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# Degree Bipartite Ramsey Numbers

Ye Wang, Yusheng Li and Yan Li*

Abstract. Let $H \overset{s}{\to} G$ denote that any edge-coloring of $H$ by $s$ colors contains a monochromatic $G$. The degree Ramsey number $r_\Delta(G;s)$ is defined to be $\min\{\Delta(H) : H \overset{s}{\to} G\}$, and the degree bipartite Ramsey number $br_\Delta(G;s)$ is defined to be min $\{\Delta(H) : H \overset{s}{\to} G \text{ and } \chi(H) = 2\}$. In this note, we show that $r_\Delta(K_{m,n};s)$ is linear on $n$ with fixed $m$. We also evaluate $br_\Delta(G;s)$ for paths and other trees.

## 1. Introduction

Ramsey theory is a fascinating branch of combinatorics. There are many difficult open problems in this area. Ramsey theory can be viewed as a generalization of the Pigeonhole Principle. A typical result in Ramsey theory states that if some mathematical object is partitioned into finite many parts, then one of the parts must contain a sub-object of particular property. The smallest size of the large object to guarantee the property is called Ramsey number. For more, see Graham, Rothschild and Spencer [11].

For graphs $G$ and $H$, let $H \overset{s}{\to} G$ denote that any edge-coloring of $H$ by $s$ colors contains a monochromatic $G$. The Ramsey number $r(G;s)$ is the smallest $N$ such that $K_N \overset{s}{\to} G$. More generally, for any monotone graph parameter $\rho$, the $\rho$-Ramsey number is defined as

$$r_\rho(G;s) = \min\{\rho(H) : H \overset{s}{\to} G\}.$$

This generalizes the Ramsey number since $r_\rho(G;s) = r(G;s)$ if $\rho(H)$ denotes the order of $H$. When $\rho(H)$ denotes the size of $H$, it becomes the size Ramsey number $\widehat{r}(G;s)$, see [2,3,6,7,10,18]. For cases $\rho(H)$ of being the clique number and the chromatic number of $H$, we refer the reader to [9,16,17] and [5,21,22], respectively.

The degree Ramsey number is defined as

$$r_\Delta(G;s) = \min\{\Delta(H) : H \overset{s}{\to} G\},$$

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*Corresponding author.

where $\Delta(H)$ is the maximum degree of $H$. Kinnersley, Milans and West [14], and Jiang, Milans and West [12] obtained bounds for degree Ramsey numbers of trees and cycles. Kang and Perarnau [13] proved that $r_\Delta(C_4; s) = \Theta(s^2)$, and Tait [19] proved that $r_\Delta(C_6; s) = \Theta(s^{3/2})$ and $r_\Delta(C_{10}; s) = \Theta(s^{5/4})$.

In this note, we define the degree bipartite Ramsey number $br_\Delta(G; s)$ as

$$br_\Delta(G; s) = \min\{\Delta(H) : H \text{ is bipartite and } H \xrightarrow{s} G\}.$$

Obviously, for any bipartite graph $G$, we have

$$r_\Delta(G; s) \le br_\Delta(G; s). \tag{1.1}$$

Note that (1.1) holds with equality for trees in Theorem 1.1. Now we consider the degree bipartite Ramsey numbers of trees, including stars, paths, and complete bipartite graphs. We show that $r_\Delta(K_{m,n}; s)$ is linear on $n$ with $m$ fixed.

**Theorem 1.1.** *If $T$ is a tree in which one vertex has degree $k$ and all others have degree at most $\lceil k/2 \rceil$, then*

$$br_\Delta(T; s) = s(k-1) + 1.$$

Kinnersley, Milans and West [14] showed $r_\Delta(T; s) = s(k-1)+1$ for any tree $T$ satisfying conditions in Theorem 1.1 with odd $k$, and thus the inequality in (1.1) is sharp. They also proved

$$r_\Delta(T; s) \le 2s(\Delta(T) - 1) \tag{1.2}$$

for any tree $T$. We shall generalize (1.2) to the bipartite version. Hence if (1.2) holds with equality, then the inequality in Theorem 1.2 becomes an equality from (1.1).

**Theorem 1.2.** *If $T$ is a tree, then*

$$br_\Delta(T; s) \le 2s(\Delta(T) - 1).$$

The above bound is sharp since Alon, Ding, Oporowski and Vertigan [1] showed that $r_\Delta(P_n; s) = 2s$ for fixed $s$ and large $n$, where $P_n$ is a path on $n$ vertices.

Let us have more notation. The Turán number of a graph $G$, denoted by $ex(N; G)$, is the maximum number of edges in a graph of order $N$ that contains no $G$. For a bipartite graph $G$, the Zarankiewicz number $z(N; G)$ is defined [20] to be the maximum number of edges in a subgraph of $K_{N,N}$ that contains no $G$. For two positive functions $f(t)$ and $g(t)$, we write that $f(t) \le O(g(t))$ or $g(t) \ge \Omega(f(t))$ if there exists a positive constant $c$ so that $f(t) \le cg(t)$ for large $t$, and $f(t) = \Theta(g(t))$ if $\Omega(g(t)) \le f(t) \le O(g(t))$.

We now turn to even cycles and complete bipartite graphs. The results $r_\Delta(C_{2m}; s) = \Theta(s^{1+1/(m-1)})$ in [13,19] and the upper bound in (1.1) imply $br_\Delta(C_{2m}; s) \ge \Omega(s^{1+1/(m-1)})$

for cycles $C_{2m}$ with $m = 2, 3, 5$ and $s \to \infty$. It is well known that $ex(2n; C_{2m}) \le O(n^{1+1/m})$ shown by Bondy and Simonovits [4] for fixed $m$ and $n \to \infty$. On the other hand, if the edges of $K_{n,n}$ are colored by $s$ colors, then at least $n^2/s$ edges are monochromatic. Therefore, if $n^2/s \ge \Omega(n^{1+1/m})$, equivalently $n \ge \Omega(s^{1+1/(m-1)})$, then $n^2/s \ge ex(2n; C_{2m})$ and $K_{n,n} \xrightarrow{s} C_{2m}$, hence $br_\Delta(C_{2m}; s) \le O(s^{1+1/(m-1)})$ for $m \ge 2$ and $s \to \infty$. Combining with the lower bound as mentioned, we have

$$br_\Delta(C_{2m}; s) = \Theta(s^{1+1/(m-1)})$$

for $m = 2, 3, 5$ and $s \to \infty$.

The following result differs from the result in [19] which pointed out $r_\Delta(K_{m,n}; s) = \Theta(s^m)$ for fixed $m$ and $n$ with $n > (m-1)!$ and $s \to \infty$.

**Theorem 1.3.** *If $m$ and $s$ are fixed integers and $\epsilon > 0$, then*

$$e^{-2} s^{(mn-1)/(m+n)} n \le r_\Delta(K_{m,n}; s) \le br_\Delta(K_{m,n}; s) \le (1+\epsilon) s^m n$$

*for all large $n$.*

## 2. Proofs of main results

**Lemma 2.1.** *For any integers $n, s \ge 2$, $br_\Delta(K_{1,n}; s) = s(n-1)+1$.*

*Proof.* Since $K_{1,s(n-1)+1} \xrightarrow{s} K_{1,n}$, we have $br_\Delta(K_{1,n}; s) \le s(n-1)+1$. For the lower bound, for any bipartite graph $H$ with maximum degree $s(n-1)$, let $H'$ be an $s(n-1)$-regular bipartite supergraph of $H$. By Hall's Theorem, $H'$ decomposes into 1-factors. Taking each of $s$ color classes to be the union of $n-1$ of these 1-factors yields an edge-coloring of $H'$ by $s$ colors with degree $n-1$ in each color at each vertex. $\square$

A classic result of Erdős and Sachs [8] for the existence of regular graphs with large degree and girth can be modified to yield a bipartite form easily.

**Lemma 2.2.** (see Erdős and Sachs [8]) *For any positive integers $g$ and $k$, there is a $k$-regular bipartite graph with girth at least $g$.*

*Proof of Theorem* 1.1. For the lower bound, for any tree $T$ in which one vertex has degree $k$, it is obtained that $K_{1,k} \subseteq T$ and $br_\Delta(K_{1,k}; s) \le br_\Delta(T; s)$, so $br_\Delta(T; s) \ge s(k-1)+1$ by Lemma 2.1.

For the upper bound, by Lemma 2.2, let $H$ be a regular bipartite graph having degree $s(k-1)+1$ and girth more than $|V(T)|$. In any edge-coloring of $H$ by $s$ colors, by the pigeonhole principle, some color class has average degree more than $k-1$, which yields a monochromatic bipartite subgraph $H_1$ with average degree more than $k-1$. If $H_1$ has a subgraph with a vertex $u$ of degree at most $r-1$ with $r = \lceil k/2 \rceil$, as $k-1 \geq 2(r-1)$, graph $H_1 \setminus \{u\}$ has average degree more than $k-1$. Thus there must be a subgraph $H_2$ in $H$ with minimum degree at least $r$ and average degree more than $k-1$. Then $H_2$ also has a vertex of degree at least $k$, denoted by $v$. In such a graph $H_2$, we can "grow" $T$ from $v$ by adding children. When we want to grow from a current leaf, it has $r-1$ neighbors in $H_2$ that (by the girth condition) are not in the tree yet, and then we get the desired tree $T$, finishing the proof. □

The following lemma is a well known fact, and we shall use it to prove Theorem 1.2. Here we sketch the proof. For graph $H$ with average degree $d > 0$, when we delete the vertices of degrees less than $d/2$ repeatedly if any, then the resulting graphs have non-decreasing average degrees and minimum degrees.

**Lemma 2.3.** *For positive integers $\delta$ and $d$ with $d \geq 2(\delta - 1)$, if graph $H$ has average degree at least $d$, then $H$ contains a subgraph with minimum degree at least $\delta$ and average degree at least $d$.*

*Proof of Theorem* 1.2. Let $r = \Delta(T)$. And we can construct a $2s(r-1)$-regular bipartite graph $H$ with girth more than $|V(T)|$ which is known to be possible in various ways. See for instance [8] for constructing a $2s(r-1)$-regular graph $G$ with girth more than $|V(T)|$, then the direct product $H = G \times K_2$ is a $2s(r-1)$-regular bipartite graph with girth $g(H) \geq g(G) \geq |V(T)|$.

Consider an edge-coloring of bipartite graph $H$ by $s$ colors, then by the pigeonhole principle, some color class has average degree at least $2(r-1)$, which yields a monochromatic bipartite subgraph $H_1$ with average degree at least $2(r-1)$. By Lemma 2.3, $H_1$ contains a subgraph $H_2$ with minimum degree at least $r$. First we choose a vertex from $V(H_2)$ as the root of tree and then "grow" $T$ from this vertex by adding children. When we want to grow from the current leaf, it has $r-1$ neighbors in $H_2$ that (by the girth condition) are not in the tree yet. Thus, we have the desired monochromatic tree $T$. □

The following lemma appeared in [13] firstly, and then it was restated by Tait [19] in a more general way. Before stating it, we need some notations. For $v \in V(G)$, denote by $N_G(v)$ the set of all neighbors of $v$ in $G$. For graphs $G$ and $H$, a homomorphism $\phi$ from $G$ to $H$ is an edge preserving mapping from $V(G)$ to $V(H)$. A homomorphism from $G$ to $H$ is locally injective if $N_G(v)$ is mapped to $N_H(\phi(v))$ injectively for every $v \in V(G)$. A graph is $L_G$-free if it does not contain any graph in $L_G$ as a subgraph, where $L_G$ is the set of all graphs $H$ such that there is a locally injective homomorphism from $G$ to $H$.

To avoid confusion, let us clarify a decomposition of a graph $G$ means a partition of the edge set of $G$ in this note.

**Lemma 2.4.** [13,19] *Let $G$ be a graph with at least one cycle and $H$ a graph of maximum degree $\Delta$. If $K_N$ can be decomposed into $O(N^{1-\xi})$ $L_G$-free graphs for fixed $\xi > 0$, then $H$ can be decomposed into $O(\Delta^{1-\xi})$ graphs which are $G$-free.*

*Proof of Theorem* 1.3. For the lower bound, we shall show that $r_\Delta(K_{m,n}; s) \geq e^{-2} s^{(mn-1)/(m+n)} n$, which is equivalent to showing that any graph of maximum degree $\Delta$ can be decomposed into $(e^2\Delta/n)^{(m+n)/(mn-1)}$ graphs which are $K_{m,n}$-free. By taking $\xi = 1 - (m+n)/(mn-1)$ and $N = e^{-2}s^{(mn-1)/(m+n)}n$, we have

$$(e^2\Delta/n)^{(m+n)/(mn-1)} = O(\Delta^{1-\xi}).$$

By Lemma 2.4, it suffices to show that $K_N$ can be decomposed into $(e^2N/n)^{(m+n)/(mn-1)} = O(N^{1-\xi})$ graphs which are $K_{m,n}$-free. Let us consider a random edge-coloring of $K_N$ by $s$ colors such that each edge is colored independently with probability $1/s$. Let $p$ be the probability that there is a monochromatic $K_{m,n}$. Then $p \leq s\binom{N}{m+n}\binom{m+n}{m}/s^{mn}$. For $s = (e^2N/n)^{(m+n)/(mn-1)}$, we have

$$\begin{aligned} p &\leq \left(\frac{eN}{m+n}\right)^{m+n}\left(\frac{e(m+n)}{m}\right)^m\left(\frac{1}{s}\right)^{mn-1} = \frac{e^{2m+n}N^{m+n}}{m^m(m+n)^n s^{mn-1}} \\ &= \frac{n^{m+n}}{m^m e^n (m+n)^n} = \frac{n^m}{m^m e^n}\left(1 - \frac{m}{m+n}\right)^n \leq \frac{n^m}{m^m e^n} e^{-mn/(m+n)} \\ &\leq \frac{n^m}{(\sqrt{e}m)^m e^n}, \end{aligned}$$

which implies $p < 1$ for finitely many $n$. Hence $K_N \overset{s}{\not\to} K_{m,n}$ and the desired lower bound follows from Lemma 2.4.

For the upper bound, let $M = br_\Delta(K_{m,n}; s) - 1$. Then there is an edge-coloring of $K_{M,M}$ by $s$ colors that contains no monochromatic $K_{m,n}$. A well known argument of Kövári, Sós and Turán [15] shows that the Zarankiewicz number

$$z(M; K_{m,n}) \leq (n-1)^{1/m} M^{2-1/m} + (m-1)M.$$

Let $c = (1+\epsilon)^{1/m} - 1 > 0$. Then we have

$$M^2 \leq s\, z(M; K_{m,n}) \leq (1+c)sn^{1/m}M^{2-1/m},$$

which yields $M \leq (1+\epsilon)s^m n$. $\square$

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Ye Wang
College of Mathematical Sciences, Harbin Engineering University, Harbin 150001, China
*E-mail address*: `wima.wy@gmail.com`

Yusheng Li
School of Mathematical Sciences, Tongji University, Shanghai 200092, China
*E-mail address*: `li_yusheng@tongji.edu.cn`

Yan Li
School of Mathematical Sciences, Tongji University, Shanghai 200092, China
and
College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China
*E-mail address*: `1610521@tongji.edu.cn`