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Homogeneous q-difference Equations and Generating Functions for the Generalized 2D-Hermite Polynomials

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Abstract. In this paper, we deduce several types of generating functions for q-2D Hermite polynomial by the method of homogeneous q-difference equations. Besides, we deduce a multilinear generating function for q-2D Hermite polynomials as a generalization of Andrew's result. Moreover, we build a transformation identity involving the generalized q-2D Hermite polynomials by the method of homogeneous q-difference equations. As an application, we give a transformation identity involving $D_q(m,n)$ and $D_q^*(m,n)$.

1. Introduction and statement of results

The 2D-Hermite polynomials $\{\widehat{H}_{m,n}(z_1,z_2)\}$ are defined by [16]

$$\widehat{H}_{m,n}(z_1,z_2) = \sum_{k=0}^{m\wedge n} \binom{m}{k}\binom{n}{k}(-1)^k k! z_1^{m-k} z_2^{n-k}, \quad \text{where } m\wedge n = \min(m,n). \tag{1.1}$$

Recently several mathematical physicists studied these types of polynomials from mathematical and physical points of view. Recent references on the 2D-Hermite polynomials are [14, 21–23].

Ismail and Zhang [15] introduced (1.2) and (1.3) as the q-analogues of (1.1):

$$\widetilde{H}_{m,n}(z_1,z_2) = \sum_{k=0}^{m\wedge n} \begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}}(q;q)_k\, z_1^{m-k} z_2^{n-k}, \tag{1.2}$$

$$h_{m,n}(z_1,z_2) = \sum_{k=0}^{m\wedge n} \begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{(m-k)(n-k)}(q;q)_k\, z_1^{m-k} z_2^{n-k}. \tag{1.3}$$

By using the raising and lowing operators, they got several types of generating functions for q-2D Hermite polynomials. This paper arose from the desire to understand the generalized q-2D Hermite polynomials through the method of homogeneous q-difference equations and to give some new applications.

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Throughout this paper, we use the standard q-notations [2, 12]. For $|q| < 1$, we define the q-shifted factorials as

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

For convenience, we also adopt the following compact notation for the multiple q-shifted factorial:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n,$$

where n is an integer or ∞. The basic hypergeometric series ${}_r\phi_s$ is defined as

$${}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n} \left((-1)^n q^{n(n-1)/2}\right)^{1+s-r} z^n.$$

The q-binomial coefficients are defined by

$$\begin{bmatrix} m \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

For any function $f(x)$, the q-derivative of $f(x)$ with respect to x, is defined as

$$\mathcal{D}_{q,x}\{f(x)\} = \frac{f(x) - f(qx)}{(1-q)x},$$

and we further define $\mathcal{D}_{q,x}^0\{f(x)\} = f(x)$, and for $n \geq 1$, $\mathcal{D}_{q,x}^n\{f(x)\} = \mathcal{D}_q\{\mathcal{D}_{q,x}^{n-1}\{f(x)\}\}$. The Leibniz rule for $D_{q,x}$ is

$$D_{q,x}^n(fg)(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} D_{q,x}^k f(x) D_{q,x}^{n-k} g(xq^k). \tag{1.4}$$

The q-binomial theorem is

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty}.$$

Two important special and limiting cases are the Euler identities

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = \frac{1}{(z;q)_\infty} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(-z)^n q^{\binom{n}{2}}}{(q;q)_n} = (z;q)_\infty. \tag{1.5}$$

The Rogers-Szegő polynomials are given by [1]

$$h_n(b,c|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} b^k c^{n-k} \quad \text{and} \quad g_n(b,c|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{n(k-n)} b^k c^{n-k}.$$

Chen, Fu and Zhang [6] introduced the following homogeneous Rogers-Szegő polynomials and gave some results

$$\overline{h}_n(x,y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} p_k(x,y) \quad \text{and} \quad p_k(x,y) = (x-y)(x-yq)\cdots(x-yq^{k-1}).$$

Motivated by Liu [19] and Cigler [9], Cao and Niu studied the extension of Cigler's polynomials by the q-difference equations [4]

$$C_n^{(\alpha)}(x,b)=\sum_{k=0}^{n}\begin{bmatrix}n+\alpha\\k\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}}(q;q)_k\,x^{n-k}b^k \tag{1.6}$$

and

$$D_n^{(\alpha)}(x,b)=\sum_{k=0}^{n}\begin{bmatrix}n+\alpha\\k\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k q^{k^2-kn}(q;q)_k\,x^{n-k}b^k. \tag{1.7}$$

Actually, it is natural to research the further extension of q-2D Hermite polynomials as follows:

$$H_{m,n}(z_1,z_2,z,a)=\sum_{k=0}^{m\wedge n}\begin{bmatrix}m\\k\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}}(a;q)_k\,z_1^{m-k}z_2^{n-k}z^k \tag{1.8}$$

and

$$Q_{m,n}(z_1,z_2,z,a)=\sum_{k=0}^{m\wedge n}\begin{bmatrix}m\\k\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k q^{-mk-nk+k^2}(a;q)_k\,z_1^{m-k}z_2^{n-k}z^k. \tag{1.9}$$

Remark 1.1. (1) When taking $a=q$, $m=n+\alpha$, $z_1=1$, $z_2=x$ and $z=b$ in (1.8), we obtain (1.6). Noting that m and n are integers, the equation $m=n+\alpha$ implies that α is an integer.

(2) When taking $a=q$, $m=n+\alpha$, $z_1=1$, $z_2=x$ and $z=bq^{n+\alpha}$ in (1.9), we obtain (1.7). Similarly, α is also an integer as in (1).

(3) When taking $z=1$ and $a=q$ in (1.8) and (1.9) respectively, $H_{m,n}(z_1,z_2,z,a)=\widetilde{H}_{m,n}(z_1,z_2)$, $Q_{m,n}(z_1,z_2,z,a)=q^{-mn}h_{m,n}(z_1,z_2)$.

(4) When taking $m=n$, $a=0$, $z=-tz_1z_2q^{2n}$ in (1.9), we have the q-Narayana polynomials [10]:

$$\widetilde{M}_n(t)=(z_1z_2)^n\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}^2 q^{k^2}t^k.$$

The method of q-difference operator has shown to be effective in solving generating functions for certain q-orthogonal polynomials. For more information, please refer to [5–8, 17]. Liu [18] established the key relation between q-exponential operator and q-difference equations by using analytic function. In [3], Cao gave the generating functions of q-hypergeometric polynomials by using the special homogeneous q-difference equations. Analytic functions expansion and q-difference equations often serve as a building block in finding the generating functions for orthogonal polynomials. Indeed, if an analytic function in several variables satisfies a system of q-difference equations, then, it can be expanded in terms of the product of some polynomials.

Liu [19] obtained several important results on Rogers-Szegő polynomials by the following q-difference equations with two variables. Liu and Zeng [20] further the relations between the q-difference equations and Rogers-Szegő polynomials.

Proposition 1.2. *Let $f(a,b)$ be a two-variable analytic function at $(0,0)\in\mathbb{C}^2$. Then*

(a) *f can be expanded in terms of $h_n(a,b|q)$ if and only if f satisfies the functional equation*

$$bf(aq,b)-af(a,bq)=(b-a)f(a,b).$$

(b) *f can be expanded in terms of $g_n(a,b|q)$ if and only if f satisfies the functional equation*

$$af(aq,b)-bf(a,bq)=(a-b)f(aq,bq).$$

The main task of the paper is to research the following homogeneous q-difference equations and the generating functions for the generalized q-2D Hermite polynomials.

Theorem 1.3. *Let $f(z_1,z_2,z,a)$ be a 4-variable analytic function at $(0,0,0,0)\in\mathbb{C}^4$. Then f can be expanded in terms of $H_{m,n}(z_1,z_2,z,a)$ if and only if f satisfies the functional equation*

$$\begin{aligned}&z[a\{f(z_1,z_2,zq^2,a)-f(z_1q,z_2,zq^2,a)-f(z_1,z_2q,zq^2,a)+f(z_1q,z_2q,zq^2,a)\}\\&-\{f(z_1,z_2,zq,a)-f(z_1q,z_2,zq,a)-f(z_1,z_2q,zq,a)+f(z_1q,z_2q,zq,a)\}]\\&=z_1z_2\{f(z_1,z_2,zq^2,a)-2f(z_1,z_2,zq,a)+f(z_1,z_2,z,a)\}.\end{aligned}\tag{1.10}$$

Proof. From the theory of several complex variables, we assume that

$$f(z_1,z_2,z,a)=\sum_{k=0}^{\infty}A_k(z_1,z_2,a)z^k.\tag{1.11}$$

Substituting the above equation into (1.10), we have

$$\begin{aligned}&z\Bigg[a\sum_{k=0}^{\infty}(zq^2)^k\{A_k(z_1,z_2,a)-A_k(z_1,z_2q,a)-A_k(z_1q,z_2,a)+A_k(z_1q,z_2q,a)\}\\&-\sum_{k=0}^{\infty}(zq)^k\{A_k(z_1,z_2,a)-A_k(z_1,z_2q,a)-A_k(z_1q,z_2,a)+A_k(z_1q,z_2q,a)\}\Bigg]\\&=z_1z_2\sum_{k=0}^{\infty}z^k(q^k-1)^2A_k(z_1,z_2,a).\end{aligned}\tag{1.12}$$

By direct calculation, equating coefficients of z^k on both sides of (1.12), we obtain

$$A_k(z_1,z_2,a)=\frac{(1-q)^2q^{k-1}(aq^{k-1}-1)}{(q^k-1)^2}D_{q,z_1}D_{q,z_2}\{A_{k-1}(z_1,z_2,a)\}.$$

Repeating this process, we have

$$A_k(z_1, z_2, a) = \frac{(1-q)^{2k}(-1)^k q^{\binom{k}{2}}(a;q)_k}{(q;q)_k^2}(D_{q,z_1}D_{q,z_2})^k\{A_0(z_1, z_2, a)\}.$$

Setting $f(z_1, z_2, 0, a) = A_0(z_1, z_2, a) = \sum_{m,n=0}^{\infty} \mu_{m,n} z_1^m z_2^n$, we have

$$\begin{aligned} A_k(z_1, z_2, a) &= \frac{(1-q)^{2k}(-1)^k q^{\binom{k}{2}}(a;q)_k}{(q;q)_k^2} \sum_{m,n=0}^{\infty} \mu_{m,n}\{D_{q,z_1}^k(z_1^m)\}\{D_{q,z_2}^k(z_2^n)\} \\ &= \frac{(1-q)^{2k}(-1)^k q^{\binom{k}{2}}(a;q)_k}{(q;q)_k^2} \sum_{m,n=k}^{\infty} \mu_{m,n} \begin{bmatrix} m \\ k \end{bmatrix} \frac{(q;q)_k}{(1-q)^k} z_1^{m-k} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(q;q)_k}{(1-q)^k} z_2^{n-k}. \end{aligned}$$

By using (1.11), we have

$$\begin{aligned} &f(z_1, z_2, z, a) \\ &= \sum_{k=0}^{\infty} \frac{(1-q)^{2k}(-1)^k z^k q^{\binom{k}{2}}(a;q)_k}{(q;q)_k^2} \sum_{m,n=k}^{\infty} \mu_{m,n} \begin{bmatrix} m \\ k \end{bmatrix} \frac{(q;q)_k}{(1-q)^k} z_1^{m-k} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(q;q)_k}{(1-q)^k} z_2^{n-k} \\ &= \sum_{m,n=0}^{\infty} \sum_{k=0}^{m\wedge n} \mu_{m,n}(-1)^k z^k q^{\binom{k}{2}}(a;q)_k \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} z_1^{m-k} z_2^{n-k} \\ &= \sum_{m,n=0}^{\infty} \mu_{m,n} H_{m,n}(z_1, z_2, z, a). \end{aligned}$$

We complete the proof of this theorem. □

Theorem 1.4. *Let $g(z_1, z_2, z, a)$ be a 4-variable analytic function at $(0,0,0,0) \in \mathbb{C}^4$. Then g can be expanded in terms of $Q_{m,n}(z_1, z_2, z, a)$ if and only if g satisfies the functional equation*

(1.13)
$$\begin{aligned} &z[a\{g(z_1q^{-1}, z_2q^{-1}, zq, a) - g(z_1q^{-1}, z_2, zq, a) - g(z_1, z_2q^{-1}, zq, a) + g(z_1, z_2, zq, a)\} \\ &\quad - \{g(z_1q^{-1}, z_2q^{-1}, z, a) - g(z_1q^{-1}, z_2, z, a) - g(z_1, z_2q^{-1}, z, a) + g(z_1, z_2, z, a)\}] \\ &= q^{-1}z_1z_2\{g(z_1, z_2, z, a) - 2g(z_1, z_2, zq, a) + g(z_1, z_2, zq^2, a)\}. \end{aligned}$$

Proof. From the theory of several complex variables, we assume that

$$g(z_1, z_2, z, a) = \sum_{k=0}^{\infty} A_k(z_1, z_2, a) z^k. \tag{1.14}$$

Substituting the above equation into (1.13), we have

(1.15)

$$q^{-1}z_1z_2\left\{\sum_{k=0}^{\infty}(q^k-1)^2z^kA_k(z_1,z_2,a)\right\}$$

$$=z\Bigg[a\sum_{k=0}^{\infty}(zq)^k\{A_k(z_1q^{-1},z_2q^{-1},a)-A_k(z_1q^{-1},z_2,a)-A_k(z_1,z_2q^{-1},a)+A_k(z_1,z_2,a)\}$$

$$-\sum_{k=0}^{\infty}z^k\{A_k(z_1q^{-1},z_2q^{-1},a)-A_k(z_1q^{-1},z_2,a)-A_k(z_1,z_2q^{-1},a)+A_k(z_1,z_2,a)\}\Bigg].$$

By direct calculation, equating coefficients of z^k on both sides of (1.15), we obtain

$$A_k(z_1,z_2,a)=\frac{(1-q)^2q^{-1}(aq^{k-1}-1)}{(q^k-1)^2}D_{q^{-1},z_1}D_{q^{-1},z_2}\{A_{k-1}(z_1,z_2,a)\}.$$

Repeating this process, we have

$$A_k(z_1,z_2,a)=\frac{(1-q)^{2k}(q^{-1})^k(-1)^k(a;q)_k}{(q;q)_k^2}(D_{q^{-1},z_1}D_{q^{-1},z_2})^k\{A_0(z_1,z_2,a)\}.$$

Setting $g(z_1,z_2,0,a)=A_0(z_1,z_2,a)=\sum_{m,n=0}^{\infty}\mu_{m,n}z_1^mz_2^n$, we have

$$\begin{aligned}A_k(z_1,z_2,a)&=\frac{(1-q)^{2k}(q^{-1})^k(-1)^k(a;q)_k}{(q;q)_k^2}\sum_{m,n=0}^{\infty}\mu_{m,n}\{D_{q^{-1},z_1}^k(z_1^m)\}\{D_{q^{-1},z_2}^k(z_2^n)\}\\&=\frac{(1-q)^{2k}(q^{-1})^k(-1)^k(a;q)_k}{(q;q)_k^2}\sum_{m,n=k}^{\infty}\mu_{m,n}\frac{z_1^{m-k}q^{-mk+\binom{k}{2}}}{(1-q^{-1})^k}\\&\quad\times\begin{bmatrix}m\\k\end{bmatrix}(q;q)_k\,(-1)^k\frac{z_2^{n-k}q^{-nk+\binom{k}{2}}}{(1-q^{-1})^k}\begin{bmatrix}n\\k\end{bmatrix}(q;q)_k\,(-1)^k.\end{aligned}$$

By using (1.14), we have

$$\begin{aligned}f(z_1,z_2,z,a)&=\sum_{m,n=0}^{\infty}\sum_{k=0}^{m\wedge n}\mu_{m,n}(-1)^kz^kq^{-mk-nk+k^2}\begin{bmatrix}m\\k\end{bmatrix}\begin{bmatrix}n\\k\end{bmatrix}(a;q)_k\,z_1^{m-k}z_2^{n-k}\\&=\sum_{m,n=0}^{\infty}\mu_{m,n}Q_{m,n}(z_1,z_2,z,a).\end{aligned}$$

We complete the proof of this theorem. □

The rest part of this paper is organized as follows. In Section 2, we give two types of generating functions for the generalized q-2D Hermite polynomials with four parameters by the method of homogeneous q-difference equations. In Section 3, we gain a mixed generating functions for the generalized q-2D Hermite polynomials and the homogeneous

Rogers-Szegő polynomials. In Section 4, we deduce multilinear generating function for the generalized q-2D Hermite polynomials as a generalization of Andrew's result. In Section 5, we obtain a dual multilinear generating functions for the generalized q-2D Hermite polynomials. In Section 6, we build a transformation identity involving the generating functions of $H_{m,n}(z_1, z_2, z, a)$. In Section 7, as an application, a transformational identity is given in regard of $D_q(m, n)$ and $D_q^*(m, n)$.

2. Generating function for the generalized q-2D Hermite polynomials

Ismail and Zhang [15] gave the following generating function for the q-2D Hermite polynomials by using the transformation and summation.

Proposition 2.1. *For* $\max\{|u|, |v|, |z_1|, |z_2|\} < 1$, *we have*

$$\sum_{m,n=0}^{\infty} \frac{u^m v^n}{(q;q)_m (q;q)_n} \widetilde{H}_{m,n}(z_1, z_2) = \frac{(uv;q)_\infty}{(uz_1, vz_2;q)_\infty} \tag{2.1}$$

and

$$\sum_{m,n=0}^{\infty} \frac{u^m}{(q;q)_m} \frac{v^n}{(q;q)_n} q^{(m-n)^2/2} h_{m,n}(z_1, z_2) = \frac{(-z_1 u q^{1/2}, -z_2 v q^{1/2}; q)_\infty}{(-uv;q)_\infty}. \tag{2.2}$$

In this section, we generalize the generating function for q-2D Hermite polynomials by the method of homogeneous q-difference equations.

Theorem 2.2. *For* $\max\{|u|, |v|, |z_1|, |z_2|, |a|, |z|\} < 1$, *we have*

$$\sum_{m,n=0}^{\infty} \frac{u^m v^n}{(q;q)_m (q;q)_n} H_{m,n}(z_1, z_2, z, a) = \frac{1}{(uz_1, vz_2;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (a;q)_k}{(q;q)_k^2} (uvz)^k \tag{2.3}$$

and

$$\begin{aligned} &\sum_{m,n=0}^{\infty} \frac{u^m}{(q;q)_m} \frac{v^n}{(q;q)_n} q^{(m^2+n^2)/2} Q_{m,n}(z_1, z_2, z, a) \\ &= (-z_1 u q^{1/2}, -z_2 v q^{1/2}; q)_\infty \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k^2} (-uvz)^k. \end{aligned} \tag{2.4}$$

Remark 2.3. For $z = 1$ and $a = q$ in the above theorem, (2.3) and (2.4) reduce to (2.1) and (2.2) by using (1.5), respectively.

Proof of Theorem 2.2. Denoting the right-hand side of (2.3) by $f(z_1,z_2,z,a)$, we verify that $f(z_1,z_2,z,a)$ satisfies (1.10):

$$\begin{aligned} f(z_1,z_2,z,a) &= \frac{1}{(uz_1uz_2;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(a;q)_k\,(uvz)^k}{(q;q)_k^2} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(a;q)_k\,(uvz)^k}{(q;q)_k^2}\sum_{n=0}^{\infty}\frac{(uz_1)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(vz_2)^m}{(q;q)_m} \\ &= \sum_{m,n=0}^{\infty}\sum_{k=0}^{m\wedge n}\frac{(1-q)^{2k}(-1)^k q^{\binom{k}{2}}(a;q)_k}{(q;q)_k^2}\frac{u^m v^n}{(q;q)_m(q;q)_n}(D_{q,z_1}D_{q,z_2})^k(z_1^n z_2^m), \end{aligned}$$

so we have

$$f(z_1,z_2,z,a) = \sum_{m,n=0}^{\infty}\mu_{m,n}H_{m,n}(z_1,z_2,z,a)$$

and

$$f(z_1,z_2,0,a) = \sum_{m,n=0}^{\infty}\mu_{m,n}z_1^m z_2^n = \frac{1}{(uz_1,vz_2;q)_\infty} = \sum_{m=0}^{\infty}\frac{(uz_1)^m}{(q;q)_m}\sum_{n=0}^{\infty}\frac{(vz_2)^n}{(q;q)_n}.$$

Thus, we have

$$f(z_1,z_2,z,a) = \sum_{m,n=0}^{\infty}\frac{u^m v^n}{(q;q)_m(q;q)_n}H_{m,n}(z_1,z_2,z,a).$$

On the other hand, rewriting the right-hand side of (2.4) as $g(z_1,z_2,z,a)$, we can verify that (2.4) satisfies (1.13), so

$$g(z_1,z_2,z,a) = \sum_{m,n=0}^{\infty}\mu_{m,n}Q_{m,n}(z_1,z_2,z,a)$$

and

$$\begin{aligned} g(z_1,z_2,0,a) = \sum_{m,n=0}^{\infty}\mu_{m,n}z_1^m z_2^n &= (-z_1uq^{1/2},-z_2vq^{1/2};q)_\infty \\ &= \sum_{m=0}^{\infty}\frac{q^{\binom{m}{2}}(z_1q^{1/2})^m}{(q;q)_m}\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(z_2q^{1/2})^n}{(q;q)_n}. \end{aligned}$$

Thus, we have

$$g(z_1,z_2,z,a) = \sum_{m,n=0}^{\infty}\frac{u^m v^n q^{m^2/2+n^2/2}}{(q;q)_m(q;q)_n}Q_{m,n}(z_1,z_2,z,a). \qquad \square$$

Taking $m=n+\alpha$, $z=b$, $z_1=1$, $z_2=x$, $a=q$ and $m=n+\alpha$, $z=bq^n$, $z_1=1$, $z_2=x$, $a=q$ respectively, we have the following corollary.

Corollary 2.4. *If $|u| < 1$, $|v| < 1$, $|b| < 1$ and $|x| < 1$, we have*

$$\sum_{n=0}^{\infty} \frac{u^{n+\alpha} v^n}{(q;q)_{n+\alpha}(q;q)_n} C_n^{(\alpha)}(x,b) = \frac{(uvb;q)_\infty}{(u,vx;q)_\infty}$$

and

$$\sum_{n=0}^{\infty} \frac{u^{n+\alpha} v^n}{(q;q)_{n+\alpha}(q;q)_n} q^{(n^2+2n\alpha+\alpha^2/2)} D_n^{(\alpha)}(x,b) = \frac{(-uq^{1/2}, -vxq^{1/2};q)_\infty}{(-uvb;q)_\infty} (-uvb;q)_{n+\alpha}.$$

3. A mixed generating function for the generalized q-2D Hermite polynomials

Using the homogeneous q-difference operator, Chen, Fu and Zhang [6] gave the generating function for the homogeneous Rogers-Szegő polynomials.

Proposition 3.1. *If $|t| < 1$ and $|x| < 1$, we have*

$$\sum_{n=0}^{\infty} \overline{h}_n(x,y|q) \frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(t,xt;q)_\infty}.$$

In this section, we obtain the following mixed generating function for the generalized q-2D Hermite polynomials and the homogeneous Rogers-Szegő polynomials.

Theorem 3.2. *For $\max\{|z_1|, |z_2|, |t|, |x|, |y|, |a|\} < 1$, we have*

$$(3.1)\quad \begin{aligned} &\sum_{m,n=0}^{\infty} \frac{u^m}{(q;q)_m} \frac{t^n}{(q;q)_n} \overline{h}_n(x,y|q) H_{m,n}(z_1,z_2,z,a) \\ &= \frac{(ytz_2;q)_\infty}{(uz_1, z_2t, xtz_2;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (a;q)_k}{(q;q)_k^2} (zut)^k \sum_{l=0}^{k} \begin{bmatrix} k \\ l \end{bmatrix} (xq^{k-l})^l \frac{(y/x, z_2t;q)_l}{(ytz_2;q)_l}. \end{aligned}$$

Remark 3.3. For $x = y$ and $t = v$ in Theorem 3.2, (3.1) reduces to (2.3).

Proof of Theorem 3.2. Denoting the right-hand side of (3.1) as $f(z_1, z_2, z, a)$, we have

$$\begin{aligned} &f(z_1,z_2,z,a) \\ &= \frac{(ytz_2;q)_\infty}{(uz_1, z_2t, xtz_2;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (a;q)_k}{(q;q)_k^2} (zut)^k \sum_{l=0}^{k} \begin{bmatrix} k \\ l \end{bmatrix} (xq^{k-l})^l \frac{(y/x, z_2t;q)_l}{(ytz_2;q)_l} \\ &= \frac{1}{(uz_1;q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (a;q)_k}{(q;q)_k^2} (zu)^k \sum_{l=0}^{k} \begin{bmatrix} k \\ l \end{bmatrix} (xt)^l (tq^l)^{k-l} (y/x;q)_l \frac{(ytz_2q^l;q)_\infty}{(xtz_2, tz_2q^l;q)_\infty} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (1-q)^{2k} (a;q)_k}{(q;q)_k^2} z^k D_{q,z_1}^k \left\{ \frac{1}{(uz_1;q)_\infty} \right\} D_{q,z_2}^k \left\{ \frac{(ytz_2;q)_\infty}{(tz_2, xtz_2;q)_\infty} \right\}. \end{aligned}$$

By using (1.4), we verify that the above equation satisfies (1.10), so we have

$$f(z_1,z_2,z,a)=\sum_{m,n=0}^{\infty}\mu_{m,n}H_{m,n}(z_1,z_2,z,a)$$

and

$$f(z_1,z_2,0,a)=\sum_{m,n=0}^{\infty}\mu_{m,n}z_1^m z_2^n=\frac{(ytz_2;q)_\infty}{(uz_1,z_2t,xtz_2;q)_\infty}=\sum_{m,n=0}^{\infty}\frac{u^m t^n}{(q;q)_m(q;q)_n}\overline{h}_n(x,y|q)z_1^m z_2^n.$$

Thus, we have

$$f(z_1,z_2,z,a)=\sum_{m,n=0}^{\infty}\frac{u^m t^n}{(q;q)_m(q;q)_n}\overline{h}_n(x,y|q)H_{m,n}(z_1,z_2,z,a). \qquad \square$$

4. Multilinear generating function for the generalized q-2D Hermite polynomials

Andrews [1] proved the following formula for the q-Lauricella function.

Proposition 4.1. *For* $\max\{|\alpha|,|r|,|y_1|,\dots,|y_k|\}<1$, *we have*

$$\sum_{n_1,n_2,\dots,n_k=0}^{\infty}\frac{(\alpha;q)_{n_1+n_2+\cdots+n_k}}{(r;q)_{n_1+n_2+\cdots+n_k}}\frac{(\beta_1;q)_{n_1}(\beta_2;q)_{n_2}\cdots(\beta_k;q)_{n_k}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_k}}y_1^{n_1}y_2^{n_2}\cdots y_k^{n_k}$$
$$=\frac{(\alpha,\beta_1y_1,\beta_2y_2,\dots,\beta_ky_k;q)_\infty}{(r,y_1,y_2,\dots,y_k;q)_\infty}\,{}_{k+1}\phi_k\left(\begin{matrix}r/\alpha,y_1,y_2,\dots,y_k\\ \beta_1y_1,\beta_2y_2,\dots,\beta_ky_k\end{matrix};q,\alpha\right).$$

By using q-Partial differential equation, Liu [19] generalized Andrew's result ($\beta_i=0$).

Proposition 4.2. *For* $\max\{|\alpha|,|r|,|x_1|,\dots,|x_k|,|y_1|,\dots,|y_k|\}<1$, *we have*

$$\sum_{n_1,n_2,\dots,n_k=0}^{\infty}\frac{(\alpha;q)_{n_1+n_2+\cdots+n_k}}{(r;q)_{n_1+n_2+\cdots+n_k}}\frac{h_{n_1}(x_1,y_1|q)h_{n_2}(x_2,y_2|q)\cdots h_{n_k}(x_k,y_k|q)}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_k}}$$
$$=\frac{(\alpha;q)_\infty}{(r,x_1,y_1,x_2,y_2,\dots,x_k,y_k;q)_\infty}\,{}_{2k+1}\phi_{2k}\left(\begin{matrix}r/\alpha,x_1,y_1,x_2,y_2,\dots,x_k,y_k\\ 0,0,\dots,0\end{matrix};q,\alpha\right).$$

In the following, we obtain our main results about the multilinear generating function by using the homogeneous q-difference equation.

Theorem 4.3. *For* $\max\{|\alpha|,|r|,|y_1|,\dots,|y_{2k}|,|m_1|,\dots,|m_k|,|a_1|,|a_2|,\dots,|a_k|\}<1$, *we have*

$$\sum_{n_1,n_2,\dots,n_{2k}=0}^{\infty}\frac{(\alpha;q)_{n_1+n_2+\cdots+n_{2k}}}{(r;q)_{n_1+n_2+\cdots+n_{2k}}}\frac{(\beta_1;q)_{n_1}(\beta_2;q)_{n_2}\cdots(\beta_{2k};q)_{n_{2k}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{2k}}}$$
$$\times H_{n_1,n_2}(y_1,y_2,m_1,a_1)H_{n_3,n_4}(y_3,y_4,m_2,a_2)\cdots H_{n_{2k-1},n_{2k}}(y_{2k-1},y_{2k},m_k,a_k)$$
$$(4.1)\qquad =\frac{(\alpha,\beta_1y_1,\beta_2y_2,\dots,\beta_{2k}y_{2k};q)_\infty}{(r,y_1,y_2,\dots,y_{2k};q)_\infty}\sum_{l=0}^{\infty}\frac{(r/\alpha,y_1,y_2,\dots,y_{2k};q)_l}{(q,\beta_1y_1,\beta_2y_2,\dots,\beta_{2k}y_{2k};q)_l}\alpha^l$$
$$\times\prod_{i=1,3,\dots,2k-1}{}_3\phi_3\left(\begin{matrix}a_i,\beta_i,\beta_{i+1}\\ q,\beta_iy_iq^l,\beta_{i+1}y_{i+1}q^l\end{matrix};q,m_iq^{2l}\right).$$

Proof. Rewrite Proposition 4.1 as

$$\begin{aligned}
&\sum_{n_1,n_2,\ldots,n_{2k}=0}^{\infty} \frac{(\alpha;q)_{n_1+n_2+\cdots+n_{2k}}}{(r;q)_{n_1+n_2+\cdots+n_{2k}}} \frac{(\beta_1;q)_{n_1}(\beta_2;q)_{n_2}\cdots(\beta_{2k};q)_{n_{2k}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{2k}}} y_1^{n_1}y_2^{n_2}\cdots y_{2k}^{n_{2k}} \\
&= \frac{(\alpha,\beta_1y_1,\beta_2y_2,\ldots,\beta_{2k}y_{2k};q)_\infty}{(r,y_1,y_2,\ldots,y_{2k};q)_\infty} {}_{2k+1}\phi_{2k}\left(\begin{matrix} r/\alpha, y_1, y_2, \ldots, y_{2k} \\ \beta_1y_1, \beta_2y_2, \ldots, \beta_{2k}y_{2k}\end{matrix}; q, \alpha\right) \qquad (4.2)\\
&= \frac{(\alpha;q)_\infty}{(r;q)_\infty} \sum_{l=0}^{\infty} \frac{(r/\alpha;q)_l}{(q;q)_l} \frac{(\beta_1y_1q^l,\beta_2y_2q^l,\ldots,\beta_{2k}y_{2k}q^l;q)_\infty}{(y_1q^l,y_2q^l,\ldots,y_{2k}q^l;q)_\infty}\alpha^l.
\end{aligned}$$

If we use $f(y_1,y_2,\ldots,y_{2k},m_1,m_2,\ldots,m_k,a_1,a_2,\ldots,a_k)$ to denote the right-hand side of (4.1), then, by direct computation, we can verify that $f(y_1,y_2,\ldots,y_{2k},m_1,m_2,\ldots,m_k,a_1,$ $a_2,\ldots,a_k)$ satisfies (1.10):

$$\begin{aligned}
&f(y_1,y_2,\ldots,y_{2k},m_1,m_2,\ldots,m_k,a_1,a_2,\ldots,a_k)\\
&= \frac{(\alpha;q)_\infty}{(r;q)_\infty} \sum_{l=0}^{\infty} \frac{(r/\alpha;q)_l}{(q;q)_l} \sum_{s_1=0}^{\infty} \frac{(\alpha_1;q)_{s_1}(1-q)^{2s_1}(-1)^{s_1}q^{\binom{s_1}{2}}m_1^{s_1}}{(q;q)_{s_1}}\\
&\quad\times \{D_{q,y_1}D_{q,y_2}\}^{s_1}\left\{\frac{(\beta_1y_1q^l,\beta_2y_2q^l;q)_\infty}{(y_1q^l,y_2q^l;q)_\infty}\right\}\\
&\quad\times \sum_{s_3=0}^{\infty} \frac{(\alpha_2;q)_{s_3}(1-q)^{2s_3}(-1)^{s_3}q^{\binom{s_3}{2}}m_3^{s_3}}{(q;q)_{s_3}}\{D_{q,y_3}D_{q,y_4}\}^{s_3}\left\{\frac{(\beta_3y_3q^l,\beta_4y_4q^l;q)_\infty}{(y_3q^l,y_4q^l;q)_\infty}\right\}\\
&\quad\times \sum_{s_{2k-1}=0}^{\infty} \frac{(\alpha_k;q)_{s_{2k-1}}(1-q)^{2s_{2k-1}}(-1)^{s_{2k-1}}q^{\binom{s_{2k-1}}{2}}m_{2k-1}^{s_{2k-1}}}{(q;q)_{s_{2k-1}}}\\
&\quad\times \{D_{q,y_2k-1}D_{q,y_2k}\}^{s_{2k-1}}\left\{\frac{(\beta_{2k-1}y_{2k-1}q^l,\beta_{2k}y_{2k}q^l;q)_\infty}{(y_{2k-1}q^l,y_{2k}q^l;q)_\infty}\right\}.
\end{aligned}$$

By using Theorem 1.3, there exists a sequence $\lambda_{n_1,\ldots,n_{2k}}$ independent of $y_1,y_2,\ldots,y_{2k},m_1,$ $m_2,\ldots,m_k,a_1,a_2,\ldots,a_k$ and that

$$\begin{aligned}
&f(y_1,y_2,\ldots,y_{2k},m_1,m_2,\ldots,m_k,a_1,a_2,\ldots,a_k)\\
&= \sum_{n_1,n_2,\ldots,n_{2k}=0}^{\infty} \lambda_{n_1,\ldots,n_{2k}} H_{n_1,n_2}(y_1,y_2,m_1,a_1)\cdots H_{n_{2k-1},n_{2k}}(y_{2k-1},y_{2k},m_k,a_k).
\end{aligned}$$

Setting $m_1=m_2=\cdots=m_k=0$ in (4.1) and using (4.2), we have

$$\begin{aligned}
&f(y_1,y_2,\ldots,y_{2k},0,0,\ldots,0,a_1,a_2,\ldots,a_k)\\
&= \sum_{n_1,n_2,\ldots,n_{2k}=0}^{\infty} \lambda_{n_1,\ldots,n_{2k}} y_1^{n_1}y_2^{n_2}\cdots y_{2k}^{n_{2k}}\\
&= \frac{(\alpha,\beta_1y_1,\beta_2y_2,\ldots,\beta_{2k}y_{2k};q)_\infty}{(r,y_1,y_2,\ldots,y_{2k};q)_\infty} \sum_{l=0}^{\infty} \frac{(r/a,y_1,y_2,\ldots,y_{2k};q)_l}{(q,\beta_1y_1,\beta_2y_2,\ldots,\beta_{2k}y_{2k};q)_l}\alpha^l
\end{aligned}$$

$$= \sum_{n_1,n_2\ldots,n_{2k}=0}^{\infty} \frac{(\alpha;q)_{n_1+n_2+\cdots+n_{2k}}}{(r;q)_{n_1+n_2+\cdots+n_{2k}}} \frac{(\beta_1;q)_{n_1}(\beta_2;q)_{n_2}\cdots(\beta_{2k};q)_{n_{2k}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{2k}}} y_1^{n_1} y_2^{n_2}\cdots y_{2k}^{n_{2k}}.$$

We deduce that $f(y_1, y_2, \ldots, y_{2k}, m_1, m_2, \ldots, m_k, a_1, a_2, \ldots, a_k)$ is equal to the left-hand side of (4.1), so we have

$$\begin{aligned} & f(y_1, y_2, \ldots, y_{2k}, m_1, m_2, \ldots, m_k, a_1, a_2, \ldots, a_k) \\ = & \sum_{n_1,n_2,\ldots,n_{2k}=0}^{\infty} \frac{(\alpha;q)_{n_1+n_2+\cdots+n_{2k}}}{(r;q)_{n_1+n_2+\cdots+n_{2k}}} \frac{(\beta_1;q)_{n_1}(\beta_2;q)_{n_2}\cdots(\beta_{2k};q)_{n_{2k}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{2k}}} \\ & \times H_{n_1,n_2}(y_1,y_2,m_1,a_1)H_{n_3,n_4}(y_3,y_4,m_2,a_2)\cdots H_{n_{2k-1},n_{2k}}(y_{2k-1},y_{2k},m_k,a_k). \end{aligned}$$

The proof is complete. □

If we take $k = 1$ and $\alpha = r$ in (4.1), we obtain the following corollary.

Corollary 4.4. *For* $\max\{|y_1|, |y_2|, |m_1|, |\beta_1|, |\beta_2|, |a_1|\} < 1$, *we have*

$$\sum_{m,n=0}^{\infty} \frac{(\beta_1;q)_m(\beta_2;q)_n}{(q;q)_m(q;q)_n} H_{m,n}(y_1,y_2,m_1,a_1) = \frac{(\beta_1 y_1, \beta_2 y_2;q)_\infty}{(y_1,y_2;q)_\infty} {}_3\phi_3\left(\begin{matrix} a_1,\beta_1,\beta_2 \\ q, y_1\beta_1, \beta_2 y_2 \end{matrix}; q, m_1\right).$$

Further, setting $\beta_1 = \alpha/u$, $\beta_2 = b/v$, $y_1 = z_1 u$, $y_2 = z_2 v$, $a_1 = q$ and $m_1 = uv$, we get the following generating function of q-2D Hermite polynomials.

Corollary 4.5. [15] *For* $\max\{|\alpha/u|, |b/v|, |z_1|, |z_2|, |z|\} < 1$, *we have*

$$\sum_{m,n=0}^{\infty} \frac{(\alpha/u;q)_m u^m (b/v;q)_n v^n}{(q;q)_m(q;q)_n} H_{m,n}(z_1,z_2) = \frac{(\alpha z_1, b z_2;q)_\infty}{(u z_1, v z_2;q)_\infty} {}_2\phi_2\left(\begin{matrix} \alpha/u, b/v \\ z_1\alpha, b z_2 \end{matrix}; q, uv\right).$$

5. A dual multilinear generating function for the generalized q-2D Hermite polynomials

Andrew [1] gave Proposition 4.1 by using basic Appell series. In this section, we gain a dual multilinear generating function for q-2D Hermite polynomials by using the homogeneous q-difference equation.

Theorem 5.1. *For* $\max\{|\alpha|, |r|, |m_1|, \ldots, |m_k|\} < 1$, *we have*

$$\begin{aligned} & \sum_{n_1,n_2,\ldots,n_{2k}=0}^{\infty} \frac{(\alpha;q)_{n_1+n_2+\cdots+n_{2k}}}{(r;q)_{n_1+n_2+\cdots+n_{2k}}} \frac{q^{(n_1^2/2+n_2^2/2+\cdots+n_{2k-1}^2/2+n_{2k}^2/2)}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{2k}}} \\ & \quad \times Q_{n_1,n_2}(y_1,y_2,m_1)Q_{n_3,n_4}(y_3,y_4,m_2)\cdots Q_{n_{2k-1},n_{2k}}(y_{2k-1},y_{2k},m_k) \\ = & \frac{(a, -y_1 q^{1/2}, -y_2 q^{1/2}, \ldots, -z_{2k} q^{1/2};q)_\infty}{(r, -m_1, -m_2, \ldots, -m_k;q)_\infty} \\ & \times \sum_{k=0}^{\infty} \frac{(r/\alpha;q)_k}{(q;q)_k} \frac{(-m_1,-m_2,\ldots,-m_k;q)_{2k}}{(-y_1 q^{1/2}, -y_2 q^{1/2}, \ldots, -y_{2k} q^{1/2};q)_k} \alpha^k. \end{aligned}$$

We will give a transformation for Basic Appell series before our main results which is a dual transformation of Proposition 4.1.

Proposition 5.2. *For* $\max\{|a|,|c|\}<1$, *we have*

$$\begin{aligned}&\sum_{m,n=0}^{\infty}\frac{(a;q)_{m+n}}{(c;q)_{m+n}}\frac{x^m}{(q;q)_m}\frac{y^n}{(q;q)_n}q^{m^2/2+n^2/2}\\&=\frac{(a;q)_\infty}{(c;q)_\infty}(-xq^{1/2},-yq^{1/2};q)_\infty\sum_{k=0}^{\infty}\frac{(c/a;q)_k}{(q,-xq^{1/2},-yq^{1/2};q)_k}a^k.\end{aligned}\tag{5.1}$$

Proof. The left-hand side of (5.1) can be rewritten as

$$\begin{aligned}&\sum_{m,n=0}^{\infty}\frac{(a;q)_{m+n}}{(c;q)_{m+n}}\frac{x^m}{(q;q)_m}\frac{y^n}{(q;q)_n}q^{m^2/2+n^2/2}\\&=\frac{(a;q)_\infty}{(c;q)_\infty}\sum_{m,n=0}^{\infty}\frac{(cq^{m+n};q)_\infty}{(aq^{m+n};q)_\infty}\frac{x^my^nq^{m^2/2+n^2/2}}{(q;q)_m(q;q)_n}\\&=\frac{(a;q)_\infty}{(c;q)_\infty}\sum_{m,n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(c/a;q)_ra^rq^{r(m+n)}}{(q;q)_r}\frac{x^my^nq^{m^2/2+n^2/2}}{(q;q)_m(q;q)_n}\\&=\frac{(a;q)_\infty}{(c;q)_\infty}(-xq^{1/2},-yq^{1/2};q)_\infty\sum_{r=0}^{\infty}\frac{(c/a;q)_r}{(q,-xq^{1/2},-yq^{1/2};q)_r}a^r.\end{aligned}$$

□

The proof of Proposition 5.2 can be directly generalized to prove

Proposition 5.3. *For* $\max\{|\alpha|,|r|,|y_1|,\dots,|y_k|\}<1$, *we have*

$$\begin{aligned}&\sum_{n_1,n_2,\dots,n_k=0}^{\infty}\frac{(\alpha;q)_{n_1+n_2+\cdots+n_k}}{(r;q)_{n_1+n_2+\cdots+n_k}}\frac{y_1^{n_1}y_2^{n_2}\cdots y_k^{n_k}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_k}}q^{(n_1^2/2+n_2^2/2+\cdots+n_k^2/2)}\\&=\frac{(\alpha,-y_1q^{1/2},-y_2q^{1/2},\dots,-y_kq^{1/2};q)_\infty}{(r;q)_\infty}{}_{k+1}\phi_k\left(\begin{matrix}r/\alpha,0,0,\dots,0\\-y_1q^{1/2},-y_2q^{1/2},\dots,-y_kq^{1/2}\end{matrix};q,\alpha\right).\end{aligned}$$

In the following, we gain a dual multilinear generating functions for q-2D Hermite polynomials by using the above proposition.

Theorem 5.4. *For* $\max\{|\alpha|,|r|,|m_1|,\dots,|m_k|\}<1$, *we have*

$$\begin{aligned}&\sum_{n_1,n_2,\dots,n_{2k}=0}^{\infty}\frac{(\alpha;q)_{n_1+n_2+\cdots+n_{2k}}}{(r;q)_{n_1+n_2+\cdots+n_{2k}}}\frac{q^{(n_1^2/2+n_2^2/2+n_3^2/2+n_4^2/2+\cdots+n_{2k-1}^2/2+n_{2k}^2/2)}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{2k}}}\\&\quad\times Q_{n_1,n_2}(y_1,y_2,m_1,a_1)Q_{n_3,n_4}(y_3,y_4,m_2,a_2)\cdots Q_{n_{2k-1},n_{2k}}(y_{2k-1},y_{2k},m_k,a_k)\\&=\frac{(\alpha,-y_1q^{1/2},-y_2q^{1/2},\dots,-y_{2k}q^{1/2};q)_\infty}{(r;q)_\infty}\sum_{l=0}^{\infty}\frac{(r/\alpha;q)_l}{(q,-y_1q^{1/2},\dots,-y_{2k}q^{1/2};q)_l}\alpha^l\\&\quad\times\prod_{i=1,3,\dots,2k-1}\sum_{n_i=0}^{\infty}\frac{(-1)^{n_i}(a_i;q)_{n_i}}{(q;q)_{n_i}^2}m_i^{n_i}q^{2ln_i}.\end{aligned}\tag{5.2}$$

Proof. If we use $f(y_1, y_2, \dots, y_{2k}, m_1, m_2, \dots, m_k, a_1, a_2, \dots, a_k)$ to denote the right-hand side of (5.2), then, we can verify that $f(y_{2i-1}, y_{2i}, m_i, a_i)$ $(i = 1, 2 \dots, k)$ satisfies (1.13). By using Theorem 1.4 and mathematical induction, there exists a sequence $\lambda_{n_1,\dots,n_{2k}}$ independent of $y_1, y_2, \dots, y_{2k}, m_1, m_2, \dots, m_k, a_1, a_2, \dots, a_k$ and that

$$\begin{aligned} &f(y_1, y_2, \dots, y_{2k}, m_1, m_2, \dots, m_k, a_1, a_2, \dots, a_k) \\ &= \sum_{n_1,n_2,\dots,n_{2k}=0}^{\infty} \lambda_{n_1,\dots,n_{2k}} Q_{n_1,n_2}(y_1, y_2, m_1, a_1) \cdots Q_{n_{2k-1},n_{2k}}(y_{2k-1}, y_{2k}, m_k, a_k). \end{aligned}$$

Setting $m_1 = m_2 = \cdots = m_k = 0$ in (5.2) and using Proposition 5.3, we have

$$\begin{aligned} &f(y_1, y_2, \dots, y_{2k}, 0, 0, \dots, 0, a_1, a_2, \dots, a_k) \\ &= \sum_{n_1,n_2,\dots,n_{2k}=0}^{\infty} \lambda_{n_1,\dots,n_{2k}} y_1^{n_1} y_2^{n_2} \cdots y_{2k}^{n_{2k}} \\ &= \frac{(\alpha, -y_1q^{1/2}, -y_2q^{1/2}, \dots, -y_{2k}q^{1/2}; q)_\infty}{(r; q)_\infty} {}_{2k+1}\phi_{2k} \left(\begin{matrix} r/a, 0, 0, \dots, 0 \\ -y_1q^{1/2}, -y_2q^{1/2}, \dots, -y_{2k}q^{1/2} \end{matrix} ; q, \alpha \right) \\ &= \sum_{n_1,n_2,\dots,n_{2k}=0}^{\infty} \frac{(\alpha; q)_{n_1+n_2+\cdots+n_{2k}}}{(r; q)_{n_1+n_2+\cdots+n_{2k}}} \frac{y_1^{n_1} y_2^{n_2} \cdots y_{2k}^{n_{2k}}}{(q; q)_{n_1} (q; q)_{n_2} \cdots (q; q)_{n_{2k}}} q^{(n_1^2/2+n_2^2/2+\cdots+n_{2k}^2/2)}. \end{aligned}$$

We deduce that $f(y_1, y_2, \dots, y_{2k}, m_1, m_2, \dots, m_k, a_1, a_2, \dots, a_k)$ is equal to the left-hand side of (5.2), so we have

$$\begin{aligned} &f(y_1, y_2, \dots, y_{2k}, m_1, m_2, \dots, m_k, a_1, a_2, \dots, a_k) \\ &= \sum_{n_1,n_2,\dots,n_{2k}=0}^{\infty} \frac{(\alpha; q)_{n_1+n_2+\cdots+n_{2k}}}{(r; q)_{n_1+n_2+\cdots+n_{2k}}} \frac{q^{(n_1^2/2+n_2^2/2+\cdots+n_{2k}^2/2)}}{(q; q)_{n_1} (q; q)_{n_2} \cdots (q; q)_{n_{2k}}} \\ &\qquad \times Q_{n_1,n_2}(y_1, y_2, m_1, a_1) Q_{n_3,n_4}(y_3, y_4, m_2, a_2) \cdots Q_{n_{2k-1},n_{2k}}(y_{2k-1}, y_{2k}, m_k, a_k). \end{aligned}$$

The proof is complete. □

Taking $a_i = q$ in the above theorem, we obtain our main result (Theorem 5.1). Taking $k = 1$ in the above theorem, we obtain the following corollary.

Corollary 5.5. *For* $\max\{|m_1|, |r|\} < 1$, *we have*

$$\begin{aligned} &\sum_{m,n=0}^{\infty} \frac{(\alpha; q)_{m+n}}{(r; q)_{m+n}} \frac{1}{(q; q)_m} \frac{1}{(q; q)_n} q^{m^2/2+n^2/2} Q_{m,n}(y_1, y_2, m_1, a) \\ &= \frac{(\alpha; q)_\infty}{(r; q)_\infty} (-y_1q^{1/2}, -y_2q^{1/2}; q)_\infty \sum_{l=0}^{\infty} \frac{(r/\alpha; q)_l}{(q, -y_1q^{1/2}, -y_2q^{1/2}; q)_l} \alpha^l \sum_{n=0}^{\infty} \frac{(-1)^n (a; q)_n}{(q; q)_n^2} m_1^n q^{2ln}. \end{aligned}$$

Remark 5.6. Taking $\alpha \to r$, $y_1 \to z_1u$, $y_2 \to z_2v$, $m_1 \to zuv$ in this corollary, we obtain (2.4).

6. A transformation identity involving generating function for the generalized q-2D Hermite polynomials

Liu [17] gave some important transformational identities by the method of q-exponential operator. Similarly, we will deduce the following transformational identity involving generating functions for the generalized q-2D Hermite polynomials by the method of homogeneous q-difference equation.

Theorem 6.1. *If two sequences* $(A_{m,n})$ *and* (B_l) *satisfy*

$$\sum_{m,n=0}^{\infty} A_{m,n} z_1^m z_2^n = \frac{(z_1\mu;q)_\infty}{(z_2\mu;q)_\infty} \sum_{l=0}^{\infty} B_l \frac{(z_2\mu;q)_l}{(z_1\mu;q)_l},$$

then we have

$$\sum_{m,n=0}^{\infty} A_{m,n} H_{m,n}(z_1,z_2,z,a) = \frac{(z_1\mu;q)_\infty}{(z_2\mu;q)_\infty} \sum_{l=0}^{\infty} B_l \frac{(z_2\mu;q)_l}{(z_1\mu;q)_l} \sum_{k=0}^{\infty} \frac{q^{k^2-k} z^k (\mu q^l)^{2k} (a;q)_k}{(z_1\mu q^l;q)_k (q;q)_k^2}. \tag{6.1}$$

Proof. Denoting the right-hand side of (6.1) as $f(z_1,z_2,z,a)$, we verify that $f(z_1,z_2,z,a)$ satisfies (1.10), by Theorem 1.3, there exists a sequence $\lambda_{m,n}$ independent of z_1, z_2, z, a and that

$$f(z_1,z_2,z,a) = \sum_{m,n=0}^{\infty} \lambda_{m,n} H_{m,n}(z_1,z_2,z,a).$$

Setting $z=0$ in (6.1), we have

$$f(z_1,z_2,0,a) = \sum_{m,n=0}^{\infty} \lambda_{m,n} z_1^m z_2^n = \sum_{l=0}^{\infty} B_l \frac{(z_1\mu q^l;q)_\infty}{(z_2\mu q^l;q)_\infty} = \sum_{m,n=0}^{\infty} A_{m,n} z_1^m z_2^n.$$

Hence, we obtain

$$f(z_1,z_2,z,a) = \sum_{m,n=0}^{\infty} A_{m,n} H_{m,n}(z_1,z_2,z,a).$$

This proof is complete. □

Taking $A_{m,n} = \frac{\mu^{n+m}(-1)^m q^{\binom{m}{2}}}{(q;q)_m (q;q)_n}$ and $B_l = (q^{-l+1};q)_l$ in (6.1) and using (1.5), we obtain the following corollary.

Corollary 6.2. *For* $\max\{|z_1|,|z_2|,|\mu|\} < 1$, *we have*

$$\sum_{m,n=0}^{\infty} \frac{\mu^{n+m}(-1)^m q^{\binom{m}{2}}}{(q;q)_m (q;q)_n} H_{m,n}(z_1,z_2,z,a) = \frac{(z_1\mu;q)_\infty}{(z_2\mu;q)_\infty} \sum_{k=0}^{\infty} \frac{(\mu^2 z)^k q^{k^2-k} (a;q)_k}{(z_1\mu;q)_k\,(q;q)_k^2}.$$

7. Application

Recall that the Delannoy numbers count lattice paths from $(0,0)$ to (n,m) consisting of horizontal $(1,0)$, vertical $(0,1)$, and diagonal $(1,1)$ steps, and have the following explicit formulas in terms of binomial coefficients [11]:

$$D(m,n) := \sum_{k=0}^{n} \binom{n}{k}\binom{n+m-k}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{m}{k} 2^k. \tag{7.1}$$

The following two natural q-analogues of Delannoy numbers were introduced in [11]:

$$D_q(m,n) := \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n+m-k \\ n \end{bmatrix}, \quad D_q^*(m,n) := \sum_{k=0}^{n} q^{\binom{k+1}{2}} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n+m-k \\ n \end{bmatrix}.$$

By using q-Chu-Vandermonde summation and q-binomial theorem, Guo, Guo and Zeng [13] gave a q-analogue of (7.1) by proving the following identities

$$D_q(m,n) = \sum_{k=0}^{m} q^{(m-k)(n-k)} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} (-1;q)_k,$$
$$D_q^*(m,n) = \sum_{k=0}^{m} q^{(m-k)(n-k)} \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} (-q;q)_k.$$

In this section, we will give a transformational identity involving $D_q(m,n)$ and $D_q^*(m,n)$ as the application of Theorem 5.4.

Theorem 7.1. *For* $\max\{|y_1|,|y_2|,|y_3|,|y_4|\} < 1$, *we have*

$$\sum_{n_1,n_2,n_3,n_4=0}^{\infty} \frac{q^{(n_1-n_2)^2/2+(n_3-n_4)^2/2}}{(q;q)_{n_1}(q;q)_{n_2}(q;q)_{n_3}(q;q)_{n_4}} y_1^{n_1} y_2^{n_2} y_3^{n_3} y_4^{n_4} D_q(n_1,n_2) D_q^*(n_3,n_4)$$
$$= (-q^{1/2}y_1, -q^{1/2}y_2, -q^{1/2}y_3, -q^{1/2}y_4; q)_\infty \, {}_2\phi_1\left(\begin{matrix} -1, 0 \\ q \end{matrix}; q, y_1y_2\right) {}_2\phi_1\left(\begin{matrix} -q, 0 \\ q \end{matrix}; q, y_3y_4\right).$$

Proof. Taking $k=2$ in (5.2), we have

$$\sum_{n_1,n_2,n_3,n_4=0}^{\infty} \frac{(\alpha;q)_{n_1+n_2+n_3+n_4}}{(r;q)_{n_1+n_2+n_3+n_4}} \frac{q^{n_1^2/2+n_2^2/2+n_3^2/2+n_4^2/2}}{(q;q)_{n_1}(q;q)_{n_2}(q;q)_{n_3}(q;q)_{n_4}}$$
$$\times Q_{n_1,n_2}(y_1,y_2,m_1,a_1) Q_{n_3,n_4}(y_3,y_4,m_2,a_2)$$
$$= \frac{(\alpha, -y_1q^{1/2}, -y_2q^{1/2}, -y_3q^{1/2}, -y_4q^{1/2}; q)_\infty}{(r;q)_\infty}$$
$$\times \sum_{l=0}^{\infty} \frac{(r/\alpha;q)_l \alpha^l}{(q, -y_1q^{1/2}, -y_2q^{1/2}, -y_3q^{1/2}, -y_4q^{1/2}; q)_l} \sum_{n_1=0}^{\infty} \frac{(-1)^{n_1}(a_1;q)_{n_1}}{(q;q)_{n_1}^2} m_1^{n_1} q^{2ln_1}$$
$$\times \sum_{n_3=0}^{\infty} \frac{(-1)^{n_3}(a_3;q)_{n_3}}{(q;q)_{n_3}^2} m_2^{n_3} q^{2ln_3}.$$

Setting $\alpha = r$, $m_1 = -y_1y_2$, $m_2 = -y_3y_4$, $a_1 = -1$ and $a_3 = -q$ in the above equation, we complete the proof of this theorem. □

If we take $y_1 = y_2 = y_3 = y_4 = q^{1/2}$ in the above theorem, we have the following result.

Corollary 7.2.

$$\sum_{n_1,n_2,n_3,n_4=0}^{\infty} \frac{q^{(n_1-n_2)^2/2+(n_3-n_4)^2/2}}{(q;q)_{n_1}(q;q)_{n_2}(q;q)_{n_3}(q;q)_{n_4}} q^{(n_1+n_2+n_3+n_4)/2} D_q(n_1,n_2) D_q^*(n_3,n_4)$$
$$= (-q;q)_\infty^4 \, {}_2\phi_1\left(\begin{matrix} -1, 0 \\ q \end{matrix}; q, q\right) {}_2\phi_1\left(\begin{matrix} -q, 0 \\ q \end{matrix}; q, q\right).$$

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