f(z) = \int_0^z f(\zeta) b'(\zeta)\, d\zeta, \quad z \in \Omega$$

for $b \in \mathcal{H}(\Omega)$ and $f \in \mathcal{H}(\Omega, X)$. The operator J_b has been studied in various $\mathbb{C}$-valued settings, see [1, 8, 12, 15, 17, 18] and the references therein. However, as far as we know, it seems that the operator J_b has not been studied in the setting of spaces of vector-valued analytic functions.

Using [18, Theorem 1.3] and the following Theorem 2.1, it is easy to show that the following are equivalent for any Banach space X, $1 \le p < \infty$ and $\alpha > -1$:

(a) $J_b\colon A^p_\alpha \to A^p_\alpha$ is bounded;

(b) $J_b\colon A^p_\alpha(X) \to A^p_\alpha(X)$ is bounded;

(c) $J_b\colon wA^p_\alpha(X) \to wA^p_\alpha(X)$ is bounded.

In the Hardy space setting, it is obvious that $J_b\colon wH^p(X) \to wH^p(X)$ is bounded if and only if $J_b\colon H^p \to H^p$ is bounded for all $1 \le p < \infty$. Similar to the Bergman space case, using [12, Theorem 3.1] and the following Theorem 4.1, it can be proved that the following are equivalent for any Banach space X, $1 \le p < \infty$ and $\alpha > -1$:

(d) $J_b\colon F^p_\alpha \to F^p_\alpha$ is bounded;

(e) $J_b\colon F^p_\alpha(X) \to F^p_\alpha(X)$ is bounded;

(f) $J_b\colon wF^p_\alpha(X) \to wF^p_\alpha(X)$ is bounded.

In this paper, we are interested in the boundedness of generalized integration operators on the vector-valued cases. More precisely, we give some estimates for the norms of the operators $I^{(m)}_{\varphi,\psi}$ from the weak type spaces $wA^p_\alpha(X)$, $wH^p(X)$ and $wF^p_\alpha(X)$ to the strong type spaces $A^p_\alpha(X)$, $H^p(X)$ and $F^p_\alpha(X)$. As applications, we obtain the boundedness of J_b on the corresponding vector-valued cases.

Our first main result is that if X is any complex infinite-dimensional Banach space, $2 \le p < \infty$ and $\alpha > -1$, then $I^{(m)}_{\varphi,\psi}\colon wA^p_\alpha(X) \to A^p_\alpha(X)$ is bounded if and only if

$$\int_{\mathbb{D}} \frac{|\psi(z)|^p(1-|z|^2)^{\alpha+p}}{(1-|\varphi(z)|^2)^{2+\alpha+mp}}\, dA(z) < \infty.$$

In particular, $J_b\colon wA^p_\alpha(X) \to A^p_\alpha(X)$ is bounded if and only if b belongs to the Besov space B_p, which is equivalent to $J_b\colon A^2_\alpha \to A^2_\alpha$ is in the Schatten class $S_p(A^2_\alpha)$.

In the Hardy space setting, we need some additional conditions for the Banach space X. A Banach space X is said p-uniformly PL-convex if there is a positive constant c such that

$$\int_{\mathbb{T}} \|x+\zeta y\|^p_X\, dm(\zeta) \ge \|x\|^p_X + c\|y\|^p_X$$

for all $x, y \in X$. For $2 \le p < \infty$ and a complex p-uniformly PL-convex infinite-dimensional Banach space X, we obtain a lower estimate for the norm of the operator $I^{(m)}_{\varphi,\psi}\colon wH^p(X) \to H^p(X)$. Furthermore, if X is a complex infinite-dimensional Hilbert space, we prove that $I^{(m)}_{\varphi,\psi}\colon wH^2(X) \to H^2(X)$ is bounded if and only if

$$\int_{\mathbb{D}} \frac{|\psi(z)|^2(1-|z|^2)}{(1-|\varphi(z)|^2)^{1+2m}}\, dA(z) < \infty.$$

In particular, if X is a complex infinite-dimensional Hilbert space, then $J_b\colon wH^2(X) \to H^2(X)$ is bounded if and only if b belongs to the Dirichlet space, which is equivalent to the operator $J_b\colon H^2 \to H^2$ is a Hilbert-Schmidt operator.

In the Fock space case, we show that if X is any complex infinite-dimensional Banach space, $2 \le p < \infty$ and $\alpha > 0$, then $I^{(m)}_{\varphi,\psi}\colon wF^p_\alpha(X) \to F^p_\alpha(X)$ is bounded if and only if

$$\int_{\mathbb{C}} \frac{|\psi(z)|^p(1+|\varphi(z)|^m)^p}{(1+|z|)^p} e^{-\frac{\alpha p}{2}(|z|^2-|\varphi(z)|^2)}\, dA(z) < \infty.$$

In particular, $J_b\colon wF^p_\alpha(X) \to F^p_\alpha(X)$ is bounded if and only if b is a linear polynomial for $2 < p < \infty$, but $J_b\colon wF^2_\alpha(X) \to F^2_\alpha(X)$ is bounded if and only if b is a constant. As a by-product, we obtain that the composition operator $C_\varphi\colon wF^p_\alpha(X) \to F^p_\alpha(X)$ $(2 \le p < \infty)$, which is defined by $C_\varphi f = f\circ\varphi$ for entire function φ, is bounded if and only if $\varphi(z) = az+d$ for some $a, d \in \mathbb{C}$ with $|a| < 1$.

Throughout this paper, the notation $A \lesssim B$ means that $A \le CB$ for some inessential constant $C > 0$. The converse relation $A \gtrsim B$ is defined in an analogous manner, and if $A \lesssim B$ and $A \gtrsim B$ both hold, we write $A \asymp B$.

2. Bergman space case

In this section we estimate the norm of the operator $I^{(m)}_{\varphi,\psi}\colon wA^p_\alpha(X) \to A^p_\alpha(X)$. To this end, we first introduce some auxiliary results that will be used in the sequel. The first gives an equivalent norm for the space $A^p_\alpha(X)$, which can be proved as that in [4, Theorem 2.5].

Theorem 2.1. *Let $f \in \mathcal{H}(\mathbb{D}, X)$, $n \in \mathbb{N}$, $1 \le p < \infty$ and $\alpha > -1$. Then $f \in A^p_\alpha(X)$ if and only if $f^{(n)} \in A^p_{\alpha+np}(X)$.*

Due to Theorem 2.1, we can define the following equivalent norm for the space $A^p_\alpha(X)$:

$$\|f\|_* = \sum_{k=0}^{n-1} \|f^{(k)}(0)\|_X + \|f^{(n)}\|_{A^p_{\alpha+np}(X)}.$$

We also need the following Dvoretzky's theorem, which can be found in [9, Chapter 19].

Theorem A. *For any $n \in \mathbb{N}$ and $\epsilon > 0$ there is $c(n,\epsilon) \in \mathbb{N}$ so that for any Banach space X of dimension at least $c(n,\epsilon)$, there is a linear embedding $T_n\colon l^n_2 \to X$ so that*

$$(1+\epsilon)^{-1}\left(\sum_{j=1}^n |a_j|^2\right)^{1/2} \le \left\|\sum_{j=1}^n a_j T_n e_j\right\|_X \le \left(\sum_{j=1}^n |a_j|^2\right)^{1/2} \tag{2.1}$$

for any $a_1, \ldots, a_n \in \mathbb{C}$. Here $(e_1, \ldots, e_n)$ is some fixed orthonormal basis of l^n_2.

The following lemma concerns bounded coefficient multipliers from A^2_α to A^p_α, see for instance [13, Theorem 12.6.10].

Lemma B. *Suppose that $1 \le p < \infty$ and $\alpha > -1$. Then the following hold.*

(i) *The sequence* $\left\{k^{(\alpha+2)/p-(\alpha+2)/2}\right\}$ *is a bounded coefficient multiplier from* A^2_α *to* A^p_α *for* $2 \le p < \infty$.

(ii) *The sequence* $\{k^\beta\}$ *is a bounded coefficient multiplier from* A^2_α *to* A^p_α *for* $1 \le p < 2$ *and* $\beta < (\alpha+1)/p - (\alpha+1)/2$.

The following well-known estimate, included here for convenience, will be used repeatedly later.

Lemma 2.2. *For any* $\beta > -1$ *and* $1/2 \le t < 1$, *one has*

$$\sum_{k=1}^{\infty} k^\beta t^k \ge \frac{c_\beta}{(1-t)^{\beta+1}},$$

where c_β *is some positive constant depending only on* β.

We are now ready to estimate the norm of $I^{(m)}_{\varphi,\psi}\colon wA^p_\alpha(X) \to A^p_\alpha(X)$. The first gives an upper bound of $\|I^{(m)}_{\varphi,\psi}\|_{wA^p_\alpha(X)\to A^p_\alpha(X)}$ for $1 \le p < \infty$.

Lemma 2.3. *Let* X *be any complex Banach space,* $1 \le p < \infty$ *and* $\alpha > -1$. *Then*

$$\|I^{(m)}_{\varphi,\psi}\|_{wA^p_\alpha(X)\to A^p_\alpha(X)} \lesssim \left(\int_{\mathbb{D}} \frac{|\psi(z)|^p(1-|z|^2)^{\alpha+p}}{(1-|\varphi(z)|^2)^{2+\alpha+mp}}\, dA(z)\right)^{1/p}.$$

Proof. For any $f \in wA^p_\alpha(X)$, by the pointwise estimate of the derivative of Bergman space functions, we get

$$\begin{aligned}\|f^{(m)}(z)\|^p_X &= \sup_{x^*\in B_{X^*}} |x^*(f^{(m)}(z))|^p = \sup_{x^*\in B_{X^*}} |(x^*\circ f)^{(m)}(z)|^p \\ &\lesssim \sup_{x^*\in B_{X^*}} \frac{\|x^*\circ f\|^p_{A^p_\alpha}}{(1-|z|^2)^{2+\alpha+mp}} = \frac{\|f\|^p_{wA^p_\alpha(X)}}{(1-|z|^2)^{2+\alpha+mp}}.\end{aligned}$$

Therefore, by Theorem 2.1,

$$\begin{aligned}\|I^{(m)}_{\varphi,\psi} f\|^p_{A^p_\alpha(X)} &\asymp \int_{\mathbb{D}} \|f^{(m)}(\varphi(z))\|^p_X |\psi(z)|^p (1-|z|^2)^{\alpha+p}\, dA(z) \\ &\lesssim \|f\|^p_{wA^p_\alpha(X)} \int_{\mathbb{D}} \frac{|\psi(z)|^p(1-|z|^2)^{\alpha+p}}{(1-|\varphi(z)|^2)^{2+\alpha+mp}}\, dA(z),\end{aligned}$$

which finishes the proof. □

The following theorem is the main result of this section, which gives a norm estimate of the operator $I^{(m)}_{\varphi,\psi}\colon wA^p_\alpha(X) \to A^p_\alpha(X)$ for $2 \le p < \infty$.

Theorem 2.4. *Let* X *be any complex infinite-dimensional Banach space,* $2 \le p < \infty$ *and* $\alpha > -1$. *Then*

$$\|I^{(m)}_{\varphi,\psi}\|_{wA^p_\alpha(X)\to A^p_\alpha(X)} \asymp \left(\int_{\mathbb{D}} \frac{|\psi(z)|^p(1-|z|^2)^{\alpha+p}}{(1-|\varphi(z)|^2)^{2+\alpha+mp}}\, dA(z)\right)^{1/p}.$$

Proof. By Lemma 2.3, we only need to proceed the lower estimate. To this end, let $n \in \mathbb{N}$ and $\epsilon > 0$. According to Theorem A, fix a linear embedding $T_n\colon l_2^n \to X$ so that (2.1) holds. Put $x_k^{(n)} = T_n e_k$ for $k = 1, 2, \ldots, n$, where $(e_1, \ldots, e_n)$ is some fixed orthonormal basis of l_2^n. Let $\lambda_k = k^{(\alpha+2)/p-1/2}$, and define $f_n\colon \mathbb{D} \to X$ by

$$f_n(z) = \sum_{k=1}^{n} \lambda_k z^k x_k^{(n)} = T_n \left(\sum_{k=1}^{n} \lambda_k z^k e_k \right), \quad z \in \mathbb{D}. \tag{2.2}$$

By Lemma B(i) and the fact that

$$\|z^k\|_{A_\alpha^2}^2 = \frac{k!\Gamma(\alpha+2)}{\Gamma(k+\alpha+2)} \asymp k^{-1-\alpha},$$

we have

$$\begin{aligned}\|f_n\|_{wA_\alpha^p(X)} &= \sup_{x^* \in B_{X^*}} \|x^* \circ f_n\|_{A_\alpha^p} = \sup_{x^* \in B_{X^*}} \left\| \sum_{k=1}^{n} \lambda_k x^*(x_k^{(n)}) z^k \right\|_{A_\alpha^p} \\ &\lesssim \sup_{x^* \in B_{X^*}} \left\| \sum_{k=1}^{n} k^{\frac{1+\alpha}{2}} x^*(x_k^{(n)}) z^k \right\|_{A_\alpha^2} \asymp \sup_{x^* \in B_{X^*}} \left(\sum_{k=1}^{n} |T_n^* x^*(e_k)|^2 \right)^{1/2} \leq 1.\end{aligned}$$

It follows from Theorem 2.1 that

$$\begin{aligned}\|I_{\varphi,\psi}^{(m)}\|_{wA_\alpha^p(X) \to A_\alpha^p(X)}^p &\gtrsim \limsup_{n\to\infty} \|I_{\varphi,\psi}^{(m)} f_n\|_{A_\alpha^p(X)}^p \\ &\asymp \limsup_{n\to\infty} \int_{\mathbb{D}} \|f_n^{(m)}(\varphi(z))\|_X^p |\psi(z)|^p (1-|z|^2)^{\alpha+p} \, dA(z).\end{aligned} \tag{2.3}$$

Since $f_n(z) = T_n\big(\sum_{k=1}^{n} \lambda_k z^k e_k\big)$, we have

$$f_n^{(m)}(z) = T_n \left(\sum_{k=1}^{n-m+1} (k)_m \lambda_{k+m-1} z^{k-1} e_{k+m-1} \right) \tag{2.4}$$

for $0 \leq m \leq n$. Here, $(k)_m = k(k+1)\cdots(k+m-1)$ for $m \geq 1$ and $(k)_0 = 1$, and $\lambda_0 = 0$. Combining (2.4) and (2.1), we establish

$$\begin{aligned}\|f_n^{(m)}(\varphi(z))\|_X^p &= \left\| T_n \Big(\sum_{k=1}^{n-m+1} (k)_m \lambda_{k+m-1} \varphi(z)^{k-1} e_{k+m-1} \Big) \right\|_X^p \\ &\geq \frac{1}{1+\epsilon} \Big(\sum_{k=1}^{n-m+1} (k)_m^2 \lambda_{k+m-1}^2 |\varphi(z)|^{2(k-1)} \Big)^{p/2} \\ &\gtrsim \Big(\sum_{k=1}^{n-m+1} k^{2m+2(\alpha+2)/p-1} |\varphi(z)|^{2(k-1)} \Big)^{p/2}.\end{aligned}$$

Inserting the above estimate into (2.3) and using monotone convergence theorem and Lemma 2.2, we obtain

$$\begin{aligned}
&\|I_{\varphi,\psi}^{(m)}\|^p_{wA^p_\alpha(X)\to A^p_\alpha(X)} \\
&\gtrsim \int_{\mathbb{D}} \left(\sum_{k=1}^{\infty} k^{2m+2(\alpha+2)/p-1}|\varphi(z)|^{2(k-1)}\right)^{p/2} |\psi(z)|^p(1-|z|^2)^{\alpha+p}\,dA(z) \\
&\geq \int_{\{z\in\mathbb{D}:|\varphi(z)|^2\geq 1/2\}} \left(\sum_{k=1}^{\infty} k^{2m+2(\alpha+2)/p-1}|\varphi(z)|^{2k}\right)^{p/2} |\psi(z)|^p(1-|z|^2)^{\alpha+p}\,dA(z) \\
&\geq c^{p/2}_{2m+2(\alpha+2)/p-1} \int_{\{z\in\mathbb{D}:|\varphi(z)|^2\geq 1/2\}} \frac{|\psi(z)|^p(1-|z|^2)^{\alpha+p}}{(1-|\varphi(z)|^2)^{\alpha+2+mp}}\,dA(z).
\end{aligned}$$

Here, $c_{2m+2(\alpha+2)/p-1}$ is the constant defined in Lemma 2.2.

In order to obtain the desired lower estimate, we need to show

$$\|I_{\varphi,\psi}^{(m)}\|^p_{wA^p_\alpha(X)\to A^p_\alpha(X)} \gtrsim \int_{\{z\in\mathbb{D}:|\varphi(z)|^2< 1/2\}} \frac{|\psi(z)|^p(1-|z|^2)^{\alpha+p}}{(1-|\varphi(z)|^2)^{\alpha+2+mp}}\,dA(z). \tag{2.5}$$

Choose $x \in X$ satisfying $\|x\|_X = 1$ and let

$$g(z) = xz^m, \quad z \in \mathbb{D}.$$

Then $g \in wA^p_\alpha(X)$ and the norm of g in $wA^p_\alpha(X)$ only depends on α, p and m. Therefore, we get

$$\begin{aligned}
\|I_{\varphi,\psi}^{(m)}\|^p_{wA^p_\alpha(X)\to A^p_\alpha(X)} &\gtrsim \|I_{\varphi,\psi}^{(m)} g\|^p_{A^p_\alpha(X)} \\
&\asymp m! \int_{\mathbb{D}} |\psi(z)|^p(1-|z|^2)^{\alpha+p}\,dA(z).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_{\{z\in\mathbb{D}:|\varphi(z)|^2< 1/2\}} \frac{|\psi(z)|^p(1-|z|^2)^{\alpha+p}}{(1-|\varphi(z)|^2)^{\alpha+2+mp}}\,dA(z) &\lesssim \int_{\mathbb{D}} |\psi(z)|^p(1-|z|^2)^{\alpha+p}\,dA(z) \\
&\lesssim \|I_{\varphi,\psi}^{(m)}\|^p_{wA^p_\alpha(X)\to A^p_\alpha(X)}.
\end{aligned}$$

Hence (2.5) holds and the lower estimate is established. The proof is therefore complete. □

For $1 \leq p < 2$, using the preceding ideas we can only establish a weaker lower bound.

Proposition 2.5. *Let X be any complex infinite-dimensional Banach space, $1 \leq p < 2$ and $\alpha > -1$. Then*

$$\|I_{\varphi,\psi}^{(m)}\|_{wA^p_\alpha(X)\to A^p_\alpha(X)} \gtrsim \left(\int_{\mathbb{D}} \frac{|\psi(z)|^p(1-|z|^2)^{\alpha+p}}{(1-|\varphi(z)|^2)^{\gamma}}\,dA(z)\right)^{1/p}$$

for $\alpha + 1 + mp < \gamma < \alpha + 1 + p/2 + mp$.

Proof. Let $\lambda_k = k^{\beta+(1+\alpha)/2}$ with $\beta < (\alpha+1)/p - (\alpha+1)/2$ and define f_n as (2.2). Then by Lemma B(ii) we have $\|f_n\|_{wA^p_\alpha(X)} \lesssim 1$ for $1 \le p < 2$. Hence Theorems 2.1, A and monotone convergence theorem yield

$$\begin{aligned}
&\|I^{(m)}_{\varphi,\psi}\|^p_{wA^p_\alpha(X)\to A^p_\alpha(X)} \\
&\gtrsim \limsup_{n\to\infty} \|I^{(m)}_{\varphi,\psi} f_n\|^p_{A^p_\alpha(X)} \\
&\gtrsim \int_{\mathbb{D}} \left(\sum_{k=1}^{\infty} (k)^2_m \lambda^2_{k+m-1} |\varphi(z)|^{2(k-1)}\right)^{p/2} |\psi(z)|^p (1-|z|^2)^{\alpha+p}\, dA(z) \\
&\gtrsim \int_{\mathbb{D}} \left(\sum_{k=1}^{\infty} k^{2m+2\beta+1+\alpha} |\varphi(z)|^{2k}\right)^{p/2} |\psi(z)|^p (1-|z|^2)^{\alpha+p}\, dA(z)
\end{aligned}$$

for $m \ge 0$. Let $\beta > (\alpha+1)/p - 1 - \alpha/2$, then $2m+2\beta+1+\alpha > -1$ and by Lemma 2.2 we have

$$\|I^{(m)}_{\varphi,\psi}\|^p_{wA^p_\alpha(X)\to A^p_\alpha(X)} \gtrsim c^{p/2}_{2m+2\beta+1+\alpha} \int_{\{z\in\mathbb{D}:|\varphi(z)|^2\ge 1/2\}} \frac{|\psi(z)|^p(1-|z|^2)^{\alpha+p}}{(1-|\varphi(z)|^2)^\gamma}\, dA(z),$$

where $\gamma = (2m+2\beta+2+\alpha)p/2$ satisfying

$$\alpha + 1 + mp < \gamma < \alpha + 1 + \frac{p}{2} + mp.$$

Similar to (2.5), we also have

$$\|I^{(m)}_{\varphi,\psi}\|^p_{wA^p_\alpha(X)\to A^p_\alpha(X)} \gtrsim \int_{\{z\in\mathbb{D}:|\varphi(z)|^2< 1/2\}} \frac{|\psi(z)|^p(1-|z|^2)^{\alpha+p}}{(1-|\varphi(z)|^2)^\gamma}\, dA(z).$$

Thus the proof is finished. □

In particular, we have the following estimates for the norm of the Volterra type integration operator $J_b\colon wA^p_\alpha(X) \to A^p_\alpha(X)$.

Corollary 2.6. *Let X be any complex infinite-dimensional Banach space, $1 \le p < \infty$, $\alpha > -1$ and $b \in \mathcal{H}(\mathbb{D})$.*

(1) *If $2 \le p < \infty$, then $J_b\colon wA^p_\alpha(X) \to A^p_\alpha(X)$ is bounded if and only if b belongs to the analytic Besov space B_p. Moreover,*

$$\|J_b\|_{wA^p_\alpha(X)\to A^p_\alpha(X)} \asymp \left(\int_{\mathbb{D}} |b'(z)|^p (1-|z|^2)^{p-2}\, dA(z)\right)^{1/p}.$$

(2) *If $1 \le p < 2$, then*

$$\begin{aligned}
\left(\int_{\mathbb{D}} |b'(z)|^p (1-|z|^2)^{\gamma}\, dA(z)\right)^{1/p} &\lesssim \|J_b\|_{wA^p_\alpha(X)\to A^p_\alpha(X)} \\
&\lesssim \left(\int_{\mathbb{D}} |b'(z)|^p (1-|z|^2)^{p-2}\, dA(z)\right)^{1/p}
\end{aligned}$$

for $p/2 - 1 < \gamma < p - 1$.

Remark 2.7. By [1, Theorem 2] (see also [18, Theorem 1.4]), we know that $J_b\colon wA^p_\alpha(X) \to A^p_\alpha(X)$ is bounded if and only if $J_b\colon A^2_\alpha \to A^2_\alpha$ is in the Schatten class $S_p(A^2_\alpha)$ when $2 \le p < \infty$.

3. Hardy space case

Let X be any complex infinite-dimensional Banach space. In this section we first give a lower bound for the norm of $I^{(m)}_{\varphi,\psi}\colon wH^p(X) \to H^p(X)$ when X is p-uniformly PL-convex and $2 \le p < \infty$. To this purpose, we need the following Littlewood-Paley inequality for $H^p(X)$, which can be found in [5, Theorem 2.3].

Theorem C. *Let $2 \le p < \infty$ and X be a Banach space. Then X is p-uniformly PL-convex if and only if there exists $c > 0$ such that*

$$\|f\|_{H^p(X)} \ge \left(\|f(0)\|^p_X + c \int_{\mathbb{D}} \|f'(z)\|^p_X (1-|z|^2)^{p-1}\, dA(z) \right)^{1/p}$$

for all $f \in H^p(X)$.

The following lemma concerns the bounded coefficient multipliers from H^2 to H^p, which is cited from [10, Theorem 1].

Lemma D. *The sequence $\{k^{1/p-1/2}\}$ is a bounded coefficient multiplier from H^2 to H^p for $2 \le p < \infty$.*

We now estimate the lower bound for $\|I^{(m)}_{\varphi,\psi}\|_{wH^p(X)\to H^p(X)}$.

Proposition 3.1. *Let $2 \le p < \infty$ and X be any complex p-uniformly PL-convex infinite-dimensional Banach space. Then*

$$\|I^{(m)}_{\varphi,\psi}\|_{wH^p(X)\to H^p(X)} \gtrsim \left(\int_{\mathbb{D}} \frac{|\psi(z)|^p (1-|z|^2)^{p-1}}{(1-|\varphi(z)|^2)^{mp+1}}\, dA(z) \right)^{1/p}.$$

Proof. For any given $n \in \mathbb{N}$ and $\epsilon > 0$, fix a linear embedding $T_n\colon l^n_2 \to X$ so that (2.1) holds. Put $x^{(n)}_k = T_n e_k$ for $k = 1, 2, \dots, n$, where $(e_1, \dots, e_n)$ is some fixed orthonormal basis of l^n_2. Consider the X-valued polynomials

$$f_n(z) = \sum_{k=1}^{n} \lambda_k z^k x^{(n)}_k, \quad z \in \mathbb{D},$$

where $\lambda_k = k^{1/p-1/2}$. Then we have

$$\begin{aligned} \|f_n\|_{wH^p(X)} &= \sup_{x^* \in B_{X^*}} \left\| \sum_{k=1}^{n} \lambda_k z^k x^*(x^{(n)}_k) \right\|_{H^p} \lesssim \sup_{x^* \in B_{X^*}} \left\| \sum_{k=1}^{n} z^k x^*(x^{(n)}_k) \right\|_{H^2} \\ &= \sup_{x^* \in B_{X^*}} \left(\sum_{k=1}^{n} |T^*_n x^*(e_k)|^2 \right)^{1/2} \le 1, \end{aligned}$$

where the inequality $\lesssim$ follows from Lemma D. Therefore,

$$\|I^{(m)}_{\varphi,\psi}\|_{wH^p(X)\to H^p(X)} \gtrsim \limsup_{n\to\infty} \|I^{(m)}_{\varphi,\psi} f_n\|_{H^p(X)}.$$

By Theorems C, A and Lemma 2.2, we obtain

$$\begin{aligned}
\|I^{(m)}_{\varphi,\psi}\|^p_{wH^p(X)\to H^p(X)} &\gtrsim \limsup_{n\to\infty} \|I^{(m)}_{\varphi,\psi} f_n\|^p_{H^p(X)} \\
&\gtrsim \limsup_{n\to\infty} \int_{\mathbb{D}} \|f_n^{(m)}(\varphi(z))\|^p_X |\psi(z)|^p (1-|z|^2)^{p-1}\, dA(z) \\
&\gtrsim \int_{\mathbb{D}} \left(\sum_{k=1}^{\infty} k^{2m+2/p-1} |\varphi(z)|^{2k} \right)^{p/2} |\psi(z)|^p (1-|z|^2)^{p-1}\, dA(z) \\
&\gtrsim c^{p/2}_{2m+2/p-1} \int_{\{z\in\mathbb{D}:|\varphi(z)|^2\geq 1/2\}} \frac{|\psi(z)|^p(1-|z|^2)^{p-1}}{(1-|\varphi(z)|^2)^{mp+1}}\, dA(z)
\end{aligned}$$

for $m \geq 0$. Let $g(z) = xz^m$ for $x \in X$ with $\|x\|_X = 1$, then $\|g\|_{wH^p(X)} = 1$. Using Theorem C again, we have

$$\begin{aligned}
\|I^{(m)}_{\varphi,\psi}\|^p_{wH^p(X)\to H^p(X)} &\geq \|I^{(m)}_{\varphi,\psi} g\|^p_{H^p(X)} \\
&\gtrsim \int_{\mathbb{D}} |\psi(z)|^p (1-|z|^2)^{p-1}\, dA(z) \\
&\gtrsim \int_{\{z\in\mathbb{D}:|\varphi(z)|^2< 1/2\}} \frac{|\psi(z)|^p(1-|z|^2)^{p-1}}{(1-|\varphi(z)|^2)^{mp+1}}\, dA(z).
\end{aligned}$$

This completes the proof. □

Remark 3.2. For the case $1 < p < 2$, there are no estimates similar to the one in Theorem C. However, we can give a weaker lower bound for the norm of the operator $I^{(m)}_{\varphi,\psi}: wH^p(X) \to H^p(X)$ via embedding Hardy spaces into Bergman spaces. If X is any complex Banach space, $1 < p < q < \infty$ and $\alpha = q/p - 2$, then $H^p(X) \subset A^q_\alpha(X)$ and the inclusion is continuous. To see this, for any $f \in H^p(X)$ and $0 < r < 1$, write $f_r(z) = f(rz)$. By [19, Corollary 4.47] and the subharmonic property of $\|f_r\|_X$, we have

$$\|f_r\|_{A^q_\alpha(X)} \leq C\|f_r\|_{H^p(X)} \leq C\|f\|_{H^p(X)}$$

for some absolute constant $C > 0$. Using Fatou's lemma, we obtain

$$\|f\|_{A^q_\alpha(X)} \leq \liminf_{r\to 1} \|f_r\|_{A^q_\alpha(X)} \lesssim \|f\|_{H^p(X)}.$$

Therefore, if X is any complex infinite-dimensional Banach space and $1 < p < 2$, then using Theorem 2.1 and the same method as in the proof of Proposition 3.1, we have

$$\|I^{(m)}_{\varphi,\psi}\|_{wH^p(X)\to H^p(X)} \gtrsim \left(\int_{\mathbb{D}} \frac{|\psi(z)|^q(1-|z|^2)^{q+q/p-2}}{(1-|\varphi(z)|^2)^{mq+q/2}}\, dA(z) \right)^{1/q}$$

for $q > p$.

If X is a complex Hilbert space, we have the following Littlewood-Paley type identity for the space $H^2(X)$.

Lemma 3.3. *Let X be a complex Hilbert space, then we have*

$$\|f - f(0)\|^2_{H^2(X)} \asymp \int_{\mathbb{D}} \|f'(z)\|^2_X (1-|z|^2)\, dA(z)$$

for any $f \in H^2(X)$.

Proof. Using the Taylor expansion of f, this can be obtained by some elementary computations. □

If X is a complex infinite-dimensional Hilbert space, we have the following estimate for the norm of the operator $I^{(m)}_{\varphi,\psi}\colon wH^2(X) \to H^2(X)$.

Theorem 3.4. *Let X be a complex infinite-dimensional Hilbert space. Then*

$$\|I^{(m)}_{\varphi,\psi}\|_{wH^2(X)\to H^2(X)} \asymp \left(\int_{\mathbb{D}} \frac{|\psi(z)|^2(1-|z|^2)}{(1-|\varphi(z)|^2)^{1+2m}}\, dA(z)\right)^{1/2}.$$

Proof. Since any Hilbert space is 2-uniformly PL-convex, the lower estimate follows from Proposition 3.1. We now consider the upper estimate. For any $f \in wH^2(X)$, by the pointwise estimate of the derivative of Hardy space functions, we have

$$\|f^{(m)}(z)\|^2_X = \sup_{x^* \in B_{X^*}} |x^*(f^{(m)}(z))|^2 = \sup_{x^* \in B_{X^*}} |(x^* \circ f)^{(m)}(z)|^2 \lesssim \frac{\|f\|^2_{wH^2(X)}}{(1-|z|^2)^{1+2m}}.$$

Therefore, by Lemma 3.3, we have

$$\begin{aligned}\|I^{(m)}_{\varphi,\psi} f\|^2_{H^2(X)} &\asymp \int_{\mathbb{D}} \|f^{(m)}(\varphi(z))\|^2_X |\psi(z)|^2 (1-|z|^2)\, dA(z) \\ &\lesssim \|f\|^2_{wH^2(X)} \int_{\mathbb{D}} \frac{|\psi(z)|^2(1-|z|^2)}{(1-|\varphi(z)|^2)^{1+2m}}\, dA(z),\end{aligned}$$

which completes the theorem. □

As applications, we have the following corollaries.

Corollary 3.5. *Let $2 \le p < \infty$ and X be any complex p-uniformly PL-convex infinite-dimensional Banach space. Then*

$$\|J_b\|_{wH^p(X)\to H^p(X)} \gtrsim \left(\int_{\mathbb{D}} |b'(z)|^p (1-|z|^2)^{p-2}\, dA(z)\right)^{1/p}.$$

Corollary 3.6. *Let X be any complex infinite-dimensional Hilbert space. Then $J_b\colon wH^2(X) \to H^2(X)$ is bounded if and only if b belongs to the Dirichlet space. Moreover,*

$$\|J_b\|_{wH^2(X)\to H^2(X)} \asymp \left(\int_{\mathbb{D}} |b'(z)|^2\, dA(z)\right)^{1/2}.$$

Remark 3.7. Due to [17, Theorem 6.7], we know that if $2 \leq p < \infty$ and X is a complex p-uniformly PL-convex infinite-dimensional Banach space, then the boundedness of $J_b\colon wH^p(X) \to H^p(X)$ implies $J_b\colon H^2 \to H^2$ is in the Schatten class $S_p(H^2)$. Furthermore, if X is a complex infinite-dimensional Hilbert space, then $J_b\colon wH^2(X) \to H^2(X)$ is bounded if and only if $J_b\colon H^2 \to H^2$ is a Hilbert-Schmidt operator.

4. Fock space case

In the last section, we investigate the boundedness of $I^{(m)}_{\varphi,\psi}\colon wF^p_\alpha(X) \to F^p_\alpha(X)$. For this purpose, we need the following result, which characterises a X-valued Fock space function by its derivatives.

Theorem 4.1. *Suppose $f \in \mathcal{H}(\mathbb{C}, X)$, $1 \leq p < \infty$, $\alpha > 0$ and $n \in \mathbb{N}$. Then*

$$\|f\|_{F^p_\alpha(X)} \asymp \sum_{k=0}^{n-1} \|f^{(k)}(0)\|_X + \left(\int_{\mathbb{C}} \left\| \frac{f^{(n)}(z)}{(1+|z|)^n} \right\|_X^p e^{-\frac{\alpha p}{2}|z|^2}\, dA(z)\right)^{1/p}.$$

In order to prove the above theorem, we need the following lemma.

Lemma 4.2. *Let $f \in \mathcal{H}(\mathbb{C}, X)$, $n \in \mathbb{N}$ and $1 \leq p < \infty$. Then for any $z \in \mathbb{C}$ and $r > 0$, we have*

$$\|f^{(n)}(z)\|_X^p \lesssim \frac{1}{r^{2+np}} \int_{D(z,r)} \|f(w)\|_X^p\, dA(w),$$

where $D(z, r) = \{w \in \mathbb{C} : |w - z| < r\}$.

Proof. We only need to consider the case $z = 0$. For any $\rho > 0$, Cauchy's integral formula yields

$$\|f^{(n)}(0)\|_X \leq \frac{n!}{2\pi} \int_0^{2\pi} \|f(\rho e^{i\theta})\|_X \rho^{-n}\, d\theta.$$

Multiplying by ρ^{n+1} and integrating with respect to ρ from $r/2$ to r, we obtain

$$\frac{r^{n+2} - (r/2)^{(n+2)}}{n+2} \|f^{(n)}(0)\|_X \leq \frac{n!}{2\pi} \int_0^r \int_0^{2\pi} \|f(\rho e^{i\theta})\|_X \rho\, d\theta d\rho.$$

Since $r^{n+2} - (r/2)^{n+2} \geq r^{n+2}/2$, we arrive at

$$\|f^{(n)}(0)\|_X \lesssim \frac{1}{r^{n+2}} \int_{D(0,r)} \|f(w)\|_X\, dA(w).$$

Hölder's inequality then gives the desired estimate. □

Proof of Theorem 4.1. By Lemma 4.2, we have

$$\|f^{(k)}(0)\|_X \lesssim \left(\int_{D(0,1)} \|f(w)\|_X^p\, dA(w)\right)^{1/p} \lesssim \|f\|_{F^p_\alpha(X)}$$

for any $0 \le k \le n-1$. Using Lemma 4.2 and the estimate (8) in [12], we obtain

$$\begin{aligned}
&\int_{\mathbb{C}} \left\| \frac{f^{(n)}(z)}{(1+|z|)^n} \right\|_X^p e^{-\frac{\alpha p}{2}|z|^2}\, dA(z) \\
&\lesssim \int_{\mathbb{C}} (1+|z|)^2 \int_{D(z,\frac{1}{1+|z|})} \|f(w)\|_X^p \, dA(w) e^{-\frac{\alpha p}{2}|z|^2}\, dA(z) \\
&\lesssim \int_{\mathbb{C}} \|f(w)\|_X^p (1+|w|)^2 \int_{D(w,\frac{2}{1+|w|})} e^{-\frac{\alpha p}{2}|z|^2}\, dA(z) dA(w) \\
&\lesssim \int_{\mathbb{C}} \|f(w)\|_X^p e^{-\frac{\alpha p}{2}|w|^2}\, dA(w),
\end{aligned}$$

where the second inequality is due to Fubini's theorem and the facts that $w \in D(z, 1/(1+|z|))$ implies $z \in D(w, 2/(1+|w|))$, and $1+|z| \lesssim 1+|w|$ if $z \in D(w, 2/(1+|w|))$. Combining the estimates above yields

$$\|f\|_{F^p_\alpha(X)} \gtrsim \sum_{k=0}^{n-1} \|f^{(k)}(0)\|_X + \left(\int_{\mathbb{C}} \left\| \frac{f^{(n)}(z)}{(1+|z|)^n} \right\|_X^p e^{-\frac{\alpha p}{2}|z|^2}\, dA(z) \right)^{1/p}.$$

Conversely, note that $\|f\|_X^p$ is subharmonic on $\mathbb{C}$ for any $1 \le p < \infty$. Consequently, $M_p(f, r)$ is increasing with r, see e.g. [11, Corollary 6.6]. We claim that

$$\int_{\mathbb{C}} \left\| \frac{f(z)}{(1+|z|)^k} \right\|_X^p e^{-\frac{\alpha p}{2}|z|^2}\, dA(z) \lesssim \int_{\mathbb{C}} \left\| \frac{f'(z)}{(1+|z|)^{k+1}} \right\|_X^p e^{-\frac{\alpha p}{2}|z|^2}\, dA(z) \tag{4.1}$$

for any fixed $1 \le p < \infty$, $k \ge 0$, and all $f \in \mathcal{H}(\mathbb{C}, X)$ with $f(0) = 0$. In fact, this can be proven by the same method as in the proof of [12, (11)]. In the case $p = 1$, for any $0 < \rho < r < \infty$, we have

$$\begin{aligned}
M_1(f, r) - M_1(f, \rho) &\le \int_{\mathbb{T}} \|f(r\zeta) - f(\rho\zeta)\|_X \, dm(\zeta) \\
&= \int_{\mathbb{T}} \left\| \int_\rho^r f'(t\zeta)\zeta \, dt \right\|_X dm(\zeta) \le (r-\rho) M_1(f', r).
\end{aligned}$$

Therefore, (4.1) holds in this case. In the case $1 < p < \infty$, vector-valued version of Lemma 2.2 in [12] is needed. Carefully examining the proof of [16, Theorem 1], we see [12, Lemma 2.2] holds for vector-valued functions. Consequently, (4.1) also holds in this case. Then for any $f \in \mathcal{H}(\mathbb{C}, X)$, due to (4.1) we obtain

$$\begin{aligned}
&\left(\int_{\mathbb{C}} \left\| \frac{f(z)}{(1+|z|)^k} \right\|_X^p e^{-\frac{\alpha p}{2}|z|^2}\, dA(z) \right)^{1/p} \\
&\le \left(\int_{\mathbb{C}} \left\| \frac{f(z) - f(0)}{(1+|z|)^k} \right\|_X^p e^{-\frac{\alpha p}{2}|z|^2}\, dA(z) \right)^{1/p} + \|f(0)\|_X \left(\int_{\mathbb{C}} \frac{e^{-\frac{\alpha p}{2}|z|^2}}{(1+|z|)^{pk}}\, dA(z) \right)^{1/p} \\
&\lesssim \|f(0)\|_X + \left(\int_{\mathbb{C}} \left\| \frac{f'(z)}{(1+|z|)^{k+1}} \right\|_X^p e^{-\frac{\alpha p}{2}|z|^2}\, dA(z) \right)^{1/p}.
\end{aligned}$$

Applying the above estimate repeatedly, we establish

$$\|f\|_{F^p_\alpha(X)} \lesssim \sum_{k=0}^{n-1} \|f^{(k)}(0)\|_X + \left(\int_{\mathbb{C}} \left\| \frac{f^{(n)}(z)}{(1+|z|)^n} \right\|_X^p e^{-\frac{\alpha p}{2}|z|^2}\, dA(z)\right)^{1/p},$$

which completes the theorem. □

The following lemma estimates the derivatives of Fock space functions.

Lemma 4.3. *Let $0 < p < \infty$ and $\alpha > 0$. For any $f \in F^p_\alpha$ and $n \geq 0$, the following estimate holds:*

$$|f^{(n)}(z)| \lesssim (1+|z|^n) e^{\frac{\alpha}{2}|z|^2} \|f\|_{F^p_\alpha}.$$

Proof. The case $n = 0$ was proved in [20, Corollary 2.8]. We consider the case $n > 0$. For $|z| \leq 1$, by Cauchy's estimate and the estimate in the case $n = 0$, we have

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \int_{|\zeta - z|=1} \frac{|f(\zeta)|}{|\zeta - z|^{n+1}} |d\zeta| \lesssim \max_{|\zeta - z|=1} |f(\zeta)| \lesssim \|f\|_{F^p_\alpha}.$$

For $|z| > 1$, arguing as above, we get

$$\begin{aligned} |f^{(n)}(z)| &\leq \frac{n!}{2\pi} \int_{|\zeta - z|=1/|z|} \frac{|f(\zeta)|}{|\zeta - z|^{n+1}} |d\zeta| \lesssim |z|^n \max_{|\zeta - z|=1/|z|} |f(\zeta)| \\ &\leq |z|^n e^{\frac{\alpha}{2}(|z| + \frac{1}{|z|})^2} \|f\|_{F^p_\alpha} \lesssim |z|^n e^{\frac{\alpha}{2}|z|^2} \|f\|_{F^p_\alpha}. \end{aligned}$$

Combining these estimates, we obtain the desired result. □

We now end this section by estimating the norm of $I^{(m)}_{\varphi,\psi}$ on the Fock type setting.

Theorem 4.4. *Let X be any complex infinite-dimensional Banach space, $2 \leq p < \infty$ and $\alpha > 0$. Then*

$$\|I^{(m)}_{\varphi,\psi}\|_{wF^p_\alpha(X) \to F^p_\alpha(X)} \asymp \left(\int_{\mathbb{C}} \frac{|\psi(z)|^p (1 + |\varphi(z)|^m)^p}{(1+|z|)^p} e^{-\frac{\alpha p}{2}(|z|^2 - |\varphi(z)|^2)}\, dA(z)\right)^{1/p}.$$

Proof. For any $f \in wF^p_\alpha(X)$, by Theorem 4.1 and the estimate in Lemma 4.3, we get

$$\begin{aligned} \|I^{(m)}_{\varphi,\psi} f\|^p_{F^p_\alpha(X)} &\asymp \int_{\mathbb{C}} \left\| \frac{f^{(m)}(\varphi(z))\psi(z)}{1+|z|} \right\|_X^p e^{-\frac{\alpha p}{2}|z|^2}\, dA(z) \\ &\lesssim \|f\|^p_{wF^p_\alpha(X)} \int_{\mathbb{C}} \frac{|\psi(z)|^p (1 + |\varphi(z)|^m)^p}{(1+|z|)^p} e^{-\frac{\alpha p}{2}(|z|^2 - |\varphi(z)|^2)}\, dA(z), \end{aligned}$$

which gives us the upper estimate.

We next consider the lower estimate. Fix $n \in \mathbb{N}$ and $\epsilon > 0$. According to Theorem A, there is a linear embedding $T_n \colon l^n_2 \to X$ so that (2.1) holds. Put $x^{(n)}_k = T_n e_k$ for $k =$

$1, 2, \ldots, n$, where $(e_1, e_2, \ldots, e_n)$ is some fixed orthonormal basis of l_2^n. Define $f_n \colon \mathbb{C} \to X$ by

$$f_n(z) = \sum_{k=0}^{n-1} \sqrt{\frac{\alpha^k}{k!}} z^k x_{k+1}^{(n)}, \quad z \in \mathbb{C}.$$

Then

$$\begin{aligned} \|f_n\|_{wF_\alpha^p(X)} &= \sup_{x^* \in B_{X^*}} \left\| \sum_{k=0}^{n-1} \sqrt{\frac{\alpha^k}{k!}} x^*(x_{k+1}^{(n)}) z^k \right\|_{F_\alpha^p} \lesssim \sup_{x^* \in B_{X^*}} \left\| \sum_{k=0}^{n-1} \sqrt{\frac{\alpha^k}{k!}} x^*(x_{k+1}^{(n)}) z^k \right\|_{F_\alpha^2} \\ &= \sup_{x^* \in B_{X^*}} \left(\sum_{k=0}^{n-1} |x^*(x_{k+1}^{(n)})|^2 \right)^{1/2} \le 1, \end{aligned}$$

where the first inequality is due to the embedding $F_\alpha^p \subset F_\alpha^q$ is bounded whenever $p \le q$. Therefore, by Theorem 4.1, we obtain

$$\begin{aligned} \|I_{\varphi,\psi}^{(m)}\|_{wF_\alpha^p(X) \to F_\alpha^p(X)}^p &\gtrsim \limsup_{n \to \infty} \|I_{\varphi,\psi}^{(m)} f_n\|_{F_\alpha^p(X)}^p \\ &\asymp \limsup_{n \to \infty} \int_{\mathbb{C}} \left\| \frac{f_n^{(m)}(\varphi(z))\psi(z)}{1+|z|} \right\|_X^p e^{-\frac{\alpha p}{2}|z|^2} \, dA(z). \end{aligned}$$

By the definition of f_n and (2.1), we have

$$\begin{aligned} \|f_n^{(m)}(\varphi(z))\|_X^p &= \left\| T_n \left(\sum_{k=0}^{n-m-1} (k+1)_m \sqrt{\frac{\alpha^{k+m}}{(k+m)!}} \varphi(z)^k e_{k+m+1} \right) \right\|_X^p \\ &\gtrsim \left(\sum_{k=0}^{n-m-1} (k+1)_m^2 \frac{\alpha^{k+m}}{(k+m)!} |\varphi(z)|^{2k} \right)^{p/2} \end{aligned}$$

for $0 \le m < n$. Therefore, by monotone convergence theorem, we arrive at

$$\|I_{\varphi,\psi}^{(m)}\|_{wF_\alpha^p(X) \to F_\alpha^p(X)}^p \gtrsim \int_{\mathbb{C}} \left(\sum_{k=0}^{\infty} (k+1)_m \frac{\alpha^k}{k!} |\varphi(z)|^{2k} \right)^{p/2} \frac{|\psi(z)|^p e^{-\frac{\alpha p}{2}|z|^2}}{(1+|z|)^p} \, dA(z).$$

It is obvious to see

$$(1 + |\varphi(z)|^m)^p e^{\frac{\alpha p}{2}|\varphi(z)|^2} \lesssim \left(\sum_{k=0}^{\infty} (k+1)_m \frac{\alpha^k}{k!} |\varphi(z)|^{2k} \right)^{p/2}.$$

Hence we establish the lower estimate for the norm of $I_{\varphi,\psi}^{(m)} \colon wF_\alpha^p(X) \to F_\alpha^p(X)$ and the proof is complete. □

Remark 4.5. The upper estimate for $\|I_{\varphi,\psi}^{(m)}\|_{wF_\alpha^p(X) \to F_\alpha^p(X)}$ in Theorem 4.4 is actually valid for all $1 \le p < \infty$ and any complex Banach space X.

In particular, the boundedness of $J_b\colon wF^p_\alpha(X)\to F^p_\alpha(X)$ and $C_\varphi\colon wF^p_\alpha(X)\to F^p_\alpha(X)$ are characterized when $2\le p<\infty$.

Corollary 4.6. *Let X be any complex infinite-dimensional Banach space and $\alpha>0$.*

(1) *$J_b\colon wF^2_\alpha(X)\to F^2_\alpha(X)$ is bounded if and only if b is a constant.*

(2) *If $2<p<\infty$, then $J_b\colon wF^p_\alpha(X)\to F^p_\alpha(X)$ is bounded if and only if $b(z)=az+d$ for some $a,d\in\mathbb{C}$. Moreover, $\|J_b\|_{wF^p_\alpha(X)\to F^p_\alpha(X)}\asymp|a|$.*

Proof. By Theorem 4.4, we have

$$\|J_b\|^p_{wF^p_\alpha(X)\to F^p_\alpha(X)}\asymp\int_{\mathbb{C}}\left|\frac{b'(z)}{1+|z|}\right|^p dA(z).$$

The subharmonicity of $|b'|^p$ implies

$$\left(\int_{D(w,1)}\left|\frac{b'(z)}{1+|z|}\right|^p dA(z)\right)^{1/p}\gtrsim\frac{|b'(w)|}{1+|w|}.$$

Hence the boundedness of $J_b\colon wF^p_\alpha(X)\to F^p_\alpha(X)$ implies

$$\frac{|b'(w)|}{1+|w|}\to 0\quad\text{as } |w|\to\infty,$$

which is equivalent to $b(z)=az+d$ for some $a,d\in\mathbb{C}$. So it is only need to prove the necessity of Case (1), since the other case is obvious.

If $J_b\colon wF^2_\alpha(X)\to F^2_\alpha(X)$ is bounded and b is not a constant, i.e., $b(z)=az+d$ for some $a\neq 0$, then by the above estimate for the norm of $J_b\colon wF^p_\alpha(X)\to F^p_\alpha(X)$, we have

$$\|J_b\|_{wF^2_\alpha(X)\to F^2_\alpha(X)}\asymp|a|\left(\int_{\mathbb{C}}\frac{dA(z)}{(1+|z|)^2}\right)^{1/2}=\infty,$$

which is a contradiction. □

Corollary 4.7. *Let X be any complex infinite-dimensional Banach space, $2\le p<\infty$ and $\alpha>0$. Then $C_\varphi\colon wF^p_\alpha(X)\to F^p_\alpha(X)$ is bounded if and only if $\varphi(z)=az+d$ for some $a,d\in\mathbb{C}$ with $|a|<1$.*

Proof. Since

$$I^{(1)}_{\varphi,\varphi'}f(z)=\int_0^z f'(\varphi(\zeta))\varphi'(\zeta)\,d\zeta=f(\varphi(z))-f(\varphi(0)),$$

we obtain that $C_\varphi\colon wF^p_\alpha(X)\to F^p_\alpha(X)$ is bounded if and only if $I^{(1)}_{\varphi,\varphi'}\colon wF^p_\varphi(X)\to F^p_\alpha(X)$ is bounded. By Theorem 4.4, the boundedness of $C_\varphi\colon wF^p_\alpha(X)\to F^p_\alpha(X)$ can be characterized by

$$\int_{\mathbb{C}}\frac{|\varphi'(z)|^p(1+|\varphi(z)|)^p}{(1+|z|)^p}e^{-\frac{\alpha p}{2}(|z|^2-|\varphi(z)|^2)}\,dA(z)<\infty. \tag{4.2}$$

If $\varphi(z) = az + d$ for some $a, d \in \mathbb{C}$ with $|a| < 1$, then it is trivial to see (4.2) holds. Conversely, the boundedness of $C_\varphi \colon wF^p_\alpha(X) \to F^p_\alpha(X)$ implies that $C_\varphi \colon F^p_\alpha \to F^p_\alpha$ is bounded. Therefore, $\varphi(z) = az + d$ with $|a| < 1$ or $\varphi(z) = az$ with $|a| = 1$ (see, for instance, Exercise 4 of page 89 in [20]). The latter case obviously contradicts (4.2). □

Remark 4.8. By Corollary 4.7 (or Corollary 4.6), we get that $F^p_\alpha(X) \subsetneq wF^p_\alpha(X)$ for any $2 \le p < \infty$, $\alpha > 0$ and complex infinite-dimensional Banach space X. In fact, if $F^p_\alpha(X) = wF^p_\alpha(X)$ as linear spaces, then $\|f\|_{F^p_\alpha(X)} \asymp \|f\|_{wF^p_\alpha(X)}$ for any $f \in \mathcal{H}(\mathbb{C}, X)$ by open mapping theorem. Hence $C_\varphi \colon wF^p_\alpha(X) \to F^p_\alpha(X)$ is bounded if and only if $C_\varphi \colon wF^p_\alpha(X) \to wF^p_\alpha(X)$ is bounded, which in turn is equivalent to the boundedness of $C_\varphi \colon F^p_\alpha \to F^p_\alpha$. However, this is impossible by Corollary 4.7.

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References

[1] A. Aleman and A. G. Siskakis, *Integration operators on Bergman spaces*, Indiana Univ. Math. J. **46** (1997), no. 2, 337–356.

[2] O. Blasco, *Boundary values of vector-valued harmonic functions considered as operators*, Studia Math. **86** (1987), no. 1, 19–33.

[3] ______, *Boundary values of functions in vector-valued Hardy spaces and geometry on Banach spaces*, J. Funct. Anal. **78** (1988), no. 2, 346–364.

[4] ______, *Introduction to vector valued Bergman spaces*, in: *Function Spaces and Operator Theory*, 9–30, Univ. Joensuu Dept. Math. Rep. Ser. **8**, Univ. Joensuu, Joensuu, 2005.

[5] O. Blasco and M. Pavlović, *Complex convexity and vector-valued Littlewood-Paley inequalities*, Bull. London Math. Soc. **35** (2003), no. 6, 749–758.

[6] J. Bonet, P. Domański and M. Lindström, *Weakly compact composition operators on analytic vector-valued function spaces*, Ann. Acad. Sci. Fenn. Math. **26** (2001), no. 1, 233–248.

[7] A. V. Bukhvalov and A. A. Danilevich, *Boundary properties of analytic and harmonic functions with values in Banach space*, Mat. Zametki **31** (1982), no. 2, 203–214.

[8] O. Constantin, *A Volterra-type integration operator on Fock spaces*, Proc. Amer. Math. Soc. **140** (2012), no. 12, 4247–4257.

[9] J. Diestel, H. Jarchow and A. Tonge, *Absolutely Summing Operators*, Cambridge Studies in Advanced Mathematics **43**, Cambridge University Press, Cambridge, 1995.

[10] P. L. Duren, *On the multipliers of H^p spaces*, Proc. Amer. Math. Soc. **22** (1969), 24–27.

[11] J. B. Garnett, *Bounded Analytic Functions*, Revised first edition, Graduate Texts in Mathematics **236**, Springer, New York, 2007.

[12] Z. Hu, *Equivalent norms on Fock spaces with some application to extended Cesaro operators*, Proc. Amer. Math. Soc. **141** (2013), no. 8, 2829–2840.

[13] M. Jevtić, D. Vukotić and M. Arsenović, *Taylor coefficients and coefficient multipliers of Hardy and Bergman-type spaces*, RSME Springer Series **2**, Springer, Cham, 2016.

[14] J. Laitila, H.-O. Tylli and M. Wang, *Composition operators from weak to strong spaces of vector-valued analytic functions*, J. Operator Theory **62** (2009), no. 2, 281–295.

[15] S. Miihkinen, J. Pau, A. Perälä and M. Wang, *Volterra type integration operators from Bergman spaces to Hardy spaces*, J. Funct. Anal. **279** (2020), no. 4, 108564, 32 pp.

[16] B. Muckenhoupt, *Hardy's inequality with weights*, Studia Math. **44** (1972), 31–38.

[17] J. Pau, *Integration operators between Hardy spaces on the unit ball of $\mathbb{C}^n$*, J. Funct. Anal. **270** (2016), no. 1, 134–176.

[18] J. Xiao, *Riemann-Stieltjes operators on weighted Bloch and Bergman spaces of the unit ball*, J. London Math. Soc. (2) **70** (2004), no. 1, 199–214.

[19] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Graduate Texts in Mathematics **226**, Springer-Verlag, New York, 2005.

[20] ——, *Analysis on Fock Spaces*, Graduate Texts in Mathematics **263**, Springer, New York, 2012.

Jiale Chen and Maofa Wang
School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China
E-mail address: `jialechen@whu.edu.cn`, `mfwang.math@whu.edu.cn`