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# Generalized Derivations and Generalization of Co-commuting Maps in Prime Rings

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Abstract. Suppose that $R$ is a prime ring of characteristic different from 2 with Utumi quotient ring $U$, $C = Z(U)$ the extended centroid of $R$, and $f(x_1, \ldots, x_n)$ a noncentral multilinear polynomial over $C$. If $F$, $G$ and $H$ are three nonzero generalized derivations of $R$ such that

$$F\big(G(f(X))f(X)\big) = f(X)H(f(X))$$

for all $X = (x_1, \ldots, x_n) \in R^n$, then we describe the nature of the maps $F$, $G$ and $H$.

## 1. Introduction

Throughout this paper $R$ denotes a prime ring with center $Z(R)$, extended centroid $C$ and $U$ its Utumi quotient ring. The definition and axiomatic formulation of Utumi quotient ring $U$ can be found in [2, 4].

We have the following properties which we need

1. $R \subseteq U$;
2. $U$ is a prime ring with unity;
3. The center of $U$ is denoted by $C$ and is called the extended centroid of $R$. $C$ is a field.

By a derivation of $R$, we mean an additive mapping $d\colon R \to R$ such that $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. An additive mapping $F\colon R \to R$ is called a generalized derivation if there exists a derivation $d$ on $R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Basic examples of generalized derivations are derivations, generalized inner derivations (i.e., maps of type $x \to ax + xb$ for some $a, b \in R$). In [16], Lee proved that any generalized derivation of $R$ can be uniquely extended to a generalized derivation of $U$ and its form will be $g(x) = ax + d(x)$ for some $a \in U$, where $d$ is the

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associated derivation. The Lie commutator of $x$ and $y$ is denoted by $[x, y]$ and also defined by $[x, y] = xy - yx$ for all $x, y \in R$; also the symbol $x \circ y$ stands for the anti-commutator $xy + yx$. By $s_4$, we denotes the standard polynomial in four variables, which is $s_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}$, where $(-1)^\sigma$ is $+1$ or $-1$ according to $\sigma$ being an even or odd permutation in symmetric group $S_4$. Let $S \subseteq R$. An additive map $F\colon R \to R$ is said to be commuting (centralizing) on $S$ if $[F(x), x] = 0$ for all $x \in S$ (resp. $[F(x), x] \in Z(R)$ for all $x \in S$). Two additive maps $F, G\colon R \to R$ are said to be co-commuting (co-centralizing) on $S$ if $F(x)x - xG(x) = 0$ for all $x \in S$ (resp. $F(x)x - xG(x) \in Z(R)$ for all $x \in S$).

In [6], De Filippis and De Vincenzo described the structure of additive mappings $d$ and $G$ satisfying $d(G(f(X))f(X) - f(X)G(f(X))) = 0$ for all $X = (x_1, \dots, x_n) \in R^n$, where $f$ is a multinear polynomial over extended centroid $C$ and $d$ is a nonzero derivation and $G$ is a nonzero generalized derivation on prime ring $R$ of $\mathrm{char}(R) \neq 2$.

In [9], the first author, Argac and Albas extended the above result by considering two generalized derivations. More precisely, they studied the situation $d(F(f(X))f(X) - f(X)G(f(X))) = 0$ for all $X = (x_1, \dots, x_n) \in R^n$, where $f$ is a multinear polynomial over extended centroid $C$ and $d$ is a nonzero derivation and $F$, $G$ are two generalized derivations on prime ring $R$ of $\mathrm{char}(R) \neq 2$. In the paper authors determined all possible forms of the additive maps $d$, $F$ and $G$.

On the other hand, Carini and De Filippis [3] proved that if $R$ is a prime ring of characteristic different from 2, $\delta$ a nonzero derivation of $R$, $G$ a nonzero generalized derivation of $R$, and $f(x_1, \dots, x_n)$ a non-central multilinear polynomial over $C$ such that $\delta(G(f(X))f(X)) = 0$ for all $X = (x_1, \dots, x_n) \in R^n$, then there exist $a, b \in U$ such that $G(x) = ax$ and $\delta(x) = [b, x]$ for all $x \in R$, with $[b, a] = 0$ and $f(x_1, \dots, x_n)^2$ is central-valued on $R$.

Further, the first author and Argac [8] extended the above result replacing derivation $\delta$ with another generalized derivation $F$, that is, $F(G(f(X))f(X)) = 0$ for all $X = (x_1, \dots, x_n) \in I^n$, and then gave the complete description of the additive maps $F$ and $G$, where $I$ is a non-zero two-sided ideal of $R$.

In another paper [1], Argaç and De Filippis studied the generalized derivations $G$ and $H$ co-commuting on $f(I) = \{f(x_1, \dots, x_n) \mid x_i \in I\}$, that is, $G(u)u - uH(u) = 0$ for all $u \in f(I)$ and then obtained the all possible forms of the maps $F$ and $G$, where $I$ is a non-zero two-sided ideal of $R$.

Motivated by the above results we prove the following theorem.

**Theorem 1.1.** *Let $R$ be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring $U$ and extended centroid $C$, and $f(x_1, \dots, x_n)$ a multilinear polynomial over $C$, which is not central valued on $R$. Suppose that $F$, $G$ and $H$ are three* *nonzero generalized derivations of R such that*

$$F\big(G(f(X))f(X)\big) = f(X)H(f(X))$$

*for all* $X = (x_1, \ldots, x_n) \in R^n$. *Then one of the following holds:*

(1) *there exist* $\lambda \in C$ *and* $a, b \in U$ *such that* $F(x) = \lambda x$, $G(x) = xa$ *and* $H(x) = \lambda ax$ *for all* $x \in R$;

(2) *there exist* $\lambda, \alpha \in C$ *and* $p, q, u, v \in U$ *such that* $F(x) = \lambda x$, $G(x) = px + xq$ *and* $H(x) = \lambda(qx + xp)$ *for all* $x \in R$ *with* $f(x_1, \ldots, x_n)^2$ *is central valued in R;*

(3) *there exist* $\lambda \in C$ *and* $a, p \in U$ *such that* $F(x) = ax$, $G(x) = px$ *and* $H(x) = \lambda x$ *for all* $x \in R$ *with* $ap = \lambda$;

(4) *there exist* $\lambda \in C$ *and* $a \in U$ *such that* $F(x) = xa$, $G(x) = \lambda x$ *and* $H(x) = \lambda xa$ *for all* $x \in R$;

(5) *there exist* $a, b, p, v \in U$ *such that* $F(x) = ax + xb$, $G(x) = px$ *and* $H(x) = xv$ *for all* $x \in R$ *with* $f(x_1, \ldots, x_n)^2$ *is central valued on R and* $ap + pb = v$.

In particular, when $G = H$, we have the following

**Corollary 1.2.** *Let R be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, and* $f(x_1, \ldots, x_n)$ *a multilinear polynomial over C, which is not central valued on R. Suppose that F and G are two nonzero generalized derivations of R such that*

$$F\big(G(f(X))f(X)\big) = f(X)G(f(X))$$

*for all* $X = (x_1, \ldots, x_n) \in R^n$. *Then one of the following holds:*

(1) *there exists* $\mu \in C$ *such that* $F(x) = x$ *and* $G(x) = \mu x$ *for all* $x \in R$;

(2) *there exist* $\alpha \in C$ *and* $p \in U$ *such that* $F(x) = x$ *and* $G(x) = px + xp + \alpha x$ *for all* $x \in R$ *with* $f(x_1, \ldots, x_n)^2$ *is central valued in R;*

(3) *there exists* $p \in U$ *such that* $F(x) = -x$ *and* $G(x) = [p, x]$ *for all* $x \in R$ *with* $f(x_1, \ldots, x_n)^2$ *is central valued in R;*

(4) *there exist* $a, b, p \in U$ *such that* $F(x) = ax + xb$ *and* $G(x) = px$ *for all* $x \in R$ *with* $f(x_1, \ldots, x_n)^2$ *is central valued on R and* $F(p) = p$.

*Proof.* By Theorem 1.1, we have the following conclusions:

(1) There exists $\mu \in C$ such that $F(x) = x$ and $G(x) = \mu x$ for all $x \in R$. This is our conclusion (1).

(2) There exist $\lambda, \alpha \in C$ and $p, q \in U$ such that $F(x) = \lambda x$, $G(x) = px + xq$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ being central valued in $R$ and $q - \lambda p = \lambda q - p = \alpha \in C$. The last relation yields $(\lambda - 1)(p + q) = 0$. This yields either $\lambda = 1$ or $p + q = 0$. (i) When $\lambda = 1$, we have $q - p = \alpha \in C$ and hence $F(x) = x$ and $G(x) = px + xp + \alpha x$ for all $x \in R$. This gives conclusion (2). (ii) When $p + q = 0$, we have $F(x) = \lambda x$, $G(x) = [p, x]$ for all $x \in R$ with $p + \lambda p = \alpha \in C$, i.e., $(1 + \lambda)p \in C$. This implies $1 + \lambda = 0$, since $p \in C$ implies $G = 0$, a contradiction. Thus $\lambda = -1$. This gives conclusion (3).

(3) There exist $\lambda \in C$ and $a, p \in U$ such that $F(x) = ax$, $G(x) = \lambda x$ for all $x \in R$ with $a\lambda = \lambda$. Since $G \neq 0$, $\lambda \neq 0$ and hence last relation gives $a = 1$. This is conclusion (1).

(4) There exist $\lambda \in C$ and $a, u \in U$ such that $F(x) = xa$, $G(x) = \lambda x$ for all $x \in R$ with $a\lambda = \lambda$. By the same argument as above, $a = 1$, as desired in (1).

(5) There exist $a, b, p, v \in U$ such that $F(x) = ax + xb$, $G(x) = px$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ being central valued on $R$ and $ap + pb = p$. This is conclusion (4). □

From Theorem 1.1(2), we conclude that when $G$ is derivation then $H$ also be a derivation. Thus following corollary is straightforward.

**Corollary 1.3.** *Let $R$ be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring $U$ and extended centroid $C$, and $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$, which is not central valued on $R$. Suppose that $F$ and $H$ are two nonzero generalized derivations of $R$ and $d$ is a derivation of $R$ such that*

$$F\big(d(f(X))f(X)\big) = f(X)H(f(X))$$

*for all $X = (x_1, \ldots, x_n) \in R^n$. Then there exist $\lambda \in C$ and $p, u \in U$ such that $F(x) = \lambda x$, $d(x) = [p, x]$ and $H(x) = -\lambda[p, x]$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ being central valued in $R$.*

**Corollary 1.4.** *Let $R$ be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring $U$ and extended centroid $C$, and $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$, which is not central valued on $R$. Suppose that $F$ and $G$ are two nonzero generalized derivations of $R$ and $d$ is a derivation of $R$ such that*

$$F\big(G(f(X))f(X)\big) = f(X)d(f(X))$$

*for all $X = (x_1, \ldots, x_n) \in R^n$. Then there exist $\lambda \in C$ and $p, u \in U$ such that $F(x) = \lambda x$, $G(x) = [p, x]$ and $d(x) = -\lambda[p, x]$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ being central valued in $R$.*

In particular, when $F$ is derivation, then we have last conclusion of Theorem 1.1.

**Corollary 1.5.** *Let $R$ be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring $U$ and extended centroid $C$, and $f(x_1,\dots,x_n)$ a multilinear polynomial over $C$, which is not central valued on $R$. Suppose that $G$ and $H$ are two nonzero generalized derivations of $R$ and $d$ is a derivation of $R$ such that*

$$d\big(G(f(X))f(X)\big) = f(X)H(f(X))$$

*for all $X = (x_1,\dots,x_n) \in R^n$. Then there exist $a,b,p,v \in U$ such that $d(x) = [a,x]$, $G(x) = px$ and $H(x) = xv$ for all $x \in R$ with $f(x_1,\dots,x_n)^2$ being central valued on $R$ and $d(p) = v$.*

**Corollary 1.6.** *Let $R$ be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring $U$ and extended centroid $C$, and $f(x_1,\dots,x_n)$ a multilinear polynomial over $C$. Suppose that $d$, $\delta$ and $h$ are three nonzero derivations of $R$ such that*

$$d\big(\delta(f(X))f(X)\big) = f(X)h(f(X))$$

*for all $X = (x_1,\dots,x_n) \in R^n$. Then $f(x_1,\dots,x_n)$ must be central valued.*

**Corollary 1.7.** *Let $R$ be a prime ring of characteristic different from 2. Suppose that $d$, $\delta$ and $h$ are three nonzero derivations of $R$ such that*

$$d\big(\delta(x)x\big) = xh(x)$$

*for all $x \in R$. Then $R$ must be commutative.*

## 2. Main results

Let $F$ $(\neq 0)$, $G$ $(\neq 0)$ and $H$ $(\neq 0)$ be all inner generalized derivations of $R$. There exist some fixed $a,b,p,q,u,v \in U$ such that $F(x) = ax+xb$, $G(x) = px+xq$ and $H(x) = ux+xv$ for all $x \in R$. Then by our hypothesis $F(G(x)x) = xH(x))$ for all $x \in f(R)$, we have

$$apx^2 + axqx + px^2b + xqxb - xux - x^2v = 0 \tag{2.1}$$

for all $x \in f(R)$.

To investigate this generalized polynomial identity (GPI) in prime ring $R$, we recall the following

**Lemma 2.1.** [1, Lemma 3] *Let $R$ be a noncommutative prime ring with Utumi quotient ring $U$ and extended centroid $C$, and $f(x_1,\dots,x_n)$ a multilinear polynomial over $C$, which is not central valued on $R$. Suppose that there exist $a,b,c,q \in U$ such that $(af(r) + f(r)b)f(r) - f(r)(cf(r) + f(r)q) = 0$ for all $r = (r_1,\dots,r_n) \in R^n$. Then one of the following holds:*

(1) $a, q \in C$ *and* $q - a = b - c = \alpha \in C$*;*

(2) $f(x_1, \ldots, x_n)^2$ *is central valued on* $R$ *and there exists* $\alpha \in C$ *such that* $q - a = b - c = \alpha$*;*

(3) $\mathrm{char}(R) = 2$ *and* $R$ *satisfies* $s_4$.

Now to investigate our generalized polynomial identity (GPI) (2.1), in all that follows, we assume $R$ a noncommutative prime ring with extended centroid $C$, $\mathrm{char}(R) \neq 2$. Moreover, we assume that $f(x_1, \ldots, x_n)$ is a multilinear polynomial over $C$ which is not central valued on $R$.

**Lemma 2.2.** *If* $a, b \in C$ *and* $R$ *satisfies* (2.1)*, then one of the following holds:*

(1) $p, v \in C$ *with* $(a + b)(p + q) = u + v$*;*

(2) $f(x_1, \ldots, x_n)^2$ *is central valued in* $R$ *with* $v - (a + b)p = (a + b)q - u = \alpha \in C$.

*Proof.* If $a, b \in C$, then by hypothesis

$$(a + b)(px + xq)x = x(ux + xv)$$

for all $x \in f(R)$. In this case by Lemma 2.1, one of the following holds:

(i) $(a + b)p, v \in C$ and $v - (a + b)p = (a + b)q - u = \alpha \in C$. Since $F \neq 0$, $a + b \neq 0$ and so $(a + b)p \in C$ implies $p \in C$.

(ii) $f(x_1, \ldots, x_n)^2$ is central valued on $R$ and there exists $\alpha \in C$ such that $v-(a+b)p = (a + b)q - u = \alpha \in C$. □

**Lemma 2.3.** *If* $q \in C$ *and* $R$ *satisfies* (2.1)*, then one of the following holds:*

(1) $b, u, v \in C$ *with* $(a + b)(p + q) = v + u$*;*

(2) $a, p, u \in C$ *with* $(a + b)(p + q) = v + u$*;*

(3) $u \in C$ *with* $f(x_1, \ldots, x_n)^2$ *being central valued on* $R$ *and* $a(p + q) + (p + q)b = u + v$.

*Proof.* If $q \in C$, then our hypothesis becomes

$$a(p + q)x^2 + px^2b + x^2(bq - v) - xux = 0$$

for all $x \in f(R)$. Then by Proposition 2.7 in [10], we conclude that $u \in C$. Then our hypothesis reduces to

$$a(p + q)x^2 + px^2b + x^2(bq - v - u) = 0$$

for all $x \in f(R)$. Then by applying Lemma 2.9 in [7], we conclude one of the following:

(i) $b, u, bq - v - u \in C$ with $a(p+q) + pb + (bq - v - u) = 0$, i.e., $(a+b)(p+q) = v+u$. Since $b, q, u \in C$, we have $v \in C$.

(ii) $a(p+q), u, p \in C$ with $a(p+q)+pb+(bq-v-u) = 0$, i.e., $(a+b)(p+q) = v+u$. In this case $G(x) = (p+q)x$ for all $x \in R$. As $G \neq 0$, thus $0 \neq p+q \in C$. Hence $a(p+q) \in C$ implies $a \in C$.

(iii) $u \in C$ and $f(x_1, \ldots, x_n)^2$ is central valued on $R$ with $a(p+q)+pb+(bq-v-u) = 0$, i.e., $a(p+q) + (p+q)b = u+v$.

Thus the lemma is proved. □

**Lemma 2.4.** *Let $R$ be a prime ring with extended centroid $C$ and $a, b, p, q, u, v \in R$. If*

$$apx^2 + axqx + px^2b + xqxb - xux - x^2v = 0$$

*for all $x \in f(R)$ is a trivial generalized polynomial identity, then either $a, b \in C$ or $q \in C$.*

*Proof.* Let $a \notin C$ and $q \notin C$. By hypothesis, we have

$$\begin{aligned}\zeta(x_1, \ldots, x_n) = {} & apf(x_1, \ldots, x_n)^2 + af(x_1, \ldots, x_n)qf(x_1, \ldots, x_n) \\ & + pf(x_1, \ldots, x_n)^2b + f(x_1, \ldots, x_n)qf(x_1, \ldots, x_n)b \\ & - f(x_1, \ldots, x_n)uf(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)^2v = 0\end{aligned}$$

for all $x_1, \ldots, x_n \in R$. Since $R$ and $U$ satisfy the same generalized polynomial identity (see [4]), $U$ satisfies $\zeta(x_1, \ldots, x_n) = 0$. By our assumption $\zeta(x_1, \ldots, x_n)$ is a trivial GPI for $U$. Let $T = U *_C C\{x_1, x_2, \ldots, x_n\}$, the free product of $U$ and $C\{x_1, \ldots, x_n\}$, the free $C$-algebra in noncommuting indeterminates $x_1, x_2, \ldots, x_n$. Then, $\zeta(x_1, \ldots, x_n)$ is zero element in $T = U *_C C\{x_1, \ldots, x_n\}$. This implies that $\{ap, a, p, 1\}$ is linearly $C$-dependent. Then there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in C$ such that $\alpha_1 ap + \alpha_2 a + \alpha_3 p + \alpha_4 \cdot 1 = 0$. If $\alpha_1 = \alpha_3 = 0$, then $\alpha_2 \neq 0$ and so $a = -\alpha_2^{-1}\alpha_4 \in C$, a contradiction. Therefore, either $\alpha_1 \neq 0$ or $\alpha_3 \neq 0$. Without loss of generality, we assume that $\alpha_1 \neq 0$. Then $ap = \alpha a + \beta p + \gamma$, where $\alpha = -\alpha_1^{-1}\alpha_2$, $\beta = -\alpha_1^{-1}\alpha_3$, $\gamma = -\alpha_1^{-1}\alpha_4$. Then

$$\begin{aligned}& (\alpha a + \beta p + \gamma)f(x_1, \ldots, x_n)^2 + af(x_1, \ldots, x_n)qf(x_1, \ldots, x_n) \\ & + pf(x_1, \ldots, x_n)^2b + f(x_1, \ldots, x_n)qf(x_1, \ldots, x_n)b \\ & - f(x_1, \ldots, x_n)uf(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)^2v = 0\end{aligned}$$

in $T$. This implies that $\{a, p, 1\}$ is linearly $C$-dependent. Then there exist $\beta_1, \beta_2, \beta_3 \in C$ such that $\beta_1 a + \beta_2 p + \beta_3 = 0$. By same argument as before, since $a \notin C$, we have $\beta_2 \neq 0$ and hence $p = \alpha' a + \beta'$ for some $\alpha', \beta' \in C$. Thus our identity becomes

$$\begin{aligned}& (\alpha a + \beta\alpha' a + \beta\beta' + \gamma)f(x_1, \ldots, x_n)^2 + af(x_1, \ldots, x_n)qf(x_1, \ldots, x_n) \\ & + (\alpha' a + \beta')f(x_1, \ldots, x_n)^2b + f(x_1, \ldots, x_n)qf(x_1, \ldots, x_n)b \\ & - f(x_1, \ldots, x_n)uf(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)^2v = 0.\end{aligned}$$

Since $\{a, 1\}$ is linearly $C$-independent, we have

$$(\alpha+\beta\alpha')af(x_1,\ldots,x_n)^2+af(x_1,\ldots,x_n)qf(x_1,\ldots,x_n)+\alpha' af(x_1,\ldots,x_n)^2b=0,$$

that is

$$af(x_1,\ldots,x_n)\big((\alpha+\beta\alpha'+q)f(x_1,\ldots,x_n)+\alpha' f(x_1,\ldots,x_n)b\big)=0$$

in $T$. Moreover, since $q \notin C$, the term $af(x_1,\ldots,x_n)qf(x_1,\ldots,x_n)$ cannot be canceled and hence $af(x_1,\ldots,x_n)qf(x_1,\ldots,x_n)=0$ in $T$ which implies $q=0$, a contradiction. Thus either $a \in C$ or $q \in C$.

Similarly, we can prove that either $b \in C$ or $q \in C$. □

**Lemma 2.5.** [6, Lemma 1] *Let $K$ be an infinite field and $m \geq 2$. If $A_1,\ldots,A_k$ are not scalar matrices in $M_m(K)$ then there exists some invertible matrix $P \in M_m(K)$ such that any matrices $PA_1P^{-1},\ldots,PA_kP^{-1}$ have all non-zero entries.*

**Proposition 2.6.** *Let $R = M_m(C)$ be the ring of all $m \times m$ matrices over the infinite field $C$ and $f(x_1,\ldots,x_n)$ a non-central multilinear polynomial over $C$. If there exist $a, b, p, q, u, v \in R$ such that*

$$apx^2+axqx+px^2b+xqxb-xux-x^2v=0$$

*for all $x \in f(R)$, then either $a$, $b$ are central or $q$ is central.*

*Proof.* By our hypothesis, $R$ satisfies the generalized polynomial identity

$$\begin{aligned}&apf(x_1,\ldots,x_n)^2+af(x_1,\ldots,x_n)qf(x_1,\ldots,x_n)\\&+pf(x_1,\ldots,x_n)^2b+f(x_1,\ldots,x_n)qf(x_1,\ldots,x_n)b\\&-f(x_1,\ldots,x_n)uf(x_1,\ldots,x_n)-f(x_1,\ldots,x_n)^2v=0.\end{aligned}$$

We assume first that $a \notin Z(R)$ and $q \notin Z(R)$. Now we shall show that this case leads to a contradiction.

Since $a \notin Z(R)$ and $q \notin Z(R)$, by Lemma 2.5 there exists a $C$-automorphism $\phi$ of $M_m(C)$ such that $\phi(a)$, $\phi(q)$ have all non-zero entries. Clearly, $R$ satisfies the generalized polynomial identity

$$\begin{aligned}&\phi(ap)f(x_1,\ldots,x_n)^2+\phi(a)f(x_1,\ldots,x_n)\phi(q)f(x_1,\ldots,x_n)\\&+\phi(p)f(x_1,\ldots,x_n)^2\phi(b)+f(x_1,\ldots,x_n)\phi(q)f(x_1,\ldots,x_n)b\\&-f(x_1,\ldots,x_n)\phi(u)f(x_1,\ldots,x_n)-f(x_1,\ldots,x_n)^2\phi(v)=0.\end{aligned}\tag{2.2}$$

By $e_{ij}$, we mean the usual matrix unit with 1 in $(i,j)$-entry and zero elsewhere. Since $f(x_1,\ldots,x_n)$ is not central valued, by [15] (see also [17]), there exist a sequence of matrices

$v_1, \ldots, v_n$ in $M_m(C)$ and $\gamma \in C - \{0\}$ such that $f(v_1, \ldots, v_n) = \gamma e_{pq}$, with $p \neq q$. Moreover, since the set $\{f(r_1, \ldots, r_n) : r_1, \ldots, r_n \in M_m(C)\}$ is invariant under the action of all $C$-automorphisms of $M_m(C)$, then for any $i \neq j$ there exist $r_1, \ldots, r_n$ in $M_m(C)$ such that $f(r_1, \ldots, r_n) = e_{ij}$. Hence by (2.2), we have

$$\phi(a)e_{ij}\phi(q)e_{ij} + e_{ij}\phi(q)e_{ij}b - e_{ij}\phi(u)e_{ij} = 0 \tag{2.3}$$

and then left multiplying by $e_{ij}$, it follows $e_{ij}\phi(a)e_{ij}\phi(q)e_{ij} = 0$, which is a contradiction, since $\phi(a)$ and $\phi(q)$ have all non-zero entries. Thus we conclude that either $a \in Z(R)$ or $q \in Z(R)$.

If we consider $b \notin Z(R)$ and $q \notin Z(R)$, then by same argument as above we have a contradiction with the fact $e_{ij}\phi(q)e_{ij}\phi(b)e_{ij} = 0$ obtained from (2.3). Thus we conclude either $b \in Z(R)$ or $q \in Z(R)$.

Thus, $q \notin Z(R)$ implies $a \in Z(R)$ and $b \in Z(R)$. Thus the conclusion follows. □

**Proposition 2.7.** *Let $R = M_m(C)$ be the ring of all matrices over the field $C$ with $\mathrm{char}(R) \neq 2$ and $f(x_1, \ldots, x_n)$ a non-central multilinear polynomial over $C$. If there exist $a, b, p, q, u, v \in R$ such that*

$$apx^2 + axqx + px^2b + xqxb - xux - x^2v = 0$$

*for all $x \in f(R)$, then either $a, b \in C \cdot I_m$ or $q \in C \cdot I_m$.*

*Proof.* In case $C$ is infinite, the conclusions follow by Proposition 2.6.

So we assume that $C$ is finite. Let $K$ be an infinite field which is an extension of the field $C$. Let $\overline{R} = M_m(K) \cong R \otimes_C K$. Notice that the multilinear polynomial $f(x_1, \ldots, x_n)$ is central-valued on $R$ if and only if it is central-valued on $\overline{R}$. Consider the generalized polynomial

$$\begin{aligned}\Psi(x_1, \ldots, x_n) = {} & apf(x_1, \ldots, x_n)^2 + af(x_1, \ldots, x_n)qf(x_1, \ldots, x_n) \\ & + pf(x_1, \ldots, x_n)^2b + f(x_1, \ldots, x_n)qf(x_1, \ldots, x_n)b \\ & - f(x_1, \ldots, x_n)uf(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)^2\end{aligned}$$

which is a generalized polynomial identity for $R$.

Moreover, it is a multi-homogeneous of multi-degree $(2, \ldots, 2)$ in the indeterminates $x_1, \ldots, x_n$. Hence the complete linearization of $\Psi(x_1, \ldots, x_n)$ yields a multilinear generalized polynomial $\Theta(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in $2n$ indeterminates, moreover

$$\Theta(x_1, \ldots, x_n, x_1, \ldots, x_n) = 2^n\Psi(x_1, \ldots, x_n).$$

Clearly the multilinear polynomial $\Theta(x_1, \ldots, x_n, y_1, \ldots, y_n)$ is a generalized polynomial identity for $R$ and $\overline{R}$ too. Since $\mathrm{char}(C) \neq 2$ we obtain $\Psi(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in \overline{R}$ and then conclusion follows from Proposition 2.6. □

In particular, we have the following

**Corollary 2.8.** *Let $R = M_m(C)$ be the ring of all matrices over the field $C$ with* $\mathrm{char}(R) \neq 2$*. If there exist $a, b, p, q, u, v \in R$ such that*

$$apx^2 + axqx + px^2b + xqxb - xux - x^2v = 0$$

*for all $x \in R$, then either $a, b \in C \cdot I_m$ or $q \in C \cdot I_m$.*

Similarly, we have the following

**Corollary 2.9.** *Let $R = M_m(C)$ be the ring of all matrices over the field $C$ with* $\mathrm{char}(R) \neq 2$*. If there exist $a', a, b, p, q, u, v \in R$ such that*

$$a'x^2 + axqx + px^2b + xqxb - xux - x^2v = 0$$

*for all $x \in R$, then either $a, b \in C \cdot I_m$ or $q \in C \cdot I_m$.*

**Lemma 2.10.** *Let $R$ be a primitive ring of* $\mathrm{char}(R) \neq 2$ *with nonzero socle* $\mathrm{Soc}(R)$*, which is isomorphic to a dense ring of linear transformations of a vector space $V$ over $C$, such that* $\dim_C V = \infty$*. Let $a', a, b, p, q, u, v \in R$. If*

$$a'x^2 + axqx + px^2b + xqxb - xux - x^2v = 0$$

*for all $x \in R$, then either $a, b \in C$ or $q \in C$.*

*Proof.* Recall that if any element $r \in R$ commutes the nonzero ideal $\mathrm{Soc}(RC)$, i.e., $[r, \mathrm{Soc}(RC)] = (0)$, then $r \in C$. Hence on contrary, we assume that there exist $h_0, h_1, h_2 \in \mathrm{Soc}(R)$ such that

(i) either $[a, h_0] \neq 0$ or $[b, h_1] \neq 0$;

(ii) $[q, h_2] \neq 0$

and prove that a number of contradiction arises. Since $V$ is infinite dimensional over $C$, for any $e = e^2 \in \mathrm{Soc}(R)$, we have $eRe \cong M_k(C)$ with $k = \dim_C Ve$. By Litoff's Theorem [12], there exists an idempotent $e \in \mathrm{Soc}(R)$ such that

- $h_0, h_1, h_2 \in eRe$;
- $h_0a, ah_0, h_1a, ah_1, h_2a, ah_2 \in eRe$;
- $h_0b, bh_0, h_1b, bh_1, h_2b, bh_2 \in eRe$;
- $h_0q, qh_0, h_1q, qh_1, h_2q, qh_2 \in eRe$,

where $eRe \cong M_k(C)$, $k = \dim_C Ve$. Since $R$ satisfies $e\{a'(exe)^2 + aexeqexe + p(exe)^2b + exeqexeb - exeuexe - (exe)^2v\}e = 0$, the subring $eRe$ satisfies $ea'ex^2 + eaexeqex + epex^2ebe + xeqexebe - xeuex - x^2eve = 0$. By Corollary 2.9, we conclude that one of the following holds:

(i) $eae, ebe \in eC$ which contradicts with the choice of $h_0$ and $h_1$;

(ii) $eqe \in eC$ which contradicts with the choices of $h_2$. □

**Lemma 2.11.** *Let $R$ be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring $U$ and extended centroid $C$, and $f(x_1, \ldots, x_n)$ a multilinear polynomial over $C$, which is not central valued on $R$. Suppose that $F$, $G$ and $H$ are three nonzero inner generalized derivations of $R$ such that $F(G(f(r))f(r)) = f(r)H(f(r))$ for all $r = (r_1, \ldots, r_n) \in R^n$, then one of the following holds:*

(1) *there exist $\lambda \in C$ and $a, b \in U$ such that $F(x) = \lambda x$, $G(x) = xa$ and $H(x) = bx$ for all $x \in R$ with $\lambda a = b$;*

(2) *there exist $\lambda, \alpha \in C$ and $p, q, u, v \in U$ such that $F(x) = \lambda x$, $G(x) = px + xq$ and $H(x) = ux + xv$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ being central valued in $R$ and $v - \lambda p = \lambda q - u = \alpha \in C$;*

(3) *there exist $\lambda \in C$ and $a, p \in U$ such that $F(x) = ax$, $G(x) = px$ and $H(x) = \lambda x$ for all $x \in R$ with $ap = \lambda$;*

(4) *there exist $\lambda \in C$ and $a, u \in U$ such that $F(x) = xa$, $G(x) = \lambda x$ and $H(x) = xu$ for all $x \in R$ with $a\lambda = u$;*

(5) *there exist $a, b, p, v \in U$ such that $F(x) = ax + xb$, $G(x) = px$ and $H(x) = xv$ for all $x \in R$ with $f(x_1, \ldots, x_n)^2$ being central valued on $R$ and $ap + pb = v$.*

*Proof.* Suppose that for some $a, b, p, q, u, v \in U$, $F(x) = ax + xb$, $G(x) = px + xq$ and $H(x) = ux + xv$ for all $x \in R$. By hypothesis, we have

$$\begin{aligned} &a\big((pf(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)q)f(x_1, \ldots, x_n)\big) \\ &\quad + \big((pf(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)q)f(x_1, \ldots, x_n)\big)b \\ &= f(x_1, \ldots, x_n)(uf(x_1, \ldots, x_n) + f(x_1, \ldots, x_n)v), \end{aligned}$$

that is,

$$\begin{aligned} &apf(x_1, \ldots, x_n)^2 + af(x_1, \ldots, x_n)qf(x_1, \ldots, x_n) \\ &+ pf(x_1, \ldots, x_n)^2b + f(x_1, \ldots, x_n)qf(x_1, \ldots, x_n)b \\ &- f(x_1, \ldots, x_n)uf(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)^2v = 0 \end{aligned}$$

for all $x_1, \dots, x_n \in R$. Since $R$ and $U$ satisfy the same generalized polynomial identities (see [4]), therefore, $U$ satisfies

$$\begin{aligned} &apf(x_1,\dots,x_n)^2 + af(x_1,\dots,x_n)qf(x_1,\dots,x_n)\\ &+ pf(x_1,\dots,x_n)^2 b + f(x_1,\dots,x_n)qf(x_1,\dots,x_n)b\\ &- f(x_1,\dots,x_n)uf(x_1,\dots,x_n) - f(x_1,\dots,x_n)^2 v = 0. \end{aligned} \tag{2.4}$$

If this is a trivial generalized polynomial identity for $U$, then by Lemma 2.4, either $a, b \in C$ or $q \in C$.

Next we assume that (2.4) is a non-trivial GPI for $U$.

Since both $U$ and $U \otimes_C \overline{C}$ are prime and centrally closed [11, Theorems 2.5 and 3.5], we may replace $R$ by $U$ or $U \otimes_C \overline{C}$ according as $C$ finite or infinite. Then $R$ is centrally closed over $C$ and $R$ satisfies (2.4). By Martindale's Theorem [18], $R$ is then a primitive ring with nonzero socle $\mathrm{soc}(R)$ and with $C$ as its associated division ring. Then, by Jacobson's Theorem [13, p. 75], $R$ is isomorphic to a dense ring of linear transformations of a vector space $V$ over $C$. Assume first that $V$ is finite dimensional over $C$, that is, $\dim_C V = m$. By density of $R$, we have $R \cong M_m(C)$. Since $f(r_1, \dots, r_n)$ is not central valued on $R$, $R$ must be noncommutative and so $m \geq 2$. In this case, by Proposition 2.7, we get that $a, b \in C$ or $q \in C$. If $V$ is infinite dimensional over $C$, then by Lemma 2.10, we conclude that either $a, b \in C$ or $q \in C$.

Thus up to now, we have proved that in any cases either $a, b \in C$ or $q \in C$.

*Case* 1*:* $a, b \in C$. In this case by Lemma 2.2, we have the following cases:

(i) $p, v \in C$ with $(a + b)(p + q) = u + v$; Thus $F(x) = ax + xb = (a + b)x$, $G(x) = px+xq = x(p+q)$ and $H(x) = ux+xv = (u+v)x$ for all $x \in R$. This is our conclusion (1).

(ii) $f(x_1, \dots, x_n)^2$ is central valued in $R$ with $v - (a + b)p = (a + b)q - u = \alpha \in C$. Thus $F(x) = ax + xb = (a + b)x$, $G(x) = px + xq$ and $H(x) = ux + xv$ for all $x \in R$. This is our conclusion (2).

*Case* 2*:* $q \in C$. In this case by Lemma 2.3, we have the following cases:

(i) $b, q, u, v \in C$ with $(a + b)(p + q) = v + u = \lambda \in C$. Thus $F(x) = (a + b)x$, $G(x) = (p + q)x$ and $H(x) = (u + v)x$ for all $x \in R$. This is our conclusion (3).

(ii) $a, u, p, q \in C$ with $(a + b)(p + q) = v + u$. Thus $F(x) = x(a + b)$, $G(x) = (p + q)x$ and $H(x) = x(u + v)$ for all $x \in R$. This is our conclusion (4).

(iii) $q, u \in C$ with $f(x_1, \dots, x_n)^2$ being central valued on $R$ and $a(p+q)+(p+q)b = u+v$. Thus $F(x) = ax + xb$, $G(x) = (p + q)x$ and $H(x) = x(u + v)$ for all $x \in R$. This is our conclusion (5). □

In particular we have

**Corollary 2.12.** *Let $R$ be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring $U$ and extended centroid $C$, and $f(x_1,\dots,x_n)$ a multilinear polynomial over $C$, which is not central valued on $R$. Suppose that $F$ is a nonzero inner generalized derivation of $R$ such that $F([p,f(r)]f(r)) = f(r)[q,f(r)]$ for all $r=(r_1,\dots,r_n)\in R^n$, then there exists $\lambda\in C$ such that $F(x)=\lambda x$ for all $x\in R$ with $f(x_1,\dots,x_n)^2$ being central valued in $R$ and $(\lambda p+q)\in C$.*

**Corollary 2.13.** *Let $R$ be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring $U$ and extended centroid $C$, and $f(x_1,\dots,x_n)$ a multilinear polynomial over $C$, which is not central valued on $R$. Suppose that $F$ is a nonzero inner generalized derivation of $R$ such that $F([p,f(r)]f(r)) = f(r)[p,f(r)]$ for all $r=(r_1,\dots,r_n)\in R^n$, then there exists $\lambda\in C$ such that $F(x)=\lambda x$ for all $x\in R$ with $f(x_1,\dots,x_n)^2$ being central valued in $R$ and $(\lambda+1)p\in C$.*

**Lemma 2.14.** *Let $R$ be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring $U$ and extended centroid $C$, and $f(x_1,\dots,x_n)$ a multilinear polynomial over $C$, which is not central valued on $R$. Suppose that $G$ and $H$ are two generalized derivations of $R$ and $F(x)=cx+xc'$ for all $x\in R$, for some $c,c'\in U$ is a nonzero inner generalized derivation of $R$, such that $F(G(f(r))f(r)) = f(r)H(f(r))$ for all $r=(r_1,\dots,r_n)\in R^n$, then one of the following holds:*

(1) *there exist $\lambda\in C$ and $a,b\in U$ such that $F(x)=\lambda x$, $G(x)=xa$ and $H(x)=bx$ for all $x\in R$ with $\lambda a=b$;*

(2) *there exist $\lambda,\alpha\in C$ and $p,q,u,v\in U$ such that $F(x)=\lambda x$, $G(x)=px+xq$ and $H(x)=ux+xv$ for all $x\in R$ with $f(x_1,\dots,x_n)^2$ being central valued in $R$ and $v-\lambda p=\lambda q-u=\alpha\in C$.*

(3) *there exist $\lambda\in C$ and $a,p\in U$ such that $F(x)=ax$, $G(x)=px$ and $H(x)=\lambda x$ for all $x\in R$ with $ap=\lambda$.*

(4) *there exist $\lambda\in C$ and $a,u\in U$ such that $F(x)=xa$, $G(x)=\lambda x$ and $H(x)=xu$ for all $x\in R$ with $a\lambda=u$.*

(5) *there exist $a,b,p,v\in U$ such that $F(x)=ax+xb$, $G(x)=px$ and $H(x)=xv$ for all $x\in R$ with $f(x_1,\dots,x_n)^2$ being central valued on $R$ and $ap+pb=v$.*

*Proof.* In view of [16, Theorem 3], we may assume that there exist $a,b\in U$ and derivations $d'$, $\delta$ of $U$ such that $G(x)=ax+d'(x)$ and $H(x)=bx+\delta(x)$. Since $R$ and $U$ satisfy the same generalized polynomial identities (see [4]) as well as the same differential identities (see [15]), we may assume that

$$c\{af(r)^2+d'(f(r))f(r)\}+\{af(r)^2+d'(f(r))f(r)\}c' = f(r)bf(r)+f(r)\delta(f(r)) \tag{2.5}$$

for all $r = (r_1, \ldots, r_n) \in U^n$, where $d'$, $\delta$ are two derivations on $U$.

If $G$ and $H$ both are inner generalized derivations of $R$, then by Lemma 2.11 we obtain our conclusions (1)–(5). Thus we assume that not both of $F$ and $G$ are inner. Then $d'$ and $\delta$ cannot be both inner derivations of $U$. Now we consider the following two cases:

*Case I:* Assume that $d'$ and $\delta$ are $C$-dependent modulo inner derivations of $U$, say $\alpha d' + \beta\delta = ad_q$, where $\alpha, \beta \in C$, $q \in U$ and $ad_q(x) = [q, x]$ for all $x \in R$.

*Subcase i: Let* $\alpha \neq 0$. Then $d'(x) = \lambda\delta(x) + [p, x]$ for all $x \in U$, for some $\lambda \in C$ and $p \in U$.

Then $\delta$ cannot be inner derivation of $U$. From (2.5), we obtain

$$\begin{aligned} &c\{af(r)^2 + \lambda\delta(f(r))f(r) + [p, f(r)]f(r)\} \\ &\quad + \{af(r)^2 + \lambda\delta(f(r))f(r) + [p, f(r)]f(r)\}c' \\ &= f(r)bf(r) + f(r)\delta(f(r)) \end{aligned} \tag{2.6}$$

for all $r = (r_1, \ldots, r_n) \in U^n$.

Since $f(r_1, \ldots, r_n)$ is a multilinear polynomial over $C$, we have $\delta(f(r_1, \ldots, r_n)) = f^\delta(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, \delta(r_i), \ldots, r_n)$, where $f^\delta(r_1, \ldots, r_n)$ is the polynomials obtained from $f(r_1, \ldots, r_n)$ replacing each coefficients $\alpha_\sigma$ with $\delta(\alpha_\sigma)$. Thus by Kharchenko's Theorem [14], we can replace $\delta(f(r_1, \ldots, r_n))$ by $f^\delta(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, y_i, \ldots, r_n)$ in (2.6) and then $U$ satisfies blended components

$$\begin{aligned} &c\left\{\lambda \sum_i f(r_1, \ldots, y_i, \ldots, r_n)f(r_1, \ldots, r_n)\right\} \\ &\quad + \left\{\lambda \sum_i f(r_1, \ldots, y_i, \ldots, r_n)f(r_1, \ldots, r_n)\right\}c' \\ &= f(r_1, \ldots, r_n)\sum_i f(r_1, \ldots, y_i, \ldots, r_n). \end{aligned} \tag{2.7}$$

Replacing $y_i$ with $[q, y_i]$ for some $q \notin C$ in (2.7), we obtain

$$c\lambda[q, f(r)]f(r) + [q, f(r)]f(r)\lambda c' = f(r)[q, f(r)].$$

By Corollary 2.13, $f(x_1, \ldots, x_n)^2$ is central valued in $R$ with $c\lambda, c'\lambda \in C$ and $(\lambda(c + c') + 1)q \in C$. Since $q \notin C$, $(\lambda(c + c') + 1)q \in C$ implies $(\lambda(c + c') + 1) = 0$, i.e., $\lambda(c + c') = -1$. Then by (2.7),

$$(c + c')\lambda \sum_i f(r_1, \ldots, y_i, \ldots, r_n)f(r_1, \ldots, r_n) = f(r_1, \ldots, r_n)\sum_i f(r_1, \ldots, y_i, \ldots, r_n)$$

which implies

$$f(r_1, \ldots, r_n)\sum_i f(r_1, \ldots, y_i, \ldots, r_n) + \sum_i f(r_1, \ldots, y_i, \ldots, r_n)f(r_1, \ldots, r_n) = 0.$$

In particular, for $y_1 = r_1$ and $y_2 = \cdots = y_n = 0$, we have $2f(r_1,\ldots,r_n)^2 = 0$ for all $r_1,\ldots,r_n \in U$, implying $f(r_1,\ldots,r_n) = 0$ for all $r_1,\ldots,r_n \in U$, a contradiction.

*Subcase ii: Let* $\alpha = 0$. Then $\delta(x) = [q', x]$ for all $x \in U$, where $q' = \beta^{-1}q$. Since $\delta$ is inner, $d'$ cannot be inner derivation. From (2.5), we obtain

$$\begin{aligned} &c\{af(r)^2 + d'(f(r))f(r)\} + \{af(r)^2 + d'(f(r))f(r)\}c' \\ &= f(r)bf(r) + f(r)[q', f(r)] \end{aligned} \tag{2.8}$$

for all $r = (r_1,\ldots,r_n) \in U^n$.

Since $d'(f(r_1,\ldots,r_n)) = f^{d'}(r_1,\ldots,r_n) + \sum_i f(r_1,\ldots,d'(r_i),\ldots,r_n)$, by Kharchenko's Theorem [14], we can replace $d'(f(r_1,\ldots,r_n))$ by $f^{d'}(r_1,\ldots,r_n) + \sum_i f(r_1,\ldots,y_i,\ldots,r_n)$ in (2.8) and then $U$ satisfies blended component

$$\begin{aligned} &c\sum_i f(r_1,\ldots,y_i,\ldots,r_n)f(r_1,\ldots,r_n) \\ &+ \sum_i f(r_1,\ldots,y_i,\ldots,r_n)f(r_1,\ldots,r_n)c' = 0. \end{aligned}$$

Replacing $y_i$ with $[a', r_i]$ for some $a' \notin C$, $U$ satisfies

$$c[a', f(r_1,\ldots,r_n)]f(r_1,\ldots,r_n) + [a', f(r_1,\ldots,r_n)]f(r_1,\ldots,r_n)c' = 0.$$

Then by Corollary 2.12, $f(x_1,\ldots,x_n)^2$ is central valued in $R$ with $c, c' \in C$ and $(c+c')a' \in C$. Since $a' \notin C$, $c + c' = 0$ implying $F = 0$, a contradiction.

*Case II:* Assume next that $d'$ and $\delta$ are $C$-independent modulo inner derivations of $U$. Then applying Kharchenko's Theorem [14], we have from (2.5) that $U$ satisfies blended components

$$\begin{aligned} &c\sum_i f(r_1,\ldots,y_i,\ldots,r_n)f(r_1,\ldots,r_n) + \sum_i f(r_1,\ldots,y_i,\ldots,r_n)f(r_1,\ldots,r_n)c' \\ &= f(r_1,\ldots,r_n)\sum_i f(r_1,\ldots,z_i,\ldots,r_n). \end{aligned}$$

In particular, for $y_1 = \cdots = y_n = 0$, $U$ satisfies $f(r_1,\ldots,r_n)\sum_i f(r_1,\ldots,z_i,\ldots,r_n) = 0$. In particular, $f(r_1,\ldots,r_n)^2 = 0$ for all $r_1,\ldots,r_n \in U$, implying $f(r_1,\ldots,r_n) = 0$, a contradiction. □

**Lemma 2.15.** *Let $R$ be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring $U$ and extended centroid $C$, and $f(x_1,\ldots,x_n)$ a multilinear polynomial over $C$, which is not central valued on $R$. Suppose that $F$ and $H$ are two generalized derivations of $R$ and $G(x) = cx + xc'$ for all $x \in R$, for some $c, c' \in U$ is a nonzero inner generalized derivation of $R$, such that $F(G(f(r))f(r)) = f(r)H(f(r))$ for all $r = (r_1,\ldots,r_n) \in R^n$, then one of the following holds:*

(1) *there exist* $\lambda \in C$ *and* $a, b \in U$ *such that* $F(x) = \lambda x$, $G(x) = xa$ *and* $H(x) = bx$ *for all* $x \in R$ *with* $\lambda a = b$;

(2) *there exist* $\lambda, \alpha \in C$ *and* $p, q, u, v \in U$ *such that* $F(x) = \lambda x$, $G(x) = px + xq$ *and* $H(x) = ux + xv$ *for all* $x \in R$ *with* $f(x_1, \ldots, x_n)^2$ *being central valued in* $R$ *and* $v - \lambda p = \lambda q - u = \alpha \in C$;

(3) *there exist* $\lambda \in C$ *and* $a, p \in U$ *such that* $F(x) = ax$, $G(x) = px$ *and* $H(x) = \lambda x$ *for all* $x \in R$ *with* $ap = \lambda$;

(4) *there exist* $\lambda \in C$ *and* $a, u \in U$ *such that* $F(x) = xa$, $G(x) = \lambda x$ *and* $H(x) = xu$ *for all* $x \in R$ *with* $a\lambda = u$;

(5) *there exist* $a, b, p, v \in U$ *such that* $F(x) = ax + xb$, $G(x) = px$ *and* $H(x) = xv$ *for all* $x \in R$ *with* $f(x_1, \ldots, x_n)^2$ *being central valued on* $R$ *and* $ap + pb = v$.

*Proof.* In view of [16, Theorem 3], we may assume that there exist $a, b \in U$ and derivations $d'$, $\delta$ of $U$ such that $F(x) = ax + d(x)$ and $H(x) = bx + \delta(x)$. Since $R$ and $U$ satisfy the same generalized polynomial identities (see [4]) as well as the same differential identities (see [15]), we may assume that

$$a\{cf(r)^2 + f(r)c'f(r)\} + d\{cf(r)^2 + f(r)c'f(r)\} = f(r)bf(r) + f(r)\delta(f(r)) \tag{2.9}$$

for all $r = (r_1, \ldots, r_n) \in U^n$, where $d$, $\delta$ are two derivations on $U$.

If $F$ and $H$ both are inner generalized derivations of $R$, then by Lemma 2.11 we obtain our conclusions (1)–(5). Thus we assume that not both of $F$ and $H$ are inner. Then $d$ and $\delta$ cannot be both inner derivations of $U$. Now we consider the following two cases:

*Case I:* Assume that $d$ and $\delta$ are $C$-dependent modulo inner derivations of $U$, say $\alpha d + \beta\delta = ad_q$, where $\alpha, \beta \in C$, $q \in U$ and $ad_q(x) = [q, x]$ for all $x \in R$. If $\beta = 0$, then $\alpha \neq 0$ and thus $d$ is inner. In this case conclusion follows by Lemma 2.14. Next we assume that $\beta \neq 0$. Then there exist some $\lambda \in C$ and $p \in U$ such that $\delta(x) = \lambda d(x) + [p, x]$ for all $x \in U$. The by (2.9), $U$ satisfies

$$\begin{aligned} &a\{cf(r)^2 + f(r)c'f(r)\} + d(c)f(r)^2 + cd(f(r))f(r) + cf(r)d(f(r)) \\ &\quad + d(f(r))c'f(r) + f(r)d(c')f(r) + f(r)c'd(f(r)) \\ &= f(r)bf(r) + f(r)\lambda d(f(r)) + f(r)[p, f(r)]. \end{aligned} \tag{2.10}$$

Since $f(r_1, \ldots, r_n)$ is a multilinear polynomial over $C$, we have $d(f(r_1, \ldots, r_n)) = f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, d(r_i), \ldots, r_n)$, where $f^d(r_1, \ldots, r_n)$ is the polynomials obtained from $f(r_1, \ldots, r_n)$ replacing each coefficients $\alpha_\sigma$ with $d(\alpha_\sigma)$. Thus by Kharchenko's

Theorem [14], we can replace $d(f(r_1,\dots,r_n))$ by $f^d(r_1,\dots,r_n)+\sum_i f(r_1,\dots,y_i,\dots,r_n)$ in (2.10) and then $U$ satisfies blended components

(2.11)
$$\begin{aligned} &c\sum_i f(r_1,\dots,y_i,\dots,r_n)f(r_1,\dots,r_n)+cf(r_1,\dots,r_n)\sum_i f(r_1,\dots,y_i,\dots,r_n)\\ &\quad+\sum_i f(r_1,\dots,y_i,\dots,r_n)c'f(r_1,\dots,r_n)+f(r_1,\dots,r_n)c'\sum_i f(r_1,\dots,y_i,\dots,r_n)\\ &=f(r_1,\dots,r_n)\lambda\sum_i f(r_1,\dots,y_i,\dots,r_n).\end{aligned}$$

In particular, for $y_1=r_1$ and $y_2=\cdots=y_n=0$, $U$ satisfies

$$(2c-\lambda)f(r_1,\dots,r_n)^2+f(r_1,\dots,r_n)(2c')f(r_1,\dots,r_n)=0,$$

which implies

$$\big((2c-\lambda)f(r_1,\dots,r_n)+f(r_1,\dots,r_n)(2c')\big)f(r_1,\dots,r_n)=0.$$

By Lemma 2.1, we conclude that $2c'=\lambda-2c\in C$. Since $\mathrm{char}(R)\neq 2$, $c,c'\in C$. Then by (2.11), $U$ satisfies

$$\begin{aligned}&(c+c')\sum_i f(r_1,\dots,y_i,\dots,r_n)f(r_1,\dots,r_n)\\ &+(c+c'-\lambda)f(r_1,\dots,r_n)\sum_i f(r_1,\dots,y_i,\dots,r_n)=0.\end{aligned}$$

Replacing $y_i$ with $[q,x_i]$ for some $q'\notin C$, we have

$$(c+c')[q',f(r_1,\dots,r_n)]f(r_1,\dots,r_n)+(c+c'-\lambda)f(r_1,\dots,r_n)[q',f(r_1,\dots,r_n)]=0,$$

that is,

$$[(c+c')q',f(r_1,\dots,r_n)]f(r_1,\dots,r_n)+f(r_1,\dots,r_n)[(c+c'-\lambda)q',f(r_1,\dots,r_n)]=0.$$

By Lemma 2.1, one of the following holds: (i) $(c+c')q',(c+c'-\lambda)q'\in C$; in this case as $q'\notin C$, $c+c'=0$, implying $G=0$, a contradiction. (ii) $f(r_1,\dots,r_n)^2$ is central valued and $(c+c'-\lambda)q'-(c+c')q'\in C$, i.e., $\lambda q'\in C$. In this case as $q'\notin C$, $\lambda=0$. Thus $\lambda=2(c+c')=0$ implying $c+c'=0$. Hence $G=0$, a contradiction. □

**Lemma 2.16.** *Let $R$ be a noncommutative prime ring of characteristic different from 2 with Utumi quotient ring $U$ and extended centroid $C$, and $f(x_1,\dots,x_n)$ a multilinear polynomial over $C$, which is not central valued on $R$. Suppose that $F$ and $G$ are two generalized derivations of $R$ and $H(x)=bx+xb'$ for all $x\in R$, for some $b,b'\in U$ is a nonzero inner generalized derivation of $R$, such that $F(G(f(r))f(r))=f(r)H(f(r))$ for all $r=(r_1,\dots,r_n)\in R^n$, then one of the following holds:*

(1) *there exist* $\lambda \in C$ *and* $a, b \in U$ *such that* $F(x) = \lambda x$, $G(x) = xa$ *and* $H(x) = bx$ *for all* $x \in R$ *with* $\lambda a = b$;

(2) *there exist* $\lambda, \alpha \in C$ *and* $p, q, u, v \in U$ *such that* $F(x) = \lambda x$, $G(x) = px + xq$ *and* $H(x) = ux + xv$ *for all* $x \in R$ *with* $f(x_1, \ldots, x_n)^2$ *being central valued in* $R$ *and* $v - \lambda p = \lambda q - u = \alpha \in C$;

(3) *there exist* $\lambda \in C$ *and* $a, p \in U$ *such that* $F(x) = ax$, $G(x) = px$ *and* $H(x) = \lambda x$ *for all* $x \in R$ *with* $ap = \lambda$;

(4) *there exist* $\lambda \in C$ *and* $a, u \in U$ *such that* $F(x) = xa$, $G(x) = \lambda x$ *and* $H(x) = xu$ *for all* $x \in R$ *with* $a\lambda = u$;

(5) *there exist* $a, b, p, v \in U$ *such that* $F(x) = ax + xb$, $G(x) = px$ *and* $H(x) = xv$ *for all* $x \in R$ *with* $f(x_1, \ldots, x_n)^2$ *being central valued on* $R$ *and* $ap + pb = v$.

*Proof.* In view of [16, Theorem 3], we may assume that there exist $a, b \in U$ and derivations $d'$, $\delta$ of $U$ such that $F(x) = cx + d(x)$ and $G(x) = ax + d'(x)$. Since $R$ and $U$ satisfy the same generalized polynomial identities (see [4]) as well as the same differential identities (see [15]), we may assume that

$$c\{af(r)^2 + d'(f(r))f(r)\} + d\{af(r)^2 + d'(f(r))f(r)\} = f(r)bf(r) + f(r)^2b' \tag{2.12}$$

for all $r = (r_1, \ldots, r_n) \in U^n$, where $d$, $d'$ are two derivations on $U$.

If $d$ or $d'$ is inner, then $F$ or $G$ is inner and then by Lemmas 2.14 and 2.15, we obtain our conclusions (1)–(5). Thus we assume that both of $d$ and $d'$ are outer. Now we consider the following two cases:

*Case I:* Assume that $d$ and $d'$ are $C$-dependent modulo inner derivations of $U$, then $d = \alpha d' + ad_{p'}$. Then (2.12) becomes

(2.13)
$$c\big\{af(r)^2 + d'(f(r))f(r)\big\} + \alpha d'\big\{af(r)^2 + d'(f(r))f(r)\big\} + [p', af(r)^2 + d'(f(r))f(r)] = f(r)bf(r) + f(r)^2b'.$$

We know that $d'(f(r_1, \ldots, r_n)) = f^{d'}(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, d'(r_i), \ldots, r_n)$, and

$$\begin{aligned} d'^2(f(r_1, \ldots, r_n)) = f^{d'^2}(r_1, \ldots, r_n) &+ 2\sum_i f^{d'}(r_1, \ldots, d'(r_i), \ldots, r_n) \\ &+ \sum_i f(r_1, \ldots, d'^2(r_i), \ldots, r_n) \\ &+ \sum_{i \neq j} f(r_1, \ldots, d'(r_i), \ldots, d'(r_j), \ldots, r_n). \end{aligned}$$

By applying Kharchenko's Theorem [14], we can replace $d(f(r_1,\dots,r_n))$ with $f^d(r_1,\dots,r_n) + \sum_i f(r_1,\dots,y_i,\dots,r_n)$ and $d'^2(f(r_1,\dots,r_n))$ with

$$f^{d'^2}(r_1,\dots,r_n) + 2\sum_i f^{d'}(r_1,\dots,y_i,\dots,r_n)$$
$$+\sum_i f(r_1,\dots,t_i,\dots,r_n) + \sum_{i\neq j} f(r_1,\dots,y_i,\dots,y_j,\dots,r_n)$$

in (2.13) and then $U$ satisfies blended component

$$\alpha\sum_i f(r_1,\dots,t_i,\dots,r_n)f(r_1,\dots,r_n) = 0.$$

This implies $\alpha f(x_1,\dots,x_n)^2 = 0$, implying $\alpha = 0$. Then $d$ is inner, a contradiction.

*Case II:* Assume that $d$ and $d'$ are $C$-independent modulo inner derivations of $U$.

Then applying Kharchenko's Theorem [14] to (2.12), we can replace

$$d'(f(r_1,\dots,r_n)) = f^{d'}(r_1,\dots,r_n) + \sum_i f(r_1,\dots,y_i,\dots,r_n),$$
$$d(f(r_1,\dots,r_n)) = f^{d}(r_1,\dots,r_n) + \sum_i f(r_1,\dots,t_i,\dots,r_n),$$

and

$$dd'(f(r_1,\dots,r_n)) = f^{dd'}(r_1,\dots,r_n) + \sum_i f^{\delta}(r_1,\dots,y_i,\dots,r_n)$$
$$+\sum_i f^{d'}(r_1,\dots,t_i,\dots,r_n) + \sum_{i\neq j} f(r_1,\dots,y_i,\dots,t_j,\dots,r_n)$$
$$+\sum_i f(r_1,\dots,w_i',\dots,r_n).$$

Then $U$ satisfies blended component $\sum_i f(r_1,\dots,w_i',\dots,r_n)f(r_1,\dots,r_n) = 0$. In particular, $f(r_1,\dots,r_n)^2 = 0$ implying $f(r_1,\dots,r_n) = 0$, a contradiction. □

*Proof of Theorem* 1.1. If any one of $F$ or $G$ or $H$ is inner, then conclusion follows by Lemmas 2.14, 2.15 and 2.16.

Thus we assume that $F$, $G$ and $H$ are all outer generalized derivations of $R$. Then by [16], we have $F(x) = cx + d(x)$, $G(x) = ax + d'(x)$ and $H(x) = bx + \delta(x)$ for some $a, b, c \in U$ and $d$, $d'$, $\delta$ are three derivations of $U$. By hypothesis, we have

$$c\{af(r)^2 + d'(f(r))f(r)\} + d\{af(r)^2 + d'(f(r)\}f(r)) = f(r)bf(r) + f(r)\delta(f(r)) \tag{2.14}$$

for all $r = (r_1,\dots,r_n) \in U^n$. Now we consider the following two cases:

*Case 1:* Let $d'$ and $\delta$ be $C$-dependent modulo inner derivations of $U$, i.e., $\alpha d' + \beta\delta = ad_{p'}$.

Now $\alpha = 0$ implies that $\delta$ is inner, a contradiction as $H$ cannot be inner. Thus $\alpha \neq 0$. Then $d' = \lambda\delta + ad_p$, where $\lambda = -\beta\alpha^{-1} \in C$ and $p = p'\alpha^{-1} \in U$. Therefore, (2.14) gives

$$\begin{aligned} &c\{af(r)^2 + \lambda\delta(f(r))f(r) + [p, f(r)]f(r)\} + d\big(af(r)^2 + \lambda\delta(f(r))f(r) + [p, f(r)]f(r)\big) \\ &= f(r)bf(r) + f(r)\delta(f(r)) \end{aligned}$$

for all $r = (r_1, \ldots, r_n) \in U^n$, that is,

$$\begin{aligned} &c\big(af(r)^2 + \lambda\delta(f(r))f(r) + [p, f(r)]f(r)\big) + d\big(af(r)^2 + [p, f(r)]f(r)\big) \\ &\quad + d(\lambda)\delta(f(r))f(r) + \lambda(d\delta)(f(r))f(r) + \lambda\delta(f(r))d(f(r)) \\ &= f(r)bf(r) + f(r)\delta(f(r)) \end{aligned} \tag{2.15}$$

for all $r = (r_1, \ldots, r_n) \in U^n$. We know that

$$d(f(r_1, \ldots, r_n)) = f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, d(r_i), \ldots, r_n)$$

and

$$\begin{aligned} \delta d(f(r_1, \ldots, r_n)) = {}& f^{\delta d}(r_1, \ldots, r_n) + \sum_i f^d(r_1, \ldots, \delta(r_i), \ldots, r_n) \\ &+ \sum_i f^\delta(r_1, \ldots, d(r_i), \ldots, r_n) + \sum_i f(r_1, \ldots, \delta d(r_i), \ldots, r_n) \\ &+ \sum_i f(r_1, \ldots, \delta(r_i), \ldots, d(r_j), \ldots, r_n). \end{aligned}$$

Let $\delta$ and $d$ be $C$-independent modulo inner derivations of $U$. By applying Kharchenko's Theorem [14] to (2.15), we can replace $d(f(r_1, \ldots, r_n))$ with $f^d(r_1, \ldots, r_n)$ $+ \sum_i f(r_1, \ldots, y_i, \ldots, r_n)$ and $\delta d(f(r_1, \ldots, r_n))$ with

$$\begin{aligned} &f^{\delta d}(r_1, \ldots, r_n) + \sum_i f^d(r_1, \ldots, s_i, \ldots, r_n) + \sum_i f^\delta(r_1, \ldots, y_i, \ldots, r_n) \\ &+ \sum_i f(r_1, \ldots, t_i, \ldots, r_n) + \sum_i f(r_1, \ldots, s_i, \ldots, y_j, \ldots, r_n) \end{aligned}$$

in (2.15) and then $U$ satisfies blended component

$$\lambda \sum_i f(r_1, \ldots, t_i, \ldots, r_n) f(r_1, \ldots, r_n) = 0. \tag{2.16}$$

In particular, for $t_1 = r_1$ and $t_2 = \cdots = t_n = 0$ in (2.16), we have $\lambda f(r_1, \ldots, r_n)^2 = 0$. If $\lambda \neq 0$, then $f(r_1, \ldots, r_n)^2 = 0$ which implies $f(r_1, \ldots, r_n) = 0$ for all $r_1, \ldots, r_n \in U$ (see [5]), a contradiction. Thus $\lambda = 0$. In this case $G$ becomes inner, a contradiction.

Now let $\delta$ and $d$ be $C$-dependent, i.e., $\alpha_1\delta + \beta_1 d = ad_{q'}$. Now, $\alpha_1 = 0$, implies $d$ is inner, a contradiction. Thus $\alpha_1 \neq 0$ and so $\delta = \mu d + [q, x]$ for some $\mu \in C$ and $q \in U$. Then by (2.15), $U$ satisfies

$$\begin{aligned}
&c\big(af(r)^2 + \lambda\mu d(f(r))f(r) + \lambda[q, f(r)]f(r) + [p, f(r)]f(r)\big)\\
&\ + d\big(af(r)^2 + [p, f(r)]f(r)\big) + d(\lambda)\mu d(f(r))f(r) + d(\lambda)[q, f(r)]f(r)\\
&\ + \lambda d\big(\mu d(f(r)) + [q, f(r)]\big)f(r) + \lambda\big(\mu d(f(r)) + [q, f(r)]\big)d(f(r))\\
&= f(r)bf(r) + f(r)\big(\mu d(f(r)) + [q, f(r)]\big)
\end{aligned}$$

for all $r = (r_1, \ldots, r_n) \in U^n$.

Since $d(f(r_1, \ldots, r_n)) = f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, d(r_i), \ldots, r_n)$ and

$$\begin{aligned}
d^2(f(r_1, \ldots, r_n)) = f^{d^2}(r_1, \ldots, r_n) &+ 2\sum_i f^d(r_1, \ldots, d(r_i), \ldots, r_n)\\
&+ \sum_i f(r_1, \ldots, d^2(r_i), \ldots, r_n)\\
&+ \sum_{i\neq j} f(r_1, \ldots, d(r_i), \ldots, d(r_j), \ldots, r_n),
\end{aligned}$$

by applying Kharchenko's Theorem [14], we can replace $d(f(r_1, \ldots, r_n))$ with $f^d(r_1, \ldots, r_n) + \sum_i f(r_1, \ldots, y_i, \ldots, r_n)$ and $d^2(f(r_1, \ldots, r_n))$ with

$$\begin{aligned}
d^2(f(r_1, \ldots, r_n)) = f^{d^2}(r_1, \ldots, r_n) &+ 2\sum_i f^d(r_1, \ldots, y_i, \ldots, r_n)\\
&+ \sum_i f(r_1, \ldots, t_i, \ldots, r_n) + \sum_{i\neq j} f(r_1, \ldots, y_i, \ldots, y_j, \ldots, r_n),
\end{aligned}$$

and then $U$ satisfies blended component

$$\lambda\mu \sum_i f(r_1, \ldots, t_i, \ldots, r_n)f(r_1, \ldots, r_n) = 0.$$

In particular, $\lambda\mu f(r_1, \ldots, r_n)^2 = 0$. This implies $\lambda\mu = 0$ and so either $\lambda = 0$ or $\mu = 0$. Now $\lambda = 0$ gives $G$ is inner, a contradiction. Again $\mu = 0$, gives $H$ is inner, a contradiction.

*Case* 2*:* Let $d'$ and $\delta$ be $C$-independent modulo inner derivations of $U$. We divide the proof into two subcases.

*Subcase i.* Let $d$, $d'$ and $\delta$ be $C$-independent modulo inner derivations of $U$. In this case we rewrite (2.14) as

$$\begin{aligned}
&c\big(af(r)^2 + d'(f(r))f(r)\big) + d(a)f(r)^2 + ad(f(r))f(r)\\
&\ + af(r)d(f(r)) + dd'(f(r))f(r) + d'(f(r))d(f(r))\\
&= f(r)bf(r) + f(r)\delta(f(r))
\end{aligned}$$

for all $r=(r_1,\ldots,r_n)\in U^n$.

By applying Kharchenko's Theorem [14], we can replace $dd'(f(x_1,\ldots,x_n))$ by

$$f^{dd'}(r_1,\ldots,r_n)+\sum_i f^{d'}(r_1,\ldots,x_i,\ldots,r_n)+\sum_i f^{d}(r_1,\ldots,t_i,\ldots,r_n)$$
$$+\sum_{i\neq j} f(r_1,\ldots,t_i,\ldots,x_j,\ldots,r_n)+\sum_i f(r_1,\ldots,w_i,\ldots,r_n)$$

in above equality and then $U$ satisfies the blended component

$$\sum_i f(r_1,\ldots,w_i,\ldots,r_n)f(r_1,\ldots,r_n)=0. \tag{2.17}$$

In particular for $w_1=r_1$ and $w_2=\cdots=w_n=0$, $U$ satisfies $f(r_1,\ldots,r_n)^2=0$ implying $f(r_1,\ldots,r_n)=0$, a contradiction.

*Subcase ii.* Let $d$, $d'$ and $\delta$ be $C$-dependent modulo inner derivations of $U$, i.e., $\alpha_1 d+\alpha_2 d'+\alpha_3\delta=ad_{a'}$ for some $\alpha_1,\alpha_2,\alpha_3\in C$. Then $\alpha_1\neq 0$, otherwise $d'$ and $\delta$ are $C$-dependent modulo inner derivation of $U$, a contradiction. Then we can write $d=\beta_1 d'+\beta_2\delta+ad_{a''}$ for some $\beta_1,\beta_2\in C$ and $a''\in U$. Then by (2.14), we have

$$\begin{aligned}&c\{af(r)^2+d'(f(r))f(r)\}+\beta_1 d'\{af(r)^2+d'(f(r))f(r)\}\\&+\beta_2\delta\{af(r)^2+d'(f(r))f(r)\}+[a'',af(r)^2+d'(f(r))f(r)]\\&=f(r)bf(r)+f(r)\delta(f(r))\end{aligned} \tag{2.18}$$

for all $r=(r_1,\ldots,r_n)\in U^n$.

Using Kharchenko's Theorem [14], we substitute the following values in (2.18)

$$\begin{aligned}d'(f(r_1,\ldots,r_n))&=f^{d'}(r_1,\ldots,r_n)+\sum_i f(r_1,\ldots,y_i,\ldots,r_n),\\
\delta(f(r_1,\ldots,r_n))&=f^{\delta}(r_1,\ldots,r_n)+\sum_i f(r_1,\ldots,t_i,\ldots,r_n),\\
\delta d'(f(r_1,\ldots,r_n))&=f^{\delta d'}(r_1,\ldots,r_n)+\sum_i f^{\delta}(r_1,\ldots,y_i,\ldots,r_n)\\
&\quad+\sum_i f^{d'}(r_1,\ldots,t_i,\ldots,r_n)+\sum_{i\neq j} f(r_1,\ldots,y_i,\ldots,t_j,\ldots,r_n)\\
&\quad+\sum_i f(r_1,\ldots,w'_i,\ldots,r_n),\\
d'^2(f(r_1,\ldots,r_n))&=f^{d'^2}(r_1,\ldots,r_n)+2\sum_i f^{d'}(r_1,\ldots,y_i,\ldots,r_n)\\
&\quad+\sum_i f(r_1,\ldots,z'_i,\ldots,r_n)+\sum_{i\neq j} f(r_1,\ldots,y_i,\ldots,y_j,\ldots,r_n).\end{aligned}$$

Therefore, $U$ satisfies the blended components

$$\beta_1\sum_i f(r_1,\ldots,z'_i,\ldots,r_n)f(r_1,\ldots,r_n)=0$$

and

$$\beta_2 \sum_i f(r_1, \ldots, w_i', \ldots, r_n) f(r_1, \ldots, r_n) = 0.$$

If $\beta_1 \neq 0$, then from above, $U$ satisfies

$$\sum_i f(r_1, \ldots, z_i', \ldots, r_n) f(r_1, \ldots, r_n) = 0.$$

This is same as (2.17) and hence by same argument as above, it leads to a contradiction. Thus we conclude that $\beta_1 = 0$. Similarly, from above relation, we conclude that $\beta_2 = 0$. Then $d$ is inner, contradicting with the fact that $F$ is outer. This complete the proof of the theorem. □

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