TAIWANESE JOURNAL OF MATHEMATICS
Vol. xx, No. x, pp. 1–13, xx 20xx
DOI: 10.11650/tjm/201205

Renormings of Nonseparable Reflexive Banach Spaces and Diametrically Complete Sets with Empty Interior

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Abstract. We prove that for each nonseparable and reflexive Banach space $(X, \|\cdot\|_X)$ with the nonstrict Opial and Kadec–Klee properties, there exists an equivalent norm $\|\cdot\|_0$ such that the Banach space $(X, \|\cdot\|_0)$ is LUR and contains a diametrically complete set with empty interior.

1. Introduction

It has recently been proved in [5] that after a suitable renorming, each infinite-dimensional, reflexive and separable Banach space is locally uniformly rotund and contains a diametrically complete set with empty interior. In the present paper we extend this result to each nonseparable and reflexive Banach space which has the nonstrict Opial and Kadec–Klee properties. The new norm is closely connected with Day's norm on $c_0(\Gamma)$. For other results of this type, see [2–4, 9, 12, 17–19, 21, 22] and the references cited therein.

Our paper is organized in the following way. In Section 2 we give basic notations and recall facts which we use in the next sections. In Section 3 we introduce a new equivalent norm $\|\cdot\|_{L,\alpha,F}$ in a Banach space $(X, \|\cdot\|_X)$ under the assumption that there exists a bounded family $F = \{f_\gamma^*\}_{\gamma\in\Gamma}$ of linear functionals in X^* such that for each $x \in X$ and for each $\epsilon > 0$, the set $\{\gamma \in \Gamma : |f_\gamma^*(x)| > \epsilon\}$ is finite. The construction of this norm is based on Day's norm on $c_0(\Gamma)$ [7]. We provide an explicit formula for this new norm which contains the original norm and functionals from the family $F = \{f_\gamma^*\}_{\gamma\in\Gamma}$, which is divided into two subfamilies. These two subfamilies are connected with indices in $\mathbb{N}$ and Γ_1, respectively, where $\Gamma = \mathbb{N} \cup \Gamma_1$ and $\mathbb{N} \cap \Gamma_1 = \emptyset$. With no other assumptions on F we prove that if the Banach space $(X, \|\cdot\|_X)$ has the nonstrict Opial property, then so does the Banach space $(X, \|\cdot\|_{L,\alpha,F})$. In Section 4 we show that if, in addition, $(X, \|\cdot\|_X)$ is reflexive and has the Kadec–Klee property, and the subfamily of functionals $\{f^*_{\gamma\in\Gamma_1}\}$ in X^* separates points in $(X, \|\cdot\|_X)$, then the Banach space $(X, \|\cdot\|_{L,\alpha,F})$ is LUR. Next, in Sections 5 and 6 we also assume that the space $(X, \|\cdot\|_X)$ is reflexive

Received March 17, 2020; Accepted December 10, 2020.
Communicated by Xiang Fang.
2020 *Mathematics Subject Classification.* 46G20, 52A05.
Key words and phrases. Day norm, diametrically complete set, Kadec–Klee property, LUR, nonstrict Opial property, reflexive Banach space.
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and find a concrete family $\widetilde{F} = \{\widetilde{f}^*_\gamma\}_{\gamma\in\Gamma}$ functionals in X^* to achieve our main results. First, applying a Schauder basis in a suitably chosen quotient space X/Y, we single out a definite subfamily $\{\widetilde{f}^*_\gamma\}_{\gamma\in\mathbb{N}}$ of the family $\widetilde{F}$ and show that the Banach space $(X, \|\cdot\|_{L,\alpha,\widetilde{F}})$ contains a diametral set. However, the choice of an appropriate subfamily $\{\widetilde{f}^*_\gamma\}_{\gamma\in\Gamma_1}$ is a real challenge in the case of nonseparable and reflexive Banach spaces. Namely, to choose a definite subfamily $\{\widetilde{f}^*_\gamma\}_{\gamma\in\Gamma_1}$ of $\widetilde{F} = \{\widetilde{f}^*_\gamma\}_{\gamma\in\Gamma}$, we employ the powerful Lindenstrauss mapping theorem. Hence, if $(X, \|\cdot\|_X)$ is reflexive and has the nonstrict Opial property, then we are able to establish the existence of a diametrically complete set with empty interior in $(X, \|\cdot\|_{L,\alpha,\widetilde{F}})$ by applying the Maluta–Papini theorem [18]. Finally, using all these results we obtain the required renorming.

2. Basic notions and facts

All the material recalled in this section without a specific citation can be found in [8, 10, 11, 13, 15, 16]. Throughout this paper all Banach spaces are real.

In our paper we always assume that Γ is an infinite set. By $c_0(\Gamma)$ we denote the Banach space (with the max norm $\|\cdot\|_{c_0(\Gamma)}$) of real-valued functions $u := \{u^\gamma\}_{\gamma\in\Gamma}$ on Γ such that $\{\gamma \in \Gamma : |u^\gamma| > \epsilon\}$ is finite for each $\epsilon > 0$, where u^γ is the value of u at $\gamma \in \Gamma$. We denote the support of $u \in c_0(\Gamma)$ by $N(u)$.

We emphasize that the Banach space $(c_0(\Gamma), \|\cdot\|_{c_0(\Gamma)})$ plays a crucial role in our construction of an equivalent norm in a Banach space $(X, \|\cdot\|_X)$.

We use the following notation connected with $c_0(\Gamma)$. Let $u = \{u^\gamma\}_{\gamma\in\Gamma} \in c_0(\Gamma)$. Then the sequence $\{\tau(j, u)\}_j$ with indices j, which are natural numbers, is defined as follows:

(1) if the support $N(u)$ of u is infinite, then $N(u)$ is enumerated as $\{\tau(j, u)\}_j$ in such a way that $|u^{\tau(j,u)}| \geq |u^{\tau(j+1,u)}|$ for $j \in \mathbb{N}$,

(2) if $N(u) = \{\widetilde{\gamma}\}$ is a singleton, then we set $\tau(1, u) = \widetilde{\gamma}$ and extend $\tau(\cdot, u)$ to all of $\mathbb{N}$ so that $\tau(\cdot, u)\colon \mathbb{N} \to \Gamma$ is an injection,

(3) if the support $N(u)$ of u is finite and consists of $k(u) \geq 2$ elements, then $N(u)$ is enumerated as $\{\tau(j, u) : j \in \{1, \ldots, k(u)\}\}$ in such a way that $|u^{\tau(j,u)}| \geq |u^{\tau(j+1,u)}|$ for $1 \leq j \leq k(u) - 1$ and we extend $\tau(\cdot, u)$ to all of $\mathbb{N}$ so that $\tau(\cdot, u)\colon \mathbb{N} \to \Gamma$ is an injection,

(4) if $u = 0$, then $\tau(\cdot, u)\colon \mathbb{N} \to \Gamma$ is an arbitrarily chosen injection.

We note the following elementary inequalities connected with the function $\tau(\cdot,\cdot)$ (see [7, 23]; see also [6, 8]).

Lemma 2.1. *Assume that*

(1) $s = \{s^i\}_i$ *is a positive and decreasing sequence,*

(2) $t = \{t^i\}_i \in c_0$,

(3) $t^i \geq 0$ *for each* $i \in \mathbb{N}$,

(4) *the function* $g \colon \mathbb{N} \to \mathbb{N}$ *is injective.*

Then

$$\sum_{j=1}^{\infty} s^j \cdot t^{g(j)} \leq \sum_{j=1}^{\infty} s^j \cdot t^{\tau(j,t)}.$$

Directly from this lemma we get the following corollary.

Corollary 2.2. *Assume that*

(1) Γ *is an infinite set,*

(2) $s = \{s^j\}_j$ *is a positive and decreasing sequence,*

(3) $t = \{t^\gamma\}_\gamma \in c_0(\Gamma)$,

(4) *the function* $\widetilde{g} \colon \mathbb{N} \to \Gamma$ *is injective.*

Then

$$\sum_{j=1}^{\infty} s^j \cdot |t^{\widetilde{g}(j)}| \leq \sum_{j=1}^{\infty} s^j \cdot |t^{\tau(j,t)}|.$$

At this point we recall that the Banach space $\ell^2(\Gamma)$ consists of all $u \in c_0(\Gamma)$ such that $\sum_{j=1}^{\infty} |u^{\tau(j,u)}|^2 < \infty$. We use the norm $\|u\|_{\ell^2(\Gamma)} := \big(\sum_{j=1}^{\infty} |u^{\tau(j,u)}|^2\big)^{1/2}$ for $u \in \ell^2(\Gamma)$.

We also recall the following definitions of a few properties which play an important role in our paper.

Definition 2.3. We say that a Banach space $(X, \|\cdot\|_X)$ is *locally uniformly rotund* (LUR) if for each $x \in X$, every sequence $\{x_n\}_n$ with $\lim_n \|x_n\|_X = \|x\|_X$ and $\lim_n \|x + x_n\|_X = 2\|x\|_X$ tends strongly to x. In this case we also say that this norm is LUR.

Definition 2.4. Let $(X, \|\cdot\|_X)$ be a Banach space. We say that $(X, \|\cdot\|_X)$ has the *Kadec–Klee property* with respect to the weak topology (the Kadec–Klee property, for short) if each sequence $\{x_n\}$ with $\lim_n \|x_n\|_X = 1$, which converges weakly to a point x with $\|x\|_X = 1$, tends strongly to x. In this case we sometimes also say that the norm $\|\cdot\|_X$ has the Kadec–Klee property.

The following theorem shows a connection between local uniform rotundity and the Kadec–Klee property.

Theorem 2.5. *Let $(X, \|\cdot\|_X)$ be a Banach space. If $(X, \|\cdot\|_X)$ is locally uniformly rotund, then $(X, \|\cdot\|_X)$ has the Kadec–Klee property with respect to the weak topology.*

We also need the Opial property of a Banach space.

Definition 2.6. A Banach space $(X, \|\cdot\|_X)$ is said to have the *Opial property* if for each weakly null sequence $\{x_n\}$ and each $x \neq 0$ in X, we have

$$\limsup_{n\to\infty} \|x_n\|_X < \limsup_{n\to\infty} \|x_n - x\|_X.$$

A Banach space $(X, \|\cdot\|_X)$ is said to have the *nonstrict Opial property* if for each weakly null sequence $\{x_n\}$ and each point x in X, we have

$$\limsup_{n\to\infty} \|x_n\|_X \leq \limsup_{n\to\infty} \|x_n - x\|_X.$$

Now we recall a few facts regarding Schauder bases in a Banach space $(X, \|\cdot\|_X)$. We always assume that for each Schauder basis $\{e_i\}_i$ we consider, there exist constants $0 < \widetilde{m} \leq \widetilde{M} < \infty$ such that $\widetilde{m} \leq \|e_i\|_X \leq \widetilde{M}$ for each $i \in \mathbb{N}$. Then for the biorthogonal functionals $\{e_i^*\}_i$ associated with the Schauder basis $\{e_i\}_i$, there also exist constants $0 < \widetilde{m}_1 \leq \widetilde{M}_1 < \infty$ such that $\widetilde{m}_1 \leq \|e_i^*\|_{X^*} \leq \widetilde{M}_1$ for each $i \in \mathbb{N}$ (in X^* we use the standard norm $\|\cdot\|_{X^*}$). In addition, we have $\lim_i e_i^*(x) = 0$ for each $x \in X$. A basis $\{e_i\}_i$ is called *normalized* if $\|e_i\|_X = 1$ for all i.

We also need the definition of a diametral set.

Definition 2.7. Let $(X, \|\cdot\|_X)$ be an infinite-dimensional Banach space and let C be a nonempty, bounded and convex subset of X. We say that the set C is *diametral* if

$$r_{\|\cdot\|_X}(C, C) := \inf\big\{\sup\{\|y - y'\|_X : y' \in C\} : y \in C\big\} = \operatorname{diam}_{\|\cdot\|_X}(C).$$

A Banach space $(X, \|\cdot\|_X)$ is said to have *normal structure* if it does not contain any diametral set, that is, if $r_{\|\cdot\|_X}(C, C) < \operatorname{diam}_{\|\cdot\|_X}(C)$ for each nonempty, bounded and convex set $C \subset X$ which is not a singleton.

The next notion is closely connected with the concept of the diameter of a set and is due to Meissner.

Definition 2.8. [20] Let $(X, \|\cdot\|_X)$ be an infinite-dimensional Banach space and let C be a nonempty, non-singleton and bounded subset of X. We say that C is a *diametrically complete* set in X if

$$\operatorname{diam}_{\|\cdot\|_X}(C \cup \{x\}) > \operatorname{diam}_{\|\cdot\|_X}(C)$$

for each $x \in X \setminus C$.

It is obvious that a diametrically complete set must be closed and convex.

Now, we mention two results which exhibit connections between the diametral property of a set and the emptiness of the interior of a diametrically complete set.

First, in [21, Theorem 3.2] Moreno, Papini and Phelps proved the following theorem.

Theorem 2.9. *Let $(X, \|\cdot\|_X)$ be an infinite-dimensional Banach space and let $C \subset X$ be diametrically complete. If the interior of C is empty, then C is diametral.*

Next, Maluta and Papini established the following result [18] which is crucial in our considerations in Section 5.

Theorem 2.10. *Each infinite-dimensional and reflexive Banach space $(X, \|\cdot\|_X)$, which has the nonstrict Opial property and lacks normal structure, contains diametrically complete sets with empty interiors.*

In our paper we use the above theorem to get diametrically complete sets with empty interiors in X with a suitably constructed equivalent norm $\|\cdot\|_{L,\alpha,F}$. To do this, as we have already mentioned in Section 1, we will need a family $F = \{f^*_\gamma\}_{\gamma\in\Gamma}$ of linear functionals in X^*. First, in Section 3 we give a general construction of the norm $\|\cdot\|_{L,\alpha,F}$ in a Banach space $(X, \|\cdot\|_X)$ and show that $(X, \|\cdot\|_{L,\alpha,F})$ has the nonstrict Opial property if $(X, \|\cdot\|_X)$ has this property. In the next section we prove that if $(X, \|\cdot\|_X)$ is reflexive, a subfamily of functionals $\{f^*_\gamma\}_{\gamma\in\Gamma_1}$ in X^* separates points in $(X, \|\cdot\|_X)$, the norm $\|\cdot\|$ has the Kadec–Klee property, then the norm $\|\cdot\|_{L,\alpha,F}$ is LUR. In Sections 5 and 6 we choose a concrete family $\widetilde{F} = \{\widetilde{f}^*_\gamma\}_{\gamma\in\Gamma}$ of this kind, which is suitable for our aims. So in Section 5 we choose a specific subfamily $\{\widetilde{f}^*_\gamma\}_{\gamma\in\mathbb{N}}$ of the family $\widetilde{F} = \{\widetilde{f}^*_\gamma\}_{\gamma\in\Gamma}$ in such a way that X with the new norm $\|\cdot\|_{L,\alpha,\widetilde{F}}$ does not have normal structure. Our choice of $\{\widetilde{f}^*_\gamma\}_{\gamma\in\mathbb{N}}$ is based on the following important and well-known theorem.

Theorem 2.11. [15, Theorem 1.b.7] *Every reflexive infinite-dimensional Banach space has an infinite-dimensional quotient space with a Schauder basis.*

It is obvious that the infinite-dimensional quotient space with a Schauder basis mentioned in Theorem 2.11 is separable. Let us observe here that the fact that even every WCG space has a separable quotient appears in [1]. However, the easiest proof of this can be found in [26, Theorem 1].

Finally, in Section 6 we choose a definite subfamily $\{\widetilde{f}^*_\gamma\}_{\gamma\in\Gamma_1}$ of $\widetilde{F} = \{\widetilde{f}^*_\gamma\}_{\gamma\in\Gamma}$ by using the Lindenstrauss mapping theorem [14] (see also [1]). Now we recall this theorem.

Theorem 2.12. *Let $(X, \|\cdot\|_X)$ be an infinite-dimensional and reflexive Banach space. Then there is a one-to-one bounded linear operator $T\colon X \to c_0(\Gamma)$ for a suitable set Γ.*

3. Construction of the equivalent norm $\|\cdot\|_{L,\alpha,F}$ and the nonstrict Opial property

We begin with the definition of the Day norm $|||\cdot|||$ on $c_0(\Gamma)$ [7]. If $u = \{u^\gamma\}_{\gamma\in\Gamma} \in c_0(\Gamma)$, then we define $D(u) := \{D^\gamma(u)\}_{\gamma\in\Gamma} \in \ell^2(\Gamma)$ by

$$D^\gamma(u) := \begin{cases} \frac{u^{\tau(j,u)}}{2^j} & \text{if } \gamma = \tau(j,u) \text{ for some } j \in \mathbb{N}, \\ 0 & \text{otherwise} \end{cases}$$

and set $|||u||| := \|D(u)\|_{\ell^2(\Gamma)}$, where $\|\cdot\|_{\ell^2(\Gamma)}$ is the standard norm in $\ell^2(\Gamma)$. It is not difficult to observe that

$$\frac{1}{2}\|u\|_{c_0(\Gamma)} \le |||u||| \le \frac{1}{\sqrt{3}}\|u\|_{c_0(\Gamma)}$$

for each $u \in c_0(\Gamma)$, where $\|\cdot\|_{c_0(\Gamma)}$ is the standard max-norm on $c_0(\Gamma)$.

At this point we recall that the local uniform rotundity of the Day norm was proved by Rainwater [23] (see also [6, 8]).

Theorem 3.1. *The Banach space* $(c_0(\Gamma), |||\cdot|||)$ *is LUR.*

Let $(X, \|\cdot\|_X)$ be an infinite-dimensional Banach space. If there exists a family of linear functionals in X^* satisfying the conditions listed below, then we can introduce a certain equivalent norm in this Banach space which is useful for our aims. The idea of our construction of this norm is closely related to the one which can be found in [24].

First, we choose an infinite family of nonzero functionals $F = \{f^*_\gamma\}_{\gamma\in\Gamma}$ in X^* which is equibounded in X^*, that is, $\|f^*_\gamma\|_{X^*} \le K$ for all $\gamma \in \Gamma$, where $1 \le K \in \mathbb{R}$ (on X^* we use the standard norm $\|\cdot\|_{X^*}$). Assume, in addition, that our choice is such that for each $x \in X$ and for each $\epsilon > 0$, the set $\{\gamma \in \Gamma : |f^*_\gamma(x)| > \epsilon\}$ is finite. Next, we divide the set Γ in the following way: $\Gamma := \Gamma_1 \cup \mathbb{N}$ and $\Gamma_1 \cap \mathbb{N} = \emptyset$ (Γ_1 can be the empty set). We need the set $\mathbb{N}$ to obtain equivalence of norms. At this point we choose and fix $\alpha \in (0,1)$. Then with each point $x \in X$, we associate the element $u(x) = \{u^\gamma(x)\}_{\gamma\in\Gamma}$ in the following way:

$$u^\gamma(x) := f^*_\gamma(x)$$

for $\gamma \in \Gamma_1$,

$$u^1(x) := \alpha\|x\|_X$$

and

$$\{u^2(x), u^3(x), \dots\} := \{f^*_1(x), f^*_2(x), f^*_2(x), \dots, f^*_j(x), \dots, f^*_j(x), \dots\},$$

where we repeat the coordinate $f^*_j(x)$ exactly j times.

Now, define $D(u(x))) = \{D^\gamma(u(x))\}_{\gamma\in\Gamma} \in \ell^2(\Gamma)$ by

$$D^\gamma(u(x)) := \begin{cases} \frac{u^{\tau(j,u(x))}(x)}{2^j} & \text{if } \gamma = \tau(j,u(x)) \text{ for some } j \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we set $|||u(x)||| := \|D(u(x))\|_{\ell^2(\Gamma)}$ and $\|x\|_{L,\alpha,F} := |||u|||$. It is not difficult to observe that

$$\frac{\alpha\|x\|_X}{2} \leq \frac{1}{2}\|u(x)\|_{c_0(\Gamma)} \leq |||u(x)||| = \|x\|_{L,\alpha,F} \leq \frac{1}{\sqrt{3}}\|u(x)\|_{c_0(\Gamma)} \leq \frac{K}{\sqrt{3}}\|x\|_X$$

for each point $x \in X$. Thus the norm $\|\cdot\|_{L,\alpha,F}$ is equivalent to the original norm $\|\cdot\|_X$.

Now, we are ready to prove the following theorem regarding the nonstrict Opial property. The proof is a modification of the proof of Theorem 5.4 in [5] (see also [3, 4]) and therefore we omit it. We also note that the main idea of the proof is due to Maluta [17].

Theorem 3.2. *Under the above assumptions and notations, if a Banach $(X, \|\cdot\|_X)$ has the nonstrict Opial property, then so does the Banach space $(X, \|\cdot\|_{L,\alpha,F})$.*

4. The Kadec–Klee property of $(X, \|\cdot\|)$ and the LUR of $(X, \|\cdot\|_{L,\alpha,F})$

Applying the Rainwater result (Theorem 3.1) and the Smith method [24], we immediately arrive at the following theorem.

Theorem 4.1. *Under the notations as in Section 3, if the Banach space $(X, \|\cdot\|_X)$ is reflexive, the subfamily of functionals $\{f^*_{\gamma\in\Gamma_1}\}$ in X^* separates points in $(X, \|\cdot\|_X)$ and the norm $\|\cdot\|_X$ has the Kadec–Klee property, then the Banach space $(X, \|\cdot\|_{L,\alpha,F})$ is LUR.*

Proof. Let x and $\{x_n\}_n$ be given such that $x \in X$, $\|x\|_{L,\alpha,F} = 1$, $x_n \in X$ for $n = 1, 2, \ldots$, $\lim_n \|x_n\|_{L,\alpha,F} = 1$ and $\lim_n \|x + x_n\|_{L,\alpha,F} = 2$. Then we also have $|||u(x)||| = 1$, $\lim_n |||u(x_n)||| = 1$ and $\lim_n |||u(x) + u(x_n)||| = 2$. Applying the local uniform rotundity (LUR) of $(c_0(\Gamma), |||\cdot|||)$ (see Theorem 3.1), we immediately obtain the strong convergence of the sequence $\{u(x_n)\}$ to $u(x)$ in the norm $|||\cdot|||$. But we have

$$\frac{1}{2}\|u(x) - u(x_n)\|_{c_0(\Gamma)} \leq |||u(x) - u(x_n)||| \leq \frac{1}{\sqrt{3}}\|u(x) - u(x_n)\|_{c_0(\Gamma)}$$

and

$$\|u(x) - u(x_n)\|_{c_0(\Gamma)} \geq \alpha \max\left\{\left|\|x\|_X - \|x_n\|_X\right|, \sup\{|u^\gamma(x) - u^\gamma(x_n)| : \gamma \in \Gamma_1\}\right\}.$$

This implies that $\lim \|x_n\|_X = \|x\|_X$ and $\lim_n f^*_\gamma(x_n) = f^*_\gamma(x)$ for each $\gamma \in \Gamma_1$. Since the subfamily of functionals $\{f^*_\gamma\}_{\gamma\in\Gamma_1}$ in X^* separates points in $(X, \|\cdot\|_X)$ and the Banach space $(X, \|\cdot\|_X)$ is reflexive, the sequence $\{x_n\}_n$ tends weakly to x. Finally, using the Kadec–Klee property of the norm $\|\cdot\|_X$, we see that $\lim_n x_n = x$ in $(X, \|\cdot\|_X)$ and therefore $\lim_n x_n = x$ in $(X, \|\cdot\|_{L,\alpha,F})$, as required. □

5. The norm $\|\cdot\|_{L,\alpha,\widetilde{F}}$ and diametral sets

Let $(X, \|\cdot\|_X)$ be an infinite-dimensional and reflexive Banach space. By Theorem 2.11, there exists a closed subspace Y of the Banach space $(X, \|\cdot\|_X)$ such that the quotient space X/Y with the canonical norm $\|\cdot\|_{X/Y}$ has a Schauder basis. For this quotient space X/Y, there exists the standard embedding $\iota\colon X \to X/Y$. Let $\{\widehat{z}_m\}_m$ be a normalized Schauder basis in $(X/Y, \|\cdot\|_{X/Y})$ and let $\{\widehat{z}^*_m\}_m$ be the sequence of biorthogonal functionals associated with this basis. Then there is a constant $K_1 > 1$ such that $\|\widehat{z}^*_m\|_{(X/Y)^*} \le K_1$ for each $m \in \mathbb{N}$. Now let Γ_1 be a set such that $\Gamma_1 \cap \mathbb{N} = \emptyset$ (Γ_1 may well be the empty set) and let $\widetilde{F} = \{\widetilde{f}^*_\gamma\}_{\gamma\in\Gamma_1}$ be a family of nonzero functionals in X^*. Assume that the family $\{\widetilde{f}^*_\gamma\}_{\gamma\in\Gamma_1}$ is bounded in X^* by $1/2$ (on X^* we use the standard norm $\|\cdot\|_{X^*}$). Assume also that for each $x \in X$ and for each $\epsilon > 0$, the set $\{\gamma \in \Gamma_1 : |\widetilde{f}^*_\gamma(x)| > \epsilon\}$ is finite. Now let $\Gamma = \Gamma_1 \cup \mathbb{N}$. Then for $\gamma = m \in \mathbb{N}$, we put

$$\widetilde{f}^*_\gamma = \widehat{z}^*_m \circ \iota.$$

Hence we get a family $\widetilde{F} = \{\widetilde{f}^*_\gamma\}_{\gamma\in\Gamma}$ of nonzero functionals in X^*. Fix $\alpha \in (0, 1)$. Then, as was shown in Section 3, for each point $x \in X$, we can define $u(x) = \{u^\gamma(x)\}_{\gamma\in\Gamma}$ and $D(u(x))) = \{D^\gamma(u(x))\} \in \ell^2(\Gamma)$. Next, using Day's norm $|||\cdot|||$ on c_0 for the element $u(x) \in c_0$, we get

$$\|x\|_{L,\alpha,\widetilde{F}} = |||u(x)||| = \|D(u(x))\|_{\ell^2(\Gamma)} \quad \text{and} \quad \frac{\alpha}{2}\|x\|_X \le \|x\|_{L,\alpha,\widetilde{F}} \le \frac{K_1}{\sqrt{3}}\|x\|_X.$$

Theorem 5.1. *Under the above assumptions and notations, if $0 < \alpha \le 1/2$, then the infinite-dimensional and reflexive Banach space $(X, \|\cdot\|_{L,\alpha,\widetilde{F}})$ lacks normal structure.*

Proof. Since $(X, \|\cdot\|_X)$ is reflexive and the basis $\{\widehat{z}_m\}_m$ is normalized, there exists a sequence $\{z_m\}_m$ in X such that $z_m \in \widehat{z}_m$ (that is, $\iota(z_m) = \widehat{z}_m$) and $\|z_m\|_X = 1$ for each $m \in \mathbb{N}$. Then for $m_2 > m_1$, we have

$$|u^\gamma(z_{m_2} - z_{m_1})| = |f^*_\gamma(z_{m_2} - z_{m_1})| \le \frac{1}{2}\|z_{m_2} - z_{m_1}\|_X \le 1$$

for $\gamma \in \Gamma_1$,

$$u^1(z_{m_2} - z_{m_1}) = \alpha\|z_{m_2} - z_{m_1}\|_X \le 2\alpha \le 1$$

and

$$\begin{aligned}
&\{u^2(z_{m_2} - z_{m_1}), u^3(z_{m_2} - z_{m_1}), u^3(z_{m_2} - z_{m_1}), \dots\} \\
&= \Big\{\widehat{z}^*_1(\iota(z_{m_2} - z_{m_1})), \widehat{z}^*_2(\iota(z_{m_2} - z_{m_1})), \widehat{z}^*_2(\iota(z_{m_2} - z_{m_1})), \dots, \\
&\quad \widehat{z}^*_{m_1-1}(\iota(z_{m_2} - z_{m_1})), \widehat{z}^*_{m_1}(\iota(z_{m_2} - z_{m_1})), \dots, \widehat{z}^*_{m_1}(\iota(z_{m_2} - z_{m_1})), \\
&\quad \widehat{z}^*_{m_1+1}(\iota(z_{m_2} - z_{m_1})), \dots, \widehat{z}^*_{m_2-1}(\iota(z_{m_2} - z_{m_1})), \\
&\quad \widehat{z}^*_{m_2}(\iota(z_{m_2} - z_{m_1})), \dots, \widehat{z}^*_{m_2}(\iota(z_{m_2} - z_{m_1})), \widehat{z}^*_{m_2+1}(\iota(z_{m_2} - z_{m_1})), \dots\Big\} \\
&= \{0, 0, 0, \dots, 0, -1, \dots, -1, 0, \dots, 0, 1, \dots, 1, 0, \dots\}.
\end{aligned}$$

Therefore

$$\left(\sum_{k=1}^{m_1+m_2} 4^{-k}\right)^{1/2} \le \|z_{m_2} - z_{m_1}\|_{L,\alpha,\widetilde{F}} \le \left(\sum_{k=1}^{\infty} 4^{-k}\right)^{1/2} = \frac{1}{\sqrt{3}}.$$

This means that $\operatorname{diam}_{\|\cdot\|_{L,\alpha,\widetilde{F}}}\{z_m\}_m = 1/\sqrt{3}$.

Now, we compute $\lim_m \operatorname{dist}_{\|\cdot\|_{L,\alpha,\widetilde{F}}}(z_{m+1}, \operatorname{conv}\{z_1,\dots,z_m\})$. To this end, suppose $\beta_1 + \cdots + \beta_m = 1$, where $0 \le \beta_k \le 1$ for $1 \le k \le m$. Then we have

$$\left|u^\gamma\left(z_{m+1} - \sum_{k=1}^m \beta_k z_k\right)\right| = \left|f_\gamma^*\left(z_{m+1} - \sum_{k=1}^m \beta_k z_k\right)\right| \le \frac{1}{2}\left\|z_{m+1} - \sum_{k=1}^m \beta_k z_k\right\|_X$$
$$\le \frac{1}{2}\sum_{k=1}^m \beta_k \|z_{m+1} - z_k\|_X \le 1$$

for $\gamma \in \Gamma_1$. Next, we observe that

$$\left|u^1\left(z_{m+1} - \sum_{k=1}^m \beta_k z_k\right)\right| = \alpha\left\|z_{m+1} - \sum_{k=1}^m \beta_k z_k\right\|_X \le 2\alpha \le 1$$

and

$$\left\{u^2\left(z_{m+1} - \sum_{k=1}^m \beta_k z_k\right), u^3\left(z_{m+1} - \sum_{k=1}^m \beta_k z_k\right), u^3\left(z_{m+1} - \sum_{k=1}^m \beta_k z_k\right), \dots\right\}$$
$$= \Bigg\{\widehat{z}_1^*\left(\iota\left(z_{m+1} - \sum_{k=1}^m \beta_k z_k\right)\right), \widehat{z}_2^*\left(\iota\left(z_{m+1} - \sum_{k=1}^m \beta_k z_k\right)\right),$$
$$\widehat{z}_2^*\left(\iota\left(z_{m+1} - \sum_{k=1}^m \beta_k z_k\right)\right), \dots, \widehat{z}_m^*\left(\iota\left(z_{m+1} - \sum_{k=1}^m \beta_k z_k\right)\right), \dots,$$
$$\widehat{z}_m^*\left(\iota\left(z_{m+1} - \sum_{k=1}^m \beta_k z_k\right)\right), \widehat{z}_{m+1}^*\left(\iota\left(z_{m+1} - \sum_{k=1}^m \beta_k z_k\right)\right), \dots,$$
$$\widehat{z}_{m+1}^*\left(\iota\left(z_{m+1} - \sum_{k=1}^m \beta_k z_k\right)\right), \widehat{z}_{m+2}^*\left(\iota\left(z_{m+1} - \sum_{k=1}^m \beta_k z_k\right)\right), \dots\Bigg\}$$
$$= \{-\beta_1, -\beta_2, -\beta_2, \dots, -\beta_m, \dots, -\beta_m, 1, \dots, 1, 0, \dots,\}.$$

Consequently, we see that

$$\left(\sum_{k=1}^{m+1} 4^{-k}\right)^{1/2} \le \left\|z_{m+1} - \sum_{k=1}^m \beta_k z_k\right\|_{L,\alpha,\widetilde{F}} \le \left(\sum_{k=1}^{\infty} 4^{-k}\right)^{1/2} = \frac{1}{\sqrt{3}}.$$

This means that

$$\lim_m \operatorname{dist}_{\|\cdot\|_{L,\alpha,\widetilde{F}}}\big(z_{m+1}, \operatorname{conv}\{z_1,\dots,z_m\}\big) = \frac{1}{\sqrt{3}} = \operatorname{diam}_{\|\cdot\|_{L,\alpha,\widetilde{F}}}\{z_m\}_m. \qquad \square$$

Remark 5.2. Our proof is based on a modification of the method employed by Smith and Turett [25].

Remark 5.3. As was mentioned in Section 3, the above theorem is valid even if the set Γ_1 is empty, but in the case of nonseparable Banach spaces a nonempty set Γ_1 appears in a natural way in our considerations in Section 6.

6. Existence of a diametrically complete set with empty interior in a reflexive Banach space

First, we have the following theorem on existence of a diametrically complete set with empty interior.

Theorem 6.1. *Let $(X, \|\cdot\|_X)$ be an infinite-dimensional and reflexive Banach space. If $(X, \|\cdot\|_X)$ has the nonstrict Opial property, then there exists an equivalent norm $\|\cdot\|_{L,\alpha,\widetilde{F}}$ such that $(X, \|\cdot\|_{L,\alpha,\widetilde{F}})$ contains a diametrically complete set the interior of which is empty.*

Proof. We use the notation from Section 5. By Theorem 2.12, there exist a set Γ_1 such that $\Gamma_1 \cap \mathbb{N} = \emptyset$ and a one-to-one bounded linear operator T from X into $c_0(\Gamma_1)$. Let $\{e_\gamma\}_{\gamma\in\Gamma_1}$ be the standard basis in $c_0(\Gamma_1)$ and let $\{e^*_\gamma\}_{\gamma\in\Gamma_1}$ be the family of biorthogonal functionals associated with this basis, that is, $e^*_\gamma(y) := y^\gamma$ for each $y = \{y^\gamma\}_{\gamma\in\Gamma_1} \in c_0(\Gamma_1)$ and

$$e^*_\gamma(e_{\gamma'}) := \begin{cases} 1 & \text{if } \gamma = \gamma', \\ 0 & \text{otherwise,} \end{cases}$$

where $\gamma, \gamma' \in \Gamma_1$. Without loss of generality we may assume that for each $\gamma \in \Gamma_1$, there exists an $x \in X$ such that $e^*_\gamma(Tx) \neq 0$. Next, we choose $0 < s < 1$ and set $\{\widetilde{f}^*_\gamma\}_{\gamma\in\Gamma_1} = \{se^*_\gamma \circ T\}_{\gamma\in\Gamma_1}$. Then $\{\widetilde{f}^*_\gamma\}_{\gamma\in\Gamma_1}$ is a subfamily of nonzero functionals in X^* which separates points of X. Assume, in addition, that $0 < s < 1$ is sufficiently small so that the family $\{\widetilde{f}^*_\gamma\}_{\gamma\in\Gamma_1}$ is bounded in X^* by $1/2$ (in X^* we use the standard norm $\|\cdot\|_{X^*}$). It is obvious that for each point $x \in X$, the set $\{\gamma \in \Gamma_1 : |\widetilde{f}^*_\gamma(x)| > \epsilon\}$ is finite for each $\epsilon > 0$. Now, let $\Gamma := \Gamma_1 \cup \mathbb{N}$. For $\gamma \in \mathbb{N}$ we construct $\widetilde{f}^*_\gamma$ as in Section 5. We also have $0 < \alpha \leq 1/2$. It is clear that the Banach space $(X, \|\cdot\|_X)$ and the family $\widetilde{F} = \{\widetilde{f}^*_\gamma\}_{\gamma\in\Gamma}$ of nonzero functionals in X^* satisfy the assumptions of Theorems 3.2 and 5.1, and therefore by Theorem 2.10, the Banach space $(X, \|\cdot\|_{L,\alpha,\widetilde{F}})$ contains a diametrically complete set the interior of which is empty, as asserted. □

We finish this paper by showing that every infinite-dimensional and reflexive Banach space with the nonstrict Opial and the Kadec–Klee properties can be equivalently renormed so that with the new norm it is LUR and contains diametrically complete sets with empty interior. Namely, we have the following theorem.

Theorem 6.2. *Each infinite-dimensional and reflexive Banach space $(X, \|\cdot\|_X)$ with the nonstrict Opial and the Kadec–Klee properties has an equivalent norm $\|\cdot\|_0$ such that $(X, \|\cdot\|_0)$ is* LUR *and contains a diametrically complete set the interior of which is empty.*

Proof. We use the notations of Theorem 6.1. Set $\|\cdot\|_0 := \|\cdot\|_{L,\alpha,\widetilde{F}}$ with $0 < \alpha \le 1/2$. Since our infinite-dimensional and reflexive Banach space $(X, \|\cdot\|_X)$ has the nonstrict Opial property, the space X with the equivalent norm $\|\cdot\|_0$ contains, by Theorem 6.1, a diametrically complete set the interior of which is empty. Next, invoking Theorem 4.1, we see that $(X, \|\cdot\|_0)$ is also LUR, as asserted. □

Remark 6.3. It is known that every separable Banach space $(X, \|\cdot\|_X)$ can be equivalently renormed in such a way that this space with the new norm has both the nonstrict Opial and the Kadec–Klee properties (see [2, 5]). An analogous result for *nonseparable* and reflexive Banach spaces is not known. So the following problem is still open.

Open problem. Let $(X, \|\cdot\|_X)$ be any infinite-dimensional and reflexive Banach space. Can it be renormed by a new norm $\|\cdot\|_0$ in such a way that $(X, \|\cdot\|_0)$ is LUR and contains a diametrically complete set with empty interior?

Acknowledgments

All the authors are grateful to two anonymous referees for their useful comments and helpful suggestions. The third author was partially supported by the Israel Science Foundation (Grant Nos. 389/12 and 820/17), the Fund for the Promotion of Research at the Technion and by the Technion General Research Fund.

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