Time Optimal Control of System Governed by a Fractional Stochastic Partial Differential Inclusion with Clarke Subdifferential

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Abstract. This paper investigates the time optimal control problems to a new class of fractional non-instantaneous impulsive stochastic partial differential inclusions with Clarke subdifferential in Hilbert spaces. Firstly, using the fractional calculus, properties of fractional resolvent operators and a fixed-point theorem, the existence of mild solutions for these systems is presented. Secondly, the existence of time optimal control of system governed by fractional stochastic control inclusions with Clarke subdifferential and non-instantaneous impulses is also obtained. Finally, an example is given to illustrate our main results.

1. Introduction

The optimal control theory has a considerable development, and one can find applications in many domains (see [9–11,13,33]). The time optimal control is an important topic in the theory of optimal controls and plays a vital role in control systems. In the past decades, much attention has been paid to time optimal control problems governed by nonlinear differential systems [14, 22, 38]. Among them, the fractional time optimal control of a distributed system is an optimal control problem for which dynamic systems are defined by means of fractional differential equations and inclusions, see for example [24, 27, 29] and references therein. In particular, it is well known that differential equations with impulsive conditions were used to describe the evolution process with abrupt perturbation at some certain time in biology, medicine and control theory [25]. However, in many cases, the models with abrupt and instantaneous impulses can not characterize many practical problems (see [2,19,20,26,28,31]). Recently, Wang et al. [37] established the time optimal control of a system governed by a class of non-instantaneous impulsive differential equations in Banach spaces.

On the other hand, stochastic optimal control theory is a stochastic generalization of classical optimal control theory. There are many interesting results on the time optimal

control of stochastic systems [5, 35]. Furthermore, impulsive stochastic optimal control systems describing these optimal control dynamical systems subject to both impulse and stochastic changes have attracted considerable attention [1, 4, 39, 42]. From a practical point of view, many physical phenomena in evolution processes are modeled as impulsive stochastic differential systems for which the impulses are non-instantesimal. Therefore, the optimal control of such systems have become an important object of investigation stimulated by their numerous applications to problems arising in many branches of physics and technical science. For example, Yan and Lu [40] considered the optimal control of a class of first-order non-instantaneous impulsive stochastic neutral evolution integro-differential equations with infinite delay. The authors in [41] established the solvability and optimal control of a class of fractional non-instantaneous impulsive stochastic partial integro-differential equations with delay-dependent states. In this paper, we study the time optimal control of non-instantaneous impulsive fractional stochastic partial differential inclusions with Clarke subdifferential of the form

$$
(1.1) \qquad {}^{c}D_t^{\alpha}\mathcal{N}(x_t) \in A\mathcal{N}(x_t) + J_t^{1-\alpha}[B(t)u(t)] + J_t^{1-\alpha}\left[\mu(\partial F(t, x_t))\frac{dw(t)}{dt}\right],
$$

$$
t \in (s_i, t_{i+1}], \ i = 0, 1, \dots, N,
$$

$$
(1.2) \qquad\qquad x(t) \in g_i(t, x_t), \quad t \in (t_i, s_i], \ i = 1, \dots, N,
$$

$$
(1.3) \qquad\qquad\qquad x(t) = \varphi(t), \quad t \in (-\infty, 0],
$$

where the state $x(\cdot)$ takes values in a separable real Hilbert space H with the inner product $\langle \cdot, \cdot \rangle_H$ and the norm $\|\cdot\|_H$, ${}^{c}D_t^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (0, 1)$, $J_t^{1-\alpha}$ is the $(1 - \alpha)$-order fractional integral operator. $A\colon D(A) \subset H \to H$ is the infinitesimal generator of resolvent operator $T_\alpha(t)$ $(t \geq 0)$. The control function u takes value from a separable Hilbert space Y, and B is a linear operator from Y into H, $p \geq 2$ is an integer. Let K be another separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle_K$ and the norm $\|\cdot\|_K$. $L(K, H)$ is the space of bounded linear operators mapping K into H equipped with the usual norm $\|\cdot\|_H$, and let $L(H)$ denote the Hilbert space of bounded linear operators from H to H. Suppose that $\{w(t) : t \geq 0\}$ is a given K-valued Wiener process with a covariance operator $Q \geq 0$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a normal filtration $\{\mathcal{F}_t\}_{t \geq 0}$, which is generated by the Wiener process w. The time history $x_t\colon (-\infty, 0] \to H$ given by $x_t(\theta) = x(t + \theta)$ belongs to some abstract phase space $\mathcal{B}$ defined axiomatically; let $0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < \cdots < t_N \leq s_N < t_{N+1} = b$ be prefixed numbers. $g_i\colon [0, b] \times \mathcal{B} \to \mathcal{P}(H)$ $(i = 1, 2, \dots, N)$ are bounded, closed, convex-valued multi-valued maps, and $G\colon [0, b] \times \mathcal{B} \to H$ is appropriate function, $\mathcal{N}(\psi) = \psi(0) - G(t, \psi)$, $\psi \in \mathcal{B}$, $\mu\colon H \to L(K, H)$ is a linear compact operator, $\partial F(t, \cdot)$ is the Clarke's subdifferential of $F(t, \cdot)$. The initial data $\{\varphi(t) : -\infty < t \leq 0\}$ is an $\mathcal{F}_0$-adapted, $\mathcal{B}$-valued random variable independent of the Wiener process w with finite pth moment.

At present, most of the literatures about differential inclusion require that the non-linear term is a convex function, which limits the study of differential inclusion. Just as said in [16], it is important to consider, in many problems, functionals on nonconvex constraints. Problems of this kind are typical of variational inequalities, optimization and optimal control theory. The study of the differential inclusions with Clarke subdifferential, which has wider applications than the subdifferential of convex function. In fact, the Clarke subdifferential has important applications in mechanics and engineering, especially in nonsmooth analysis and optimization (see [6,7]). Therefore, it is necessary and important to study differential inclusions with Clarke subdifferential type. Very recently, the papers [18, 23] studied the optimal control results for fractional evolution inclusion and fractional impulsive evolution inclusion with Clarke subdifferential in Banach spaces. To the best of our knowledge no work has been reported in the present literature regarding the time optimal control for fractional impulsive stochastic partial differential inclusions in Hilbert spaces, which is expressed in the form (1.1)–(1.3). Motivated by the above consideration, we study this interesting problem, which is natural generalization of time optimal control concept well known in the theory of infinite dimensional control systems. In the present paper, we investigate the time optimal control of (1.1)–(1.3) under the mixed Lipschitz and Carathéodory conditions. The results are obtained by using theory of fractional resolvent operators, stochastic analysis, the properties of Clarke subdifferential with a fixed-point theorem. The known results appeared in [14,18,22–24,27,29,37,38] are generalized to the impulsive stochastic inclusions settings and the case of with infinite delay and multi-valued non-instantaneous impulses.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and necessary preliminaries. In Section 3, we give the existence of solutions for control system. In Section 4, we establish the existence of time optimal control. In Section 5, an example is given to illustrate our results.

2. Preliminaries

We assume that there exists a complete orthonormal system $\{e_n\}_{n=1}^{\infty}$ in K, a bounded sequence of nonnegative real numbers $\{\lambda_n\}_{n=1}^{\infty}$ such that $Qe_n = \lambda_n e_n$, and a sequence α_n of independent Brownian motions such that

$$\langle w(t), e \rangle_K = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \langle e_n, e \rangle_K \alpha_n(t), \quad e \in K, \ t \geq 0.$$

Let $L_2^0 = L_2(Q^{1/2}K, H)$ be the space of all Hilbert-Schmidt operators from $Q^{1/2}K$ to H with the norm $\|\psi\|_{L_2^0}^2 = \mathrm{Tr}((\psi Q^{1/2})(\psi Q^{1/2})^*)$ for any $\psi \in L_2^0$. Clearly for any bounded operators $\psi \in L(K, H)$ this norm reduces to $\|\psi\|_{L_2^0}^2 = \mathrm{Tr}(\psi Q \psi^*)$. Let $L_{\mathcal{F}_t}^p(\Omega, H)$ be the

Banach space of all $\mathcal{F}_t$-measurable pth power integrable random variables with values in the Hilbert space H. Let $C([0,b]; L^p(\Omega, H))$ be the Banach space of continuous maps from $[0,b]$ into $L^p(\Omega, H)$ satisfying the condition $\sup_{t \in [0,b]} E\|x(t)\|_H^p < \infty$.

We use the notations $\mathcal{P}(H) = \{x \in 2^H : x \neq \emptyset\}$. Let V and Z be two Hausdorff topological spaces. A multi-valued map $\Phi \colon V \to 2^Z \setminus \{\emptyset\}$ is said to be upper semicontinuous (for short u.s.c.) $x_0 \in V$, if for any neighborhood $O(\Phi(x_0))$ of $\Phi(x_0)$, there exists a neighborhood $O(x_0)$ of x_0 such that $\Phi(x) \subset O(\Phi x_0)$ for all $x \in O(x_0)$; compact if $\Phi(D)$ is relatively compact for every any bounded subset D of V; quasicompact if its restriction to any compact subset $D \subset V$ is compact. Consider $H_d \colon \mathcal{P}(H) \times \mathcal{P}(H) \to \mathbb{R}^+ \cup \{\infty\}$ given by $H_d(\widetilde{A}, \widetilde{B}) = \max\left\{ \sup_{\widetilde{a} \in \widetilde{A}} d(\widetilde{a}, \widetilde{B}), \sup_{\widetilde{b} \in \widetilde{B}} d(\widetilde{A}, \widetilde{a}) \right\}$, where $d(\widetilde{A}, \widetilde{b}) = \inf_{\widetilde{a} \in \widetilde{A}} d(\widetilde{a}, \widetilde{b})$, $d(\widetilde{a}, \widetilde{B}) = \inf_{\widetilde{b} \in \widetilde{B}} d(\widetilde{a}, \widetilde{b})$. Then, $(\mathcal{P}_{\mathrm{bd,cl}}(H), H_d)$ is a metric space and $(\mathcal{P}_{\mathrm{cl}}(H), H_d)$ is a generalized metric space. For more details, one can see [12].

We introduce the space $\mathcal{PC}(H)$ formed by all $\mathcal{F}_t$-adapted measurable, H-valued stochastic processes $\{x(t) : t \in [0,b]\}$ such that x is continuous at $t \neq t_i$, $x(t_i) = x(t_i^-)$ and $x(t_i^+)$ exists for all $i = 1, \ldots, N$. In this paper, we always assume that $\mathcal{PC}(H)$ is endowed with the norm $\|x\|_{\mathcal{PC}} = \left(\sup_{0 \leq t \leq b} E\|x(t)\|_H^p \right)^{1/p}$. Then $(\mathcal{PC}(H), \|\cdot\|_{\mathcal{PC}})$ is a Banach space. The notation $B_r(x, H)$ stands for the closed ball with center at x and radius $r > 0$ in H.

In this paper, we assume that the phase space $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a semi-normed linear space of $\mathcal{F}_0$-measurable functions mapping $(-\infty, 0]$ into H, and satisfying the following fundamental axioms due to Hale and Kato (see, e.g., [17]):

(A) If $x \colon (-\infty, \sigma + b] \to H$, $b > 0$, is such that $x|_{[\sigma, \sigma+b]} \in C([\sigma, \sigma+b], H)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + b]$ the following conditions hold:

   (i) x_t is in $\mathcal{B}$;

   (ii) $\|x(t)\|_H \leq \widetilde{H} \|x_t\|_{\mathcal{B}}$;

   (iii) $\|x_t\|_{\mathcal{B}} \leq K(t-\sigma) \sup\{\|x(s)\|_H : \sigma \leq s \leq t\} + M(t-\sigma)\|x_\sigma\|_{\mathcal{B}}$, where $\widetilde{H} \geq 0$ is a constant; $K, M \colon [0, \infty) \to [1, \infty)$, K is continuous and M is locally bounded, and $\widetilde{H}, K, M$ are independent of $x(\cdot)$.

(B) For the function $x(\cdot)$ in (A), x_t is a $\mathcal{B}$-valued function on $[\sigma, \sigma + b]$.

(C) The space $\mathcal{B}$ is complete.

In the following, let Y be a separable Hilbert space from which the controls u take the values. Operator $B \in L_\infty([0,b], L(Y,H))$, $\|B\|_\infty$ stands for the norm of operator B on Banach space $L_\infty([0,b], L(Y,H))$, where $L_\infty([0,b], L(Y,H))$ denotes the space of operator valued functions which are measurable in the strong operator topology and uniformly bounded on the interval $[0,b]$. Let $L_{\mathcal{F}}^p([0,b], Y)$ be the closed subspace of $L_{\mathcal{F}}^p([0,b] \times$

Ω, Y), consisting of all measurable and $\mathcal{F}_t$-adapted, Y-valued stochastic processes satisfying the condition $E \int_0^b \|u(t)\|_Y^p \, dt < \infty$, and endowed with the norm $\|u\|_{L_{\mathcal{F}}^p([0,b],Y)} = \left(E \int_0^b \|u(t)\|_Y^p \, dt \right)^{1/p}$.

Let U be a nonempty closed bounded convex subset of Y. We define the admissible control set

$$U_{\mathrm{ad}} = \{\varpi(\cdot) \in L_{\mathcal{F}}^p([0,b],Y) : \varpi(t) \in U \text{ a.e. } t \in [0,b]\}.$$

Then, $Bu \in L^p([0,b],H)$ for all $u \in U_{\mathrm{ad}}$.

We recall some basic definitions about fractional integral and derivative, we refer to [34].

Definition 2.1. [7] Let X be a Banach space with the dual space X^* and $F \colon X \to \mathbb{R}$ be a locally Lipschitz functional on X. The Clarke's generalized directional derivative of F at the point $x \in X$ in the direction $v \in X$, denoted by $F^0(x,v)$, is defined by

$$F^0(x,v) = \lim_{\lambda \to 0^+} \sup_{y \to x} \frac{F(y + \lambda v) - F(y)}{\lambda}.$$

The Clarke's generalized gradient of F at $x \in X$, denoted by $\partial F(x)$, is a subset of X^* given by

$$\partial F(x) = \{x^* \in X^* : F^0(x,v) \geq \langle x^*, v \rangle, \forall v \in X\}.$$

Definition 2.2. [36] A family $\{T_\alpha(t)\}_{t \geq 0} \subseteq L(H)$ of bounded linear operators in H is called an α-order fractional resolvent operator (or α-order fractional solution operator) if it satisfies the following conditions:

(i) $T_\alpha(t)$ is strongly continuous for $t \geq 0$ and $T_\alpha(0) = I$.

(ii) $T_\alpha(t)D(A) \subset D(A)$ and $AT_\alpha(t)x = T_\alpha(t)Ax$ for all $x \in D(A)$, $t \geq 0$.

(iii) The resolvent equation holds for all $x \in D(A)$, $t \geq 0$,

$$T_\alpha(t)x = x + \int_0^t g_\alpha(t - s)AT_\alpha(s)x \, ds,$$

where $g_\alpha(t - s) = \frac{(t-s)^\alpha}{\Gamma(\alpha)}$.

Definition 2.3. [36] Let $0 < \theta_0 < \pi/2$ and $\omega_0 \in \mathbb{R}$. An α-order fractional resolvent operator $T_\alpha(t)$ ($t \geq 0$) is called analytic if it admits an analytic extension to a sector $\Sigma_{\theta_0} := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta_0\}$ and the analytic extension is strongly continuous on Σ_θ for every $\theta \in (0, \theta_0)$. An analytic α-order fractional resolvent operator $T_\alpha(z)$ ($z \in \Sigma_\theta$) is said to be of analyticity type (ω_0, θ_0) if for each $\theta \in (0, \theta_0)$ and $\omega > \omega_0$, there exists a positive constant $M = M(\omega, \theta)$ such that

$$\|T_\alpha(z)\|_H \leq Me^{\omega \operatorname{Re} z}, \quad z \in \Sigma_\theta.$$

An analytic α-order fractional resolvent operator $T_\alpha(z)$ $(z \in \Sigma_\theta)$ is said to be bounded if for each $\theta \in (0, \theta_0)$, there exists a positive constant M_θ such that

$$\|T_\alpha(z)\|_H \le M_\theta, \quad z \in \Sigma_\theta.$$

Let us consider the linear fractional stochastic differential inclusions of the form

$$(2.1) \qquad {}^cD_t^\alpha x(t) \in Ax(t) + J_t^{1-\alpha}[B(t)u(t)] + J_t^{1-\alpha}\left[\mu(\partial F(t))\frac{dw(t)}{dt}\right], \quad t \in [0, b],$$

$$(2.2) \qquad\qquad\qquad\qquad x(0) = \varphi(0) \in \mathcal{B},$$

where function $\partial F \in L^p([0, b], H)$.

If x satisfies the inclusions (2.1)–(2.2), then for $t \in [0, b]$, we have

$$x(t) = \varphi(0) + A(g_\alpha * x)(t) + \int_0^t B(s)u(s)\,ds + \int_0^t \mu f(s)\,dw(s),$$

where $f \in L^p([0, b], H)$, $f(t) \in \partial F(t)$ a.e. $t \in [0, b]$. Using the definition of an α-resolvent and the definition of an integral solution, it follows that

$$\begin{aligned}
I * x &= (T_\alpha - Ag_\alpha * T_\alpha) * x = T_\alpha * [x - (Ag_\alpha * x)] \\
&= T_\alpha * [x - (x - \varphi(0) - I * Bu - I * \mu f)] = T_\alpha * [\varphi(0) + I * Bu + I * \mu f] \\
&= T_\alpha * \varphi(0) + I * T_\alpha * Bu + I * T_\alpha * \mu f,
\end{aligned}$$

where $*$ is the convolution. Then we have for all $t \in [0, b]$,

$$(2.3) \quad x(t) = T_\alpha(t)\varphi(0) + \int_0^t T_\alpha(t-s)B(s)u(s)\,ds + \int_0^t T_\alpha(t-s)\mu f(s)\,dw(s), \quad t \in [0, b]$$

is the variation of constant formula satisfying (2.1)–(2.2). Now, we define the mild solution for (2.1)–(2.2) as follows:

Definition 2.4. A function $x \in C([0, b], H)$ is called a mild solution of (2.1)–(2.2) if $x_0 = \varphi \in \mathcal{B}$ on $(-\infty, 0]$, and (2.3) holds for every $u \in U_{\mathrm{ad}}$, where $f \in L^p([0, b], H)$, $f(t) \in \partial F(t)$ a.e. $t \in [0, b]$.

Let $x \colon (-\infty, b] \to H$ be a function such that $x \in \mathcal{PC}(H)$. If x is a solution of (1.1)–(1.3), from Definition 2.4 and the partial neutral differential inclusions theory, we obtain

$$x(t) = T_\alpha(t)[\varphi(0) - G(0, \varphi)] + G(t, x_t) + \int_0^t T_\alpha(t-s)B(s)u(s)\,ds$$

$$+ \int_0^t T_\alpha(t-s)\mu f(s)\,dw(s), \quad t \in [0, t_1],$$

where $f \in L^p([0,b], H)$, $f(t) \in \partial F(t, x_t)$ a.e. $t \in [0, b]$. If $t = s_1$, then $x(s_1) = \tilde{g}_1(s_1)$, where $\tilde{g}_1(s_1) \in g_1(s_1, x_{s_1})$. For $t \in (s_1, t_2]$, we have

$$x(t) = T_\alpha(t - s_1)[x(s_1) - G(s_1, x_{s_1})] + G(t, x_t) + \int_{s_1}^t T_\alpha(t - s)B(s)u(s)\,ds$$

$$+ \int_{s_1}^t T_\alpha(t - s)\mu f(s)\,dw(s)$$

$$= T_\alpha(t - s_1)[\tilde{g}_1(s_1) - G(s_1, x_{s_1})] + G(t, x_t) + \int_{s_1}^t T_\alpha(t - s)B(s)u(s)\,ds$$

$$+ \int_{s_i}^t T_\alpha(t - s)\mu f(s)\,dw(s).$$

By repeating the same procedure, we can easily deduce that

$$x(t) = T_\alpha(t - s_i)[\tilde{g}_i(s_i) - G(s_i, x_{s_i})] + G(t, x_t) + \int_{s_i}^t T_\alpha(t - s)B(s)u(s)\,ds$$

$$+ \int_{s_i}^t T_\alpha(t - s)\mu f(s)\,dw(s)$$

holds for all $t \in (s_i, t_{i+1}]$, where $\tilde{g}_i(t) \in g_i(t, x_t)$, $i = 1, \ldots, N$.

Next we will derive the appropriate definition of mild solutions of (1.1)–(1.3).

Definition 2.5. An $\mathcal{F}_t$-adapted stochastic process $x \colon (-\infty, b] \to H$ is called a mild solution of the system (1.1)–(1.3) if $x_0 = \varphi \in \mathcal{B}$ on $(-\infty, 0]$, $x|_{[0,b]} \in \mathcal{PC}([0, b], H)$, and

(i) $x(t)$ is measurable and adapted to $\mathcal{F}_t$, $t \geq 0$.

(ii) $x(t) \in H$ has càdlàg paths on $t \in [0, b]$ a.s and for each $t \in [0, b]$, $x(t)$ satisfies $x(t) = \tilde{g}_j(t)$ for all $t \in (t_j, s_j]$, $j = 1, \ldots, N$, and

$$x(t) = T_\alpha(t)[\varphi(0) - G(0, \varphi)] + G(t, x_t) + \int_0^t T_\alpha(t - s)B(s)u(s)\,ds$$

$$+ \int_0^t T_\alpha(t - s)\mu f(s)\,dw(s)$$

for all $t \in [0, t_1]$, and

$$x(t) = S_\alpha(t - s_i)[\tilde{g}_i(s_i) - G(s_i, x_{s_i})] + G(t, x_t) + \int_{s_i}^t T_\alpha(t - s)B(s)u(s)\,ds$$

$$+ \int_{s_i}^t T_\alpha(t - s)\mu f(s)\,dw(s)$$

for all $t \in (s_i, t_{i+1}]$, $i = 1, \ldots, N$, where $u \in U_{\mathrm{ad}}$, $f \in L^p([0, b], H)$, $f(t) \in \partial F(t, x_t)$ a.e. $t \in [0, b]$, and $\tilde{g}_i(t) \in g_i(t, x_t)$, $i = 1, \ldots, N$.

Definition 2.6. [15] A multi-valued map $\Phi\colon H \to 2^H$ is referred to as condensing if, for any bounded subset $D \subseteq H$, which is not relatively compact, one has $\beta(\Phi(D)) < \beta(D)$, where $\beta(\cdot)$ stands for the Kuratowski measure of noncompactness.

Lemma 2.7. [15] *Let X and V be two Hausdorff topological spaces. Assume that $\Phi\colon X \to 2^V \setminus \{\emptyset\}$ is a multi-valuedmap such that $\Phi(X) \subset \widetilde{K}$ and the graph of Φ is closed, where $\widetilde{K}$ is a compact set. Then Φ is u.s.c.*

The next result is a consequence of the phase space axioms.

Lemma 2.8. *Let $x\colon (-\infty, b] \to H$ be an $\mathcal{F}_t$-adapted measurable process such that the $\mathcal{F}_0$-adapted process $x_0 = \varphi(t) \in L^p_{\mathcal{F}_0}(\Omega, \mathcal{B})$ and $x|_{[0,b]} \in \mathcal{PC}([0, b], H)$, then*

$$\|x_s\|_{\mathcal{B}} \le M_b E\|\varphi\|_{\mathcal{B}} + K_b \sup_{0 \le s \le b} E\|x(s)\|_H,$$

where $K_b = \sup\{K(t) : 0 \le t \le b\}$, $M_b = \sup\{M(t) : 0 \le t \le b\}$.

Lemma 2.9. [8] *For any $p \ge 1$ and for arbitrary $L^0_2(K, H)$-valued predictable process $\phi(\cdot)$ such that*

$$\sup_{s \in [0,t]} E \left\| \int_0^s \phi(v)\, dw(v) \right\|_H^{2p} \le (p(2p-1))^p \left(\int_0^t \left(E\|\phi(s)\|_{L^0_2}^{2p} \right)^{1/p} ds \right)^p, \quad t \in [0, \infty).$$

Lemma 2.10. [12] *Let D be a nonempty, bounded, closed subset of a Banach space H, and $\Phi\colon D \to \mathcal{P}(D)$ be a u.s.c. condensing multi-valued map. If for every $x \in D$, $\Phi(x)$ is a closed and convex subset of D, then Φ admits a fixed point in D.*

3. Existence of mild solutions

In this section, we prove the existence of mild solutions for the system (1.1)–(1.3). Assume that the following hypotheses:

(H1) $T_\alpha(t)$ is an analytic resolvent of analyticity type (ω_0, θ_0) and $M = \sup_{t \in [0,b]} \|T_\alpha(t)\|_H < \infty$.

(H2) The function $G\colon [0, b] \times \mathcal{B} \to H$ is continuous, and there exists a positive constant L_G such that

$$E\|G(t_1, \psi_1) - G(t_2, \psi_2)\|_H^p \le L_G[|t_1 - t_2| + \|\psi_1 - \psi_2\|_{\mathcal{B}}^p], \qquad t \in [0, b],\ \psi_1, \psi_2 \in \mathcal{B},$$
$$E\|G(t, \psi)\|_H^p \le L_G(\|\psi\|_{\mathcal{B}}^p + 1), \qquad\qquad t \in [0, b],\ \psi \in \mathcal{B}.$$

(H3) The functional $F\colon [0, b] \times \mathcal{B} \to \mathbb{R}$ satisfies the following conditions:

  (i) $F(t, \cdot)\colon [0, b] \to \mathbb{R}$ is measurable for all $\psi \in \mathcal{B}$.

(ii) $F(t, \psi) \colon \mathcal{B} \to \mathbb{R}$ is locally Lipschitz continuous for a.e. $t \in [0, b]$.

(iii) There exist functions $\{\Lambda_{n^*}\} \subset L^1([0, b], \mathbb{R}^+)$ such that

$$\sup_{\|\psi\|_{\mathcal{B}}^p \leq n^*} E\|\partial F(t, \psi)\|_H^p \leq \Lambda_{n^*}(t)$$

for a.e. $t \in [0, b]$, $n^* > 0$ with

$$\liminf_{n^* \to +\infty} \frac{\|\Lambda_{n^*}\|_{L^1}}{n^*} = 0,$$

where $E\|\partial F(t, \psi)\|_H^p = \sup\{E\|f\|_H^p : f \in \partial F(t, \psi)\}$.

(H4) There exist $\gamma_i > 0$, $i = 1, \ldots, N$, such that

$$EH_d^p(g_i(t_1, \psi_1), g_i(t_2, \psi_2)) \leq \gamma_i[|t_1 - t_2| + \|\psi_1 - \psi_2\|_{\mathcal{B}}^p], \qquad t \in (t_i, s_i], \ \psi_1, \psi_2 \in \mathcal{B},$$
$$EH_d^p(g_i(t, \psi), 0) \leq \gamma_i(\|\psi\|_{\mathcal{B}}^p + 1), \qquad\qquad t \in (t_i, s_i], \ \psi \in \mathcal{B}.$$

Let q satisfy $1/p + 1/q = 1$ and an operator $\Psi \colon L^q([0, b], H) \to \mathcal{P}(L^p([0, b], H))$ be defined by

$$\Psi(\psi) = \{f \in L^p([0, b], H) : f(t) \in \partial F(t, \psi) \text{ a.e. } t \in [0, b]\}, \quad \psi \in L^q([0, b], H).$$

We also need some auxiliary results.

Lemma 3.1. (Compare with [30]) *Let condition* (H3) *hold and the operator* Ψ *satisfy: if* $x_n \to x \in L^q([0, b], H)$, $f_n \xrightarrow{w} f \in L^p([0, b], H)$ *and* $f_n \in \Psi(x_n)$. *Then one has* $f \in \Psi(x)$.

Lemma 3.2. (Compare with [23]) *Let condition* (H3) *hold. Then for every* $x \in L^q([0, b], H)$, *the set* $\Psi(x)$ *is nonempty weakly compact convex.*

Now, we are in a position to state the existence theorem

Theorem 3.3. *If assumptions* (H1)–(H4) *are satisfied, then for each* $u \in U_{\mathrm{ad}}$, *the system* (1.1)–(1.3) *has at least one mild solution on* $[0, b]$, *provided that*

$$(3.1) \qquad 2^{p-1} K_b^p \max_{1 \leq i \leq N} \left[(8^{p-1} M^p + 1)\gamma_i + 4^{p-1}(2^{p-1} M^p + 1)L_G\right] < 1.$$

Proof. Consider the space $\mathcal{Y} = \{x \colon (-\infty, b] \to H; x(0) = \varphi(0), x|_{[0,b]} \in \mathcal{PC}(H)\}$ endowed with the uniform convergence topology and consider the operator $\Phi \colon \mathcal{Y} \to \mathcal{P}(\mathcal{Y})$ defined by

$$\Phi(x) = \{v \in \mathcal{Y} : v(t) \text{ satifies } (3.2)\},$$

where (3.2) is given by

$$
(3.2) \qquad v(t) = \begin{cases}
T_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, \overline{x}_t) \\
\quad + \int_0^t T_\alpha(t-s)B(s)u(s)\,ds \\
\quad + \int_0^t T_\alpha(t-s)\mu f(s)\,dw(s), & t \in [0, t_1],\ i = 0, \\
\widetilde{g}_i(t), & t \in (t_i, s_i],\ i \geq 1, \\
T_\alpha(t-s_i)[\widetilde{g}_i(s_i) - G(s_i, \overline{x}_{s_i})] \\
\quad + G(t, \overline{x}_t) + \int_{s_i}^t T_\alpha(t-s)B(s)u(s)\,ds \\
\quad + \int_{s_i}^t T_\alpha(t-s)\mu f(s)\,dw(s), & t \in (s_i, t_{i+1}],\ i \geq 1,
\end{cases}
$$

and $\overline{x} \colon (-\infty, 0] \to H$ is such that $\overline{x}_0 = \varphi$ and $\overline{x} = x$ on $[0, b]$, where $f(t) \in \Psi(\overline{x}_t) = \{f \in L^p([0,b], H) : f(t) \in \partial F(t, \overline{x}_t)$ a.e. $t \in [0,b]\}$, $\overline{x}_t \in \mathcal{B}$, and $\widetilde{g}_i(t) \in g_i(t, \overline{x}_t)$, $i = 1, \dots, N$. From (H1) and Hölder's inequality, the following inequality holds:

$$
E\left\| \int_{s_i}^t T_\alpha(t-s)B(s)u(s)\,ds \right\|_H^p \leq E\left[\int_{s_i}^t \|T_\alpha(t-s)\|_H \|B(s)u(s)\|_H\,ds \right]^p
$$
$$
\leq M^p \|B\|_\infty^p (t_{i+1} - s_i)^{p-1} E \int_{s_i}^t \|u(s)\|_Y^p\,ds
$$
$$
\leq M^p \|B\|_\infty^p (t_{i+1} - s_i)^{p-1} \|u\|_{L_{\mathcal{F}}^p([0,b],Y)}^p.
$$

Then from Bochner's Theorem, it follows that $T_\alpha(t-s)B(s)u(s)$ are integrable on (s_i, t), $i = 0, 1, \dots, N$. It is clear that Φ is a well-defined operator on $\mathcal{Y}$. We shall show that Φ satisfies the hypotheses of Lemma 2.10. The proof will be given in several steps.

Step 1: There exists $n \in \mathbb{N}$ such that $\Phi(B_n) \subset B_n$, where $B_n = \{x \in \mathcal{Y} : \|x\|_{\mathcal{PC}}^p \leq n\}$. To the contrary, assume that there exist $\{x_n\}, \{v_n\} \subset \mathcal{Y}$ such that $x_n \in B_n$, $v_n \in \Phi(x_n)$ and $v_n \notin B_n$, $n \in \mathbb{N}$. Then, there exist sequences $\{f_n\}$ and $\{\widetilde{g}_{i,n}\}$ with $f_n(t) \in \Psi(\overline{x}_{n,t})$ and $\widetilde{g}_{i,n}(t) \in g_i(t, \overline{x}_{n,t})$, such that

$$
v_n(t) = \begin{cases}
T_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, \overline{x}_{n,t}) \\
\quad + \int_0^t T_\alpha(t-s)B(s)u(s)\,ds \\
\quad + \int_0^t T_\alpha(t-s)\mu f_n(s)\,dw(s), & t \in [0, t_1],\ i = 0, \\
\widetilde{g}_{i,n}(t), & t \in (t_i, s_i],\ i \geq 1, \\
T_\alpha(t-s_i)[\widetilde{g}_{i,n}(s_i) - G(s_i, \overline{x}_{n,s_i})] \\
\quad + G(t, \overline{x}_{n,t}) + \int_{s_i}^t T_\alpha(t-s)B(s)u(s)\,ds \\
\quad + \int_{s_i}^t T_\alpha(t-s)\mu f_n(s)\,dw(s), & t \in (s_i, t_{i+1}],\ i \geq 1.
\end{cases}
$$

If $x \in B_n$, from Lemma 2.8, it follows that

$$
\|\overline{x}_s\|_{\mathcal{B}}^p \leq 2^{p-1}[M_b\|\varphi\|_{\mathcal{B}}]^p + 2^{p-1}K_b^p n := n^*.
$$

By (H1)–(H3), we have for $t \in [0, t_1]$,

$$E\|v_n(t)\|_H^p \leq 4^{p-1}E\|T_\alpha(t)[\varphi(0) - G(0, \varphi)]\|_H^p + 4^{p-1}E\|G(t, \overline{x}_{n,t})\|_H^p$$
$$+ 4^{p-1}E\left\|\int_0^t T_\alpha(t-s)B(s)u(s)\,ds\right\|_H^p$$
$$+ 4^{p-1}E\left\|\int_0^t T_\alpha(t-s)\mu f_n(s)\,dw(s)\right\|_H^p$$
$$\leq 8^{p-1}M^p[E\|\varphi(0)\|_H^p + L_G(\|\varphi\|_{\mathcal{B}}^p + 1)] + 4^{p-1}L_G(\|\overline{x}_t\|_{\mathcal{B}}^p + 1)$$
$$+ 4^{p-1}M^p\|B\|_\infty^p t_1^{p-1}\|u\|_{L_{\mathcal{F}}^p([0,t_1],Y)}^p$$
$$+ 4^{p-1}C_pM^p\|\mu\|_{L(H)}^p \left[\int_0^t [E\|f_n(s)\|_{L_2^0}^p]^{2/p}\,ds\right]^{p/2}$$
$$\leq 8^{p-1}M^p[E\|\varphi(0)\|_H^p + L_G(\|\varphi\|_{\mathcal{B}}^p + 1)] + 4^{p-1}L_G(n^* + 1)$$
$$+ 4^{p-1}M^p\|B\|_\infty^p t_1^{p-1}\|u\|_{L_{\mathcal{F}}^p([0,t_1],Y)}^p$$
$$+ 4^{p-1}C_pM^p\|\mu\|_{L(K,H)}^p t_1^{p/2-1}\|\Lambda_{n^*}\|_{L^1([0,t_1],\mathbb{R}^+)}.$$

For any $t \in (t_i, s_i]$, $i = 1, \ldots, N$; by (H4), we have

$$E\|v_n(t)\|_H^p \leq \gamma_i(\|\overline{x}_t\|_{\mathcal{B}}^p + 1) \leq \gamma_i(n^* + 1).$$

Similarly, for any $t \in (s_i, t_{i+1}]$, $i = 1, \ldots, N$, we have

$$E\|v_n(t)\|_H^p \leq 4^{p-1}E\|T_\alpha(t - s_i)[\widetilde{g}_{i,n}(s_i) - G(s_i, \overline{x}_{n,s_i})]\|_H^p$$
$$+ E\|G(t, \overline{x}_{n,t})\|_H^p + E\left\|\int_{s_i}^t T_\alpha(t-s)B(s)u(s)\,ds\right\|_H^p$$
$$+ E\left\|\int_{s_i}^t T_\alpha(t-s)\mu f_n(s)\,dw(s)\right\|_H^p$$
$$\leq 8^{p-1}M^p[\gamma_i(n^* + 1) + L_G(n^* + 1)] + 4^{p-1}L_G(n^* + 1)$$
$$+ 4^{p-1}M^p\|\mu\|_{L(H)}^p\|B\|_\infty^p(t_{i+1} - s_i)^{p-1}\|u\|_{L_{\mathcal{F}}^p([s_i,t_{i+1}],Y)}^p$$
$$+ 4^{p-1}C_pM^p\|\mu\|_{L(K,H)}^p(t_{i+1} - s_i)^{p/2-1}\|\Lambda_{n^*}\|_{L^1([s_i,t_{i+1}],\mathbb{R}^+)}.$$

Then, for all $t \in [0, b]$, we have

$$n < E\|v_n(t)\|_H^p \leq M_* + \max_{1 \leq i \leq N}\left\{(8^{p-1}M^p + 1)\gamma_i + 4^{p-1}(2^{p-1}M^p + 1)L_G\right\}n^*$$
$$+ 4^{p-1}C_pM^p\|\mu\|_{L(K,H)}^p b^{p/2-1}\|\Lambda_{n^*}\|_{L^1([0,b],\mathbb{R}^+)},$$

where

$$M_* = 8^{p-1}M^p[E\|\varphi(0)\|_H^p + L_G(\|\varphi\|_{\mathcal{B}}^p + 1)]$$
$$+ \max_{1 \leq i \leq N}\left\{(8^{p-1}M^p + 1)\gamma_i + 4^{p-1}(2^{p-1}M^p + 1)L_G\right\}$$
$$+ 4^{p-1}M^p\|B\|_\infty^p b^{p-1}\|u\|_{L_{\mathcal{F}}^p([0,b],Y)}^p.$$

Thus,

$$n^* < 2^{p-1}[M_b\|\varphi\|_{\mathcal{B}}]^p + 2^{p-1}K_b^p\Big[M_* + \max_{1\le i\le N}\big\{(8^{p-1}M^p+1)\gamma_i + 4^{p-1}(2^{p-1}M^p+1)L_G\big\}n^*$$

$$+ 4^{p-1}C_pM^p\|\mu\|_{L(K,H)}^p b^{p/2-1}\|\Lambda_{n^*}\|_{L^1([0,b],\mathbb{R}^+)}\Big].$$

Dividing on both sides by n^* and taking the lower limit as $n^* \to \infty$, we get

$$2^{p-1}K_b^p\max_{1\le i\le N}\big\{(8^{p-1}M^p+1)\gamma_i + 4^{p-1}(2^{p-1}M^p+1)L_G\big\} \ge 1.$$

This is a contradiction with the formula (3.1). Hence for some $n \in \mathbb{N}$, $\Phi(B_n) \subset B_n$.

Step 2: $\Phi(x)$ is convex for each $x \in \mathcal{Y}$. In fact, if v^1, v^2 belong to $\Phi(x)$, then there exist $f_1(t), f_2(t) \in \Psi(\overline{x}_t)$ and $\widetilde{g}_{i,1}(t), \widetilde{g}_{i,2}(t) \in g_i(t,\overline{x}_t)$ for $t \in [0,t_1]$ such that

$$v^k(t) = T_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t,\overline{x}_t)$$

$$+ \int_0^t T_\alpha(t-s)B(s)u(s)\,ds + \int_0^t T_\alpha(t-s)\mu f_k(s)\,dw(s), \quad k=1,2.$$

Let $0 \le \lambda \le 1$. For $t \in [0,t_1]$ we have

$$(\lambda v^1 + (1-\lambda)v^2)(t) = T_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t,\overline{x}_t) + \int_0^t T_\alpha(t-s)B(s)u(s)\,ds$$

$$+ \int_0^t T_\alpha(t-s)\mu[\lambda f_1(s) + (1-\lambda)f_2(s)]\,dw(s).$$

For any $t \in (t_i, s_i]$, $i = 1, \ldots, N$, we have

$$(\lambda v^1 + (1-\lambda)v^2)(t) = [\lambda\widetilde{g}_{i,1}(t) + (1-\lambda)\widetilde{g}_{i,2}(t)].$$

Similarly, for any $t \in (s_i, t_{i+1}]$, $i = 1, \ldots, N$, we have

$$v^k(t) = T_\alpha(t-s_i)[\widetilde{g}_{i,k}(s_i) - G(s_i,\overline{x}_{s_i})] + G(t,\overline{x}_t)$$

$$+ \int_{s_i}^t T_\alpha(t-s)B(s)u(s)\,ds + \int_{s_i}^t T_\alpha(t-s)\mu f_k(s)\,dw(s), \quad k=1,2.$$

Let $0 \le \lambda \le 1$. For each $t \in (s_i, t_{i+1}]$, $i = 1, \ldots, N$, we have

$$(\lambda v^1 + (1-\lambda)v^2)(t) = T_\alpha(t-s_i)\big\{[\lambda\widetilde{g}_{i,1}(s_i) + (1-\lambda)\widetilde{g}_{i,2}(s_i)] - G(s_i,\overline{x}_{s_i})\big\} + G(t,\overline{x}_t)$$

$$+ \int_{s_i}^t T_\alpha(t-s)B(s)u(s)\,ds$$

$$+ \int_{s_i}^t T_\alpha(t-s)\mu[\lambda f_1(s) + (1-\lambda)f_2(s)]\,dw(s).$$

In view of Lemma 3.2 and g_i $(i = 1, 2, \ldots, N)$ are convex values, we have $(\lambda v^1 + (1-\lambda)v^2) \in \Phi(x)$.

Step 3: $\Phi(x)$ is closed for each $x \in \mathcal{Y}$. Let $\{v_n\}_{n\geq 0} \in \Phi(x)$ such that $v_n \to v$ in $\mathcal{Y}$. Then there exists $f_n(t) \in \Psi(\overline{x}_{n,t})$ and $\widetilde{g}_{i,n}(t) \in g_i(t, \overline{x}_{n,t})$ such that

$$
v_n(t) = \begin{cases}
T_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, \overline{x}_t) \\
\quad + \int_0^t T_\alpha(t-s)B(s)u(s)\,ds \\
\quad + \int_0^t T_\alpha(t-s)\mu f_n(s)\,dw(s), & t \in [0, t_1],\ i = 0, \\
\widetilde{g}_{i,n}(t), & t \in (t_i, s_i],\ i \geq 1, \\
T_\alpha(t-s_i)[\widetilde{g}_{i,n}(s_i) - G(s_i, \overline{x}_{s_i})] \\
\quad + G(t, \overline{x}_t) + \int_{s_i}^t T_\alpha(t-s)B(s)u(s)\,ds \\
\quad + \int_{s_i}^t T_\alpha(t-s)\mu f_n(s)\,dw(s), & t \in (s_i, t_{i+1}],\ i \geq 1.
\end{cases}
$$

By Lemma 3.2, passing to a subsequence if necessary, we can deduce that $f_n \xrightarrow{w} f$ in $L^p([0,b], H)$. By applying condition (H3), we know that $\{f_n\} \subset L^p([0,b], H)$ is bounded. Thus, there exists a subsequence, denoted it again by $\{f_n\}$, which converges weakly to f^* in $L^p([0,b], H)$. By the uniqueness of the limit, it is easy to have $f = f^*$. Since g_i, $i = 1, \ldots, N$, are closed operators, it follows that $\widetilde{g}_{i,n}(t) \to \widetilde{g}_i(t) \in g_i(t, \overline{x}_t)$ for every $t \in (t_i, s_i]$, $i = 1, \ldots, N$. Therefore, we have for each $t \in [0, b]$,

$$
v_n(t) \to v(t) = \begin{cases}
T_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, \overline{x}_t) \\
\quad + \int_0^t T_\alpha(t-s)B(s)u(s)\,ds \\
\quad + \int_0^t T_\alpha(t-s)\mu f(s)\,dw(s), & t \in [0, t_1],\ i = 0, \\
\widetilde{g}_i(t), & t \in (t_i, s_i],\ i \geq 1, \\
T_\alpha(t-s_i)[\widetilde{g}_i(s_i) - G(s_i, \overline{x}_{s_i})] \\
\quad + G(t, \overline{x}_t) + \int_{s_i}^t T_\alpha(t-s)B(s)u(s)\,ds \\
\quad + \int_{s_i}^t T_\alpha(t-s)\mu f(s)\,dw(s), & t \in (s_i, t_{i+1}],\ i \geq 1.
\end{cases}
$$

This shows that $v \in \Phi(x)$.

Step 4: $\Phi(x)$ is u.s.c. and condensing. For this purpose, we decompose Φ as $\Phi = \Phi_1 + \Phi_2$, where the operators Φ_1, Φ_2 are defined on B_n respectively by

$$
(\Phi_1 x)(t) = \begin{cases}
T_\alpha(t)[\varphi(0) - G(0,\varphi)] + G(t, \overline{x}_t), & t \in [0, t_1],\ i = 0, \\
\widetilde{g}_i(t), & t \in (t_i, s_i],\ i \geq 1, \\
T_\alpha(t-s_i)[\widetilde{g}_i(s_i, \overline{x}_{s_i}) - G(s_i, \overline{x}_{s_i})] + G(t, \overline{x}_t), & t \in (s_i, t_{i+1}],\ i \geq 1
\end{cases}
$$

and

$$
\Phi_2(x) = \{\zeta \in \mathcal{Y} : v(t) \text{ satifies (3.3)}\},
$$

where (3.3) is given by

$$(3.3) \qquad \zeta(t) = \begin{cases} \int_0^t T_\alpha(t-s)B(s)u(s)\,ds \\ \quad + \int_0^t T_\alpha(t-s)\mu f(s)\,dw(s), & t \in [0,t_1],\ i=0, \\ 0, & t \in (t_i,s_i],\ i \geq 1, \\ \int_{s_i}^t S_\alpha(t-s)B(s)u(s)\,ds \\ \quad + \int_{s_i}^t T_\alpha(t-s)\mu f(s)\,dw(s), & t \in (s_i,t_{i+1}],\ i \geq 1, \end{cases}$$

where $f(t) \in \Psi(\overline{x}_t)$ and $\widetilde{g}_i(t) \in g_i(t,\overline{x}_t)$. By Lemma 2.10, we need only show that Φ_1 is contractive and Φ_2 is u.s.c. and compact.

Claim 1: Φ_1 is a contraction map on B_n. Let $t \in [0,t_1]$ and $x^*, x^{**} \in \mathcal{BPC}$. From (H2) and Lemma 2.8, we have

$$\begin{aligned} & EH_d^p((\Phi_1 x^*)(t), (\Phi_1 x^{**})(t)) \\ &\leq L_G \|(\overline{x^*})_t - (\overline{x^{**}})_t\|_{\mathcal{B}}^p \\ &\leq 2^{p-1} K_b^p L_G \sup \left\{ E\|\overline{x^*}(\tau) - \overline{x^{**}}(\tau)\|_H^p, 0 \leq \tau \leq t \right\} \\ &\leq 2^{p-1} K_b^p L_G \sup_{s \in [0,b]} E\|\overline{x^*}(s) - \overline{x^{**}}(s)\|_H^p \\ &= 2^{p-1} K_b^p L_G \sup_{s \in [0,b]} E\|x^*(s) - x^{**}(s)\|_H^p \quad (\text{since } \overline{x} = x \text{ on } [0,b]) \\ &= 2^{p-1} K_b^p L_G \|x^* - x^{**}\|_{\mathcal{PC}}^p. \end{aligned}$$

For any $t \in (t_i,s_i]$, $i=1,\ldots,N$, we have

$$EH_d^p((\Phi_1 x^*)(t), (\Phi_1 x^{**})(t)) \leq 2^{p-1} K_b^p \gamma_i \|x^* - x^{**}\|_{\mathcal{PC}}^p.$$

Similarly, for any $t \in (t_i,t_{i+1}]$, $i=1,\ldots,N$, we have

$$\begin{aligned} EH_d^p((\Phi_1 x^*)(t), (\Phi_1 x^{**})(t)) &\leq 4^{p-1} M^p \big[\gamma_i \|(\overline{x^*})_t - (\overline{x^{**}})_t\|_{\mathcal{B}}^p + L_G \|(\overline{x^*})_{s_i} - (\overline{x^{**}})_{s_i}\|_{\mathcal{B}}^p \big] \\ &\quad + 2^{p-1} L_G \|(\overline{x^*})_t) - (\overline{x^{**}})_t\|_{\mathcal{B}}^p \\ &\leq 4^{p-1} K_b^p [2^{p-1} M^p \gamma_i + (1 + 2^{p-1} M^p) L_G] \|x^* - x^{**}\|_{\mathcal{PC}}^p. \end{aligned}$$

Thus, for all $t \in [0,b]$, we have

$$EH_d^p((\Phi_1 x^*)(t), (\Phi_1 x^{**})(t)) \leq L_0 \|x^* - x^{**}\|_{\mathcal{PC}}^p.$$

Taking supremum over t,

$$EH_d^p(\Phi_1 x^*, \Phi_1 x^{**}) \leq L_0 \|x^* - x^{**}\|_{\mathcal{PC}}^p,$$

where $L_0 = 2^{p-1} K_b^p \max_{1 \leq i \leq N} [(4^{p-1} M^p + 1)\gamma_i + 2^{p-1}(1 + M^p) L_G]$. By (3.1), we see that $L_0 < 1$. Hence, Φ_1 is a contraction.

Claim 2: $\Phi_2 x$ *is clearly bounded for all* $x \in B_n$.

Claim 3: Φ_2 *is equicontinuous on* B_n. Since $T_\alpha(\cdot)$ is strongly continuous, the function $s \to T_\alpha(s)$ is continuous in the uniform operator topology on $(0, b]$. Let $s_i < \varepsilon < t \le t_{i+1}$, $i = 0, 1, \ldots, N$, and $\delta > 0$ such that $\|T_\alpha(\tau_1) - T_\alpha(\tau_2)\|_H^p < \varepsilon$ for every $\tau_1, \tau_2 \in (s_i, t_{i+1}]$ with $|\tau_1 - \tau_2| < \delta$. For each $x \in B_n$, $0 < |\widetilde{\tau}| < \delta$, $t, t + \widetilde{\tau} \in [s_i, t_{i+1}]$, $i = 1, \ldots, N$, and $\zeta \in \Phi_2 x$, there exists $f \in \Psi(\overline{x}_t)$, we have

$$E\|\zeta(t + \widetilde{\tau}) - \zeta(t)\|_H^p$$

$$\le 4^{p-1} E \left\| \int_{s_i}^t [T_\alpha(t + \widetilde{\tau} - s) - T_\alpha(t - s)] B(s) u(s) \, ds \right\|_H^p$$

$$+ 4^{p-1} E \left\| \int_t^{t+\widetilde{\tau}} T_\alpha(t + \widetilde{\tau} - s) B(s) u(s)(s) \, ds \right\|_H^p$$

$$+ 4^{p-1} E \left\| \int_{s_i}^t [T_\alpha(t + \widetilde{\tau} - s) - T_\alpha(t - s)] \mu f(s) \, dw(s) \right\|_H^p$$

$$+ 4^{p-1} E \left\| \int_t^{t+\widetilde{\tau}} T_\alpha(t + \widetilde{\tau} - s) \mu f(s) \, dw(s) \right\|_H^p$$

$$\le 4^{p-1} \|B\|_\infty^p (t_{i+1} - s_i)^{p-1} \int_{s_i}^t \|S_\alpha(t + \widetilde{\tau} - s) - S_\alpha(t - s)\|_H^p E\|u(s)\|_H^p \, ds$$

$$+ 4^{p-1} \|B\|_\infty^p M^p \widetilde{\tau}^{p-1} \int_t^{t+\widetilde{\tau}} E\|u(s)\|_H^p \, ds$$

$$+ 4^{p-1} C_p \left[\int_{s_i}^t \|T_\alpha(t + \widetilde{\tau} - s) - T_\alpha(t - s)\|_H^p [E\|f(s)\|_H^p]^{2/p} \right]^{p/2} ds$$

$$+ 4^{p-1} C_p M^p \left[\int_t^{t+\widetilde{\tau}} [E\|f(s)\|_H^p]^{2/p} \, ds \right]^{p/2} ds$$

$$\le 4^{p-1} \varepsilon \|B\|_\infty^p (t_{i+1} - s_i)^{p-1} \|u\|_{L_{\mathcal{F}}^p((s_i, t_{i+1}], Y)}^p$$

$$+ 4^{p-1} \|B\|_\infty^p M^p \widetilde{\tau}^{p-1} \int_t^{t+\widetilde{\tau}} E\|u(s)\|_H^p \, ds$$

$$+ 4^{p-1} C_p \|\mu\|_{L(K,H)}^p (t_{i+1} - s_i)^{p/2-1} \varepsilon \int_{s_i}^t \Lambda_{n^*}(s) \, ds$$

$$+ 4^{p-1} C_p M^p \|\mu\|_{L(K,H)}^p \widetilde{\tau}^{p/2-1} \int_t^{t+\widetilde{\tau}} \Lambda_{n^*}(s) \, ds.$$

The right-hand side tends to zero as $\widetilde{\tau} \to 0$, and sufficiently small positive number ε. Hence Φ_2 maps B_n into an equicontinuous family of functions.

Claim 4: $(\Phi_2 B_n)(t)$ *is relatively compact in* H *for each* $t \in [0, b]$, where $(\Phi_2 B_n)(t) = \{\zeta(t) : \zeta \in \Phi_2 B_n\}$, $t \in [0, b]$. If $\tau, \tau' \in [\tau_k, \tau_{k+1}]$ for some $k = 1, 2, \ldots, n-1$, we can select points $s_i = \tau_1 < \tau_2 < \cdots < \tau_n = t$ such that $\|T_\alpha(t - \tau) - T_\alpha(t - \tau')\| \le \varepsilon$. For $x \in B_n$,

and $\zeta \in \Phi_2 x$, there exists $f(t) \in \Psi(\overline{x}_t)$, we have

$$E \left\| \int_{s_i}^t f(s)\, dw(s) \right\|_H^p \leq C_p (t_{i+1} - s_i)^{p/2-1} \int_{s_i}^{t_{i+1}} \Lambda_n(s)\, ds = l_i^*.$$

Then we find that

$$\int_{s_i}^t T_\alpha(t-s)\mu f(s)\, dw(s) = \sum_{k=1}^{n-1} \int_{\tau_k}^{\tau_{k+1}} [T_\alpha(t-s) - T_\alpha(t-\tau_k)]\mu f(s)\, dw(s)$$
$$+ \sum_{k=1}^{n-1} T_\alpha(t-\tau_k)\mu \int_{\tau_k}^{\tau_{k+1}} f(s)\, dw(s)$$
$$\in \mathcal{D}_{i,\varepsilon} + \sum_{k=1}^{n-1} T_\alpha(t-\tau_k)\mu B_{l_i^*}(0, H).$$

By the compactness of μ, we have $\left\{ \int_{s_i}^t T_\alpha(t-s)\mu f(s)\, dw(s) : x \in B_n \right\} \in \mathcal{D}_{i,\varepsilon} + \mathcal{J}_{i,\varepsilon}$, where $\operatorname{diam}(\mathcal{D}_{i,\varepsilon}) \leq \varepsilon \|\mu\|_{L(K,H)}^p l_i^*$ and $\mathcal{J}_{i,\varepsilon}$ are relatively compact, which permit us concluding that the set $\left\{ \int_{s_i}^t T_\alpha(t-s)\mu f(s)\, dw(s) : x \in B_n \right\}$ is relatively compact in H since ε is arbitrary. On the other hand, the set $\left\{ \int_{s_i}^t T_\alpha(t-s)B(s)u(s)\, ds \right\}$ is bounded and independent of x. Thus $(\Phi_2 B_n)(t)$ is relatively compact in H for each $t \in [0, b]$.

Step 5: Φ_2 has a closed graph. Let $x^{(n)} \to x^*$ $(n \to \infty)$, $\zeta^{(n)} \in \Phi_2 x^{(n)}$, $x^{(n)} \in B_n$ and $\zeta^{(n)} \to \zeta^*$. From Axiom (A), it is easy to see that $(\overline{x^{(n)}})_s \to (\overline{x^*})_s$ uniformly for $s \in (-\infty, b]$ as $n \to \infty$. We shall prove that $\zeta^* \in \Phi_2 \overline{x^*}$. Now $\zeta^{(n)} \in \Phi \overline{x^{(n)}}$ means that there exists $f^{(n)}(t) \in \Psi((\overline{x^{(n)}})_t)$ such that, for each $t \in [s_i, t_{i+1}]$, $i = 1, \ldots, N$,

$$\zeta^{(n)}(t) = \int_{s_i}^t T_\alpha(t-s)B(s)u(s)\, ds + \int_{s_i}^t T_\alpha(t-s)\mu f^{(n)}(s)\, dw(s).$$

According to (H3), we obtain that the boundedness of f^n. In view of the reflexivity of $L^p([0, b], H)$, without loss of generality, we can suppose that $f^n \xrightarrow{w} f^* \in L^p([0, b], H)$. A continuous linear operator $\Upsilon \colon L^p([0, b], H) \to PC([0, b], H)$ is defined as

$$\Upsilon(f)(\cdot) = \int_{s_i}^{\cdot} T_\alpha(\cdot - s)B(s)u(s)\, ds + \int_{s_i}^{\cdot} T_\alpha(\cdot - s)\mu f^*(s)\, dw(s).$$

By the compactness of μ, we have

$$\|\Upsilon(f^n)(t) - \Upsilon(f^*)(t)\|_H^p \leq C_p M^p \left[\int_{s_i}^t \left[E\|\mu[f^n(s) - f^*(s)]\|_H^p \right]^{2/p} ds \right]^{p/2}$$
$$\leq C_p M^p (t_{i+1} - s_i)^{p/2-1} \int_{s_i}^t \|\mu f^m(s) - \mu f^*(s)\|_H^p\, ds$$
$$\leq C_p M^p (t_{i+1} - s_i)^{p/2} \|\mu f^m - \mu f^*\|_{PC}^p \quad \text{as } n \to \infty.$$

Because $x^{(n)} \to x$ in $\mathcal{Y} \subset L^q([0, b], H)$, from Lemma 3.1, it follows that

$$\zeta^*(t) = \int_{s_i}^{t} T_\alpha(t - s)B(s)u(s)\, ds + \int_{s_i}^{t} T_\alpha(t - s)\mu f^*(s)\, dw(s)$$

for $f^*(t) \in \Psi((\overline{x^*})_t)$. Therefore, Φ_2 has a closed graph and Φ_2 is a completely continuous multi-valued map with compact value. So Φ_2 is u.s.c. Hence $\Phi = \Phi_1 + \Phi_2$ is u.s.c. and condensing. By Lemma 2.10, Φ admits a fixed point on B_n. Therefore, the system (1.1)–(1.3) has at least one mild solution x in B_n. The proof is complete. $\qquad\square$

4. Existence of time optimal controls

In this section, we consider a control problem and present a result on the existence of fractional impulsive stochastic time optimal controls.

Let x^u denote the mild solution of system (1.1)–(1.3) corresponding to the control $u \in U_{\mathrm{ad}}$. Let $x_0, x_1 \in H$ with $x_0 \neq x_1$. For some $t > 0$, we suppose that there exists an admissible control u satisfying $x(t, f, u) = x_1$ and $x_0 = x(0) = \varphi(0)$. Let us define the transition time, which is the first time t^u such that $x(t^u, f^u, u) = x_1$. The optimal time is defined by low limit t^0 of t^u such that $x(t^u, f^u, u) = x_1$ for admissible control u. If the control $u^0 \in U_{\mathrm{ad}}$ such that $x(t^0, f^0, u^0) = x_1$, then $u^0 \in U_{\mathrm{ad}}$ is called the time optimal control. It is sufficient to prove that the existence of the admissible control satisfies $x(t^0, f^0, u^0) = x_1$ with respect to (x_0, x_1).

Theorem 4.1. *Assume that all the hypotheses of Theorem* 3.3 *hold. Suppose that B is a strongly continuous operator. Then there exists a time optimal control with respect to (x_0, x_1).*

Proof. Let

$$t^0 = \inf\{t : x(t, f, u) = x_1, \text{ where } u \text{ is an admissible control}\}.$$

Then, there exists a monotone decreasing sequence $t^n \to t^0$ as $n \to \infty$. Assume that $u \in U_{\mathrm{ad}}$ is the corresponding control such that the trajectories

$$x^n(t^n, f^n, u^n) = \begin{cases} T_\alpha(t^n)[\varphi(0) - G(0, \varphi) + G(t^n, (\overline{x^n})_{t^n})] \\ \quad + \int_0^{t^n} T_\alpha(t^n - s)B(s)u^n(s)\, ds \\ \quad + \int_0^{t^n} T_\alpha(t^n - s)\mu f^n(s)\, dw(s), & t \in [0, t_1],\ i = 0, \\ \widetilde{g}_i^n(t^n), & t \in (t_i, s_i],\ i \geq 1, \\ T_\alpha(t^n - s_i)[\widetilde{g}_i^n(s_i) - G(s_i, (\overline{x^n})_{s_i})] \\ \quad + G(t^n, (\overline{x^n})_{t^n}) + \int_{s_i}^{t^n} T_\alpha(t^n - s)B(s)u^n(s)\, ds \\ \quad + \int_{s_i}^{t^n} T_\alpha(t^n - s)\mu f^n(s)\, dw(s), & t \in (s_i, t_{i+1}],\ i \geq 1 \end{cases}$$

satisfying $x^n(t^n, f^n, u^n) = x_1$, $n = 1, 2, \ldots$, and $f^n(s) \in \Psi((\overline{x^n})_s)$, $\widetilde{g}_i^n(t^n) \in g_i(t^n, (\overline{x^n})_{t^n})$. Notice that $x^n(t^n, f^n, u^n) \in B_n$. Because $\{u^n\} \subseteq U_{ad}$, $\{u^n\}$ is bounded in $L_{\mathcal{F}}^p([0, b], Y)$, so there exists a subsequence, relabeled as $\{u^n\}$, and $u^0 \in L_{\mathcal{F}}^p([0, b], Y)$ such that

$$u^n \xrightarrow{w} u^0 \quad \text{in } L_{\mathcal{F}}^p([0, b], Y) \quad \text{as } n \to \infty.$$

Since U_{ad} is closed and convex, we can use Mazur's lemma to conclude that there exists a convex combination of $\{u_n\}$, denoted by $\{\overline{u}_n\}$, which strongly converges to $u^0 \in U_{ad}$.

For every $t \in [0, b]$, we know that $x^n(t^n, f^n, u^n)$ can be rewritten as

$$x^n(t^n, f^n, u^n) = \begin{cases} T_\alpha(t^n)[\varphi(0) - G(0, \varphi) + G(t^n, (\overline{x^n})_{t^n})] \\ \quad + \int_0^{t^0} T_\alpha(t^n - s)B(s)u^n(s)\,ds \\ \quad + \int_0^{t^0} T_\alpha(t^n - s)\mu f^n(s)\,dw(s) \\ \quad + \int_{t^0}^{t^n} T_\alpha(t^n - s)B(s)u^n(s)\,ds \\ \quad + \int_{t^0}^{t^n} T_\alpha(t^n - s)\mu f^n(s)\,dw(s), & t \in [0, t_1],\ i = 0, \\ \widetilde{g}_i^n(t^n), & t \in (t_i, s_i],\ i \geq 1, \\ T_\alpha(t^n - s_i)[\widetilde{g}_i^n(s_i) - G(s_i, (\overline{x^n})_{s_i})] \\ \quad + G(t^n, (\overline{x^n})_{t^n}) + \int_{s_i}^{t^0} T_\alpha(t^n - s)B(s)u^n(s)\,ds \\ \quad + \int_{s_i}^{t^0} T_\alpha(t^n - s)f^n(s)\,dw(s) \\ \quad + \int_{t^0}^{t^n} T_\alpha(t^n - s)B(s)u^n(s)\,ds \\ \quad + \int_{t^0}^{t^n} T_\alpha(t^n - s)\mu f^n(s)\,dw(s), & t \in (s_i, t_{i+1}],\ i \geq 1. \end{cases}$$

By (H1), (H2) and (H4), we have

$$T_\alpha(t^n)[\varphi(0) - G(0, \varphi)] \to T_\alpha(t^0)[\varphi(0) - G(0, \varphi)] \quad \text{as } n \to \infty,$$
$$G(t^n, (\overline{x^n})_{t^n}) \to G(t^0, (\overline{x^0})_{t^0}) \quad \text{as } n \to \infty,$$
$$\widetilde{g}_i^n(t^n) \to \widetilde{g}_i^0(t^0) \quad \text{as } n \to \infty,$$
$$T_\alpha(t^n - s_i)[\widetilde{g}_i^n(s_i) - G(s_i, (\overline{x^n})_{s_i})] \to T_\alpha(t^0 - s_i)[\widetilde{g}_i^0(s_i) - G(s_i, (\overline{x^0})_{s_i})] \quad \text{as } n \to \infty,$$

where $\widetilde{g}_i^0(t^0) \in g_i(t^0, (\overline{x^0})_{t^0})$. It follows from condition (H3), Lemma 2.9 and Höder's inequality that

$$E\left\|\int_{t^0}^{t^n} T_\alpha(t^n - s)B(s)u^n(s)\,ds + \int_{t^0}^{t^n} T_\alpha(t^n - s)\mu f^n(s)\,dw(s)\right\|_H^p$$
$$\leq 2^{p-1}M^p\|B\|_\infty^p(t^n - t^0)^{p-1}\|u\|_{L_{\mathcal{F}}^p([0,t_1],Y)}^p$$
$$\quad + 2^{p-1}C_pM^p\|\mu\|_{L(K,H)}^p(t^n - t^0)^{p/2-1}\|\Lambda_{n^*}\|_{L^1([0,t_1],\mathbb{R}^+)}.$$

One has

$$\int_{t^0}^{t^n} T_\alpha(t^n - s)B(s)u^n(s)\,ds \to 0 \quad \text{as } n \to \infty,$$

$$\int_{t^0}^{t^n} T_\alpha(t^n - s)\mu f^n(s)\,dw(s) \to 0 \quad \text{as } n \to \infty.$$

By strongly continuity of B, we have

$$\|Bu^n - Bu^0\|_H^p \to 0 \quad \text{as } n \to \infty.$$

It follows that

$$E\left\|\int_{s_i}^t S_\alpha(t - s)[B(s)u^n(s) - B(s)u^*(s)]\,ds\right\|_H^p \leq M^p(t_{i+1} - s_i)^p\|Bu^n - Bu^0\|_H^p,$$

which implies that

$$\int_{s_i}^t S_\alpha(t - s)B(s)u^n(s)\,ds \to \int_{s_i}^t S_\alpha(t - s)B(s)u^0(s)\,ds \quad \text{as } n \to \infty.$$

By condition (H3), we can deduce that $\{f^n(s)\} \subseteq \Psi((\overline{x^n})_s)$ is bounded on $L^p([0, b], H)$. Therefore, by the reflexivity of $L^p([0, b], H)$, then there exist a subsequence of $\{f^n\}$, denoted again by $\{f^n\}$, and $f^0 \in L^p([0, b], H)$, such that

$$f^n \xrightarrow{\ w\ } f^0 \quad \text{in } L^p([0, b], H) \quad \text{as } n \to \infty.$$

By the compactness of μ, we have

$$\|\mu[f^n(s) - f^0(s)]\|_H^p \to 0 \quad \text{as } n \to \infty.$$

It follows that

$$E\left\|\int_{s_i}^t T_\alpha(t - s)[\mu f^n(s) - \mu f^0(s)]\,dw(s)\right\|_H^p \leq C_p M^p(t_{i+1} - s_i)^{p/2}\|\mu[f^n(s) - f^0(s)]\|_H^p.$$

Thus, from Lemma 3.1, it follows that

$$\int_{s_i}^t T_\alpha(t - s)\mu f^n(s)\,dw(s) \to \int_{s_i}^t T_\alpha(t - s)\mu f^0(s)\,dw(s) \quad \text{as } n \to \infty,$$

where $f^0 \in \Psi((\overline{x^0})_s)$. Hence,

$$x_1 = x(t^0, f^0, u^0)$$

$$
= \begin{cases}
T_\alpha(t^0)[\varphi(0) - G(0,\varphi) + G(t^0, (\overline{x^0})_{t^0})] \\
\quad + \int_0^{t^n} T_\alpha(t^0 - s)B(s)u^0(s)\,ds \\
\quad + \int_0^{t^0} T_\alpha(t^0 - s)\mu f^0(s)\,dw(s), & t \in [0, t_1],\ i = 0, \\
\widetilde{g}_i^0(t^0), & t \in (t_i, s_i],\ i \geq 1, \\
T_\alpha(t^0 - s_i)[\widetilde{g}_i^0(s_i) - G(s_i, (\overline{x^0})_{s_i})] + G(t^n, (\overline{x^0})_{t^n}) \\
\quad + \int_{s_i}^{t^0} T_\alpha(t^0 - s)B(s)u^0(s)\,ds \\
\quad + \int_{s_i}^{t^0} T_\alpha(t^0 - s)\mu f^0(s)\,dw(s), & t \in (s_i, t_{i+1}],\ i \geq 1,
\end{cases}
$$

where $f^0(s) \in \Psi((\overline{x^0})_s)$, $\widetilde{g}_i^0(t^0) \in g_i(t^0, (\overline{x^0})_{t^0})$, that is, u^0 is the time optimal control, and $x(t^0, f^0, u^0)$ is just the trajectory corresponding to the control u^0 and the proof is complete. $\qquad \square$

5. An example

In this section, we provide an example which comes from nonconvex problems for semipermeable media and leads to a fractional stochastic differential inclusions with Clarke's subdifferential of type (1.1)–(1.3). We remark that the nonmonotone semipermeability conditions are realized by various types of membranes, natural and artificial ones. These conditions arise in electrostatics, hydraulics and in the description of the flow of Binghams fluids. Here, a practical method to describe the system with fractional neutral functional differential inclusions with non-instantaneous impulses and Clarke subdifferential aims at reality in nonconvex semipermeable system identification and control, as real systems are distributed phenomena. We study this type of system with hereditary control variable and obtained the time optimal control of the system.

Consider the following fractional non-instantaneous impulsive stochastic partial differential systems of the form

(5.1)
$$
\begin{aligned}
{}^c D_t^{2/3} &\left[z(t,x) - \int_{-\infty}^0 e^{2s} \frac{|z(t+s,x)|}{1 + |z(t+s,x)|}\,ds \right] \\
&\in \left(\sum_{|\varsigma| \leq 2m} a_\varsigma(x)D^\varsigma \right) \left[z(t,x) - \int_{-\infty}^0 e^{2s} \frac{|z(t+s,x)|}{1 + |z(t+s,x)|}\,ds \right] \\
&\quad + J_t^{1/3} \int_\Pi \sin(x,y)u(t,y)\,dy \\
&\quad + J_t^{1/3} \left[\left(\int_\Pi \cos(x,y)\,dy \right) \left(\partial \int_{-\infty}^0 e^{4s} \sin(|z(t+s,x)|)\,ds \right) \right] \frac{dw(t)}{dt}, \\
&\quad t \in [0, 1/2) \cup [3/4, 1],\ x \in \Pi,
\end{aligned}
$$

(5.2) $$z(t, x) = 0, \quad (t, x) \in [0, 1] \times \partial \Pi,$$

(5.3) $$z(\tau, x) = \varphi(\tau, x), \quad (\tau, x) \in (-\infty, 0] \times \Pi,$$

(5.4) $$z(t, x) \in \int_{-\infty}^{0} t^2 e^{2s} |z(t + s, x)| \, ds, \quad t \in (1/2, 3/4], \ x \in \Pi,$$

where ${}^{c}D_t^{2/3}$ is a Caputo fractional partial derivative of order $\alpha = 2/3 \in (0, 1)$, φ is continuous, and Π is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial \Pi$, $a_\varsigma(x)$ is a smooth real function on $\overline{\Pi}$, $D^\varsigma = D_{x_1}^\varsigma \cdots D_{x_n}^\varsigma$, $D_{x_k} = \frac{\partial}{\partial x_k}$, $\varsigma = (\varsigma_1, \ldots, \varsigma_n)$ is an n-dimensional multi-index, $|\varsigma| = \varsigma_1 + \cdots + \varsigma_n$. $0 = t_0 = s_0$, $t_1 = 1/2$, $s_1 = 3/4$, $t_2 = 1$ are fixed real numbers, and $w(t)$ denotes a one-dimensional standard Wiener process.

The differential operator $L_m = \sum_{|\varsigma| \le 2m} a_\varsigma(x) D^\varsigma$ is strongly elliptic (see [32]). Let $H = Y = \{\omega : \omega \in C(\overline{\Pi}), \omega = 0 \text{ on } \partial \Pi\}$ with the norm $\| \cdot \|_{C(\overline{\Pi})}$ and $D(A) = \{\omega : \omega \in D(A_\infty), L_m \omega \in C(\overline{\Pi}), L_m \omega = 0 \text{ on } \partial \Pi\}$ define the operator $A : D(A) \to H$ by $Av = L_m v$, where $D(A_\infty) = \{\omega : \omega \in W^{2m,p}(\Pi)\}$ for all $p > n$. It is well known that A generates an analytic, noncompact semigroup $(T(t))_{t \ge 0}$ on H. From the subordination principle [3, Theorems 3.1 and 3.3], we know that A is the infinitesimal generator of an α-order fractional bounded and analytic resolvent operator $T_\alpha(t)$ ($t \ge 0$) of analyticity type (ω_0, θ_0). Therefore, the linear system corresponding to (5.1)–(5.4) has an associated solution operator $T_\alpha(\cdot)$ on H. We define the admissible control set $U_{\mathrm{ad}} = \{u(\cdot, y) \mid \mathbb{R} \to Y \text{ measurable}, \mathcal{F}_t\text{-adapted stochastic processes, and } \|u\|_{L_{\mathcal{F}}^p(\mathbb{R}, Y)} \le 1\}$.

Let the phase space $\mathcal{B}$ be

$$\mathcal{B} = C_{\overline{\gamma}} = \left\{ \psi \in C((-\infty, 0], H) : \lim_{\theta \to -\infty} e^{\overline{\gamma} \theta} \psi(\theta) \text{ exists in } H \right\},$$

where $\overline{\gamma} > 0$ and set

$$\|\psi\|_{\mathcal{B}} = \sup_{-\infty < \theta \le 0} e^{\overline{\gamma} \theta} \|\psi(\theta)\|_{C(\overline{\Pi})}, \quad \psi \in C_{\overline{\gamma}}.$$

It is well known that $C_{\overline{\gamma}}$ satisfies the Axioms (A)–(C) with $\widetilde{H} = 1$, $\widetilde{K}(t) = \max\{1, e^{-\overline{\gamma} t}\}$ and $\widetilde{M}(t) = e^{-\overline{\gamma} t}$ (see [21, Theorem 1.3.7] for details).

Additionally, we choose $0 < \overline{\gamma} < 2$ and take $\psi(\theta)(x) = \psi(\theta, x)$, $(\theta, x) \in (-\infty, 0] \times \mathcal{B}$. Let $z(s)(x) = z(s, x)$, $G : [0, 1] \times \mathcal{B} \to H$, $F : [0, 1] \times \mathcal{B} \to \mathbb{R}$, $g_i : (1/2, 3/4) \times \mathcal{B} \to \mathcal{P}(H)$ be the operators defined by

$$G(t, \psi)(x) = \int_{-\infty}^{0} e^{2s} \frac{|\psi(s)(x)|}{1 + |\psi(s)(x)|} \, ds, \quad \mathcal{N}(t, \psi)(x) = \psi(0)x - G(t, \psi)(x),$$

$$F(t, \psi)(x) = \int_{-\infty}^{0} e^{4s} \sin(|\psi(s)(x)|) \, ds, \quad g_1(t, \psi)(x) = \int_{-\infty}^{0} t^2 e^{2s} |\psi(s)(x)| \, ds.$$

For all $u \in C([0, 1] \times \overline{\Pi})$, we define an operator B as follows:

$$(Bu)(t)(x) = \int_{\Pi} \sin(x, y) u(t, y) \, dy, \quad \mu(x) = \int_{\Pi} \cos(x, y) \, dy.$$

Using these definitions, we can represent the system (5.1)–(5.4) in the abstract form (1.1)–(1.3). Moreover, we have for all $(t, \psi), (t_1, \psi_1) \in [0, 1] \times \mathcal{B}$,

$$
\begin{aligned}
E\|G(t, \psi) - G(t_1, \psi_1)\|_{C(\overline{\Pi})}^p &= E\left[\sup_{x\in\overline{\Pi}} \int_{-\infty}^0 e^{2s}\left(\frac{|\psi(s)(x)|}{1+|\psi(s)(x)|} - \frac{|\psi_1(s)(x)|}{1+|\psi_1(s)(x)|}\right) ds\right]^p \\
&\leq E\left[\int_{-\infty}^0 e^{(2-\overline{\gamma})s}e^{\overline{\gamma}s}\|\psi(s) - \psi_1(s)\|_{C(\overline{\Pi})}\, ds\right]^p \\
&\leq \left(\int_{-\infty}^0 e^{(2-\overline{\gamma})s}\, ds\right)^p \|\psi - \psi_1\|_{\mathcal{B}}^p \\
&\leq \widetilde{L}_G \|\psi - \psi_1\|_{\mathcal{B}}^p,
\end{aligned}
$$

and $E\|h(t, \psi)\|^p \leq \widetilde{L}_G\|\psi\|_{\mathcal{B}}^p$, where $\widetilde{L}_G = \frac{1}{(2-\overline{\gamma})^p}$. For all $(t, \psi), (t, \psi_1) \in \mathbb{R} \times \mathcal{B}$, we have

$$
\begin{aligned}
|F(t, \psi)(x) - F(t, \psi_1)(x)| &\leq \int_{-\infty}^0 e^{4s}|\sin(|\psi(s)(x)|) - \sin(|\psi_1(s)(x)|)|\, ds \\
&\leq \int_{-\infty}^0 e^{4s}|\psi(s)(x) - \psi_1(s)(x)|\, ds \\
&\leq \int_{-\infty}^0 e^{(4-\overline{\gamma})s}e^{\overline{\gamma}s}\|\psi(s) - \psi_1(s)\|_{C(\overline{\Pi})}\, ds \\
&\leq \frac{1}{4-\overline{\gamma}}\|\psi - \psi_1\|_{\mathcal{B}}.
\end{aligned}
$$

For any $\lambda > 0$, and $\psi, \psi_1 \in \mathcal{B}$, $t \in \mathbb{R}$, we have

$$
\begin{aligned}
&\left|\frac{F(t, \psi_1 + \lambda\psi)(x) - F(t, \psi_1)(x)}{\lambda}\right| \\
&= \left|\frac{\int_{-\infty}^0 e^{4s}[\sin(|\psi_1(s)(x) + \lambda\psi(s)(x)|) - \sin(|\psi_1(s)(x)|)]\, ds}{\lambda}\right| \\
&\leq \left|\int_{-\infty}^0 e^{4s}|\psi(s)(x)|\, ds\right|.
\end{aligned}
$$

Let $F^0(t, \psi)(x) = \int_{-\infty}^0 e^{4s}|\psi(s)(x)|\, ds$. Then it is easy to verify that

$$
\begin{aligned}
E\|\partial F(t, \psi)\|_{C(\overline{\Pi})}^p &= E\left[\sup_{x\in\overline{\Pi}} \int_{-\infty}^0 e^{4s}|\psi(s)(x)|\, ds\right]^p \leq E\left[\int_{-\infty}^0 e^{(4-\overline{\gamma})s}e^{\widetilde{\gamma}s}\|\psi(s)\|_{C(\overline{\Pi})}\, ds\right]^p \\
&\leq \left(\int_{-\infty}^0 e^{(4-\overline{\gamma})s}\, ds\right)^p \|\psi\|_{\mathcal{B}}^p \leq \widetilde{L}_F\|\psi\|_{\mathcal{B}}^p
\end{aligned}
$$

for all $(t, \psi) \in [0, 1] \times \mathcal{B}$, where $\widetilde{L}_F = \frac{1}{(4-\overline{\gamma})^p}$. For all $(t, \psi), (t_1, \psi_1) \in (1/2, 3/4] \times \mathcal{B}$, we

have

$$E\|g_1(t,\psi) - g_1(t_1,\psi_1)\|_{\overline{\Pi}}^p$$

$$= E\left[\sup_{x\in\overline{\Pi}}\left(\int_{-\infty}^0 t^2 e^{2s}|\psi(s)(x)|\,ds - \int_{-\infty}^0 t_1^2 e^{2s}|\psi_1(s)(x)|\,ds\right)\right]^p$$

$$\leq E\left[|t^2 - t_1^2|\int_{-\infty}^0 e^{\overline{\gamma}s}\,ds + |t_1|^2\int_{-\infty}^0 e^{(2-\overline{\gamma})s}e^{\overline{\gamma}s}\|\psi(s) - \psi_1(s)\|_{C(\overline{\Pi})}\,ds\right]^p$$

$$\leq 2^{p-1}\left[|t^2 - t_1^2|^p\left(\int_{-\infty}^0 e^{\overline{\gamma}s}\,ds\right)^p + |t_1|^{2p}\left(\int_{-\infty}^0 e^{(2-\overline{\gamma})s}\,ds\right)^p\|\psi - \psi_1\|_{\mathcal{B}}^p\right]$$

$$\leq \widetilde{\gamma}_1[|t - t_1| + \|\psi - \psi_1\|_{\mathcal{B}}^p],$$

and $\|g_1(t,\psi)\|_{\overline{\Pi}}^p \leq \widetilde{\gamma}_1\|\psi\|_{\mathcal{B}}^p$, where $\widetilde{\gamma}_1 = 2^{p-1}\left[(\frac{3}{2\overline{\gamma}})^p + (\frac{3}{4})^{2p}\frac{1}{(2-\overline{\gamma})^p}\right]$. Similarly, we have $\|\mu\|_{C(\overline{\Pi})}^p \leq \left(\sup_{x\in\overline{\Pi}}\int_{\overline{\Pi}}\cos(x,y)\,dy\right)^p$. Therefore, Assumptions (H1)–(H4) all hold in Section 3. On the other hand, for $u \in C([0,1]\times\overline{\Pi})$, we have

$$\|Bu\|_{C([0,1]\times\overline{\Pi})}^p = \left[\sup_{x\in\overline{\Pi}}\left(\int_{\overline{\Pi}}\sin(x,y)u(t,y)\,dy\right)\right]^p \leq \widetilde{L}_u\|u\|_{C([0,1]\times\overline{\Pi})}^p,$$

where $\widetilde{L}_u = \left(\sup_{x\in\overline{\Pi}}\int_{\overline{\Pi}}\sin(x,y)\,dy\right)^p$. Then, we can conclude that $B \in L^\infty([0,1], L(H))$. Let's take $K_b = M_b = b = 1$, $L_G = \widetilde{L}_G$, $\gamma_i = \widetilde{\gamma}_1$, $i = 1,\ldots,N$. It is easy to see that

$$2^{p-1}\left[(4^{p-1}M^p + 1)\widetilde{\gamma}_1 + 2^{p-1}(1 + M^p)\widetilde{L}_G\right] < 1,$$

whenever $\widetilde{L}_G$ and $\widetilde{\gamma}_1$ are small enough. Now the condition (3.1) holds. Further, all the conditions in Theorem 4.1 satisfied. Hence by Theorem 4.1, the system (5.1)–(5.4) has at least one time optimal pair.

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