

Symmetry of Positive Solutions to Quasilinear Fractional Systems

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Abstract. Using the method of moving planes, we establish the radial symmetry of positive solutions to the fractional system

$$\begin{cases} (-\Delta)_p^s u = f(u, v), \\ (-\Delta)_q^t v = g(u, v) \end{cases}$$

in the entire Euclidean space \mathbb{R}^n and in the unit ball, where $0 < s, t < 1$ and $p, q \geq 2$. In particular, our result can be applied to the nonlinearities $f(u, v) \equiv u^a v^b$ and $g(u, v) \equiv u^c v^d$, where $a, d \in \mathbb{R}$ and $b, c > 0$.

1. Introduction

This paper is concerned with the symmetry of positive solutions to the quasilinear fractional systems

$$(1.1) \quad \begin{cases} (-\Delta)_p^s u = f(u, v) & \text{in } B_1, \\ (-\Delta)_q^t v = g(u, v) & \text{in } B_1, \\ u = v = 0 & \text{in } \mathbb{R}^n \setminus B_1, \end{cases}$$

and

$$(1.2) \quad \begin{cases} (-\Delta)_p^s u = f(u, v) & \text{in } \mathbb{R}^n, \\ (-\Delta)_q^t v = g(u, v) & \text{in } \mathbb{R}^n, \end{cases}$$

where B_1 denotes the unit ball in \mathbb{R}^n , $0 < s, t < 1$ and $p, q \geq 2$.

The fractional p -Laplacian $(-\Delta)_p^s$, which appears in (1.1) and (1.2), is a nonlinear nonlocal operator. It is defined as

$$\begin{aligned} (-\Delta)_p^s u(x) &= C_{n,s,p} \text{PV} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2} [u(x) - u(y)]}{|x - y|^{n+sp}} dy \\ &= C_{n,s,p} \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2} [u(x) - u(y)]}{|x - y|^{n+sp}} dy, \end{aligned}$$

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where PV is an abbreviation of the Cauchy principal value, $C_{n,s,p}$ is a normalization constant and $B_\varepsilon(x)$ denotes the ball of center $x \in \mathbb{R}^n$ and radius $\varepsilon > 0$. Clearly, $(-\Delta)_p^s u(x)$ is well-defined for $u \in L_{s,p} \cap C_{\text{loc}}^{1,1}(\Omega)$ and $x \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is an open domain and

$$L_{s,p} = \left\{ u \in L_{\text{loc}}^{p-1}(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \frac{|1 + u(x)|^{p-1}}{1 + |x|^{n+sp}} dx < \infty \right\},$$

see [9, 24] for more details. The operator $(-\Delta)_q^t$ can be defined in the same way.

The fractional p -Laplacian arises in some important applications such as the non-local ‘‘Tug-of-War’’ game [3, 4]. It is also a generalization of the well-known fractional Laplacian, which has been widely used to model various physical phenomena, such as turbulence, water waves, anomalous diffusion, phase transitions, flame propagation and quasi-geostrophic flows, see [5, 7, 14]. The fractional Laplacian also appears in probability, optimization and finance, see [1, 2, 6].

In order to study the symmetry properties of solutions to equations involving the fractional Laplacian, one may exploit some notable methods such as the extension method by Caffarelli and Silvestre [6], the integral equation method by Chen, Li and Ou [11] and the direct method of moving planes by Chen, Li and Li [10] (see also [15, 16]). Several works in literature follow these methods to establish symmetry, monotonicity and nonexistence results for various types of systems involving the fractional Laplacian, see [13, 17, 18, 20, 21, 23, 25, 27, 28] and the references therein.

However, due to its degeneracy, the fractional p -Laplacian cannot be studied by the above methods. Recently, Chen and Li [9] introduced a variant of the direct moving plane method. Using this new tool, they are able to establish the symmetry of positive solutions to some fractional p -Laplacian equations in \mathbb{R}^n and B_1 .

More precisely, in [9, Theorem 4], Chen and Li showed that, if u is a positive solution of the equation

$$\begin{cases} (-\Delta)_p^s u = f(u) & \text{in } B_1, \\ u = 0 & \text{in } \mathbb{R}^n \setminus B_1, \end{cases}$$

where $f(u) \equiv u^q$ with $q \geq p - 1$, then u must be radially symmetric and monotone decreasing about the origin. Later, Wu and Niu [26] extend the result to any Lipschitz continuous function f (see [26, Theorem 1.2]). A symmetry result for system (1.1) was recently established in [8] under some technical assumptions on f and g . One of our purposes in writing this paper is to improve [8] by establishing the symmetry of positive solutions to (1.1) for a very large class of f and g . Our first result reads as follows.

Theorem 1.1. *Let $f, g \in C_{\text{loc}}^{0,1}([0, \infty) \times [0, \infty))$ be such that $f(z, \cdot)$ and $g(\cdot, z)$ are non-decreasing in $(0, \infty)$ for all $z > 0$. Assume that $(u, v) \in C^{1,1}(\overline{B}_1) \times C^{1,1}(\overline{B}_1)$ is a positive solution of (1.1). Then u and v must be radially symmetric and monotone decreasing about the origin.*

The second objective of the present paper is to establish the symmetry of positive solutions to system (1.2). In [9, Theorem 5], Chen and Li studied the equation

$$(1.3) \quad (-\Delta)_p^s u = f(u) \quad \text{in } \mathbb{R}^n.$$

They showed that, if $f'(z) \leq 0$ for z sufficiently small and u is a decaying positive solution, then u must be radially symmetric and monotone decreasing about some point. Notice that an analogous result for system (1.2) was currently obtained by Chen and Liu in [12]. They assumed $f, g \in C^1([0, \infty) \times [0, \infty))$ be such that

- (i) $f'_v(u, v) > 0, g'_u(u, v) > 0$ for all $u, v > 0$,
- (ii) $f'_u(0, 0) < 0, g'_v(0, 0) < 0, f'_u(0, 0)g'_v(0, 0) - f'_v(0, 0)g'_u(0, 0) > 0$.

Then they proved that u and v must be radially symmetric and monotone decreasing about the same point if (u, v) is a decaying positive solution to (1.2) (see [12, Theorem 1]). One may observe that, because of the restriction (ii), Chen and Liu's result cannot be applied to the fractional Schrödinger system

$$(1.4) \quad \begin{cases} (-\Delta)_p^s u = u^a v^b & \text{in } \mathbb{R}^n, \\ (-\Delta)_q^t v = u^c v^d & \text{in } \mathbb{R}^n. \end{cases}$$

One may ask whether we can obtain the symmetry of some kind of decaying positive solutions to system (1.4) with $a, b, c, d > 0$, or more generally, to system (1.2) without the conditions $f'_u \leq 0$ and $g'_v \leq 0$ for $(u, v) \approx (0, 0)$. We give a positive answer to this question in the following theorem.

Theorem 1.2. *Let $\alpha, \beta > 0$ and $f, g \in C^1((0, \infty) \times (0, \infty))$ be such that*

- (i) $f'_v(u, v) > 0, g'_u(u, v) > 0$ for all $u, v > 0$,
- (ii) $f'_u(u, v) \leq 0$ or $f'_u(u, v) = o(u^{a_1} v^{b_1})$ as $(u, v) \rightarrow (0^+, 0^+)$, where $a_1, b_1 \geq 0$ satisfy $\alpha a_1 + \beta b_1 \geq \alpha(p-2) + sp$,
- (iii) $f'_v(u, v) = o(u^{a_2} v^{b_2})$ as $(u, v) \rightarrow (0^+, 0^+)$, where $a_2, b_2 \geq 0$ satisfy $\alpha a_2 + \beta b_2 \geq \alpha(p-2) + sp$,
- (iv) $g'_u(u, v) = o(u^{c_1} v^{d_1})$ as $(u, v) \rightarrow (0^+, 0^+)$, where $c_1, d_1 \geq 0$ satisfy $\alpha c_1 + \beta d_1 \geq \beta(q-2) + tq$,
- (v) $g'_v(u, v) \leq 0$ or $g'_v(u, v) = o(u^{c_2} v^{d_2})$ as $(u, v) \rightarrow (0^+, 0^+)$, where $c_2, d_2 \geq 0$ satisfy $\alpha c_2 + \beta d_2 \geq \beta(q-2) + tq$.

Assume that $(u, v) \in (L_{s,p} \cap C_{\text{loc}}^{1,1}(\mathbb{R}^n)) \times (L_{t,q} \cap C_{\text{loc}}^{1,1}(\mathbb{R}^n))$ is a positive solution of (1.2) such that

$$u(x) \sim \frac{1}{|x|^\alpha} \quad \text{and} \quad v(x) \sim \frac{1}{|x|^\beta} \quad \text{as } |x| \rightarrow \infty.$$

Then u and v must be radially symmetric and monotone decreasing about the same point.

Remark 1.3. Since the Kelvin transform is no longer valid for the fractional p -Laplacian, we need to impose the additional assumptions on the behavior of u and v at infinity. This is the same for other works involving the symmetry results for fractional p -Laplacian in the literature to the best of our knowledge. Theorem 1.2 can be seen as an extension of Theorems 1 and 4 in [21] to the quasilinear case $p, q \geq 2$.

Theorem 1.2 can be applied to system (1.4) with $a, d \in (-\infty, 0] \cup [1, +\infty)$ and $b, c \in [1, +\infty)$ but cannot deal with the case a, b, c or $d \in (0, 1)$. However, by conducting a finer analysis, we are able to cover also the case $a, b, c, d \in (0, 1)$ for system (1.4) as follows.

Theorem 1.4. *Let $\alpha, \beta > 0$ and $a, b, c, d \in \mathbb{R}$ be such that*

- (i) $a \leq 0$, or $a > 0$ and $\alpha(a - 1) + \beta b > \alpha(p - 2) + sp$,
- (ii) $b > 0$ and $\alpha a + \beta(b - 1) > \alpha(p - 2) + sp$,
- (iii) $c > 0$ and $\alpha(c - 1) + \beta d > \beta(q - 2) + tq$,
- (iv) $d \leq 0$, or $d > 0$ and $\alpha c + \beta(d - 1) > \beta(q - 2) + tq$.

Assume that $(u, v) \in (L_{s,p} \cap C_{\text{loc}}^{1,1}(\mathbb{R}^n)) \times (L_{t,q} \cap C_{\text{loc}}^{1,1}(\mathbb{R}^n))$ is a positive solution of (1.4) such that

$$u(x) \sim \frac{1}{|x|^\alpha} \quad \text{and} \quad v(x) \sim \frac{1}{|x|^\beta} \quad \text{as } |x| \rightarrow \infty.$$

Then u and v must be radially symmetric and monotone decreasing about the same point.

In particular, when $a = r - 1$, $b = 1$, $c = r$, $d = 0$ and $q = 2$, Theorem 1.4 reduces to a known result by Ma and Zhang [22], which involves the symmetry of positive solutions to the static Choquard type equation

$$(1.5) \quad (-\Delta)_p^s u = \left(\frac{1}{|x|^{n-2t}} * u^r \right) u^{r-1} \quad \text{in } \mathbb{R}^n.$$

More precisely, Ma and Zhang transformed (1.5) to the fractional system

$$(1.6) \quad \begin{cases} (-\Delta)_p^s u = u^{r-1} v & \text{in } \mathbb{R}^n, \\ (-\Delta)^t v = u^r & \text{in } \mathbb{R}^n. \end{cases}$$

Then they proved the following result, which is exactly a corollary of Theorem 1.4.

Corollary 1.5. [22, Theorem 1.1] *Let $r > p - 1$ be such that*

$$\alpha > \max \left\{ \frac{sp}{r-p+1}, \frac{2t}{r-1} \right\} \quad \text{and} \quad \alpha(r-p) + \beta > sp.$$

Assume that $(u, v) \in (L_{s,p} \cap C_{\text{loc}}^{1,1}(\mathbb{R}^n)) \times (L_{t,2} \cap C_{\text{loc}}^{1,1}(\mathbb{R}^n))$ is a positive solution of (1.6) such that

$$u(x) \sim \frac{1}{|x|^\alpha} \quad \text{and} \quad v(x) \sim \frac{1}{|x|^\beta} \quad \text{as } |x| \rightarrow \infty.$$

Then u and v must be radially symmetric and monotone decreasing about the same point.

Notice that a better symmetry result for (1.5) was obtained recently by the author in [19] using more direct analysis.

Finally, we may state a corollary of Theorem 1.2 for (1.3) as follows.

Corollary 1.6. *Let $f \in C^1((0, \infty))$ be such that*

$$f'(z) = O(z^{a-1}) \quad \text{as } z \rightarrow 0^+,$$

where $a > p - 1$. Assume that $u \in L_{s,p} \cap C_{\text{loc}}^{1,1}(\mathbb{R}^n)$ is a positive solution of (1.3) such that

$$u(x) \sim \frac{1}{|x|^\alpha} \quad \text{as } |x| \rightarrow \infty$$

with $\alpha > \frac{sp}{a-p+1}$. Then u must be radially symmetric and monotone decreasing about some point.

Remark 1.7. To see Corollary 1.6 follows from Theorem 1.2, one just notice that, if u is a positive solution of (1.3), then (u, u) is a positive solution of the system

$$\begin{cases} (-\Delta)_p^s u = f(u) - u^a + v^a & \text{in } \mathbb{R}^n, \\ (-\Delta)_p^s v = f(v) - v^a + u^a & \text{in } \mathbb{R}^n. \end{cases}$$

Remark 1.8. Corollary 1.6 has been proved in [9, Theorem 6] for the special case $f(z) \equiv z^a$ with $a > p - 1$. Here we only assume that $f'(z)$ is bounded by Cz^{a-1} for sufficiently small z .

The remainder of the present paper is organized as follows. In Section 2, we introduce some notations and lemmas which will be used later. In Section 3, we prove Theorem 1.1. In Section 4, we derive a decay at infinity principle and combine it with the moving plane method to prove Theorems 1.2 and 1.4.

For brevity, we use C to denote various positive constants which may varies from place to place. In some situations, we append subscripts to C to specify its dependence on the subscript parameters. We also denote $B_\varepsilon = B_\varepsilon(0)$, $z^+ = \max\{z, 0\}$ and $z^- = \min\{z, 0\}$ for any $\varepsilon > 0$ and $z \in \mathbb{R}$.

2. Preliminaries

Throughout the paper, we use the following notations to facilitate our description. For arbitrary $\lambda \in \mathbb{R}$, let

- $T_\lambda = \{x \in \mathbb{R}^n \mid x_1 = \lambda\}$ be the moving plane,
- $\Sigma_\lambda = \{x \in \mathbb{R}^n \mid x_1 < \lambda\}$ be the left region of the plane,
- $x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$ be the reflection of the point $x = (x_1, x_2, \dots, x_n)$ about the plane T_λ .

We also define $u_\lambda(x) = u(x^\lambda)$, $v_\lambda(x) = v(x^\lambda)$ and introduce functions

$$U_\lambda(x) = u_\lambda(x) - u(x), \quad V_\lambda(x) = v_\lambda(x) - v(x).$$

One important ingredient of the moving plane method for the fractional p -Laplacian is the following lemma.

Lemma 2.1 (A boundary estimate [9]). *Assume that $U_{\lambda_0} \geq 0$ in Σ_{λ_0} and $U_{\lambda_0} \not\equiv 0$ in Σ_{λ_0} . Suppose $\lambda_k \searrow \lambda_0$ and $x^k \in \Sigma_{\lambda_k}$ such that*

$$U_{\lambda_k}(x^k) = \min_{\Sigma_{\lambda_k}} U_{\lambda_k} \leq 0 \quad \text{and} \quad x^k \rightarrow x^0 \in T_{\lambda_0}.$$

Let $\delta_k = \text{dist}(x^k, T_{\lambda_k}) = \lambda_k - x_1^k$. Then

$$\limsup_{\delta_k \rightarrow 0} \frac{1}{\delta_k} [(-\Delta)_p^s u_{\lambda_k}(x^k) - (-\Delta)_p^s u(x^k)] < 0.$$

Remark 2.2. In the original statement (see [9, Theorem 2.3]), U_{λ_0} is assumed to be positive in all of Σ_{λ_0} . However, it is clear from its proof in [9] that this assumption can be weakened as $U_{\lambda_0} \geq 0$ in Σ_{λ_0} and $U_{\lambda_0} \not\equiv 0$ in Σ_{λ_0} , like we stated above. In applications, however, one usually need $U_{\lambda_0} > 0$ to guarantee that the limit of $\{x^k\}$ belongs to T_{λ_0} .

The following elementary inequality was also proved in [9].

Lemma 2.3. [9, Lemma 5.1] *Let $G(z) = |z|^{p-2}z$, then G is strictly increasing in \mathbb{R} and*

$$G(w) - G(z) \leq C|z|^{p-2}(w - z)$$

for all $w \leq z$.

To deal with problems in \mathbb{R}^n , we need the following estimation.

Lemma 2.4. *Let $u \in L_{s,p} \cap C_{\text{loc}}^{1,1}(\mathbb{R}^n)$ be a positive function such that*

$$u(x) \sim \frac{1}{|x|^\alpha} \quad \text{as } |x| \rightarrow \infty$$

for some $\alpha > 0$. Assume $\lambda < \bar{\lambda} < 0$ and $U_\lambda(x^*) = \min_{\Sigma_\lambda} U_\lambda \leq 0$ for some $x^* \in \Sigma_\lambda$. Then we have

$$(-\Delta)_p^s u_\lambda(x^*) - (-\Delta)_p^s u(x^*) \leq \frac{CU_\lambda(x^*)}{|x^*|^{\alpha(p-2)+sp}},$$

where C depends on $\bar{\lambda}$, but is independent of λ .

Proof. Denoting $G(z) = |z|^{p-2}z$ and using the definition of fractional p -Laplacian, we can compute

$$\begin{aligned}
& (-\Delta)_p^s u_\lambda(x^*) - (-\Delta)_p^s u(x^*) \\
&= C_{n,s,p} \text{PV} \int_{\mathbb{R}^n} \frac{G(u_\lambda(x^*) - u_\lambda(y)) - G(u(x^*) - u(y))}{|x^* - y|^{n+sp}} dy \\
&= C_{n,s,p} \text{PV} \int_{\Sigma_\lambda} \frac{G(u_\lambda(x^*) - u_\lambda(y)) - G(u(x^*) - u(y))}{|x^* - y|^{n+sp}} dy \\
&\quad + C_{n,s,p} \int_{\Sigma_\lambda} \frac{G(u_\lambda(x^*) - u(y)) - G(u(x^*) - u_\lambda(y))}{|x^* - y^\lambda|^{n+sp}} dy \\
(2.1) \quad &= C_{n,s,p} \text{PV} \int_{\Sigma_\lambda} \left(\frac{1}{|x^* - y|^{n+sp}} - \frac{1}{|x^* - y^\lambda|^{n+sp}} \right) \\
&\quad \times [G(u_\lambda(x^*) - u_\lambda(y)) - G(u(x^*) - u(y))] dy \\
&\quad + C_{n,s,p} \int_{\Sigma_\lambda} \frac{G(u_\lambda(x^*) - u_\lambda(y)) - G(u(x^*) - u_\lambda(y))}{|x^* - y^\lambda|^{n+sp}} dy \\
&\quad + C_{n,s,p} \int_{\Sigma_\lambda} \frac{G(u_\lambda(x^*) - u(y)) - G(u(x^*) - u(y))}{|x^* - y^\lambda|^{n+sp}} dy.
\end{aligned}$$

Clearly,

$$(2.2) \quad \frac{1}{|x^* - y|^{n+sp}} - \frac{1}{|x^* - y^\lambda|^{n+sp}} > 0.$$

By the mean value theorem, we also have

$$\begin{aligned}
(2.3) \quad & G(u_\lambda(x^*) - u_\lambda(y)) - G(u(x^*) - u(y)) = G'(\eta_1)[U_\lambda(x^*) - U_\lambda(y)] \leq 0, \\
& G(u_\lambda(x^*) - u_\lambda(y)) - G(u(x^*) - u_\lambda(y)) = G'(\eta_2)U_\lambda(x^*) \leq 0, \\
& G(u_\lambda(x^*) - u(y)) - G(u(x^*) - u(y)) = G'(\eta_3)U_\lambda(x^*) \leq 0,
\end{aligned}$$

where

$$\begin{aligned}
u_\lambda(x^*) - u_\lambda(y) &\leq \eta_1 \leq u(x^*) - u(y), \\
u_\lambda(x^*) - u_\lambda(y) &\leq \eta_2 \leq u(x^*) - u_\lambda(y), \\
u_\lambda(x^*) - u(y) &\leq \eta_3 \leq u(x^*) - u(y).
\end{aligned}$$

Combining (2.1), (2.2) and (2.3), we deduce

$$(-\Delta)_p^s u_\lambda(x^*) - (-\Delta)_p^s u(x^*) \leq C_{n,s,p} \int_{\Sigma_\lambda} \frac{G(u_\lambda(x^*) - u(y)) - G(u(x^*) - u(y))}{|x^* - y^\lambda|^{n+sp}} dy.$$

Then we may apply the Lemma 2.3 to obtain

$$(2.4) \quad (-\Delta)_p^s u_\lambda(x^*) - (-\Delta)_p^s u(x^*) \leq CU_\lambda(x^*) \int_{\Sigma_\lambda} \frac{|u(x^*) - u(y)|^{p-2}}{|x^* - y^\lambda|^{n+sp}} dy.$$

From the decay property of u , we have

$$(2.5) \quad \frac{C_1}{|y|^\alpha} < u(y) < \frac{C_2}{|y|^\alpha} \quad \text{for all } y \in \mathbb{R}^n \setminus B_{|\bar{\lambda}|},$$

where constants $C_1, C_2 > 0$ depend only on $\bar{\lambda}$. We denote $M = \left(\frac{2C_2}{C_1}\right)^{1/\alpha}$. Then for any $x, y \in \mathbb{R}^n$ such that $|y| > M|x|$ and $|x| \geq |\bar{\lambda}|$, we deduce from (2.5)

$$u(y) < \frac{C_2}{|y|^\alpha} < \frac{C_1}{2|x|^\alpha} < \frac{u(x)}{2}.$$

Next, we choose $z = (-(M+1)|x^*|, 0, \dots, 0)$, then $|z| = (M+1)|x^*|$ and $B_{|x^*|}(z) \subset \Sigma_\lambda$. Moreover, $u(y) < \frac{1}{2}u(x^*)$ for all $y \in B_{|x^*|}(z)$. Therefore,

$$(2.6) \quad \begin{aligned} \int_{\Sigma_\lambda} \frac{|u(x^*) - u(y)|^{p-2}}{|x^* - y^\lambda|^{n+sp}} dy &\geq \frac{u^{p-2}(x^*)}{2^{p-2}} \int_{B_{|x^*|}(z)} \frac{dy}{|x^* - y^\lambda|^{n+sp}} \\ &\geq \frac{u^{p-2}(x^*)}{2^{p-2}} \int_{B_{|x^*|}(z)} \frac{dy}{[(M+3)|x^*|]^{n+sp}} \\ &\geq C \frac{u^{p-2}(x^*)}{|x^*|^{sp}}, \end{aligned}$$

where we have used the fact

$$|x^* - y^\lambda| \leq |x^*| + |-z^\lambda| + |z^\lambda - y^\lambda| \leq |x^*| + |z| + |z - y| \leq (M+3)|x^*|$$

for all $y \in B_{|x^*|}(z)$. Collecting (2.4), (2.5) and (2.6), we obtain

$$(-\Delta)_p^s u_\lambda(x^*) - (-\Delta)_p^s u(x^*) \leq CU_\lambda(x^*) \frac{u^{p-2}(x^*)}{|x^*|^{sp}} \leq \frac{CU_\lambda(x^*)}{|x^*|^{\alpha(p-2)+sp}}.$$

This proves the lemma. \square

3. Symmetry of solutions to fractional systems in the unit ball

We prove Theorem 1.1 in this section. A usual application of the moving plane method in the unit ball consists of two steps: showing that the plane can move from $\lambda = -1$ a little bit to the right, then moving the plane to its limiting position. However, since the plane can move from $\lambda = -\infty$ to $\lambda = -1$ already, we find that Step 1 is unnecessary. Therefore, we go directly to Step 2. Hence, our proof is simpler and requires fewer technical assumptions on f and g comparing to [8, 9].

Proof of Theorem 1.1. Clearly,

$$U_\lambda \geq 0, \quad V_\lambda \geq 0 \quad \text{in } \Sigma_\lambda$$

for all $\lambda \leq -1$. Therefore, we may define

$$\lambda_0 = \sup\{\lambda \leq 0 \mid U_\mu \geq 0, V_\mu \geq 0 \text{ in } \Sigma_\mu \text{ for all } \mu \leq \lambda\}.$$

We claim that

$$(3.1) \quad \lambda_0 = 0.$$

By contradiction, suppose $-1 \leq \lambda_0 < 0$. Since U_λ and V_λ depend on λ continuously, we have

$$U_{\lambda_0} \geq 0, \quad V_{\lambda_0} \geq 0 \quad \text{in } \Sigma_{\lambda_0}.$$

If $-1 < \lambda_0 < 0$, we must have

$$(3.2) \quad U_{\lambda_0} > 0, \quad V_{\lambda_0} > 0 \quad \text{in } \Sigma_{\lambda_0} \cap \bar{B}_1.$$

To prove (3.2), we may assume in contrary that $U_{\lambda_0}(x^*) = 0$ for some $x^* \in \Sigma_{\lambda_0} \cap \bar{B}_1$. Then

$$(3.3) \quad \begin{aligned} (-\Delta)_p^s u_{\lambda_0}(x^*) - (-\Delta)_p^s u(x^*) &= f(u_{\lambda_0}(x^*), v_{\lambda_0}(x^*)) - f(u(x^*), v(x^*)) \\ &= f(u(x^*), v_{\lambda_0}(x^*)) - f(u(x^*), v(x^*)) \geq 0. \end{aligned}$$

Moreover, since

$$U_{\lambda_0}(x^*) = \min_{\Sigma_{\lambda_0}} U_{\lambda_0} = 0,$$

the identity (2.1) and (2.3) indicate

$$(3.4) \quad \begin{aligned} &(-\Delta)_p^s u_{\lambda_0}(x^*) - (-\Delta)_p^s u(x^*) \\ &= C_{n,s,p} \text{PV} \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x^* - y|^{n+sp}} - \frac{1}{|x^* - y^{\lambda_0}|^{n+sp}} \right) \\ &\quad \times [G(u_{\lambda_0}(x^*) - u_{\lambda_0}(y)) - G(u(x^*) - u(y))] dy \\ &< 0, \end{aligned}$$

where in the last estimation we have used the strict monotonicity of G and the fact

$$[u_{\lambda_0}(x^*) - u_{\lambda_0}(y)] - [u(x^*) - u(y)] = -u_{\lambda_0}(y) < 0 \quad \text{for } y \in B_1(0^{\lambda_0}) \setminus B_1.$$

Since (3.4) contradicts (3.3), we must have $U_{\lambda_0} > 0$ in $\Sigma_{\lambda_0} \cap \bar{B}_1$. The positivity of V_{λ_0} in $\Sigma_{\lambda_0} \cap \bar{B}_1$ can be obtained in the same way. Hence (3.2) is proved.

By the definition of λ_0 , there exists a sequence $\lambda_k \searrow \lambda_0$ and $x^k \in \Sigma_{\lambda_k} \cap B_1$ such that $\lambda_k < 0$ and

$$U_{\lambda_k}(x^k) = \min_{\Sigma_{\lambda_k}} U_{\lambda_k} < 0 \quad \text{or} \quad V_{\lambda_k}(x^k) = \min_{\Sigma_{\lambda_k}} V_{\lambda_k} < 0.$$

Without loss of generality, by passing to a subsequence, we may assume

$$U_{\lambda_k}(x^k) = \min_{\Sigma_{\lambda_k}} U_{\lambda_k} < 0 \quad \text{and} \quad x^k \rightarrow x^0 \in \bar{\Sigma}_{\lambda_0} \cap \bar{B}_1.$$

Moreover, since $x^k \in \Sigma_{\lambda_k} \cap B_1$ is the interior minimum point, we have

$$\nabla U_{\lambda_k}(x^k) = 0.$$

From the continuity of $U_\lambda(x)$ and its derivative with respect to both x and λ , we deduce

$$U_{\lambda_0}(x^0) \leq 0 \quad \text{and} \quad \nabla U_{\lambda_0}(x^0) = 0.$$

If $\lambda_0 = -1$, then $x^0 = (-1, 0, \dots, 0) \in T_{\lambda_0}$. If $-1 < \lambda_0 < 0$, then we also have $x^0 \in T_{\lambda_0}$ thanks to (3.2). Define δ_k as in Lemma 2.1. It follows that

$$(3.5) \quad \frac{U_{\lambda_k}(x^k)}{\delta_k} \rightarrow -\frac{\partial U_{\lambda_0}(x^0)}{\partial x_1} = 0 \quad \text{as } k \rightarrow \infty.$$

We also have

$$(3.6) \quad \frac{V_{\lambda_k}(x^k)}{\delta_k} \rightarrow -\frac{\partial V_{\lambda_0}(x^0)}{\partial x_1} \geq 0 \quad \text{as } k \rightarrow \infty.$$

On the other hand, we have the formulae

$$(3.7) \quad \begin{aligned} & (-\Delta)_p^s u_\lambda(x) - (-\Delta)_p^s u(x) \\ &= f(u_\lambda(x), v_\lambda(x)) - f(u(x), v(x)) \\ &= [f(u_\lambda(x), v(x)) - f(u(x), v(x))] + [f(u_\lambda(x), v_\lambda(x)) - f(u_\lambda(x), v(x))] \\ &\geq F_\lambda(x)U_\lambda(x) + G_\lambda(x)V_\lambda(x)^-, \end{aligned}$$

where

$$F_\lambda(x) = \begin{cases} \frac{f(u_\lambda(x), v(x)) - f(u(x), v(x))}{u_\lambda(x) - u(x)} & \text{if } u_\lambda(x) \neq u(x), \\ 0 & \text{if } u_\lambda(x) = u(x), \end{cases}$$

$$G_\lambda(x) = \begin{cases} \frac{f(u_\lambda(x), v_\lambda(x)) - f(u_\lambda(x), v(x))}{v_\lambda(x) - v(x)} & \text{if } v_\lambda(x) < v(x), \\ 0 & \text{if } v_\lambda(x) \geq v(x). \end{cases}$$

Notice that $u_{\lambda_k}(x^k) < u(x^k) \leq \max_{\bar{B}_1} u$ and $v_{\lambda_k}(x^k), v(x^k) \leq \max_{\bar{B}_1} v$. Then we may use the local Lipschitz continuity of f to deduce that $F_{\lambda_k}(x^k) \leq C$ and $G_{\lambda_k}(x^k) \leq C$.

Therefore, by combining (3.5), (3.6) and (3.7), we obtain

$$\frac{1}{\delta_k} [(-\Delta)_p^s u_{\lambda_k}(x^k) - (-\Delta)_p^s u(x^k)] \geq C \left(\frac{U_{\lambda_k}(x^k)}{\delta_k} + \frac{V_{\lambda_k}(x^k)^-}{\delta_k} \right) \rightarrow 0.$$

However, this contradicts Lemma 2.1. Hence (3.1) is proved.

Since the direction x_1 can be chosen arbitrarily, we deduce that u and v are radially symmetric and monotone decreasing about the origin. This completes the proof of Theorem 1.1. \square

4. Symmetry of solutions to fractional systems in \mathbb{R}^n

We assume that all assumptions of Theorem 1.2 are satisfied.

Lemma 4.1 (Decay at infinity principle). *Let $\bar{\lambda} < 0$. There exists $R_0 > 0$ such that for all $\lambda \leq \bar{\lambda}$, we have the following claims:*

(i) *If $x^* \in \Sigma_\lambda \setminus B_{R_0}$ and $U_\lambda(x^*) = \min_{\Sigma_\lambda} U_\lambda < 0$, then*

$$V_\lambda(x^*) < 2U_\lambda(x^*).$$

(ii) *If $x^* \in \Sigma_\lambda \setminus B_{R_0}$ and $V_\lambda(x^*) = \min_{\Sigma_\lambda} V_\lambda < 0$, then*

$$U_\lambda(x^*) < 2V_\lambda(x^*).$$

Proof. Let us assume

$$U_\lambda(x^*) = \min_{\Sigma_\lambda} U_\lambda < 0$$

for some $x^* \in \Sigma_\lambda$. Clearly, if there exists some R_0 independent of λ such that $|x^*| < R_0$, then (i) is true. Therefore, in what follows, we may assume that $|x^*|$ is sufficiently large. We have

$$\begin{aligned} & (-\Delta)_p^s u_\lambda(x) - (-\Delta)_p^s u(x) \\ (4.1) \quad & = f(u_\lambda(x), v_\lambda(x)) - f(u(x), v(x)) \\ & = [f(u_\lambda(x), v(x)) - f(u(x), v(x))] + [f(u_\lambda(x), v_\lambda(x)) - f(u_\lambda(x), v(x))] \\ & = f'_u(\xi_\lambda(x), v(x))U_\lambda(x) + f'_v(u_\lambda(x), \zeta_\lambda(x))V_\lambda(x), \end{aligned}$$

where $\xi_\lambda(x)$ is between $u_\lambda(x)$ and $u(x)$; $\zeta_\lambda(x)$ is between $v_\lambda(x)$ and $v(x)$. Combining (4.1) and Lemma 2.4, we obtain

$$(4.2) \quad \left(\frac{C}{|x^*|^{\alpha(p-2)+sp}} - f'_u(\xi_\lambda(x^*), v(x^*)) \right) U_\lambda(x^*) \geq f'_v(u_\lambda(x^*), \zeta_\lambda(x^*))V_\lambda(x^*).$$

If $|x^*|$ is sufficiently large, by the assumptions (ii) and (iii), we have

$$f'_u(\xi_\lambda(x^*), v(x^*)) \leq 0$$

or

$$f'_u(\xi_\lambda(x^*), v(x^*)) = o(1)\xi_\lambda(x^*)^{a_1}v(x^*)^{b_1} = \frac{o(1)}{|x^*|^{\alpha a_1 + \beta b_1}},$$

where $o(1) \rightarrow 0$ as $|x^*| \rightarrow \infty$. Hence for $|x^*|$ sufficiently large, the left-hand side of (4.2) is negative. This implies $V_\lambda(x^*) < 0$ and $\zeta_\lambda(x^*) < v(x^*)$, which is small. Then we have

$$f'_v(u_\lambda(x^*), \zeta_\lambda(x^*)) = o(1)\xi_\lambda(x^*)^{a_2}v(x^*)^{b_2} = \frac{o(1)}{|x^*|^{\alpha a_2 + \beta b_2}}.$$

Therefore, from (4.2), we conclude that

$$V_\lambda(x^*) < 2U_\lambda(x^*)$$

if $|x^*| \geq R_0$ for some $R_0 > 0$ sufficiently large. Hence, (i) holds for such an R_0 .

In the same way, we can choose R_0 , which is larger if necessary, such that (ii) also holds. \square

Proof of Theorem 1.2. The asymptotic properties of u and v indicate that, if U_λ or V_λ is negative somewhere in Σ_λ , then its negative minima are attained in the interior of Σ_λ . We carry out the method of moving planes along the x_1 direction in two steps.

Step 1: Move the plane along x_1 direction from near $-\infty$. In this step, we show that for λ sufficiently negative,

$$(4.3) \quad U_\lambda \geq 0, \quad V_\lambda \geq 0 \quad \text{in } \Sigma_\lambda.$$

Indeed, we may choose $\lambda \leq -R_0$, where R_0 is defined in Lemma 4.1 with $\bar{\lambda} = -1$. If (4.3) does not hold for such a λ , then there exists $x^* \in \Sigma_\lambda$ such that $U_\lambda(x^*) = \min_{\Sigma_\lambda} U_\lambda < 0$ or $V_\lambda(x^*) = \min_{\Sigma_\lambda} V_\lambda < 0$. Without loss of generality, we may assume

$$U_\lambda(x^*) = \min_{\Sigma_\lambda} U_\lambda < 0.$$

Since $|x^*| > |\lambda| \geq R_0$, we may use Lemma 4.1 to deduce

$$(4.4) \quad V_\lambda(x^*) < 2U_\lambda(x^*) < 0.$$

Hence, there exists $y^* \in \Sigma_\lambda$ such that

$$V_\lambda(y^*) = \min_{\Sigma_\lambda} V_\lambda < 0.$$

Using Lemma 4.1 again, we obtain

$$(4.5) \quad U_\lambda(y^*) < 2V_\lambda(y^*).$$

Combining (4.4) and (4.5), we get

$$V_\lambda(x^*) < 2U_\lambda(x^*) \leq 2U_\lambda(y^*) < 4V_\lambda(y^*) \leq 4V_\lambda(x^*),$$

which implies $V_\lambda(x^*) > 0$. However, this contradicts (4.4). Hence (4.3) holds for all $\lambda \leq -R_0$.

Step 2: Move the plane to the limiting position. Step 1 provides a starting point, from which we can now move the plane T_λ to the right as long as (4.3) holds to its limiting position. Define

$$\lambda_0 = \sup\{\lambda \leq 0 \mid U_\mu \geq 0, V_\mu \geq 0 \text{ in } \Sigma_\mu \text{ for all } \mu \leq \lambda\}.$$

In what follows, we assume $\lambda_0 < 0$. Since U_λ and V_λ depend on λ continuously, we already have

$$U_{\lambda_0} \geq 0, \quad V_{\lambda_0} \geq 0 \quad \text{in } \Sigma_{\lambda_0}.$$

There are two possible cases:

Case 1: $U_{\lambda_0} > 0$ and $V_{\lambda_0} > 0$ in Σ_{λ_0} . By the definition of λ_0 , there exists a sequence $\lambda_k \searrow \lambda_0$ and $x^k \in \Sigma_{\lambda_k}$ such that $\lambda_k < \lambda_0/2$ and

$$U_{\lambda_k}(x^k) = \min_{\Sigma_{\lambda_k}} U_{\lambda_k} < 0 \quad \text{or} \quad V_{\lambda_k}(x^k) = \min_{\Sigma_{\lambda_k}} V_{\lambda_k} < 0.$$

Up to a subsequence, we may assume

$$U_{\lambda_k}(x^k) = \min_{\Sigma_{\lambda_k}} U_{\lambda_k} < 0.$$

Let R_0 be defined as in Lemma 4.1 with $\bar{\lambda} = \lambda_0/2$. There are two possibilities.

Possibility (i): Up to a subsequence, $|x^k| < R_0$. We may assume that $x^k \rightarrow x^0 \in \bar{\Sigma}_{\lambda_0}$. Moreover, since $x^k \in \Sigma_{\lambda_k}$ is the interior minimum point, we have

$$\nabla U_{\lambda_k}(x^k) = 0.$$

From the continuity of $U_\lambda(x)$ and its derivative with respect to both x and λ , we have

$$U_{\lambda_0}(x^0) \leq 0 \quad \text{and} \quad \nabla U_{\lambda_0}(x^0) = 0.$$

Hence $x^0 \in T_{\lambda_0}$. It follows that

$$(4.6) \quad \frac{U_{\lambda_k}(x^k)}{\delta_k} \rightarrow -\frac{\partial U_{\lambda_0}(x^0)}{\partial x_1} = 0 \quad \text{as } k \rightarrow \infty.$$

We also have

$$(4.7) \quad \frac{V_{\lambda_k}(x^k)}{\delta_k} \rightarrow -\frac{\partial V_{\lambda_0}(x^0)}{\partial x_1} \geq 0 \quad \text{as } k \rightarrow \infty.$$

On the other hand, we may use (4.1) to obtain

$$\begin{aligned} & \frac{1}{\delta_k} [(-\Delta)_p^s u_{\lambda_k}(x^k) - (-\Delta)_p^s u(x^k)] \\ &= f'_u(\xi_{\lambda_k}(x^k), v(x^k)) \frac{U_{\lambda_k}(x^k)}{\delta_k} + f'_v(u_{\lambda_k}(x^k), \zeta_{\lambda_k}(x^k)) \frac{V_{\lambda_k}(x^k)}{\delta_k} \\ &\geq \left(\max_{\Omega} f'_u \right) \frac{U_{\lambda_k}(x^k)}{\delta_k} + \left(\max_{\Omega} f'_v \right) \frac{V_{\lambda_k}(x^k)^-}{\delta_k} \end{aligned}$$

for sufficiently large k , where

$$\Omega = \left[\min_{\overline{B}_1(x^0)} u, \max_{\overline{B}_1(x^0)} u \right] \times \left[\min_{\overline{B}_1(x^0)} v, \max_{\overline{B}_1(x^0)} v \right].$$

Combining (4.6) and (4.7) with the above estimation, we obtain

$$\liminf_{\delta_k \rightarrow 0} \frac{1}{\delta_k} [(-\Delta)_p^s u_{\lambda_k}(x^k) - (-\Delta)_p^s u(x^k)] \geq 0.$$

However, this contradicts Lemma 2.1.

Possibility (ii): $|x^k| \geq R_0$ for all $k \geq k_0$. By Lemma 4.1, $V_{\lambda_k}(x^k) < 2U_{\lambda_k}(x^k) < 0$. Hence, there exists $y^k \in \Sigma_{\lambda_k}$ such that

$$V_{\lambda_k}(y^k) = \min_{\Sigma_{\lambda_k}} V_{\lambda_k} < 0.$$

If $|y^k| \geq R_0$, then we get a contradiction as in Step 1.

Thus, we must have $|y^k| < R_0$ for all $k \geq k_0$. Repeating the arguments in Possibility (i) for V_{λ_k} and y^k , we can derive a contradiction with Lemma 2.1. Therefore, Case 1 cannot happen.

Case 2: $U_{\lambda_0}(x^*) = 0$ or $V_{\lambda_0}(x^*) = 0$ for some $x^* \in \Sigma_{\lambda_0}$. We may assume $U_{\lambda_0}(x^*) = 0$. From (4.1), we have

$$(4.8) \quad (-\Delta)_p^s u_{\lambda_0}(x^*) - (-\Delta)_p^s u(x^*) = f'_v(u_{\lambda_0}(x^*), \zeta_{\lambda_0}(x^*)) V_{\lambda_0}(x^*) \geq 0.$$

Moreover, since

$$U_{\lambda_0}(x^*) = \min_{\Sigma_{\lambda_0}} U_{\lambda_0} = 0,$$

the identity (2.1) and (2.3) imply

$$\begin{aligned} & (-\Delta)_p^s u_{\lambda_0}(x^*) - (-\Delta)_p^s u(x^*) \\ (4.9) \quad &= C_{n,s,p} \text{PV} \int_{\Sigma_{\lambda_0}} \left(\frac{1}{|x^* - y|^{n+sp}} - \frac{1}{|x^* - y^{\lambda_0}|^{n+sp}} \right) \\ & \quad \times [G(u_{\lambda_0}(x^*) - u_{\lambda_0}(y)) - G(u(x^*) - u(y))] dy \\ & \leq 0. \end{aligned}$$

Due to the strict monotonicity of G , (4.8) and (4.9) imply $U_{\lambda_0} \equiv 0$ in Σ_{λ_0} . Then using (4.1) again, we have

$$0 = (-\Delta)_p^s u_{\lambda_0}(x) - (-\Delta)_p^s u(x) = f'_v(u_{\lambda_0}(x), \zeta_{\lambda_0}(x)) V_{\lambda_0}(x) \geq 0.$$

Since $f'_v > 0$, we must have $V_{\lambda_0} \equiv 0$ in Σ_{λ_0} . In summary, if $\lambda_0 < 0$, then u and v are symmetric and monotone decreasing about plane T_{λ_0} .

It remains to consider the case $\lambda_0 = 0$. In this case, $U_0 \geq 0$, $V_0 \geq 0$ in Σ_0 . By moving the plane along x_1 direction from near $+\infty$, we also deduce that either u and v are symmetric and monotone decreasing about a plane T_{λ_1} for some $\lambda_1 > 0$ or $U_0 \leq 0$, $V_0 \leq 0$ in Σ_0 . In the latter case, u and v must be symmetric and monotone decreasing about plane T_0 .

We may repeat the above arguments for any direction to conclude that u and v are radially symmetric and monotone decreasing about some point $x^0 \in \mathbb{R}^n$. This completes the proof of Theorem 1.2. \square

Proof of Theorem 1.4. We claim that Lemma 4.1 still holds for system (1.4) under assumptions of Theorem 1.4.

Indeed, we may proceed as in the proof of Lemma 4.1. Instead of (4.1) and (4.2), we now have

$$\begin{aligned} & (-\Delta)_p^s u_\lambda(x^*) - (-\Delta)_p^s u(x^*) \\ (4.10) \quad &= [u_\lambda(x^*)^a - u(x^*)^a] v(x^*)^b + [v_\lambda(x^*)^b - v(x^*)^b] u_\lambda(x^*)^a \\ &\geq [u_\lambda(x^*)^a - u(x^*)^a] v(x^*)^b + [v_\lambda(x^*)^b - v(x^*)^b]^- u_\lambda(x^*)^a. \end{aligned}$$

If $a \leq 0$, then $[u_\lambda(x^*)^a - u(x^*)^a] v(x^*)^b \geq 0$. If $a > 0$, then

$$\begin{aligned} [u_\lambda(x^*)^a - u(x^*)^a] v(x^*)^b &\geq C [u_\lambda(x^*)^a - u(x^*)^a] u(x^*) |x^*|^{\alpha-\beta b} \\ &\geq C [u_\lambda(x^*)^{a+1} - u(x^*)^{a+1}] |x^*|^{\alpha-\beta b} \\ &= C U_\lambda(x^*) \xi_\lambda(x^*)^a |x^*|^{\alpha-\beta b} \\ &\geq C U_\lambda(x^*) u(x^*)^a |x^*|^{\alpha-\beta b} \\ &\geq \frac{C U_\lambda(x^*)}{|x^*|^{\alpha(a-1)+\beta b}}. \end{aligned}$$

Hence for all $a \in \mathbb{R}$, we have

$$(4.11) \quad [u_\lambda(x^*)^a - u(x^*)^a] v(x^*)^b \geq \frac{C a^+ U_\lambda(x^*)}{|x^*|^{\alpha(a-1)+\beta b}}.$$

Similarly, since $b > 0$,

$$(4.12) \quad [v_\lambda(x^*)^b - v(x^*)^b]^- u_\lambda(x^*)^a \geq \frac{C V_\lambda(x^*)^-}{|x^*|^{\alpha a + \beta(b-1)}}.$$

Substituting (4.11) and (4.12) into (4.10) and using Lemma 2.4, we obtain

$$C \left(\frac{1}{|x^*|^{\alpha(p-2)+sp}} - \frac{a^+}{|x^*|^{\alpha(a-1)+\beta b}} \right) U_\lambda(x^*) \geq \frac{CV_\lambda(x^*)^-}{|x^*|^{\alpha a + \beta(b-1)}}.$$

From this and assumptions (i), (ii), we deduce that there exists R_0 sufficiently large such that

$$V_\lambda(x^*) < 2U_\lambda(x^*)$$

if $|x^*| \geq R_0$. This proves Lemma 4.1.

The remainder of the proof is similar to that of Theorem 1.2 and hence will be omitted. \square

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