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Counting the Number of Solutions to Certain Infinite Diophantine Equations

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Abstract. Let r, v, n be positive integers. This paper investigate the number of solutions $s_{r,v}(n)$ of the following infinite Diophantine equations

$$n = 1^r \cdot |k_1|^v + 2^r \cdot |k_2|^v + 3^r \cdot |k_3|^v + \cdots$$

for $\boldsymbol{k} = (k_1, k_2, k_3, \ldots) \in \mathbb{Z}^\infty$. For each $(r, v) \in \mathbb{N} \times \{1, 2\}$, a generating function and some asymptotic formulas of $s_{r,v}(n)$ are established.

1. Introduction and statement of results

Let r, n be positive integers. A partition into r-th powers of an integer n is a sequence of non-increasing r-th powers of positive integers whose sum equals n. Such a partition corresponds to a solution of the following infinite Diophantine equation:

$$n = 1^r \cdot k_1 + 2^r \cdot k_2 + 3^r \cdot k_3 + \cdots \tag{1.1}$$

for $\boldsymbol{k} = (k_1, k_2, k_3, \ldots) \in \mathbb{N}_0^\infty$. Let $p_r(n)$ be the number of partitions of n into r-th powers and let $p_r(0) := 1$, we have the generating function

$$\sum_{n \geq 0} p_r(n) q^n = \prod_{n \geq 1} \frac{1}{1 - q^{n^r}},$$

where $q \in \mathbb{C}$ with $|q| < 1$.

Determining the values of $p_r(n)$ has a long history and can be traced back to the work of Euler. In the famous paper [3], Hardy and Ramanujan proved an asymptotic expansion for $p_1(n)$ as $n \to \infty$. They [3, p. 111] also gave an asymptotic formula for $p_r(n)$, $r \geq 2$, without proof. In [7, Theorem 2], Wright confirmed their asymptotic formula

$$p_r(n) \sim \frac{c_r n^{\frac{1}{r+1} - \frac{3}{2}}}{\sqrt{(2\pi)^{1+r}(1 + 1/r)}} e^{(r+1) c_r n^{\frac{1}{r+1}}}$$

as integer $n \to \infty$, where $c_r = \left(r^{-1}\zeta(1 + 1/r)\Gamma(1 + 1/r)\right)^{\frac{r}{r+1}}$, $\zeta(\cdot)$ is the Riemann zeta function and $\Gamma(\cdot)$ is the classical Euler Gamma function.

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In this paper we investigate certain infinite Diophantine equation analogous to (1.1). For given positive integers n and r, we use $s_{r,v}(n)$ to denote the number of solutions of the following infinite Diophantine equation

$$n = 1^r \cdot |k_1|^v + 2^r \cdot |k_2|^v + 3^r \cdot |k_3|^v + \cdots \tag{1.2}$$

for $\boldsymbol{k} = (k_1, k_2, k_3, \ldots) \in \mathbb{Z}^\infty$. The first result of this paper is about the generating function for $s_{r,v}(n)$.

Proposition 1.1. *Let $s_{r,v}(0) := 1$ and $q \in \mathbb{C}$ with $|q| < 1$. We have*

$$G_{r,1}(q) := \sum_{n\geq 0} s_{r,1}(n)q^n = \prod_{n\geq 1} \frac{1+q^{n^r}}{1-q^{n^r}}$$

and

$$G_{r,2}(q) := \sum_{n\geq 0} s_{r,2}(n)q^n = \prod_{j\geq 1}\prod_{n\geq 1} \frac{1-(-1)^n q^{nj^r}}{1+(-1)^n q^{nj^r}}.$$

Remark 1.2. From the proof of this proposition (see Subsection 2.1), the above infinite product expansion for $G_{r,s}(q)$ ($r \in \mathbb{N}$, $s = 1, 2$) follows the identities

$$\sum_{n\in\mathbb{Z}} q^{|n|} = \frac{1+q}{1-q} \quad \text{and} \quad \sum_{n\in\mathbb{Z}} q^{|n|^2} = \prod_{n\geq 1} \frac{1-(-q)^n}{1+(-q)^n}.$$

They actually follow from the geometric sequence sum formula and the Jacobi triple product identity. However, any useful expansion for the sum $\sum_{n\in\mathbb{Z}} q^{|n|^v}$ with each integer $v > 2$ is still not found yet. Therefore, whether there are infinite product formulas which is similar to Proposition 1.1 for $s_{r,v}(n)$ ($r \in \mathbb{N}$, $v \in \mathbb{Z}_{>2}$) is still a question to be settled.

Thanks to the infinite product expansion in Proposition 1.1, we can determine the asymptotic behavior of $G_{r,v}(q)$ when $|q| \to 1^-$. From which we can further determine the asymptotics of $s_{r,v}(n)$ ($(r, v) \in \mathbb{N} \times \{1, 2\}$) as $n \to \infty$. More precisely, we prove

Theorem 1.3. *For any given positive integers r and p, we have*

$$s_{r,1}(n) = \frac{\kappa_r^{3/2}}{\sqrt{2^{r+1}\pi^r}} \left(\frac{1}{n}\right)^{\frac{1+1/2}{1+1/r}} W_{\frac{1}{r},\frac{1}{2}}\left(\kappa_r n^{\frac{1}{1+r}}\right)\left(1 + O\left(\frac{1}{n^p}\right)\right)$$

and

$$s_{r,2}(n) = \frac{\kappa_r^{5/4}}{\sqrt[4]{2^r\pi^{r+1}}} \left(\frac{\eta(1/r)}{n}\right)^{\frac{1+1/4}{1+1/r}} W_{\frac{1}{r},\frac{1}{4}}\left(\kappa_r \eta(1/r)\left(\frac{n}{\eta(1/r)}\right)^{\frac{1}{1+r}}\right)\left(1 + O\left(\frac{1}{n^p}\right)\right)$$

as integer $n \to \infty$. Here $\kappa_r > 0$ is given by

$$\kappa_r^{1+1/r} = 2r^{-1}(1 - 2^{-1-1/r})\zeta(1 + 1/r)\Gamma(1 + 1/r),$$

$\eta(s) = \sum_{n\geq 1}(-1)^{n-1}n^{-s}$ *is the Dirichlet eta function, and*

$$W_{\alpha,\beta}(\lambda) = \frac{1}{2\pi}\int_{-1}^{1}(1+\mathrm{i}u)^{\beta}\exp\left(\lambda(\alpha^{-1}(1+\mathrm{i}u)^{-\alpha}+(1+\mathrm{i}u))\right)du$$

for all $\alpha,\beta,\lambda > 0$.

Using the standard saddle-point method, such as referring to [5, p. 127, Theorem 7.1], we can derive an asymptotic expansion for $W_{\alpha,\beta}(\lambda)$ as $\lambda\to+\infty$. Hence it is possible to derive full asymptotic expansions for $s_{r,v}(n)$ ($(r,v)\in\mathbb{N}\times\{1,2\}$). In particular, we have the following leading asymptotics.

Corollary 1.4. *For any given positive integer* r, *we have*

$$s_{r,1}(n) \sim 2^{-(r+2)/2}\pi^{-(r+1)/2}(1+1/r)^{-1/2}\kappa_r n^{-\frac{3r+1}{2+2r}}e^{(1+r)\kappa_r n^{\frac{1}{1+r}}}$$

and

$$s_{r,2}(n) \sim 2^{-(r+2)/4}\pi^{-(r+3)/4}(1+1/r)^{-1/2}\eta(1/r)^{\frac{3r}{4r+4}}\kappa_r^{3/4}n^{-\frac{5r+2}{4+4r}}e^{(1+r)\kappa_r\eta(1/r)^{\frac{r}{1+r}}n^{\frac{1}{1+r}}}$$

as $n\to\infty$.

2. Some results of the generating function

2.1. Proof of Proposition 1.1

We shall proceed in a formal manner to prove Proposition 1.1. Formally, using (1.2) we have

$$\begin{aligned}\sum_{n\geq 0}s_{r,v}(n)q^n &= \sum_{n\geq 0}q^n\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^\infty\\ \sum_{j\geq 1}j^r|k_j|^s=n}}1\\ &= \sum_{\boldsymbol{k}\in\mathbb{Z}^\infty}q^{\sum_{j\geq 1}j^r|k_j|^s} = \prod_{j\geq 1}\left(\sum_{k_j\in\mathbb{Z}}q^{j^r|k_j|^s}\right).\end{aligned}$$

Now, for $q\in\mathbb{C}$ with $|q|<1$, by noting that

$$\sum_{n\in\mathbb{Z}}q^{|n|} = 1+2\sum_{n\geq 1}q^n = \frac{1+q}{1-q}$$

and an identity of Gauss (see Andrews [1, Corollary 2.10])

$$\sum_{n\in\mathbb{Z}}q^{n^2} = \prod_{n\geq 1}\frac{1-(-q)^n}{1+(-q)^n},$$

we have

$$G_{r,1}(q) := \sum_{n\geq 0} s_{r,1}(n)q^n = \prod_{n\geq 1} \frac{1+q^{n^r}}{1-q^{n^r}}$$

and

$$G_{r,2}(q) := \sum_{n\geq 0} s_{r,2}(n)q^n = \prod_{j\geq 1}\prod_{n\geq 1} \frac{1-(-1)^n q^{nj^r}}{1+(-1)^n q^{nj^r}}.$$

Clearly, the product for $G_{r,1}(q)$ is absolute convergence for all $q \in \mathbb{C}$ with $|q| < 1$. For the product for $G_{r,2}(q)$, since

$$\left|\prod_{j\geq 1}\prod_{n\geq 1} \frac{1-(-1)^n q^{nj^r}}{1+(-1)^n q^{nj^r}}\right| \leq \prod_{j\geq 1}\prod_{n\geq 1} \frac{1+|q|^{nj^r}}{1-|q|^{nj^r}} = \prod_{\ell\geq 1}\left(\frac{1+|q|^{\ell}}{1-|q|^{\ell}}\right)^{\sigma_{1,r}(\ell)},$$

where

$$\sigma_{1,r}(\ell) = \#\{(n,j) \in \mathbb{N}^2 : nj^r = \ell\} \leq \ell;$$

and hence the product is absolute convergence for all $q \in \mathbb{C}$ with $|q| < 1$. This completes the proof of Proposition 1.1.

2.2. Asymptotics of the generating function

To give a proof of Theorem 1.3, we need to determine asymptotics of the generating function in Proposition 1.1 at $q = 1$.

Proposition 2.1. *Let r be a given positive integer, $z = x+\mathrm{i}y$ with $x, y \in \mathbb{R}$ and $|\arg(z)| \leq \pi/4$. As $z \to 0$,*

$$G_{r,1}(e^{-z}) = \frac{z^{1/2}\exp\left(r\kappa_r^{1+1/r} z^{-1/r}\right)}{\sqrt{2^{r+1}\pi^r}}\left(1+O(|z|^p)\right)$$

and

$$G_{r,2}(e^{-z}) = \frac{z^{1/4}\exp\left(r\eta(1/r)\kappa_r^{1+1/r} z^{-1/r}\right)}{\sqrt[4]{2^r\pi^{r+1}}}\left(1+O(|z|^p)\right)$$

holds for any given $p > 0$. Here $\kappa_r > 0$ such that

$$\kappa_r^{1+1/r} = 2r^{-2}(1-2^{-1-1/r})\zeta(1+1/r)\Gamma(1/r).$$

Proof. The proof of the result for $G_{r,1}(e^{-z})$ is similar to $G_{r,2}(e^{-z})$, hence we only prove the later one. We shall follow the proof of [1, p. 89, Lemma 6.1]. The series for the Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1} n^{-s}$$

and the Dirichlet eta function

$$\eta(s)=\sum_{n\geq 1}(-1)^{n-1}n^{-s}$$

converge absolutely and uniformly for $s\in\mathbb{C}$ when $\Re(s)\geq c>1$. Therefore, by using Mellin's transform,

$$\begin{aligned}\log G_{r,2}(e^{-z})&=2\sum_{\substack{\ell\geq 1\\ \ell\text{ odd}}}\frac{1}{\ell}\sum_{j\geq 1}\sum_{n\geq 1}(-1)^{n-1}e^{-n\ell j^r z}\\&=2\sum_{\substack{\ell\geq 1\\ \ell\text{ odd}}}\frac{1}{\ell}\sum_{j\geq 1}\sum_{n\geq 1}(-1)^{n-1}\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}(n\ell j^r z)^{-s}\Gamma(s)\,ds\\&=\frac{2}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\Bigg(\sum_{\substack{\ell\geq 1\\ \ell\text{ odd}}}\frac{1}{\ell^{s+1}}\sum_{n\geq 1}\frac{(-1)^{n-1}}{n^s}\sum_{j\geq 1}\frac{1}{j^{rs}}\Bigg)\Gamma(s)z^{-s}\,ds,\end{aligned}$$

that is

$$\log G_{r,2}(e^{-z})=\frac{2}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}(1-2^{-1-s})\zeta(s+1)\eta(s)\zeta(rs)\Gamma(s)z^{-s}\,ds \tag{2.1}$$

for all $z\in\mathbb{C}$ with $\Re(z)>0$. Since the only poles of gamma function $\Gamma(s)$ are at $s=-k$ ($k\in\mathbb{Z}_{\geq 0}$), and all are simple; $\eta(s)$ is an entire function on $\mathbb{C}$; all $s=-2k$ ($k\in\mathbb{N}$) are zeros of zeta function $\zeta(s)$, and $s=1$ is the only pole of $\zeta(s)$ and is simple. Thus, it is easy to check that the only possible poles of the integrand

$$g_r(s)z^{-s}:=(1-2^{-1-s})\zeta(s+1)\eta(s)\zeta(rs)\Gamma(s)z^{-s}$$

are at $s=0$ and $1/r$. For all $\sigma\in[a,b]$, $a,b\in\mathbb{R}$ and real number t, $|t|\geq 1$, we have the well-known classical facts (see [6, p. 38, p. 92]) that

$$\Gamma(\sigma+\mathrm{i}t)\ll_{a,b}|t|^{\sigma-1/2}\exp\Big(-\frac{\pi}{2}|t|\Big)\quad\text{and}\quad\zeta(\sigma+\mathrm{i}t)\ll_{a,b}|t|^{|\sigma|+1/2}.$$

Hence we have $g_r(s)\ll_{a,b}|t|^{O(1)}\exp\big(-\frac{\pi}{2}|t|\big)$. Thus, using the residue theorem, moving the line of integration (2.1) to the $\Re(s)=-p$ with any given $p>0$, and taking into account the possible pole at $s=0$ and $s=1/r$ of $g(s)$, we obtain

$$\log G_{r,2}(e^{-z})=2\sum_{s\in\{0,1/r\}}\operatorname{Res}\big(g_r(s)z^{-s}\big)+O(|z|^p) \tag{2.2}$$

as $z\to 0$ with $|\arg(z)|<\pi/4$. By Laurent expansion of $\zeta(s+1)$ and $\Gamma(s)$ at $s=0$, we have

$$\zeta(s+1)=1/s+\gamma+O(|s|)\quad\text{and}\quad\Gamma(s)=1/s-\gamma+O(|s|)$$

as $s \to 0$. Therefore,

$$\operatorname*{Res}_{s=1/r}\left(g_r(s)z^{-s}\right) = \frac{(1-2^{-1-1/r})\zeta(1/r+1)\eta(1/r)\Gamma(1/r)}{rz^{1/r}}$$

and

$$\operatorname*{Res}_{s=0}\left(g_r(s)z^{-s}\right) = \frac{1}{8}\log\left(\frac{z}{2^r\pi^{r+1}}\right).$$

Combining (2.2) with above results, we obtain the proof of this proposition. □

We also need the following upper bound results.

Lemma 2.2. *Let* $(r,v) \in \mathbb{N} \times \{1,2\}$ *be given,* $z = x + \mathrm{i}y$ *with* $x \in \mathbb{R}_+$ *and* $y \in (-\pi,\pi] \setminus (-x,x)$*. As* $x \to 0$*,*

$$\Re\left(\log\frac{G_{r,v}(e^{-x})}{G_{r,v}(e^{-z})}\right) \gg x^{-1/r}.$$

Proof. By using Proposition 1.1 with $q \in \mathbb{C}$ and $|q| < 1$, we have

$$\begin{aligned}
\log G_{r,1}(q) &= \sum_{j\ge1}\log\left(\frac{1+q^{j^r}}{1-q^{j^r}}\right)\\
&= \sum_{j\ge1}\left(\sum_{\ell\ge1}(-1)^{\ell-1}\frac{q^{\ell j^r}}{\ell} + \sum_{\ell\ge1}\frac{q^{\ell j^r}}{\ell}\right)\\
&= \sum_{\ell\ge1}\frac{1}{\ell}\sum_{j\ge1}\left((-1)^{\ell-1}+1\right)q^{\ell j^r} = 2\sum_{\substack{\ell\ge1\\ \ell \text{ odd}}}\frac{1}{\ell}\sum_{j\ge1}q^{j^r\ell}
\end{aligned}$$

and

$$\begin{aligned}
\log G_{r,2}(q) &= \sum_{n,j\ge1}\log\left(\frac{1-(-1)^n q^{nj^r}}{1+(-1)^n q^{nj^r}}\right)\\
&= \sum_{n,j\ge1}\sum_{\ell\ge1}\frac{1}{\ell}\left(-(-1)^{n\ell}q^{nj^r\ell} + (-1)^{\ell}(-1)^{n\ell}q^{nj^r\ell}\right)\\
&= \sum_{\ell,j\ge1}\frac{(-1)^{\ell}-1}{\ell}\frac{(-q^{j^r})^{\ell}}{1-(-q^{j^r})^{\ell}} = 2\sum_{\substack{\ell\ge1\\ \ell \text{ odd}}}\frac{1}{\ell}\sum_{j\ge1}\frac{q^{j^r\ell}}{1+q^{j^r\ell}}.
\end{aligned}$$

Furthermore,

$$\Re\left(\log\frac{G_{r,1}(e^{-x})}{G_{r,1}(e^{-z})}\right) = 2\sum_{\substack{\ell\ge1\\ \ell \text{ odd}}}\frac{1}{\ell}\sum_{j\ge1}e^{-j^r\ell x}\Re\left(1-\exp\left(2\pi\mathrm{i}\ell j^r\frac{y}{2\pi}\right)\right)$$

and

$$\Re\left(\log\frac{G_{r,2}(e^{-x})}{G_{r,2}(e^{-z})}\right)=2\sum_{\substack{\ell\geq 1\\ \ell \text{ odd}}}\frac{1}{\ell}\sum_{j\geq 1}\Re\left(\frac{e^{-j^r\ell x}}{1+e^{-j^r\ell x}}-\frac{e^{-j^r\ell z}}{1+e^{-j^r\ell z}}\right)$$
$$=2\sum_{\substack{\ell\geq 1\\ \ell \text{ odd}}}\frac{1}{\ell}\sum_{j\geq 1}\frac{\tanh\left(j^r\ell\frac{x}{2}\right)}{\cosh(j^r\ell x)+\cos(j^r\ell y)}\sin^2\left(j^r\ell\frac{y}{2}\right).$$

By noting that all summand in above sums are nonnegative we have

$$\Re\left(\log\frac{G_{r,1}(e^{-x})}{G_{r,1}(e^{-z})}\right)\geq 2\sum_{j\geq 1}e^{-j^r x}\Re\left(1-\exp\left(2\pi \mathrm{i} j^r\frac{y}{2\pi}\right)\right)$$
$$\gg\sum_{(2\pi/x)^{1/r}<j\leq 2(2\pi/x)^{1/r}}\Re\left(1-\exp\left(2\pi \mathrm{i} j^r\frac{y}{2\pi}\right)\right)$$

and

$$\Re\left(\log\frac{G_{r,2}(e^{-x})}{G_{r,2}(e^{-z})}\right)\geq 2\sum_{j\geq 1}\frac{\tanh\left(j^r\frac{x}{2}\right)}{\cosh(j^r x)+\cos(j^r y)}\sin^2\left(j^r\frac{y}{2}\right)$$
$$\gg\sum_{(2\pi/x)^{1/r}<j\leq 2(2\pi/x)^{1/r}}\Re\left(1-\exp\left(2\pi \mathrm{i} j^r\frac{y}{2\pi}\right)\right).$$

Thus by using Lemma 2.3 with $L=(2\pi/x)^{1/r}$, we find that

$$\Re\left(\log\frac{G_{r,v}(e^{-x})}{G_{r,v}(e^{-z})}\right)\gg\delta_r(2\pi/x)^{1/r}\gg x^{-1/r}$$

holds for all sufficiently small $x>0$ and $v\in\{1,2\}$. This finishes the proof. □

Lemma 2.3. *Let $r\in\mathbb{N}$, $y\in\mathbb{R}$ and $L\in\mathbb{R}_+$ such that $L^{-r}<|y|\leq 1/2$. Then there exists a constant $\delta_r\in(0,1)$ depending only on r such that*

$$\left|\sum_{L<n\leq 2L}e^{2\pi \mathrm{i} n^r y}\right|\leq(1-\delta_r)L$$

holds for all positive sufficiently large L.

Proof. The lemma for $r=1$ is easy and we shall focus on the cases of $r\geq 2$. By the well-known Dirichlet's approximation theorem, for any $y\in\mathbb{R}$ and $L>0$ being sufficiently large, then there exist integers d and h with $0<h\leq L^{r-1}$ and $\gcd(h,d)=1$ such that

$$\left|y-\frac{d}{h}\right|<\frac{1}{hL^{r-1}}. \tag{2.3}$$

The use of [4, Equation 20.32] implies that

$$\sum_{L<n\le 2L} e^{2\pi \mathrm{i} n^r y} = \frac{1}{h}\sum_{1\le j\le h} e^{2\pi \mathrm{i} j^r \frac{d}{h}} \int_L^{2L} e^{2\pi \mathrm{i} u^r \left(y-\frac{d}{h}\right)}\, du + O(h). \tag{2.4}$$

If the real number y satisfies $L^{-r} < |y| \le L^{1-r}$, then y satisfies the approximation (2.3) with $(h,d) = (1,0)$. This means that

$$\begin{aligned} \left|\sum_{L<n\le 2L} e^{2\pi \mathrm{i} n^r y}\right| &= \left|\int_L^{2L} e^{2\pi \mathrm{i} u^r y}\, du + O(1)\right| \\ &\le 2\cdot \frac{1}{2\pi r|y|L^{r-1}}(1+2^{1-r}) + O(1) \le \frac{1+2^{1-r}}{\pi r} L + O(1). \end{aligned} \tag{2.5}$$

If the real number y satisfies $1/2 \ge |y| \ge L^{1-r}$, then y satisfies the approximation (2.3) with $h \ge 2$. Further, by using [2, Lemma 2.1] in (2.4), we find that there exists a positive constant δ_{r1} depending only on r such that

$$\left|\sum_{L<n\le 2L} e^{2\pi \mathrm{i} n^r y}\right| \le (1-\delta_{r1})L + O(h). \tag{2.6}$$

On the other hand, the use of Weyl's inequality (see [4, Lemma 20.3]) implies that

$$\sum_{L<j\le 2L} e^{2\pi \mathrm{i} j^r y} \ll_\varepsilon L^{1+\varepsilon}(h^{-1} + L^{-1} + hL^{-r})^{2^{1-r}} \ll L^{1-2^{-r-1/2}} \tag{2.7}$$

holds for all integers $h \in (L^{1/2}, L^{r-1}]$. By using (2.5), (2.6) and (2.7), it is not difficult to obtain the proof of the lemma. □

3. Proof of the main theorem

From Proposition 2.1 and Lemma 2.2, we can check that the sequences $\{s_{r,1}(n)\}_{n\ge 0}$ and $\{s_{r,2}(n)\}_{n\ge 0}$ satisfy the conditions of Proposition 3.1 below. Therefore, applying the following proposition, Theorem 1.3 and Corollary 1.4 follow.

Proposition 3.1. *For a sequence $\{c_n\}_{n\ge 0}$ of real numbers, we let $G(q) := \sum_{n\ge 0} c_n q^n$. Suppose that for $x \in \mathbb{R}_+$ and $y \in (-\pi, \pi]$,*

$$G(e^{-x-\mathrm{i}y}) - \gamma(x+\mathrm{i}y)^\beta e^{\kappa\alpha^{-1}(x+\mathrm{i}y)^{-\alpha}} \ll x^p G(e^{-x}), \quad x \to 0$$

holds for any given $p > 0$, where $\kappa, \gamma, \beta, \alpha \in \mathbb{R}_+$. Then, for any given $p > 0$ we have

$$c_n = \gamma \left(\frac{\kappa}{n}\right)^{\frac{1+\beta}{1+\alpha}} W_{\alpha,\beta}\big(\kappa^{\frac{1}{1+\alpha}} n^{\frac{\alpha}{1+\alpha}}\big)\big(1 + O(n^{-p})\big)$$

as integer $n \to \infty$. In particular,

$$c_n \sim 2^{-1/2}\pi^{-1/2}(1+\alpha)^{-1/2}\gamma\kappa^{\frac{\beta+1/2}{1+\alpha}} n^{-\frac{1+\beta+\alpha/2}{1+\alpha}} e^{(1+\alpha^{-1})\kappa^{\frac{1}{1+\alpha}} n^{\frac{\alpha}{1+\alpha}}}, \quad n \to \infty.$$

Proof. For any given positive integer n sufficiently large, by using the orthogonality we have

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{-x-\mathrm{i}y})e^{nx+niy}\,dy.$$

We split the above integral as

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-x}^{x} \gamma(x+\mathrm{i}y)^{\beta} e^{\kappa\alpha^{-1}(x+\mathrm{i}y)^{-\alpha}+n(x+\mathrm{i}y)}\,dy \\ &\quad + \frac{1}{2\pi}\int_{-\pi}^{\pi}\big(G(e^{-x-\mathrm{i}y}) - \gamma(x+\mathrm{i}y)^{\beta}e^{\kappa\alpha^{-1}(x+\mathrm{i}y)^{-\alpha}}\big)e^{n(x+\mathrm{i}y)}\,dy \\ &=: I(n) + E(n). \end{aligned} \tag{3.1}$$

Let $x = \left(\frac{\kappa}{n}\right)^{\frac{1}{\alpha+1}}$. For $E(n)$, we estimate that

$$\begin{aligned} E(n) &\ll \int_{-\pi}^{\pi} x^{(1+\alpha)p} G(e^{-x})e^{nx}\,dy \\ &\ll \int_{-\pi}^{\pi} x^{p} e^{\kappa\alpha^{-1}x^{-\alpha}+nx}\,dy \ll n^{-p}e^{(\alpha^{-1}+1)\kappa^{\frac{1}{1+\alpha}}n^{\frac{\alpha}{1+\alpha}}} \end{aligned} \tag{3.2}$$

holds for any given $p > 0$. For $I(n)$, we compute that

$$\begin{aligned} I(n) &= \frac{\gamma}{2\pi\mathrm{i}}\int_{x-\mathrm{i}x}^{x+\mathrm{i}x} z^{\beta}e^{\kappa\alpha^{-1}z^{-\alpha}+nz}\,dz \\ &= \frac{\gamma x^{1+\beta}}{2\pi\mathrm{i}}\int_{1-\mathrm{i}}^{1+\mathrm{i}} u^{\beta}e^{\kappa\alpha^{-1}x^{-\alpha}u^{-\alpha}+nxu}\,du \\ &= \gamma\left(\frac{\kappa}{n}\right)^{\frac{1+\beta}{1+\alpha}}\frac{1}{2\pi\mathrm{i}}\int_{1-\mathrm{i}}^{1+\mathrm{i}} u^{\beta}e^{\kappa^{\frac{1}{1+\alpha}}n^{\frac{\alpha}{1+\alpha}}(\alpha^{-1}u^{-\alpha}+u)}\,du, \end{aligned}$$

that is

$$I(n) = \gamma\left(\frac{\kappa}{n}\right)^{\frac{1+\beta}{1+\alpha}} W_{\alpha,\beta}\big(\kappa^{\frac{1}{1+\alpha}}n^{\frac{\alpha}{1+\alpha}}\big). \tag{3.3}$$

By using the standard Laplace saddle-point method (see, for example, [5, p. 127, Theorem 7.1]), since the integral

$$W_{\alpha,\beta}(\lambda) = \frac{1}{2\pi}\int_{-1}^{1}(1+\mathrm{i}u)^{\beta}\exp\big(\lambda(\alpha^{-1}(1+\mathrm{i}u)^{-\alpha} + (1+\mathrm{i}u))\big)\,du$$

has a simple saddle point $u = 0$, it is not difficult to prove that

$$W_{\alpha,\beta}(\lambda) \sim \frac{1}{\sqrt{2\pi(1+\alpha)}}\frac{e^{(1+\alpha^{-1})\lambda}}{\lambda^{1/2}} \tag{3.4}$$

as $\lambda \to +\infty$. The proof of Proposition 3.1 follows from (3.1)–(3.3) and (3.4). This completes the proof. □

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