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A Variational Approach to the Problem of Continuous Dependence of Solutions to Second Order Periodic System

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Abstract. In the paper a second order differential system with a periodic boundary data is considered. Using some variational methods sufficient conditions for the continuous dependence of trajectories on controls are proved. The obtained results are applied then to the optimal control problem governed by the above system and the cost functional of a Bolza-type. In the end of the paper, an example of periodic optimal control problem demonstrating the applicability of the results is presented.

1. Introduction

In this paper we investigate the optimal control problem governed by the second order differential equation

$$\ddot{x}(t) = f^1(t, x(t))u(t) + f^2(t, x(t)) \tag{1.1}$$

with periodic boundary conditions

$$x(0) = x(T) \quad \text{and} \quad \dot{x}(0) = \dot{x}(T), \tag{1.2}$$

where $t \in [0, T]$, $T > 0$, $f^1\colon [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $f^2\colon [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $u\colon [0, T] \to M \subset \mathbb{R}^m$ is an admissible control. We assume that the matrix field f^1 and the vector field f^2 are potential, i.e., there exist a vector function $F^1 = F^1(t, x)$ and a scalar function $F^2 = F^2(t, x)$ such that $F^1_x = f^1$ and $F^2_x = f^2$. Any solution to system (1.1)–(1.2) can be extended to the periodic function defined for all $t \in \mathbb{R}$.

The main results of the paper are theorems on the existence and continuous dependence on the functional parameter u of solutions to (1.1)–(1.2). In addition, we consider a Bolza-type optimal control problem (see e.g., [7]) by using the above properties, optimal control problem associated to (1.1)–(1.2) and the following cost functional

$$J(x, u) = \int_0^T f(t, x(t), \dot{x}(t), u(t))\, dt + l(x(T)), \tag{1.3}$$

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where f and l are given scalar functions. The cost functional consists of the "integral part" and the endpoint cost function.

We use a variational approach to investigate the existence and continuous dependence problem. To be more specific, on the space H_T^1 we define a functional of action for system (1.1)–(1.2) whose critical points are, by the well-known Fundamental Lemma (see [20]), Carathéodory solutions to (1.1)–(1.2). The existence of the above critical points is obtained by the classical minimization of the functional of action or by the Saddle Point Theorem, depending on assumptions (see Theorems 3.2 and 3.4). The continuous dependence problem, due to the fact that critical point is not unique in both cases, is described with the aid of properties of some multifunction or the notion of the upper limit in the Kuratowski–Painlevé sense (see Definition 2.4). Finally, the existence problem of optimal solutions for optimal control for problem (1.1)–(1.3) is investigated in Section 4. Using Theorem 3.2 and the well-known sufficient condition for the lower semicontinuity of integral functionals, we prove the existence of optimal processes for system (1.1)–(1.3) (see Theorem 4.1). We also add an illustrative example at the end of this section.

Periodic systems are important due to their numerous applications and physical interpretations. The most famous periodic phenomenon is the movement of the pendulum or—more generally—all kinds of oscillators. Probably the first paper applying variational methods to study periodic systems (more precisely, the forced pendulum system) was published by Hamel [12] in 1922. The existence problem of solutions for nonlinear periodic systems using variational approach was considered in many papers and monographs (see [20, 21, 25, 28], and references therein). An excellent review of variational methods for periodic systems, with interesting historical comments, can be found in the well-known book of Mawhin and Willem [20]. In fact, a variational approach presented in our paper is based on this book.

It is difficult to overestimate the importance of continuous dependence on differential problems. This is again due to many applications, interpretations and potential implications for numerical solutions to differential problems. In the classic book [8] of Courant and Hilbert, we read: *"A mathematical problem which is to correspond to physical reality should satisfy the following basic requirements:*

(1) *The solution must exist.*

(2) *The solution should be uniquely determined.*

(3) *The solution should depend continuously on the data.*

(...) The third requirement, particularly incisive, is necessary if the mathematical formulation is to describe observable natural phenomena. Data in nature cannot possibly be conceived as rigidly fixed; the mere process of measuring them involves small errors."

Certainly, the continuous dependence problem of solutions to differential problems is less studied than the existence problem of solutions. It has been investigated for both periodic and non-periodic systems among others in [3, 4, 9, 11, 13, 14, 16, 18, 26]. However, if we compare it to our result, either a different type of equation or a different method is used. To be more specific, in [11] the authors investigate a quasilinear Hamiltonian system. They approximate a quasilinear differential equation by a sequence of semilinear differential equations. Finally, the existence problem of solutions is solved using the mountain pass theorem applied to an approximated functional. In [16] the author studies the continuity of the dependence of a so-called Bohr almost periodic solution of the nonlinear control system. Our paper concerns periodic boundary problem, however, the continuous dependence problem was also studied for problems with other boundary conditions: Dirichlet [29], Neumann [17, 23, 27], or Robin [19].

It has to be noted that system (1.1)–(1.2) has a variational form—it is an Euler–Lagrange equation for a specific functional of action. This justifies the application of variational methods to study it. Moreover, this approach seems to be appropriate for the continuous dependence problem with the lack of the uniqueness of solutions. The set-valued analysis tools (among others the notion of the Kuratowski–Painlevé limit) allow for a clear study of this issue. This idea was used first by Walczak [29] and was continued by his students and us. Similar methods were used in papers [3, 4]. However, in [4] the considered system is of the fourth order, while in [3] only the classical variational method was used with more restrictive assumptions. Some improvement concerning the nature of the minimizing sequence which can be strongly convergent in most cases is given in [10]. For a comprehensive introduction to nonlinear analysis with an up to date approach, we refer to [24].

The results presented in [20] concern the problem without parameter

$$\ddot{x}(t) = \nabla F(t, x(t)), \quad x(0) = x(T) \quad \text{and} \quad \dot{x}(0) = \dot{x}(T),$$

where

$$|\nabla F(t, x(t))| \leq g(t) \tag{1.4}$$

with $g \in L^1$. In paper [28], the author has weakened assumption (1.4) assuming that

$$|\nabla F(t, x(t))| \leq f(t)|x|^{\alpha} + g(t)$$

with $f, g \in L^1$ and $\alpha \in [0, 1)$. Following this type of assumption, we consider a periodic system with a functional parameter u. In fact, the existence of solutions to (1.1)–(1.2) for a given u can be deduced from [28]. As far as we know, the result concerning the continuous dependence for the second order periodic systems via variational approach is new.

2. Formulation of the problem

Let H^1_T be the space of absolutely continuous functions $x\colon [0,T] \to \mathbb{R}^n$ such that $\dot{x} \in L^2([0,T],\mathbb{R}^n)$ and $x(0)=x(T)$, where $T>0$. It is clear that H^1_T is a Hilbert space with the inner product

$$\langle x,y\rangle_{H^1_T} = \int_0^T \big(\langle x(t),y(t)\rangle_{\mathbb{R}^n} + \langle \dot{x}(t),\dot{y}(t)\rangle_{\mathbb{R}^n}\big)\,dt$$

and the corresponding norm

$$\|x\|_{H^1_T} = \bigg(\int_0^T |x(t)|^2\,dt + \int_0^T |\dot{x}(t)|^2\,dt\bigg)^{1/2} = \big(\|x\|^2_{L^2} + \|\dot{x}\|^2_{L^2}\big)^{1/2},$$

where $\langle\,\cdot\,,\cdot\,\rangle_{\mathbb{R}^n}$ denotes the classical scalar product in $\mathbb{R}^n$. In what follows, we shall denote by $\langle\,\cdot\,,\cdot\,\rangle$ the dual pair between $(H^1_T)^*$ and H^1_T.

It is easy to check that the norm

$$\|x\| = |\overline{x}| + \|\dot{x}\|_{L^2}, \quad \text{where } \overline{x} = \frac{1}{T}\int_0^T x(s)\,ds \tag{2.1}$$

is equivalent to $\|\cdot\|_{H^1_T}$. In our considerations we shall use norm (2.1) in H^1_T.

Let $M \subset \mathbb{R}^m$ be a given convex and compact subset of $\mathbb{R}^m$ and let $p \in [1,\infty]$. Define the set

$$\mathcal{U} = \big\{u \in L^p([0,T],\mathbb{R}^m) : u(t) \in M \text{ for all } t \in [0,T] \text{ a.e.}\big\}.$$

The set $\mathcal{U}$ will be referred to as the set of admissible controls.

We consider a control system described by the second order differential equations with periodic boundary data

$$\ddot{x}(t) = \big(F^1_x(t,x(t))\big)^T u(t) + F^2_x(t,x(t)), \quad t \in [0,T] \text{ a.e.}, \tag{2.2}$$

$$x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T), \tag{2.3}$$

where $u \in \mathcal{U}$, $x \in H^1_T$, $F^1\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}^m$ and $F^2\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}$.

It is easy to see that the functional of action for equation (2.2) is of the form

$$\varphi_u(x) = \int_0^T \bigg(\frac{1}{2}|\dot{x}(t)|^2 + \langle F^1(t,x(t)),u(t)\rangle_{\mathbb{R}^m} + F^2(t,x(t))\bigg)\,dt. \tag{2.4}$$

We will make the following assumptions:

(A1) $F^1(t,x)$ and $F^2(t,x)$ are measurable with respect to $t \in [0,T]$ for any $x \in \mathbb{R}^n$ and continuously differentiable in x for $t \in [0,T]$ a.e., (A2) there are functions $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1([0,T], \mathbb{R}^+)$ such that

$$|F^i(t,x)| \leq a(|x|)b(t)$$

for all $x \in \mathbb{R}^n$, $t \in [0,T]$ a.e. and $i = 1, 2$,

(A3) there exist a function $g \in L^1([0,T], \mathbb{R}^+)$ and a number $\alpha \in [0,1)$ such that

$$|F^i_x(t,x)| \leq g(t)(|x|^\alpha + 1)$$

for all $x \in \mathbb{R}^n$, $i = 1, 2$. Here we consider such a norm for the matrix $F^1_x(t,x)$ that $|(F^1_x(t,x))^T u| \leq |F^1_x(t,x)| \cdot |u|$ (e.g., Euclidean norm $|[a_{ij}]| = \sum_{i,j} |a_{ij}|$).

Remark 2.1. Assumptions (A1) and (A2) are of a technical nature and are typical for variational approach for such a type of problems (see [20, Sections 1.4 and 4.3]). Generally speaking, Assumptions (A1)–(A3) allow to define the functional of action (2.4) and ensure its basic properties related to differentiability. As already mentioned, (A3) means that problem (2.2)–(2.3) is sublinear and is a weakening of the assumption considered in book [20].

Remark 2.2 (On physical interpretation of the problem). Equation (2.2) can be interpreted physically as a model of motion of a point in a potential force field. In our case one of the field components can be controlled in a linear way. The assumptions adopted in the paper represent the sublinear (with respect to the point location) character of the force. Considering problem (2.2)–(2.3), we are looking for periodic orbits of a point. The functional of action is then the integral of the difference between kinetic and potential energy, i.e., the classical Lagrange functional. A classical Hamilton's principle of least action says that motion of the mechanical system (2.2) coincides with extremal of Lagrange functional. In fact, the above principle is the physical justification of the direct variational methods on which the approach presented in the paper is based.

A special case of system (2.2) is an autonomous system with one degree of freedom (i.e., when $n = 1$ and F^i does not depend on t). In this case, there is a clear method of analyzing solutions by drawing phase curves (see [1]). However, it is rather difficult to compare this method with the results obtained in our paper.

Proposition 2.3. *Assume* (A1)–(A3) *hold. Then for any $u \in \mathcal{U}$, functional φ_u defined by* (2.4) *is of class $C^1(H^1_T, \mathbb{R})$. Moreover, for any $u \in \mathcal{U}$ and $x, h \in H^1_T$, we have*

$$\langle \varphi'_u(x), h \rangle = \int_0^T \big(\langle \dot{x}(t), \dot{h}(t) \rangle_{\mathbb{R}^n} + \langle F^1_x(t, x(t))h(t), u(t) \rangle_{\mathbb{R}^m} + \langle F^2_x(t, x(t)), h(t) \rangle_{\mathbb{R}^n} \big)\, dt. \tag{2.5}$$

For the convenience of the reader, we provide several definitions of notions used in the paper, in particular for the concept of the upper limit of sets in the Kuratowski–Painlevé sense.

Definition 2.4. We say that the sequence of sets $Z_k \subset H^1_T$ tends to Z_0 in H^1_T if any sequence $(x_k)_k$, $x_k \in Z_k$, $k \in \mathbb{N}$, possesses cluster points (in the sense of the norm topology of H^1_T) in the set Z_0 only, i.e., $\operatorname{LimSup} Z_k \subset Z_0$, where the symbol $\operatorname{LimSup} Z_k$ denotes the set of all cluster points of the sequence (x_k), $x_k \in Z_k$, $k \in \mathbb{N}$. The set $\operatorname{LimSup} Z_k$ is referred to as the upper limit of the sequence of sets in the Painlevé–Kuratowski sense (see [2]). In the case when Z_k are singletons, $Z_k = \{x_k\}$, $k \in \mathbb{N}_0$, then the convergence of sets is identical with the strong convergence of points in H^1_T.

Definition 2.5. We say that the functional φ_u is uniformly coercive with respect to $u \in \mathcal{U}$ if for each $K > 0$, there exists an $R > 0$ such that $\varphi_u(x) > K$ for all $u \in \mathcal{U}$ and $|x| > R$.

Definition 2.6. A sequence $(x_k)_k$ in a Banach space X is called a Palais–Smale sequence ((PS) sequence) for functional $\varphi\colon X \to \mathbb{R}$ if there is a constant $R > 0$ such that $\|\varphi(x_k)\| \leq R$ and $\varphi'(x_k) \to 0$ as $k \to \infty$. We say that φ satisfies Palais–Smale condition ((PS) condition) if any (PS) sequence admits a convergent subsequence.

3. Continuous dependence on controls of solutions to periodic systems

In this section, we will prove some sufficient conditions under which the set of minimizers and the set of saddle points of the functional of action semicontinuously depends on controls. As a consequence, we can formulate similar results concerning solutions to periodic system (2.2)–(2.3).

Whenever we consider a sequence $(u_k)_k \subset \mathcal{U}$ and a function $u_0 \in \mathcal{U}$, then φ_k stands for the functional of action defined by (2.4) corresponding to u_k, $k \in \mathbb{N}_0$, i.e.,

$$\varphi_k(x) := \varphi_{u_k}(x) = \int_0^T \left(\frac{1}{2}|\dot{x}(t)|^2 + \langle F^1(t, x(t)), u_k(t)\rangle_{\mathbb{R}^m} + F^2(t, x(t)) \right) dt$$

for $x \in H^1_T$ and $k \in \mathbb{N}$.

First, we prove

Lemma 3.1. *Assume* (A1)–(A3) *hold. If a sequence $(u_k)_k \subset \mathcal{U}$, $k \in \mathbb{N}$, tends to $u_0 \in \mathcal{U}$ weakly in L^p when $p \in [1, \infty)$ or weakly–* when $p = \infty$, then sequences $(\varphi_k)_k$, $(\varphi'_k)_k$ converge to φ_0, φ'_0, respectively, uniformly on any ball $B_r \subset H^1_T$.*

Proof. Suppose that functionals φ_k do not tend to the functional φ_0 uniformly on some ball $B_\rho \subset H^1_T$. So there exist $a > 0$ and a sequence $(x_k)_k \subset B_\rho$ such that

$$|\varphi_k(x_k) - \varphi_0(x_k)| \geq a \tag{3.1}$$

for $k \in \mathbb{N}$. Since B_ρ is weakly compact and the weak convergence in H^1_T implies the uniform convergence (see [20, Proposition 1.2]), we may assume that the sequence $(x_k)_k$

tends to some x_0 uniformly on $[0,T]$. It is easy to calculate that

$$\begin{aligned}\varphi_k(x_k)-\varphi_0(x_k) &= \int_0^T \big(\langle F^1(t,x_k(t)),u_k(t)\rangle_{\mathbb{R}^m} - \langle F^1(t,x_k(t)),u_0(t)\rangle_{\mathbb{R}^m}\big)\,dt \\ &= \int_0^T \langle F^1(t,x_k(t)) - F^1(t,x_0(t)), u_k(t)-u_0(t)\rangle_{\mathbb{R}^m}\,dt \\ &\quad + \int_0^T \langle F^1(t,x_0(t)), u_k(t)-u_0(t)\rangle_{\mathbb{R}^m}\,dt.\end{aligned}$$

Let $\varepsilon>0$. Since u_k tends to u_0 weakly in L^p (or weakly–$*$ in L^∞) and x_k converges to x_0 uniformly on $[0,T]$, we obtain from (A1)–(A2) that

$$\left|\int_0^T \langle F^1(t,x_k(t)) - F^1(t,x_0(t)), u_k(t)-u_0(t)\rangle_{\mathbb{R}^m}\,dt\right| < \frac{\varepsilon}{2}$$

and

$$\left|\int_0^T \langle F^1(t,x_0(t)), u_k(t)-u_0(t)\rangle_{\mathbb{R}^m}\,dt\right| < \frac{\varepsilon}{2}$$

for sufficiently large k. This is a contradiction to (3.1), which means that φ_k converges to φ_0 uniformly on any ball $B_\rho\subset H_T^1$, $\rho>0$.

For the proof of uniform convergence φ_k' to φ_0', assume on the contrary that there are $a>0$ and a sequence $(x_k)_k\subset B_\rho$ such that

$$\sup_{\|h\|\le 1} |\langle \varphi_k'(x_k)-\varphi_0'(x_k),h\rangle| \ge a$$

for $k\in\mathbb{N}$. Next, we note by (2.5) that

$$\langle \varphi_k'(x_k)-\varphi_0'(x_k),h\rangle = \int_0^T \langle F_x^1(t,x_k(t))h(t), u_k(t)-u_0(t)\rangle_{\mathbb{R}^m}.$$

Hence we get a contradiction by Assumptions (A1), (A3) and reason as before. □

3.1. The case of minimizers

In what follows, Y_k denotes the set of minimizers for φ_k for $k\in\mathbb{N}_0$, i.e.,

$$Y_k = \left\{ x\in H_T^1 : \varphi_k(x) = \min_{y\in H_T^1}\varphi_k(y)\right\}.$$

Theorem 3.2. *If*

(1) *functions* F^1, F^2 *satisfy Assumptions* (A1)–(A3),

(2) $|r|^{-2\alpha}\int_0^T\big(\langle F^1(t,r),u(t)\rangle + F^2(t,r)\big)\,dt\to\infty$ *uniformly with respect to* $u\in\mathcal{U}$ *when* $|r|\to\infty$ *for* $\alpha\in[0,1)$ *mentioned in* (A3),

(3) *the sequence* $(u_k)_k \subset \mathcal{U}$ *of admissible controls tends to* $u_0 \in \mathcal{U}$ *in the weak topology of* L^p *when* $p \in [1, \infty)$ *or the weak–* topology of* L^∞ *when* $p = \infty$,

then

(a) *for any* k, *the set* Y_k *of minimizers of the functional* φ_k *is nonempty,*

(b) *there exists a ball* $B(0, \rho) \subset H^1_T$ *such that* $Y_k \subset B(0, \rho)$ *for* $k \in \mathbb{N}_0$,

(c) *any sequence* $(x_k)_k \subset H^1_T$ *such that* $x_k \in Y_k$ *for* $k \in \mathbb{N}$ *possesses a strong cluster point* $x_0 \in Y_0$, *in particular,*

$$\emptyset \neq \operatorname{LimSup} Y_k \subset Y_0.$$

If the sets Y_k *are singletons, i.e.,* $Y_k = \{x_k\}$ *for* $k \in \mathbb{N}_0$, *then* x_k *tends to* x_0 *in* H^1_T.

Proof. The proof will be divided into three steps.

First step. We will show that the set Y_k of minimizers of the functional φ_k is not empty for $k \in \mathbb{N}_0$. Note that the functional $H^1_T \ni x \mapsto \int_0^T \frac{1}{2}|\dot{x}(t)|^2\, dt$ is convex and continuous, therefore it is weakly lower semicontinuos. On the other hand, the functional $H^1_T \ni x \mapsto \int_0^T \big(\langle F^1(t, x(t)), u(t)\rangle_{\mathbb{R}^m} + F^2(t, x(t))\big)\, dt$ is weakly continuous for any $u \in \mathcal{U}$ (see [20, Proposition 1.2]). Thus the functional φ_k is weakly lower semicontinuous for $k \in \mathbb{N}_0$.

We now fix $u \in \mathcal{U}$. For $x \in H^1_T$, we put $\widetilde{x} = x - \overline{x}$, where $\overline{x} = \int_0^T x(s)\, ds$. By (A3) and Sobolev inequality [20], we get

$$\begin{aligned}
&\left|\int_0^T \langle F^1(t, x(t)) - F^1(t, \overline{x}), u(t)\rangle_{\mathbb{R}^m}\, dt\right| \\
&\le \int_0^T |F^1(t, x(t)) - F^1(t, \overline{x})| \cdot |u(t)|\, dt \\
&\le \int_0^T \left(\int_0^1 |F^1_x(t, \overline{x} + s\widetilde{x}(t))\widetilde{x}(t)|\, ds\right) \cdot |u(t)|\, dt \\
&\le \int_0^T \left(\int_0^1 g(t)\big(|\overline{x} + s\widetilde{x}(t)|^\alpha + 1\big)\, ds\right) \cdot |\widetilde{x}(t)| \cdot |u(t)|\, dt \\
&\le 2\int_0^T \left(|u(t)||\widetilde{x}(t)| \int_0^1 \big(g(t)|\overline{x}|^\alpha + g(t)|s\widetilde{x}(t)|^\alpha + g(t)\big)\, ds\right) dt \qquad (3.2)\\
&\le 2C \int_0^T g(t)\big(|\overline{x}|^\alpha|\widetilde{x}(t)| + |\widetilde{x}(t)|^{\alpha+1} + |\widetilde{x}(t)|\big)\, dt \\
&\le 2C\|\widetilde{x}\|_\infty|\overline{x}|^\alpha \int_0^T g(t)\, dt + 2C\|\widetilde{x}\|_\infty^{\alpha+1}\int_0^T g(t)\, dt + 2C\|\widetilde{x}\|_\infty \int_0^T g(t)\, dt \\
&\le \frac{3}{2T}\|\widetilde{x}\|_\infty^2 + \frac{2TC^2}{3}|\overline{x}|^{2\alpha}\|g\|_{L^1}^2 + 2C\|\widetilde{x}\|_\infty^{\alpha+1}\|g\|_{L^1} + 2C\|\widetilde{x}\|_\infty\|g\|_{L^1}
\end{aligned}$$

$$\leq \frac{1}{8}\|\dot{x}\|_{L^2}^2 + C_1|\overline{x}|^{2\alpha} + C_2\|\dot{x}\|_{L^2}^{\alpha+1} + C_3\|\dot{x}\|_{L^2}.$$

where C_1, C_2 and C_3 are some positive constants not depending on u. In an analogous way, we obtain that

$$\begin{aligned}\left|\int_0^T \left(F^2(t,x(t)) - F^2(t,\overline{x})\right) dt\right| &\leq \int_0^T \left(\int_0^1 |F_x^2(t,\overline{x}+s\widetilde{x}(t))\widetilde{x}(t)|\, ds\right) dt \\ &\leq \int_0^T \left(\int_0^1 g(t)(|\overline{x}+s\widetilde{x}(t)|^\alpha + 1)\, ds\right) \cdot |\widetilde{x}(t)|\, dt\end{aligned}$$

and conclude similarly that

$$\left|\int_0^T \left(F^2(t,x(t)) - F^2(t,\overline{x})\right) dt\right| \leq \frac{1}{8}\|\dot{x}\|_{L^2}^2 + K_1|\overline{x}|^{2\alpha} + K_2\|\dot{x}\|_{L^2}^{\alpha+1} + K_3\|\dot{x}\|_{L^2}.$$

Therefore

$$\begin{aligned}\varphi_u(x) &= \int_0^T \left(\frac{1}{2}|\dot{x}(t)|^2 + \langle F^1(t,\overline{x}), u(t)\rangle_{\mathbb{R}^m} + F^2(t,\overline{x})\right) dt \\ &\quad + \int_0^T \left(\langle F^1(t,x(t)) - F^1(t,\overline{x}), u(t)\rangle_{\mathbb{R}^m} + \left(F^2(t,x(t)) - F^2(t,\overline{x})\right)\right) dt \\ &\geq \frac{1}{4}\|\dot{x}\|_{L^2}^2 + A\|\dot{x}\|_{L^2}^{\alpha+1} + B\|\dot{x}\|_{L^2} \\ &\quad + |\overline{x}|^{2\alpha}\left(\int_0^T |\overline{x}|^{-2\alpha}\left(\langle F^1(t,\overline{x}), u(t)\rangle_{\mathbb{R}^m} + F^2(t,\overline{x})\right) dt + D\right),\end{aligned} \tag{3.3}$$

where A, B, D are some constants not depending on u. Thus, if $\|x\| \to \infty$, then $\varphi_u(x) \to \infty$ uniformly with respect to u by Assumption (2). So the functional φ_u is weakly lower semicontinuous and (uniformly) coercive with respect to u. This implies that the set Y_k of minimizers of the functional φ_k is not empty for $k \in \mathbb{N}_0$.

Second step. We now prove that $Y_k \subset B(0,\rho)$ for $k \in \mathbb{N}_0$ for some $\rho > 0$. Suppose on the contrary that there exists a sequence $(x_k)_k$ such that $x_k \in Y_k$ for $k \in \mathbb{N}_0$ and $\|x_k\| \to \infty$ if $k \to \infty$ (note that each of the set Y_k is bounded by (3.3)). From (A2) and the facts that x_k is a minimizer for φ_k and M is compact, it follows that

$$\begin{aligned}\varphi_k(x_k) \leq \varphi_k(0) &\leq \int_0^T \left(\langle F^1(t,0), u_k(t)\rangle_{\mathbb{R}^m} + F^2(t,0)\right) dt \\ &\leq \int_0^T \left(a(0)b(t)|u_k(t)| + a(0)b(t)\right) dt \leq D\end{aligned}$$

for $k \in \mathbb{N}_0$, but this contradicts the fact that $\varphi_k(x_k) \to \infty$ if $k \to \infty$ (the first step).

Third step. Let $(x_k)_k \subset H_T^1$ be any sequence such that $x_k \in Y_k$ for $k \in \mathbb{N}$. Passing, if necessary, to a subsequence, we may assume that x_k tends weakly to some x_0 on H_T^1. Now, we shall show that $x_0 \in Y_0$. For $k \in \mathbb{N}_0$, let

$$m_k := \varphi_k(x_k) = \min\left\{\varphi_k(y) : y \in H_T^1\right\} = \min\{\varphi_k(y) : y \in B(0,\rho)\}.$$

Since φ_k tends to φ_0 uniformly on the ball $B(0,\rho)$ by Lemma 3.1, we have

$$m_k \to m_0. \tag{3.4}$$

Suppose that x_0 does not belong to Y_0. The set Y_0 is not empty, thus there exists $v \in Y_0$ such that $x_0 \neq v$ and

$$\begin{aligned} m_k - m_0 &= \varphi_k(x_k) - \varphi_0(v) \\ &= (\varphi_k(x_k) - \varphi_0(x_k)) + [\varphi_0(x_k) - \varphi_0(x_0)] + (\varphi_0(x_0) - \varphi_0(v)). \end{aligned} \tag{3.5}$$

Because $x_0 \notin Y_0$, we have $\varphi_0(x_0) - \varphi_0(v) > 0$. Passing with $k \to \infty$ in (3.5), we get a contradiction to (3.4) (note that φ_0 is weakly lower semicontinuous). Consequently, we have proved that $(x_k)_k$ possess a weak cluster point $x_0 \in Y_0$. We will prove that the above weak cluster point is also a strong cluster point. Weak convergence in H_T^1 implies uniform convergence on $[0,T]$ (see [20, Proposition 1.2]), therefore we may assume that $x_k \rightrightarrows x_0$. Note that φ_k is Gâteaux differentiable and x_k is a minimizer for φ_k, thus $\varphi_k'(x_k) = 0$ for $k = 0, 1, \ldots$. Consequently by (A3),

$$\begin{aligned} 0 &= \langle \varphi_k'(x_k) - \varphi_0'(x_0), x_k - x_0 \rangle \\ &\geq \|\dot{x}_k - \dot{x}\|_{L^2}^2 - \int_0^T g(t)\big(|x_k(t)|^\alpha + 1\big)\big(|u_k(t)| + 1\big)|x_k(t) - x_0(t)|\, dt \\ &\quad - \int_0^T g(t)\big(|x_0(t)|^\alpha + 1\big)\big(|u_0(t)| + 1\big)|x_k(t) - x_0(t)|\, dt. \end{aligned}$$

Hence $\|\dot{x}_k - \dot{x}\|_{L^2} \to 0$ as $k \to \infty$ and consequently $x_k \to x_0$ in H_T^1.

The proof is then completed. □

Remark 3.3. The key assumption of Theorem 3.2 is Assumption (2). It describes the type of the convergence at infinity of the average value of the potential. Similar to (A3), this is a generalization of the assumption from a book by Mawhin and Willem [20]. This implies a property of the functional of action (coercivity, in this case) that is essential for the existence of solutions (as well as for continuous dependence). An interesting problem is whether we can allow $\alpha = 1$ in Assumptions (A3) and (2) (which would mean that we also allow a linear case). First of all, we point out that assuming (A3) with $\alpha = 1$ it is impossible to obtain estimation (3.2) and consequently (3.3) in the proof of Theorem 3.2. On the other hand, the linear system $\ddot{x} = x$ does not have periodic solutions (except stationary point), while the system $\ddot{x} = -x$ has only periodic solutions (except stationary point). In both cases, Theorem 3.2(2) is not satisfied (even if we allow $\alpha = 1$) which means that this theorem is only a specific necessary condition for the existence of solutions to the considered problem. A similar observation applies to the case of saddle points and Theorem 3.4.

3.2. The case of saddle points

Let

$$X^+ := \left\{ x \in H_T^1 : \int_0^T x(t) = 0 \right\} \quad \text{and} \quad X^- := \left\{ x \in H_T^1 : x(t) = \text{constant} \right\}.$$

It is easy to see that $H_T^1 = X^+ \oplus X^-$.

Next, for $R > 0$, let

$$B_R^- := \{x \in X^- : \|x\| \le R\} \quad \text{and} \quad \Gamma := \left\{ \gamma \in C(B_R^-, H_T^1) : \gamma(x) = x \text{ for } x \in \partial B_R^- \right\}.$$

Let $(u_k)_k \subset \mathcal{U}$ be an arbitrary sequence and let $u_0 \in \mathcal{U}$. Denote by

$$S_k = \{x \in H_T^1 : \varphi_k(x) = c_k \text{ and } \varphi_k'(x) = 0\}$$

the set of critical points of the functional $\varphi_k = \varphi_{u_k}$ for $k \in \mathbb{N}_0$, which corresponds to the critical value

$$c_k = c_{u_k} = \inf_{\gamma \in \Gamma} \max_{s \in B_R^-} \varphi_k(\gamma(s)).$$

Theorem 3.4. *If*

(1) *functions F^1, F^2 satisfy Assumptions* (A1)–(A3),

(2) $|r|^{-2\alpha} \int_0^T \left(\langle F^1(t,r), u(t) \rangle + F^2(t,r) \right) dt \to -\infty$ *uniformly with respect to $u \in \mathcal{U}$ when $|r| \to \infty$, for $\alpha \in [0,1)$ mentioned in* (A3),

(3) *the sequence $(u_k)_k \subset \mathcal{U}$ of admissible controls tends to $u_0 \in \mathcal{U}$ in the weak topology of L^p when $p \in [1,\infty)$, or the weak–$*$ topology of L^∞ when $p = \infty$,*

then

(a) *for any k, the set S_k of saddle points of the functional φ_k is not empty,*

(b) *there exists a ball $B(0,\rho) \subset H_T^1$ such that $S_k \subset B(0,\rho)$ for $k \in \mathbb{N}_0$,*

(c) *any sequence $(x_k)_k \subset H_T^1$ such that $x_k \in S_k$ for $k \in \mathbb{N}$ possesses a strong cluster point $x_0 \in S_0$, in particular,*

$$\emptyset \ne \operatorname{LimSup} S_k \subset S_0.$$

If the sets S_k are singletons, i.e., $S_k = \{x_k\}$ for $k \in \mathbb{N}_0$, then x_k tends to x_0 in H_T^1.

Proof. The proof is based on Rabinowitz's Saddle Point Theorem (cf. [25]) and will be divided into three steps.

First step. We notice first that the functional φ_u is unbounded from above and below for any $u \in \mathcal{U}$.

In an analogous way as in Theorem 3.2 we can prove that there are constants A, B, C such that for every $x \in X^+$ and every $u \in \mathcal{U}$,

$$\varphi_u(x) \geq \frac{1}{4}\|x\|^2 + A\|x\|^{\alpha+1} + B\|x\| + C.$$

This means that φ_u is unbounded from above and

$$\inf_{x\in X^+} \varphi_u(x) > -\infty.$$

Next, for $x \in X^-$, we have that

$$\varphi_u(x) = \int_0^T \left(\langle F^1(t,x), u(t)\rangle_{\mathbb{R}^m} + F^2(t,x)\right) dt$$

and by Assumption (2), $\varphi_u(x) \to -\infty$ uniformly with respect to $u \in \mathcal{U}$ as $\|x\| \to \infty$, $x \in X^-$. Therefore φ_u is unbounded from below and there exists $R > 0$ such that for every $u \in \mathcal{U}$ we have that

$$\max_{x\in\partial B_R^-} \varphi_u(x) < \inf_{x\in X^+} \varphi_u(x).$$

Fix $u \in \mathcal{U}$, we will show that φ_u satisfies (PS) condition. Let $(x_k)_k \subset H_T^1$ be such that $\varphi_u'(x_k) \to 0$ as $k \to \infty$ and $(\varphi_u(x_k))_k$ is bounded. Let $x_k = \overline{x}_k + \widetilde{x}_k$ where $\overline{x}_k = (1/T)\int_0^T x(t)\,dt$. Similarly as in the proof of Theorem 3.2, we get that

$$\begin{aligned}&\left|\int_0^T \left(\langle (F_x^1(t,x_k(t)))^T \widetilde{x}_k(t), u(t)\rangle_{\mathbb{R}^m} + \langle F_x^2(t,x_k(t)), \widetilde{x}_k(t)\rangle_{\mathbb{R}^n}\right) dt\right| \\ &\leq \frac{1}{4}\|\dot{x}_k\|_{L^2}^2 + C_1|\overline{x}_k|^{2\alpha} + C_2\|\dot{x}_k\|_{L^2}^{\alpha+1} + C_3\|\dot{x}_k\|_{L^2}\end{aligned}$$

for all k and some positive constants C_1, C_2, C_3. This shows that

$$|\langle \varphi_u'(x_k), \widetilde{x}_k\rangle| \geq \frac{3}{4}\|\dot{x}_k\|_{L^2}^2 - C_1|\overline{x}_k|^{2\alpha} - C_2\|\dot{x}_k\|_{L^2}^{\alpha+1} - C_3\|\dot{x}_k\|_{L^2}. \tag{3.6}$$

Since $\varphi_u'(x_k) \to 0$ as $k \to \infty$, for sufficiently large k

$$|\langle \varphi_u'(x_k), \widetilde{x}_k\rangle| \leq \|\widetilde{x}_k\|. \tag{3.7}$$

Consequently, by inequalities (3.6) and (3.7), we get for sufficiently large k that,

$$\|\widetilde{x}_k\| \geq \frac{3}{4}\|\dot{x}_k\|_{L^2}^2 - C_1|\overline{x}_k|^{2\alpha} - C_2\|\dot{x}_k\|_{L^2}^{\alpha+1} - C_3\|\dot{x}_k\|_{L^2},$$

thus

$$C_1|\overline{x}_k|^{2\alpha} \geq \frac{3}{4}\|\dot{x}_k\|_{L^2}^2 - C_2\|\dot{x}_k\|_{L^2}^{\alpha+1} - (C_3+1)\|\dot{x}_k\|_{L^2}. \tag{3.8}$$

Analysis similar to that in the proof of Theorem 3.2 with application of (3.8) and the fact that $(x_k)_k$ is (PS) sequence, we show that

$$\begin{aligned}
C_5 \leq \varphi_u(x_k) &= \int_0^T \left(\frac{1}{2}|\dot{x}_k(t)|^2 + \langle F^1(t,\overline{x}_k), u(t)\rangle_{\mathbb{R}^m} + F^2(t,\overline{x}_k)\right) dt \\
&\quad + \int_0^T \langle F^1(t,x_k(t)) - F^1(t,\overline{x}_k), u(t)\rangle_{\mathbb{R}^m}\, dt + \int_0^T \big(F^2(t,x_k(t)) - F^2(t,\overline{x}_k)\big)\, dt \\
&\leq |\overline{x}_k|^{2\alpha}\left(|\overline{x}_k|^{-2\alpha}\int_0^T \big(\langle F^1(t,\overline{x}_k), u(t)\rangle_{\mathbb{R}^m} + F^2(t,\overline{x}_k)\big)\, dt + C_6\right)
\end{aligned}$$

for some constants C_5, C_6 and sufficiently large k. Hence, by Assumption (2), $(|\overline{x}_k|)_k$ is bounded. Consequently, by (3.8), the sequence $(x_k)_k$ is bounded in H_T^1. Passing to a subsequence, we can assume that $(x_k)_k$ tends weakly to some $x \in H_T^1$ and therefore strongly in $C([0,T],\mathbb{R}^n)$ (see [20, Proposition 1.2]).

So $\langle \varphi_u'(x_k) - \varphi_u'(x), x_k - x\rangle \to 0$ as $k \to \infty$ and since

$$\begin{aligned}
\langle \varphi_u'(x_k) - \varphi_u'(x), x_k - x\rangle = \|\dot{x}_k - \dot{x}\|_{L^2}^2 \\
&+ \int_0^T \langle F_x^1(t,x_k(t))(x_k(t)-x(t)), u(t)\rangle_{\mathbb{R}^m}\, dt \\
&+ \int_0^T \langle F_x^2(t,x_k(t)), (x_k(t)-x(t))\rangle_{\mathbb{R}^m}\, dt \\
&- \int_0^T \big(\langle F_x^1(t,x(t))(x_k(t)-x(t)), u(t)\rangle_{\mathbb{R}^m}\big)\, dt \\
&- \int_0^T \big(\langle F_x^2(t,x(t)), (x_k(t)-x(t))\rangle_{\mathbb{R}^m}\big)\, dt,
\end{aligned}$$

it follows from (A2) that $\|\dot{x}_k - \dot{x}\|_{L^2} \to 0$ as $k \to \infty$. Consequently, $x_k \to x$ in H_T^1 and φ_u satisfies (PS) condition.

Applying Rabinowitz's Saddle Point Theorem [25] shows that there exists $R > 0$ such that for any $u \in \mathcal{U}$, the functional φ_u possesses a critical point which corresponds to the critical value $c_u = \inf_{\gamma\in\Gamma}\max_{x\in B_R^-}\varphi_u(\gamma(x))$.

Second step. Let $(u_k)_k \subset \mathcal{U}$ be a sequence as in Assumption (3). First, note that all c_k are uniformly bounded from below for $k \in \mathbb{N}_0$. Indeed, for $k \in \mathbb{N}_0$, we have by (A2) that

$$\max_{x\in B_R^-}\varphi_k(\gamma(x)) \geq \varphi_k(\gamma(R)) = \varphi_k(R) \geq -K$$

for all functions $\gamma \in \Gamma$ and some K.

We will now show that the sets S_k are uniformly bounded for $k \in \mathbb{N}_0$. Let $x_k \in S_k$ for $k \in \mathbb{N}_0$. Similarly as in the first step, we have

$$0 = \langle \varphi_k'(x_k), \widetilde{x}_k \rangle \geq \frac{3}{4}\|\dot{x}_k\|_{L^2}^2 - C_1|\overline{x}_k|^{2\alpha} - C_2\|\dot{x}_k\|_{L^2}^{\alpha+1} - C_3\|\dot{x}_k\|_{L^2},$$

where $x_k = \overline{x}_k + \widetilde{x}$, $\overline{x}_k = (1/T)\int_0^T x_k(t)\,dt$ and $C_1, C_2, C_3 > 0$. Hence

$$C_1|\overline{x}_k|^{2\alpha} \geq \frac{3}{4}\|\dot{x}_k\|_{L^2}^2 - C_2\|\dot{x}_k\|_{L^2}^{\alpha+1} - C_3\|\dot{x}_k\|_{L^2} \tag{3.9}$$

for $k \in \mathbb{N}_0$ and some constant $C > 0$. Since all c_k are uniformly bounded from below for $k \in \mathbb{N}_0$, it follows that

$$-K \leq \varphi_k(x_k) \leq |\overline{x}_k|^{2\alpha}\bigg(|\overline{x}_k|^{-2\alpha}\int_0^T \big(\langle F^1(t,\overline{x}_k), u_k(t)\rangle_{\mathbb{R}^m} + F^2(t,\overline{x}_k)\big)\,dt + 2C_1\bigg)$$

for $k \in \mathbb{N}_0$. Hence $(|\overline{x}_k|)_k$ is bounded by Assumption (2). Consequently, by (3.9), the sequence $(x_k)_k$ is also bounded. In the same way, we can prove that every sequence $(x_k)_k \in S_{k_0}$ is bounded for every k_0. This means that there exists $\rho > 0$ such that $S_k \subset B(0,\rho)$ for all $k \in \mathbb{N}_0$.

Third step. Let $(x_k)_k \subset H_T^1$ be such a sequence such that $x_k \in S_k$ for $k \in \mathbb{N}$ (that is, $(x_k)_k$ is a sequence of saddle points). Let $B(0,\rho) \subset H_T^1$ be the ball referred to as in the previous step. By Assumption (2) and Lemma 3.1, there is $\varepsilon > 0$ such that for sufficiently large k,

$$\begin{aligned} c_k &= \inf_{\gamma\in\Gamma}\max_{x\in B_R^-}\big(\varphi_k(\gamma(x)) - \varphi_0(\gamma(x)) + \varphi_0(\gamma(x))\big) \\ &\leq \inf_{\gamma\in\Gamma}\max_{x\in B_R^-}\big(\varepsilon + \varphi_0(\gamma(x))\big) = \varepsilon + c_0. \end{aligned}$$

Following this and using Lemma 3.1 again, we get

$$\lim_{k\to\infty}\varphi_0(x_k) = \lim_{k\to\infty}\big((\varphi_0(x_k) - \varphi_k(x_k)) + \varphi_k(x_k)\big) = \lim_{k\to\infty} c_k = c_0 \tag{3.10}$$

and

$$0 = \lim_{k\to\infty}\big(\varphi_0'(x_k) - \varphi_k'(x_k)\big) = \lim_{k\to\infty}\varphi_0'(x_k). \tag{3.11}$$

This means that $(x_k)_k$ is a (PS) sequence for φ_0 which, as we have proved in the previous step, satisfies (PS) condition. Let x_0 be a cluster point of this sequence. By (3.10) and (3.11), $c_0 = \lim_{k\to\infty}\varphi_0(x_k) = \varphi_0(x_0)$ and $0 = \lim_{k\to\infty}\varphi_0'(x_k) = \varphi_0'(x_0)$. Hence $x_0 \in S_0$. The proof is thus completed. □

Remark 3.5. It is easy to see that Conditions (a)–(c) imply that the set-valued mapping $\mathcal{U} \ni u \to S_u \subset H_T^1$ is upper semicontinuous with respect to the weak topology in L^p when $p \in [1,\infty)$, or the weak–* topology of L^∞ when $p = \infty$, and the strong topology in H_T^1.

3.3. Remarks on periodic solutions

Remark 3.6. Theorems 3.2 and 3.4 describe some properties of critical points of functional φ_u defined by (2.4). However, applying the Fundamental Lemma [20] gives us that every critical point x_u of φ_u is a solution to problem (2.2)–(2.3), and vice versa.

Remark 3.7. If the functional of action φ_u given by (2.4) is convex, then the set of critical points Y_u is identical with the set of solutions to periodic problem (2.2)–(2.3). If the functional φ_u is strictly convex, then Y_u is a singleton.

Consequently, we can formulate the following

Theorem 3.8. *Assume that the functions F^1, F^2 satisfy Assumptions* (A1)–(A3) *and that*

(a) $|r|^{-2\alpha}\int_0^T \big(\langle F^1(t,r),u(t)\rangle_{\mathbb{R}^m}+F^2(t,r)\big)\,dt \to \infty$ *uniformly with respect to* $u \in \mathcal{U}$ *when* $|r| \to \infty$ *for some* $\alpha \in [0,1)$, *or*

(b) $|r|^{-2\alpha}\int_0^T \big(\langle F^1(t,r),u(t)\rangle_{\mathbb{R}^m}+F^2(t,r)\big)\,dt \to -\infty$ *uniformly with respect to* $u \in \mathcal{U}$ *when* $|r| \to \infty$ *for some* $\alpha \in [0,1)$.

Then, for any $u \in \mathcal{U}$, there exists at least one solution x_u to (2.2)–(2.3) *corresponding to u. If the sequence $(u_k)_k \subset \mathcal{U}$ of admissible controls tends to $u_0 \in \mathcal{U}$ in the weak topology of L^p when $p \in [1,\infty)$, or the weak–* topology of L^∞ when $p = \infty$, then the sequence $(x_k)_k \subset H^1_T$ of solutions to* (2.2)–(2.3) *corresponding to u_k and such that they are critical points of suitable type (minimizer or saddle point), possesses a cluster point x_0 in the strong topology of H^1_T which is a solution to* (2.2)–(2.3) *corresponding to u_0, being a critical point of the same type.*

Moreover, in the case of Assumption (a), if additionally φ_u is strictly convex then the solution x_u corresponding to u is a unique solution which depends continuously on u.

4. Existence of optimal controls and example

Consider optimal control problem governed by the second order differential system with the periodic boundary conditions

$$\ddot{x}(t) = \big(F^1_x(t,x(t))\big)^T u(t) + F^2_x(t,x(t)), \tag{4.1}$$

$$x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T) \tag{4.2}$$

and with the cost functional

$$J(x,u) = \int_0^T f(t,x(t),\dot{x}(t),u(t))\,dt + l(x(T)), \tag{4.3}$$

where $x \in H^1_T$, $u \in \mathcal{U} = \{w \in L^2([0,T],\mathbb{R}^m) : w(t) \in M\}$, M is a subset of $\mathbb{R}^m$.

Further, we shall assume

(A4) the vector function F^1 and the scalar function F^2 satisfy Conditions (A1)–(A3) and the functional of action given by (2.4) is convex,

(A5) $|r|^{-2\alpha} \int_0^T \big(\langle F^1(t,r), u(t) \rangle_{\mathbb{R}^m} + F^2(t,r) \big)\, dt \to \infty$ uniformly with respect to $u \in \mathcal{U}$ when $|r| \to \infty$ for $\alpha \in [0,1)$ mentioned in (A3), or $|r|^{-2\alpha} \int_0^T \big(\langle F^1(t,r), u(t) \rangle_{\mathbb{R}^m} + F^2(t,r) \big)\, dt \to -\infty$ uniformly with respect to $u \in \mathcal{U}$ for $\alpha \in [0,1)$ mentioned in (A3),

(A6) the integrand $f = f(t,x,\dot{x},u)\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times M \to \mathbb{R}$ is measurable with respect to t, continuous with respect to $(x,\dot{x},u)$ and convex with respect to $(\dot{x},u)$. Moreover, we assume that, for any ball $B(0,\rho) \subset \mathbb{R}^n$, there exists a constant $C > 0$ such that

$$|f(t,x,\dot{x},u)| \le C(1 + |\dot{x}|^2)$$

for $t \in [0,T]$ a.e. $x \in B(0,\rho)$, $\dot{x} \in \mathbb{R}^n$ and $u \in M$ and the function $l\colon \mathbb{R}^n \to \mathbb{R}$ is lower semicontinuous.

Applying the results of Section 3, we prove the following existence theorem for the periodic optimal control problem (4.1)–(4.3).

Theorem 4.1. *If the set $M \subset \mathbb{R}^n$ is convex and compact, and the functions F^1, F^2, f and l satisfy Conditions* (A4)–(A6)*, then periodic optimal control system* (4.1)–(4.3) *possesses at least one optimal process (x^0,u^0) where $u^0 \in \mathcal{U}$ and $x^0 \in H^1_T$.*

Proof. Let $(x_k,u_k)_k$ be a minimizing sequence for the problem considered, i.e., $\lim_{k\to\infty} J(x_k,u_k) = \inf J(x,u) = \mu$, $u_k \in \mathcal{U}$ and x_k is a solution of the periodic system (4.1)–(4.3) with $u = u_k$. Since $u_k(t) \in M$ and M is compact and convex, the sequence $(u_k)_k$ is compact in the weak topology of $L^2([0,T],\mathbb{R}^m)$. Passing, if necessary, to a subsequence, we may assume that u_k tends to some $u_0 \in \mathcal{U}$ weakly in $L^2([0,T],\mathbb{R}^m)$. By Theorem 3.2, we may assume that x_k tends to $x_0 \in H^1_T$ in the norm topology of H^1_T and the pair (x_0,u_0) is admissible, i.e., $u_0 \in \mathcal{U}$ and x_0 satisfies problem (4.1)–(4.2) with $u = u_0$. This implies that $x_k \to x_0$ uniformly on $[0,T]$ and $\dot{x}_k \to \dot{x}_0$ in $L^2([0,T],\mathbb{R}^n)$. Without loss of generality, we may assume that $\dot{x}_k$ tends to $\dot{x}_0$ pointwise on $[0,T]$ and there exists a function $h \in L^2([0,T],\mathbb{R}^n)$ such that $|\dot{x}_k(t)| \le h(t)$ for $t \in [0,T]$ a.e. and $k \in \mathbb{N}$ (see [6]).

Assumption (A4) and the well-known theorem on lower semicontinuity of integral functionals (see e.g., [5, 15, 22]) imply that

$$\begin{aligned}
\mu = \lim_{k\to\infty} J(x_k,u_k) &= \lim_{k\to\infty} \int_0^T f(t,x_k(t),\dot{x}_k(t),u_k(t))\,dt \\
&\ge \lim_{k\to\infty} \int_0^T f(t,x_0(t),\dot{x}_0(t),u_0(t))\,dt = J(x_0,u_0) \ge \mu.
\end{aligned}$$

Consequently, we have shown that the pair (x_0,u_0) is admissible and $\inf J(x,u) = J(x_0,u_0)$. Thus (x_0,u_0) is an optimal process. □

We now consider an illustrative calculation example of an optimal control periodic system. Although the example is theoretical, it shows a specific situation for which it is possible to apply theorems proved in the paper.

Example 4.2. Consider the following optimal control problem

$$J(x,u) = \int_0^T \left[(x(t))^4 (u^1(t))^2 - \dot{x}(t)u^2(t) + \dot{x}(t)x(t)\right] dt \to \min \tag{4.4}$$

subject to

$$\ddot{x}(t) - |x(t)|^\alpha = \langle G_x(t,x(t)), u(t)\rangle_{\mathbb{R}^2}, \quad x(0) = x(T), \quad \dot{x}(0) = \dot{x}(T), \tag{4.5}$$

where $x \in H^1_T$, $u = (u^1, u^2) \in \mathcal{U} = \{u \in L^2([0,T],\mathbb{R}^2) : |u^1| \le 1 \text{ and } |u^2| \le 1\}$, $\alpha \in [0,1)$, and $G\colon [0,T] \times \mathbb{R} \to \mathbb{R}^2$ is of the form

$$G(t,v) = \big(t^2 v^{3/2} + \cos v, \sin v - tv\big).$$

The functional of action related to system (4.5) is of the form

$$\varphi_u(x) = \int_0^T \left(\frac{1}{2}|\dot{x}(t)|^2 + \langle G(t,x(t)), u(t)\rangle_{\mathbb{R}^2} + \frac{1}{\alpha+1}|x(t)|^{\alpha+1}\right) dt.$$

It is easy to see that when

$$F^1(t,x) = G(t,x) \quad \text{and} \quad F^2(t,x) = \frac{1}{\alpha+1}|x|^{\alpha+1},$$

all assumption of Theorem 3.8 are satisfied. In particular, the functional φ_u is strictly convex on H^1_T. Thus, for any admissible control $u \in \mathcal{U}$, system (4.5) has exactly one solution $x_u \in H^1_T$ which depends continuously on u. Next, notice that the cost functional given by (4.4) and the functions

$$F^1(t,x) = G(t,x), \quad F^2(t,x) = \frac{1}{\alpha+1}|x|^{\alpha+1}$$

satisfy Assumptions (A1)–(A6). Thus Theorem 4.1 implies that periodic optimal control problem (4.4)–(4.5) possesses at least one solution.

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