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A System of Coupled Two-sided Sylvester-type Tensor Equations over the Quaternion Algebra

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Abstract. We establish some necessary and sufficient conditions for the solvability to a system of a pair of coupled two-sided Sylvester-type tensor equations over the quaternion algebra. We also give an expression of the general solution to the system when it is solvable. As applications, we derive some solvability conditions and expressions of the η-Hermitian solutions to some systems of coupled two-sided Sylvester-type quaternion tensor equations. Moreover, we provide an example to illustrate the main results of this paper.

1. Introduction

Quaternions was first proposed by Hamilton [14] in 1843. Nowadays quaternions and quaternion matrices have been widely used in many fields such as quantum computing, mechanics, signal and color image processing (e.g., [7, 9, 18, 19, 28, 31, 32, 39]).

Sylvester-type matrix equations have a number of applications in system and control theory, neural network [40], robust control [33], statistics and probability (e.g., [1, 20]) and eigenvalue assignment problems [3]. The investigations on Sylvester-type equations over quaternions have attracted more and more attentions in recent years (e.g., [5, 16, 34, 35]).

In recent decades, tensor conceived by Tullio Levi-Civita [21], has attracted a lot of scholars to study, and research results are widely applied in mechanics, data mining, general relativity and so on (e.g., [2, 4, 6, 10–13, 23–27, 29, 30, 36–38]). The two-sided Sylvester-type tensor equation

$$\mathcal{A} *_N \mathcal{X} *_M \mathcal{B} + \mathcal{C} *_N \mathcal{Y} *_M \mathcal{D} = \mathcal{E} \tag{1.1}$$

plays an important role in discretization of a linear partial differential equation of high dimension [22]. He, Navasca and Wang [17] investigated the solvability conditions and the expression of the general solution to the equation (1.1) over the quaternion algebra. He [15] defined the η-Hermitian quaternion tensor as follows. For any $\eta \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, a

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quaternion tensor $\mathcal{A}$ with the order $2N$ dimension $I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N$ is said to be η-Hermitian if $\mathcal{A} = \mathcal{A}^{\eta^*}$, where $\mathcal{A}^{\eta^*} = -\eta \mathcal{A}^* \eta$, and $\mathcal{A}^*$ represents for the conjugate and transpose of $\mathcal{A}$. He also studied the solvability and the general η-Hermitian solution to a system of quaternion Sylvester-type tensor equations.

To our best knowledge, there has been little information on the two-sided coupled Sylvester-type tensor equations over the quaternion algebra.

Motivated by increasing interest in quaternions, Sylvester-type equations, tensor equations, and in order to improve theoretical understanding of Sylvester-type tensor equations over the quaternion algebra, we in this paper will consider the solvability conditions and the expressions of the solution to the following two systems of coupled two-sided Sylvester-type tensor equations over the quaternion algebra:

$$\begin{aligned} &\mathcal{A}_1 *_N \mathcal{X}_1 *_M \mathcal{B}_1 + \mathcal{C}_1 *_N \mathcal{Y}_1 *_M \mathcal{D}_1 = \mathcal{E}_1, \\ &\mathcal{A}_2 *_N \mathcal{X}_2 *_M \mathcal{B}_2 + \mathcal{C}_2 *_N \mathcal{Y}_1 *_M \mathcal{D}_2 = \mathcal{E}_2, \end{aligned} \tag{1.2}$$

where the operation $*_N$ is the Einstein product, and $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{C}_i$, $\mathcal{E}_i$, $\mathcal{D}_i$ $(i = 1, 2)$ are given quaternion tensors and $\mathcal{X}_i$ $(i = 1, 2)$, $\mathcal{Y}_1$ are unknown tensors; and

$$\begin{aligned} &\mathcal{A}_1 *_N \mathcal{X}_1 *_N \mathcal{A}_1^{\eta^*} + \mathcal{B}_1 *_N \mathcal{Y}_1 *_N \mathcal{B}_1^{\eta^*} = \mathcal{E}_1, \\ &\mathcal{A}_2 *_N \mathcal{X}_2 *_N \mathcal{A}_2^{\eta^*} + \mathcal{B}_2 *_N \mathcal{Y}_1 *_N \mathcal{B}_2^{\eta^*} = \mathcal{E}_2, \end{aligned} \tag{1.3}$$

where $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{E}_i$ are given and $\mathcal{E}_i$ are η-Hermitian quaternion tensors $(i = 1, 2)$, and $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{Y}_1$ are unknown η-Hermitian tensors.

The rest of this paper is organized as follows. In Section 2, we introduce some related definitions and properties about a quaternion tensor. In Section 3, we establish some necessary and sufficient conditions for the existence of a solution to the system (1.2) and give an expression of the general solution when it is solvable. Moreover we present an example to illustrate our results. In Section 4, as applications of the system (1.2), we derive some solvable conditions for the existence of η-Hermitian solutions to the equation (1.1) and the system (1.3) and give expressions of such solutions when the solvability conditions are met. In Section 5, we give a conclusion to end this paper.

2. Preliminaries

Let $\mathbb{R}$ and $\mathbb{H}^{L_1 \times \cdots \times L_N}$ stand, respectively, for the real number field and the set of order N and dimension $L_1 \times \cdots \times L_N$ tensors over the real quaternion algebra

$$\mathbb{H} = \big\{ a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \mid \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, a_0, a_1, a_2, a_3 \in \mathbb{R} \big\}.$$

It is well known that the quaternion algebra is an associative and noncommutative division algebra. For more details about quaternion matrices, please see [28, 39].

Given $\mathcal{A} = (a_{i_1\cdots i_N j_1\cdots j_M}) \in \mathbb{H}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_M}$, the tensor $\mathcal{A}^* = (\overline{a}_{j_1\cdots j_M i_1\cdots i_N}) \in \mathbb{H}^{J_1\times\cdots\times J_M\times I_1\times\cdots\times I_N}$ is called the conjugate transpose of $\mathcal{A}$. A tensor $\mathcal{A} \in \mathbb{H}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ is said to be Hermitian if $\mathcal{A} = \mathcal{A}^*$. A tensor $\mathcal{D} = (d_{i_1\cdots i_N i_1\cdots i_N}) \in \mathbb{H}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ is known as a diagonal tensor if its all entries are zeros except for $d_{i_1\cdots i_N i_1\cdots i_N}$. When all the diagonal entries $d_{i_1\cdots i_N i_1\cdots i_N} = 1$, $\mathcal{D}$ is the unit tensor.

For $\mathcal{A} \in \mathbb{H}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$ and $\mathcal{B} \in \mathbb{H}^{J_1\times\cdots\times J_N\times K_1\times\cdots\times K_M}$, the Einstein product [8] of the two tensors is defined by the operation $*_N$ via

$$(\mathcal{A} *_N \mathcal{B})_{i_1\cdots i_N k_1\cdots k_M} = \sum_{j_1\cdots j_N} a_{i_1\cdots i_N j_1\cdots i_N} b_{j_1\cdots j_N k_1\cdots k_M}.$$

Hence $\mathcal{A} *_N \mathcal{B} \in \mathbb{H}^{I_1\times\cdots\times I_N\times K_1\times\cdots\times K_M}$.

Now we introduce the definition of the Moore-Penrose inverse of a tensor over $\mathbb{H}$ via the Einstein product, which is a generation of the Moore-Penrose inverse of a matrix.

Definition 2.1. [17] Let $\mathcal{A} \in \mathbb{H}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$. The tensor $\mathcal{X} \in \mathbb{H}^{J_1\times\cdots\times J_N\times I_1\times\cdots\times I_N}$ satisfying the following four quaternion tensor equalities:

(1) $\mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = \mathcal{A}$,

(2) $\mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X}$,

(3) $(\mathcal{A} *_N \mathcal{X})^* = \mathcal{A} *_N \mathcal{X}$, and

(4) $(\mathcal{X} *_N \mathcal{A})^* = \mathcal{X} *_N \mathcal{A}$

is called the Moore-Penrose inverse of $\mathcal{A}$, denoted by $\mathcal{A}^\dagger$.

By [17], for an arbitrary tensor $\mathcal{A} \in \mathbb{H}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$, the Moore-Penrose inverse exists and is unique. For $\mathcal{C} \in \mathbb{H}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$, if there is $\mathcal{B} \in \mathbb{H}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$ satisfying the equation $\mathcal{C} *_N \mathcal{B} = \mathcal{B} *_N \mathcal{C} = \mathcal{I}$, then $\mathcal{B}$ is regarded as the inverse of $\mathcal{C}$, denoted by $\mathcal{C}^{-1}$. If tensor $\mathcal{C}$ is invertible, then $\mathcal{C}^\dagger = \mathcal{C}^{-1}$.

Let the symbols $\mathcal{L}_\mathcal{A}$ and $\mathcal{R}_\mathcal{A}$ stand for $\mathcal{L}_\mathcal{A} = \mathcal{I} - \mathcal{A}^\dagger *_N \mathcal{A}$, $\mathcal{R}_\mathcal{A} = \mathcal{I} - \mathcal{A} *_N \mathcal{A}^\dagger$, respectively. Then we have the following properties.

Proposition 2.2. *Let* $\mathcal{A} \in \mathbb{H}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$. *Then*

(1) $(\mathcal{A}^{\eta*})^\dagger = (\mathcal{A}^\dagger)^{\eta*}$,

(2) $\mathcal{L}_\mathcal{A} = \mathcal{L}_\mathcal{A}^*$, $\mathcal{R}_\mathcal{A} = \mathcal{R}_\mathcal{A}^*$,

(3) $\mathcal{L}_\mathcal{A}^{\eta*} = \mathcal{R}_{\mathcal{A}^{\eta*}}$, $\mathcal{R}_\mathcal{A}^{\eta*} = \mathcal{L}_{\mathcal{A}^{\eta*}}$,

(4) $(\mathcal{A}^\dagger)^\dagger = \mathcal{A}$,

(5) $(\mathcal{A}^*)^\dagger = (\mathcal{A}^\dagger)^*$,

(6) $(\mathcal{A}^* *_N \mathcal{A})^\dagger = \mathcal{A}^\dagger *_N (\mathcal{A}^*)^\dagger$, $(\mathcal{A} *_N \mathcal{A}^*)^\dagger = (\mathcal{A}^*)^\dagger *_N \mathcal{A}^\dagger$,

(7) $\mathcal{L}_\mathcal{A} *_N \mathcal{A}^\dagger = \mathcal{A} *_N \mathcal{L}_\mathcal{A} = 0$, $\mathcal{R}_\mathcal{A} *_N \mathcal{A} = \mathcal{A}^\dagger *_N \mathcal{R}_\mathcal{A} = 0$.

Now we introduce the block quaternion tensors just as the definitions of block complex tensors [30]. Let $\mathcal{A} = (a_{i_1\cdots i_N j_1\cdots j_M}) \in \mathbb{H}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_M}$ and $\mathcal{B} = (b_{i_1\cdots i_N k_1\cdots k_M}) \in \mathbb{H}^{I_1\times\cdots\times I_N\times K_1\times\cdots\times K_M}$. Then the *row block tensor* of $\mathcal{A}$ and $\mathcal{B}$ is denoted by

$$(\mathcal{A}\ \ \mathcal{B}) \in \mathbb{H}^{I_1\times\cdots\times I_N\times L_1\times\cdots\times L_M}, \tag{2.1}$$

where $L_i = J_i + K_i$, $i = 1,\ldots,M$ and

$$(\mathcal{A}\ \ \mathcal{B})_{i_1\cdots i_N l_1\cdots l_M} = \begin{cases} a_{i_1\cdots i_N l_1\cdots l_M} & \text{if } i_1\cdots i_N \in [I_1]\times\cdots\times[I_N],\ l_1\cdots l_M \in [J_1]\times\cdots\times[J_M], \\ b_{i_1\cdots i_N l_1\cdots l_M} & \text{if } i_1\cdots i_N \in [I_1]\times\cdots\times[I_N],\ l_1\cdots l_M \in \Gamma_1\times\cdots\times\Gamma_M, \\ 0 & \text{otherwise,} \end{cases}$$

where $\Gamma_i = \{J_i+1,\ldots,J_i+K_i\}$, $i = 1,\ldots,M$. For given tensors $\mathcal{C} = (c_{j_1\cdots j_M i_1\cdots i_N}) \in \mathbb{H}^{J_1\times\cdots\times J_M\times I_1\times\cdots\times I_N}$ and $\mathcal{D} = (d_{k_1\cdots k_M i_1\cdots i_N}) \in \mathbb{H}^{K_1\times\cdots\times K_M\times I_1\times\cdots\times I_N}$, the *column block tensor* of $\mathcal{C}$ and $\mathcal{D}$ is denoted by

$$\begin{pmatrix} \mathcal{C} \\ \mathcal{D} \end{pmatrix} \in \mathbb{H}^{L_1\times\cdots\times L_M\times I_1\times\cdots\times I_N}, \tag{2.2}$$

where $L_i = J_i + K_i$, $i = 1,\ldots,M$ and

$$\begin{pmatrix} \mathcal{C} \\ \mathcal{D} \end{pmatrix}_{l_1\cdots l_M i_1\cdots i_N} = \begin{cases} c_{l_1\cdots l_M i_1\cdots i_N} & \text{if } l_1\cdots l_M \in [J_1]\times\cdots\times[J_M],\ i_1\cdots i_N \in [I_1]\times\cdots\times[I_N], \\ d_{l_1\cdots l_M i_1\cdots i_N} & \text{if } l_1\cdots l_M \in \Gamma_1\times\cdots\times\Gamma_M,\ i_1\cdots i_N \in [I_1]\times\cdots\times[I_N], \\ 0 & \text{otherwise,} \end{cases}$$

where $\Gamma_i = \{J_i+1,\ldots,J_i+K_i\}$, $i = 1,\ldots,M$.

The product of *block tensors* is given as follows.

Proposition 2.3. *Let* $(\mathcal{A}\ \ \mathcal{B})$ *and* $\left(\begin{smallmatrix} \mathcal{C} \\ \mathcal{D} \end{smallmatrix}\right)$ *be of the form in* (2.1) *and* (2.2), *respectively. Then*

(1) $\mathcal{F} *_N (\mathcal{A}\ \ \mathcal{B}) = (\mathcal{F} *_N \mathcal{A}\ \ \mathcal{F} *_N \mathcal{B}) \in \mathbb{H}^{I_1\times\cdots\times I_N\times L_1\times\cdots\times L_M}$,

(2) $\left(\begin{smallmatrix} \mathcal{C} \\ \mathcal{D} \end{smallmatrix}\right) *_N \mathcal{F} = \left(\begin{smallmatrix} \mathcal{C} *_N \mathcal{F} \\ \mathcal{D} *_N \mathcal{F} \end{smallmatrix}\right) \in \mathbb{H}^{L_1\times\cdots\times L_M\times I_1\times\cdots\times I_N}$,

(3) $(\mathcal{A}\ \ \mathcal{B}) *_M \left(\begin{smallmatrix} \mathcal{C} \\ \mathcal{D} \end{smallmatrix}\right) = \mathcal{A} *_M \mathcal{C} + \mathcal{B} *_M \mathcal{D} \in \mathbb{H}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$,

where $\mathcal{F} \in \mathbb{H}^{I_1\times\cdots\times I_N\times I_1\times\cdots\times I_N}$.

In order to prove our main result in the next section, we start with the two-sided Sylvester-type quaternion tensor equation (1.1). The following lemma provides the solvability conditions and general solution to the equation (1.1).

Lemma 2.4. [17] *Assume that* $\mathcal{A} \in \mathbb{H}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$, $\mathcal{B} \in \mathbb{H}^{K_1\times\cdots\times K_M\times L_1\times\cdots\times L_M}$, $\mathcal{C} \in \mathbb{H}^{I_1\times\cdots\times I_N\times G_1\times\cdots\times G_N}$, $\mathcal{D} \in \mathbb{H}^{H_1\times\cdots\times H_M\times L_1\times\cdots\times L_M}$ *and* $\mathcal{E} \in \mathbb{H}^{I_1\times\cdots\times I_N\times L_1\times\cdots\times L_M}$. *Set*

$$\mathcal{P} = \mathcal{R}_{\mathcal{A}} *_N \mathcal{C}, \quad \mathcal{Q} = \mathcal{D} *_M \mathcal{L}_{\mathcal{B}}, \quad \mathcal{S} = \mathcal{C} *_N \mathcal{L}_{\mathcal{P}}.$$

Then the equation (1.1) *is consistent if and only if*

$$\mathcal{R}_{\mathcal{P}} *_N \mathcal{R}_{\mathcal{A}} *_N \mathcal{E} = 0, \qquad \mathcal{E} *_M \mathcal{L}_{\mathcal{B}} *_M \mathcal{L}_{\mathcal{Q}} = 0,$$
$$\mathcal{R}_{\mathcal{A}} *_N \mathcal{E} *_M \mathcal{L}_{\mathcal{D}} = 0, \qquad \mathcal{R}_{\mathcal{C}} *_N \mathcal{E} *_M \mathcal{L}_{\mathcal{B}} = 0.$$

In that case, the general solution to the equation (1.1) *can be expressed as follows:*

$$\begin{aligned}
\mathcal{X} &= \mathcal{A}^\dagger *_N \mathcal{E} *_M \mathcal{B}^\dagger - \mathcal{A}^\dagger *_N \mathcal{C} *_N \mathcal{P}^\dagger *_N \mathcal{E} *_M \mathcal{B}^\dagger \\
&\quad - \mathcal{A}^\dagger *_N \mathcal{S} *_N \mathcal{C}^\dagger *_N \mathcal{E} *_M \mathcal{Q}^\dagger *_M \mathcal{D} *_M \mathcal{B}^\dagger - \mathcal{A}^\dagger *_N \mathcal{S} *_N \mathcal{U}_2 *_M \mathcal{R}_{\mathcal{Q}} *_M \mathcal{D} *_M \mathcal{B}^\dagger \\
&\quad + \mathcal{L}_{\mathcal{A}} *_N \mathcal{U}_4 + \mathcal{U}_5 *_M \mathcal{R}_{\mathcal{B}}, \\
\mathcal{Y} &= \mathcal{P}^\dagger *_N \mathcal{E} *_M \mathcal{D}^\dagger + \mathcal{S}^\dagger *_N \mathcal{S} *_N \mathcal{C}^\dagger *_N \mathcal{E} *_M \mathcal{Q}^\dagger + \mathcal{L}_{\mathcal{P}} *_N \mathcal{L}_{\mathcal{S}} *_N \mathcal{U}_1 \\
&\quad + \mathcal{L}_{\mathcal{P}} *_N \mathcal{U}_2 *_M \mathcal{R}_{\mathcal{Q}} + \mathcal{U}_3 *_M \mathcal{R}_{\mathcal{D}},
\end{aligned}$$

where $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$, $\mathcal{U}_4$, $\mathcal{U}_5$ *are arbitrary quaternion tensors with suitable orders.*

3. The general solution to the system (1.2)

In this section, we consider the solvability conditions and the general solution to (1.2) where $\mathcal{A}_1 \in \mathbb{H}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$, $\mathcal{A}_2 \in \mathbb{H}^{L_1\times\cdots\times L_N\times K_1\times\cdots\times K_N}$, $\mathcal{C}_1 \in \mathbb{H}^{I_1\times\cdots\times I_N\times P_1\times\cdots\times P_N}$, $\mathcal{C}_2 \in \mathbb{H}^{L_1\times\cdots\times L_N\times P_1\times\cdots\times P_N}$, $\mathcal{B}_1 \in \mathbb{H}^{R_1\times\cdots\times R_M\times S_1\times\cdots\times S_M}$, $\mathcal{B}_2 \in \mathbb{H}^{T_1\times\cdots\times T_M\times Q_1\times\cdots\times Q_M}$, $\mathcal{D}_1 \in \mathbb{H}^{G_1\times\cdots\times G_M\times S_1\times\cdots\times S_M}$, $\mathcal{D}_2 \in \mathbb{H}^{G_1\times\cdots\times G_M\times Q_1\times\cdots\times Q_M}$, $\mathcal{E}_1 \in \mathbb{H}^{I_1\times\cdots\times I_N\times S_1\times\cdots\times S_M}$, $\mathcal{E}_2 \in \mathbb{H}^{L_1\times\cdots\times L_N\times Q_1\times\cdots\times Q_M}$ are given quaternion tensors. For convenience, we set

$$\mathcal{M}_1 = \mathcal{R}_{\mathcal{A}_1} *_N \mathcal{C}_1, \quad \mathcal{N}_1 = \mathcal{D}_1 *_M \mathcal{L}_{\mathcal{B}_1}, \quad \mathcal{S}_1 = \mathcal{C}_1 *_N \mathcal{L}_{\mathcal{M}_1},$$
$$\mathcal{M}_2 = \mathcal{R}_{\mathcal{A}_2} *_N \mathcal{C}_2, \quad \mathcal{N}_2 = \mathcal{D}_2 *_M \mathcal{L}_{\mathcal{B}_2}, \quad \mathcal{S}_2 = \mathcal{C}_2 *_N \mathcal{L}_{\mathcal{M}_2},$$

$$\mathcal{A}_{11} = (\mathcal{L}_{\mathcal{M}_1} *_N \mathcal{L}_{\mathcal{S}_1} \ \ \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{L}_{\mathcal{S}_2}), \quad \mathcal{B}_{11} = \begin{pmatrix} \mathcal{R}_{\mathcal{D}_1} \\ \mathcal{R}_{\mathcal{D}_2} \end{pmatrix}, \tag{3.1}$$

$$\begin{aligned}\mathcal{E}_{11} &= \mathcal{M}_2^\dagger *_N \mathcal{E}_2 *_M \mathcal{D}_2^\dagger + \mathcal{S}_2^\dagger *_N \mathcal{S}_2 *_N \mathcal{C}_2^\dagger *_N \mathcal{E}_2 *_M \mathcal{N}_2^\dagger \\ &\quad - \mathcal{M}_1^\dagger *_N \mathcal{E}_1 *_M \mathcal{D}_1^\dagger - \mathcal{S}_1^\dagger *_N \mathcal{S}_1 *_N \mathcal{C}_1^\dagger *_N \mathcal{E}_1 *_M \mathcal{N}_1^\dagger,\end{aligned} \tag{3.2}$$

$$\mathcal{A} = \mathcal{R}_{\mathcal{A}_{11}} *_N \mathcal{L}_{\mathcal{M}_1}, \ \ \mathcal{B} = \mathcal{R}_{\mathcal{N}_1} *_M \mathcal{L}_{\mathcal{B}_{11}}, \ \ \mathcal{C} = \mathcal{R}_{\mathcal{A}_{11}} *_N \mathcal{L}_{\mathcal{M}_2}, \ \ \mathcal{D} = \mathcal{R}_{\mathcal{N}_2} *_M \mathcal{L}_{\mathcal{B}_{11}}, \tag{3.3}$$

$$\mathcal{E} = \mathcal{R}_{\mathcal{A}_{11}} *_N \mathcal{E}_{11} *_M \mathcal{L}_{\mathcal{B}_{11}}, \quad \mathcal{M} = \mathcal{R}_{\mathcal{A}} *_N \mathcal{C}, \quad \mathcal{N} = \mathcal{D} *_M \mathcal{L}_{\mathcal{B}}, \quad \mathcal{S} = \mathcal{C} *_N \mathcal{L}_{\mathcal{M}}. \tag{3.4}$$

Theorem 3.1. *The system* (1.2) *is consistent if and only if*

$$\mathcal{R}_{\mathcal{A}_1} *_N \mathcal{E}_1 *_M \mathcal{L}_{\mathcal{D}_1} = 0, \quad \mathcal{R}_{\mathcal{C}_1} *_N \mathcal{E}_1 *_M \mathcal{L}_{\mathcal{B}_1} = 0,$$

$$\mathcal{R}_{\mathcal{M}_1} *_N \mathcal{R}_{\mathcal{A}_1} *_N \mathcal{E}_1 = 0, \quad \mathcal{E}_1 *_M \mathcal{L}_{\mathcal{B}_1} *_M \mathcal{L}_{\mathcal{N}_1} = 0, \tag{3.5}$$

$$\mathcal{R}_{\mathcal{A}_2} *_N \mathcal{E}_2 *_M \mathcal{L}_{\mathcal{D}_2} = 0, \quad \mathcal{R}_{\mathcal{C}_2} *_N \mathcal{E}_2 *_M \mathcal{L}_{\mathcal{B}_2} = 0,$$

$$\mathcal{R}_{\mathcal{M}_2} *_N \mathcal{R}_{\mathcal{A}_2} *_N \mathcal{E}_2 = 0, \quad \mathcal{E}_2 *_M \mathcal{L}_{\mathcal{B}_2} *_M \mathcal{L}_{\mathcal{N}_2} = 0, \tag{3.6}$$

$$\mathcal{R}_{\mathcal{M}} *_N \mathcal{R}_{\mathcal{A}} *_N \mathcal{E} = 0, \quad \mathcal{E} *_M \mathcal{L}_{\mathcal{B}} *_M \mathcal{L}_{\mathcal{N}} = 0,$$

$$\mathcal{R}_{\mathcal{A}} *_N \mathcal{E} *_M \mathcal{L}_{\mathcal{D}} = 0, \quad \mathcal{R}_{\mathcal{C}} *_N \mathcal{E} *_M \mathcal{L}_{\mathcal{B}} = 0. \tag{3.7}$$

In this case, the general solution to the system (1.2) *can be expressed as*

$$\begin{aligned} \mathcal{X}_1 = {} & \mathcal{A}_1^\dagger *_N \mathcal{E}_1 *_M \mathcal{B}_1^\dagger - \mathcal{A}_1^\dagger *_N \mathcal{S}_1 *_N \mathcal{C}_1^\dagger *_N \mathcal{E}_1 *_M \mathcal{N}_1^\dagger *_N \mathcal{D}_1 *_M \mathcal{B}_1^\dagger \\ & - \mathcal{A}_1^\dagger *_N \mathcal{C}_1 *_N \mathcal{M}_1^\dagger *_N \mathcal{E}_1 *_M \mathcal{B}_1^\dagger - \mathcal{A}_1^\dagger *_N \mathcal{S}_1 *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} *_M \mathcal{D}_1 *_M \mathcal{B}_1^\dagger \\ & + \mathcal{L}_{\mathcal{A}_1} *_N \mathcal{V}_4 + \mathcal{V}_5 *_M \mathcal{R}_{\mathcal{B}_1}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \mathcal{X}_2 = {} & \mathcal{A}_2^\dagger *_N \mathcal{E}_2 *_M \mathcal{B}_2^\dagger - \mathcal{A}_2^\dagger *_N \mathcal{S}_2 *_N \mathcal{C}_2^\dagger *_N \mathcal{E}_2 *_M \mathcal{N}_2^\dagger *_N \mathcal{D}_2 *_M \mathcal{B}_2^\dagger \\ & - \mathcal{A}_2^\dagger *_N \mathcal{C}_2 *_N \mathcal{M}_2^\dagger *_N \mathcal{E}_2 *_M \mathcal{B}_2^\dagger - \mathcal{A}_2^\dagger *_N \mathcal{S}_2 *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2} *_M \mathcal{D}_2 *_M \mathcal{B}_2^\dagger \\ & + \mathcal{L}_{\mathcal{A}_2} *_N \mathcal{T}_4 + \mathcal{T}_5 *_M \mathcal{R}_{\mathcal{B}_2}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \mathcal{Y}_1 = {} & \mathcal{M}_1^\dagger *_N \mathcal{E}_1 *_M \mathcal{D}_1^\dagger + \mathcal{S}_1^\dagger *_N \mathcal{S}_1 *_N \mathcal{C}_1^\dagger *_N \mathcal{E}_1 *_M \mathcal{N}_1^\dagger + \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{L}_{\mathcal{S}_1} *_N \mathcal{V}_1 \\ & + \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} + \mathcal{V}_3 *_M \mathcal{R}_{\mathcal{D}_1}, \end{aligned} \tag{3.10}$$

or

$$\begin{aligned} \mathcal{Y}_1 = {} & \mathcal{M}_2^\dagger *_N \mathcal{E}_2 *_M \mathcal{D}_2^\dagger + \mathcal{S}_2^\dagger *_N \mathcal{S}_2 *_N \mathcal{C}_2^\dagger *_N \mathcal{E}_2 *_M \mathcal{N}_2^\dagger - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{L}_{\mathcal{S}_2} *_N \mathcal{T}_1 \\ & - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2} - \mathcal{T}_3 *_M \mathcal{R}_{\mathcal{D}_2}, \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} \mathcal{V}_1 = {} & (\mathcal{I} \ \ 0) *_N (\mathcal{A}_{11}^\dagger *_N (\mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2}) \\ & + \mathcal{W}_1 *_M \mathcal{B}_{11} + \mathcal{L}_{\mathcal{A}_{11}} *_N \mathcal{W}_2), \end{aligned} \tag{3.12a}$$

$$\begin{aligned} \mathcal{V}_2 = {} & \mathcal{A}^\dagger *_N \mathcal{E} *_M \mathcal{B}^\dagger - \mathcal{A}^\dagger *_N \mathcal{S} *_N \mathcal{C}^\dagger *_N \mathcal{E} *_M \mathcal{N}^\dagger *_M \mathcal{D} *_M \mathcal{B}^\dagger \\ & - \mathcal{A}^\dagger *_N \mathcal{C} *_N \mathcal{M}^\dagger *_N \mathcal{E} *_M \mathcal{B}^\dagger - \mathcal{A}^\dagger *_N \mathcal{S} *_N \mathcal{W}_4 *_M \mathcal{R}_{\mathcal{N}} *_M \mathcal{D} *_M \mathcal{B}^\dagger \\ & + \mathcal{L}_{\mathcal{A}} *_N \mathcal{W}_5 + \mathcal{W}_6 *_M \mathcal{R}_{\mathcal{B}}, \end{aligned} \tag{3.12b}$$

$$\begin{aligned} \mathcal{V}_3 = {} & (\mathcal{R}_{\mathcal{A}_{11}} *_N (\mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2}) *_M \mathcal{B}_{11}^\dagger \\ & - \mathcal{A}_{11} *_N \mathcal{W}_1 - \mathcal{W}_3 *_M \mathcal{R}_{\mathcal{B}_{11}}) *_M \begin{pmatrix} \mathcal{I} \\ 0 \end{pmatrix}, \end{aligned} \tag{3.12c}$$

$$\begin{aligned} \mathcal{T}_1 = {} & (0 \ \ \mathcal{I}) *_N (\mathcal{A}_{11}^\dagger *_N (\mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2}) \\ & + \mathcal{W}_1 *_M \mathcal{B}_{11} + \mathcal{L}_{\mathcal{A}_{11}} *_N \mathcal{W}_2), \end{aligned} \tag{3.12d}$$

$$\begin{aligned}\mathcal{T}_2 = {} & \mathcal{M}^{\dagger} *_N \mathcal{E} *_M \mathcal{D}^{\dagger} + \mathcal{S} *_N \mathcal{S}^{\dagger} *_N \mathcal{C}^{\dagger} *_N \mathcal{E} *_M \mathcal{N}^{\dagger} + \mathcal{L}_{\mathcal{M}} *_N \mathcal{L}_{\mathcal{S}} *_N \mathcal{W}_7 \\ & + \mathcal{L}_{\mathcal{M}} *_N \mathcal{W}_4 *_M \mathcal{R}_{\mathcal{N}} + \mathcal{W}_8 *_M \mathcal{R}_{\mathcal{D}},\end{aligned} \tag{3.12e}$$

$$\begin{aligned}\mathcal{T}_3 = {} & (\mathcal{R}_{\mathcal{A}_{11}} *_N (\mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2}) *_M \mathcal{B}_{11}^{\dagger} \\ & - \mathcal{A}_{11} *_N \mathcal{W}_1 - \mathcal{W}_3 *_M \mathcal{R}_{\mathcal{B}_{11}}) *_M \begin{pmatrix} 0 \\ \mathcal{I} \end{pmatrix},\end{aligned} \tag{3.12f}$$

and $\mathcal{V}_4$, $\mathcal{V}_5$, $\mathcal{T}_4$, $\mathcal{T}_5$, $\mathcal{W}_1, \ldots, \mathcal{W}_8$ *are arbitrary tensors over* $\mathbb{H}$ *with appropriate sizes.*

Proof. We first prove that if the system (1.2) has a solution, then the solvable conditions in (3.5)–(3.7) are true and the general solution can be expressed as (3.8)–(3.11).

If the system (1.2) is solvable, then

$$\mathcal{A}_1 *_N \mathcal{X}_1 *_M \mathcal{B}_1 + \mathcal{C}_1 *_N \mathcal{Y}_1 *_M \mathcal{D}_1 = \mathcal{E}_1 \tag{3.13}$$

and

$$\mathcal{A}_2 *_N \mathcal{X}_2 *_M \mathcal{B}_2 + \mathcal{C}_2 *_N \mathcal{Y}_1 *_M \mathcal{D}_2 = \mathcal{E}_2 \tag{3.14}$$

are consistent. It follows from Lemma 2.4 that the system (3.13) is consistent if and only if (3.5) is satisfied. In this case, the general solution to the system (3.13) can be expressed as

$$\begin{aligned}\mathcal{X}_1 = {} & \mathcal{A}_1^{\dagger} *_N \mathcal{E}_1 *_M \mathcal{B}_1^{\dagger} - \mathcal{A}_1^{\dagger} *_N \mathcal{S}_1 *_N \mathcal{C}_1^{\dagger} *_N \mathcal{E}_1 *_M \mathcal{N}_1^{\dagger} *_N \mathcal{D}_1 *_M \mathcal{B}_1^{\dagger} \\ & - \mathcal{A}_1^{\dagger} *_N \mathcal{C}_1 *_N \mathcal{M}_1^{\dagger} *_N \mathcal{E}_1 *_M \mathcal{B}_1^{\dagger} - \mathcal{A}_1^{\dagger} *_N \mathcal{S}_1 *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} *_M \mathcal{D}_1 *_M \mathcal{B}_1^{\dagger} \\ & + \mathcal{L}_{\mathcal{A}_1} *_N \mathcal{V}_4 + \mathcal{V}_5 *_M \mathcal{R}_{\mathcal{B}_1}, \\ \mathcal{Y}_1 = {} & \mathcal{M}_1^{\dagger} *_N \mathcal{E}_1 *_M \mathcal{D}_1^{\dagger} + \mathcal{S}_1^{\dagger} *_N \mathcal{S}_1 *_N \mathcal{C}_1^{\dagger} *_N \mathcal{E}_1 *_M \mathcal{N}_1^{\dagger} + \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{L}_{\mathcal{S}_1} *_N \mathcal{V}_1 \\ & + \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} + \mathcal{V}_3 *_M \mathcal{R}_{\mathcal{D}_1},\end{aligned} \tag{3.15}$$

where $\mathcal{V}_1, \ldots, \mathcal{V}_5$ are arbitrary quaternion tensors with suitable orders. This means $(\mathcal{X}_1, \mathcal{Y}_1)$ can be expressed by (3.8) and (3.10).

Similarly, the system (3.14) is consistent if and only if (3.6) is satisfied. Then the general solution can be expressed as

$$\begin{aligned}\mathcal{X}_2 = {} & \mathcal{A}_2^{\dagger} *_N \mathcal{E}_2 *_M \mathcal{B}_2^{\dagger} - \mathcal{A}_2^{\dagger} *_N \mathcal{S}_2 *_N \mathcal{C}_2^{\dagger} *_N \mathcal{E}_2 *_M \mathcal{N}_2^{\dagger} *_N \mathcal{D}_2 *_M \mathcal{B}_2^{\dagger} \\ & - \mathcal{A}_2^{\dagger} *_N \mathcal{C}_2 *_N \mathcal{M}_2^{\dagger} *_N \mathcal{E}_2 *_M \mathcal{B}_2^{\dagger} - \mathcal{A}_2^{\dagger} *_N \mathcal{S}_2 *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2} *_M \mathcal{D}_2 *_M \mathcal{B}_2^{\dagger} \\ & + \mathcal{L}_{\mathcal{A}_2} *_N \mathcal{T}_4 + \mathcal{T}_5 *_M \mathcal{R}_{\mathcal{B}_2}, \\ \mathcal{Y}_1 = {} & \mathcal{M}_2^{\dagger} *_N \mathcal{E}_2 *_M \mathcal{D}_2^{\dagger} + \mathcal{S}_2^{\dagger} *_N \mathcal{S}_2 *_N \mathcal{C}_2^{\dagger} *_N \mathcal{E}_2 *_M \mathcal{N}_2^{\dagger} - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{L}_{\mathcal{S}_2} *_N \mathcal{T}_1 \\ & - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2} - \mathcal{T}_3 *_M \mathcal{R}_{\mathcal{D}_2},\end{aligned} \tag{3.16}$$

where $\mathcal{T}_1, \ldots, \mathcal{T}_5$ are arbitrary tensors over $\mathbb{H}$ with appropriate sizes. This means $(\mathcal{X}_2, \mathcal{Y}_1)$ can be expressed by (3.9) and (3.11).

Equating $\mathcal{Y}_1$ in (3.15) and $\mathcal{Y}_1$ in (3.16), we have

$$\mathcal{A}_{11} *_N \begin{pmatrix} \mathcal{V}_1 \\ \mathcal{T}_1 \end{pmatrix} + (\mathcal{V}_3 \ \ \mathcal{T}_3) *_M \mathcal{B}_{11} = \mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2}. \tag{3.17}$$

It follows from Lemma 2.4 that the system (3.17) is consistent if and only if

$$\mathcal{R}_{\mathcal{A}_{11}} *_N (\mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2}) *_M \mathcal{L}_{\mathcal{B}_{11}} = 0,$$

which is equal to

$$\mathcal{A} *_N \mathcal{V}_2 *_M \mathcal{B} + \mathcal{C} *_N \mathcal{T}_2 *_M \mathcal{D} = \mathcal{E}, \tag{3.18}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are shown in (3.3) and (3.4). According to Lemma 2.4, the system (3.18) is consistent if and only if (3.7) is satisfied and its general solution can be expressed as

$$\begin{aligned} \mathcal{V}_2 = {} & \mathcal{A}^\dagger *_N \mathcal{E} *_M \mathcal{B}^\dagger - \mathcal{A}^\dagger *_N \mathcal{S} *_N \mathcal{C}^\dagger *_N \mathcal{E} *_M \mathcal{N}^\dagger *_M \mathcal{D} *_M \mathcal{B}^\dagger \\ & - \mathcal{A}^\dagger *_N \mathcal{C} *_N \mathcal{M}^\dagger *_N \mathcal{E} *_M \mathcal{B}^\dagger - \mathcal{A}^\dagger *_N \mathcal{S} *_N \mathcal{W}_4 *_M \mathcal{R}_{\mathcal{N}} *_M \mathcal{D} *_M \mathcal{B}^\dagger \\ & + \mathcal{L}_{\mathcal{A}} *_N \mathcal{W}_5 + \mathcal{W}_6 *_M \mathcal{R}_{\mathcal{B}}, \end{aligned} \tag{3.19}$$

$$\begin{aligned} \mathcal{T}_2 = {} & \mathcal{M}^\dagger *_N \mathcal{E} *_M \mathcal{D}^\dagger + \mathcal{S} *_N \mathcal{S}^\dagger *_N \mathcal{C}^\dagger *_N \mathcal{E} *_M \mathcal{N}^\dagger + \mathcal{L}_{\mathcal{M}} *_N \mathcal{L}_{\mathcal{S}} *_N \mathcal{W}_7 \\ & + \mathcal{L}_{\mathcal{M}} *_N \mathcal{W}_4 *_M \mathcal{R}_{\mathcal{N}} + \mathcal{W}_8 *_M \mathcal{R}_{\mathcal{D}}, \end{aligned} \tag{3.20}$$

where $\mathcal{W}_4, \ldots, \mathcal{W}_8$ are arbitrary quaternion tensors. Therefore, (3.19) is just as (3.12b) and (3.20) is the same as (3.12e).

Back to (3.17), we can deduce from Lemma 2.4 that

$$\begin{aligned} \begin{pmatrix} \mathcal{V}_1 \\ \mathcal{T}_1 \end{pmatrix} = {} & \mathcal{A}_{11}^\dagger *_N (\mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2}) \\ & + \mathcal{W}_1 *_M \mathcal{B}_{11} + \mathcal{L}_{\mathcal{A}_{11}} *_N \mathcal{W}_2, \end{aligned} \tag{3.21}$$

$$\begin{aligned} (\mathcal{V}_3 \ \ \mathcal{T}_3) = {} & \mathcal{R}_{\mathcal{A}_{11}} *_N (\mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2}) *_M \mathcal{B}_{11}^\dagger \\ & - \mathcal{A}_{11} *_N \mathcal{W}_1 - \mathcal{W}_3 *_M \mathcal{R}_{\mathcal{B}_{11}}, \end{aligned} \tag{3.22}$$

where $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$ are arbitrary quaternion tensors with suitable sizes.

By applying Proposition 2.3, we obtain that

$$\begin{aligned} \mathcal{V}_1 = {} & (\mathcal{I} \ \ 0) *_N (\mathcal{A}_{11}^\dagger *_N (\mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2}) \\ & + \mathcal{W}_1 *_M \mathcal{B}_{11} + \mathcal{L}_{\mathcal{A}_{11}} *_N \mathcal{W}_2), \\ \mathcal{V}_3 = {} & (\mathcal{R}_{\mathcal{A}_{11}} *_N (\mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2}) *_M \mathcal{B}_{11}^\dagger \end{aligned}$$

$$- \mathcal{A}_{11} *_N \mathcal{W}_1 - \mathcal{W}_3 *_M \mathcal{R}_{\mathcal{B}_{11}}) *_M \begin{pmatrix} \mathcal{I} \\ 0 \end{pmatrix},$$

$$\begin{aligned}
\mathcal{T}_1 &= (0\ \ \mathcal{I}) *_N (\mathcal{A}_{11}^{\dagger} *_N (\mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2}) \\
&\quad + \mathcal{W}_1 *_M \mathcal{B}_{11} + \mathcal{L}_{\mathcal{A}_{11}} *_N \mathcal{W}_2), \\
\mathcal{T}_3 &= (\mathcal{R}_{\mathcal{A}_{11}} *_N (\mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2}) *_M \mathcal{B}_{11}^{\dagger} \\
&\quad - \mathcal{A}_{11} *_N \mathcal{W}_1 - \mathcal{W}_3 *_M \mathcal{R}_{\mathcal{B}_{11}}) *_M \begin{pmatrix} 0 \\ \mathcal{I} \end{pmatrix}.
\end{aligned}$$

Therefore, $\mathcal{V}_1$, $\mathcal{V}_3$, $\mathcal{T}_1$, $\mathcal{T}_3$ can be expressed by (3.12a), (3.12c), (3.12d), (3.12f), respectively.

Now we show that tensors $\mathcal{X}_i$ $(i = 1, 2)$, $\mathcal{Y}_1$ having the form of (3.8)–(3.11), respectively, constitute a set of solution to the system (1.2) under the conditions (3.5)–(3.7).

If (3.5) is satisfied, then we can deduced from Lemma 2.4 that (3.8) and (3.10) are solutions to the system (3.13). Similarly, we know that (3.9) and (3.11) are a set of solutions to the system (3.14) if (3.6) is met.

Now we show that $\mathcal{Y}_1$ in (3.10) and $\mathcal{Y}_1$ in (3.11) are equal, that is

$$\begin{aligned}
&\mathcal{M}_1^{\dagger} *_N \mathcal{E}_1 *_M \mathcal{D}_1^{\dagger} + \mathcal{S}_1^{\dagger} *_N \mathcal{S}_1 *_N \mathcal{C}_1^{\dagger} *_N \mathcal{E}_1 *_M \mathcal{N}_1^{\dagger} + \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{L}_{\mathcal{S}_1} *_N \mathcal{V}_1 \\
&+ \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} + \mathcal{V}_3 *_M \mathcal{R}_{\mathcal{D}_1} \\
= &\mathcal{M}_2^{\dagger} *_N \mathcal{E}_2 *_M \mathcal{D}_2^{\dagger} + \mathcal{S}_2^{\dagger} *_N \mathcal{S}_2 *_N \mathcal{C}_2^{\dagger} *_N \mathcal{E}_2 *_M \mathcal{N}_2^{\dagger} - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{L}_{\mathcal{S}_2} *_N \mathcal{T}_1 \\
&- \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2} - \mathcal{T}_3 *_M \mathcal{R}_{\mathcal{D}_2}.
\end{aligned} \tag{3.23}$$

Using (3.1) and (3.2), we can simplify (3.23) into the form as in (3.17). So we need to show that (3.17) is true. Substituting (3.21) and (3.22) into (3.17) yields

$$\begin{aligned}
&\mathcal{A}_{11} *_N (\mathcal{A}_{11}^{\dagger} *_N (\mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2}) + \mathcal{W}_1 *_M \mathcal{B}_{11} + \mathcal{L}_{\mathcal{A}_{11}} *_N \mathcal{W}_2) \\
&+ (\mathcal{R}_{\mathcal{A}_{11}} *_N (\mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2}) *_M \mathcal{B}_{11}^{\dagger} \\
&- \mathcal{A}_{11} *_N \mathcal{W}_1 - \mathcal{W}_3 *_M \mathcal{R}_{\mathcal{B}_{11}}) *_M \mathcal{B}_{11} \\
= &\mathcal{R}_{\mathcal{A}_{11}} *_N (\mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2}) *_M \mathcal{L}_{\mathcal{B}_{11}} \\
&+ \mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2} \\
= &\mathcal{E} - \mathcal{A} *_N \mathcal{V}_2 *_M \mathcal{B} - \mathcal{C} *_N \mathcal{T}_2 *_M \mathcal{D} + \mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_M \mathcal{R}_{\mathcal{N}_1} - \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{T}_2 *_M \mathcal{R}_{\mathcal{N}_2}.
\end{aligned}$$

Since (3.7) is satisfied, it follows from Lemma 2.4 that $\mathcal{V}_2$ in (3.19) and $\mathcal{T}_2$ in (3.20) constitute a solution to the system (3.18), that is to say

$$\mathcal{E} - \mathcal{A} *_N \mathcal{V}_2 *_M \mathcal{B} - \mathcal{C} *_N \mathcal{T}_2 *_M \mathcal{D} = 0,$$

where $\mathcal{V}_2$ and $\mathcal{T}_2$, respectively, are the form of (3.19) and (3.20). Then (3.17) is satisfied. Therefore, $\mathcal{Y}_1$ in (3.9) and $\mathcal{Y}_1$ in (3.11) are equal. This means that tensors $\mathcal{X}_i$, $\mathcal{Y}_1$ $(i = 1, 2)$ as the form of (3.8)–(3.11), respectively, constitute a solution to the system (1.2) when (3.5)–(3.7) are satisfied. $\square$

We now give an example to illustrate our results.

Example 3.2. For the system (1.2), $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{C}_i$, $\mathcal{D}_i$ and $\mathcal{E}_i$ $(i = 1, 2)$ are quaternion tensors given by

$$\mathcal{A}_1(:,:,1,1) = \begin{pmatrix} \mathbf{i} & \mathbf{j} \\ -\mathbf{i} & -\mathbf{j} \end{pmatrix}, \quad \mathcal{A}_1(:,:,1,2) = \begin{pmatrix} \mathbf{i}-\mathbf{j} & -\mathbf{i} \\ \mathbf{i}+\mathbf{k} & -\mathbf{j} \end{pmatrix}, \quad \mathcal{A}_1(:,:,2,1) = \begin{pmatrix} \mathbf{i}+\mathbf{k} & -\mathbf{k} \\ \mathbf{i}+\mathbf{j} & -\mathbf{j} \end{pmatrix},$$

$$\mathcal{A}_1(:,:,2,2) = \begin{pmatrix} 1+\mathbf{i} & 1+\mathbf{j} \\ \mathbf{i} & -\mathbf{k} \end{pmatrix}, \quad \mathcal{A}_2(:,:,1,1) = \begin{pmatrix} \mathbf{i}+\mathbf{j}-\mathbf{k} & \mathbf{i}+\mathbf{k} \\ \mathbf{j} & \mathbf{i}+\mathbf{j} \end{pmatrix}, \quad \mathcal{A}_2(:,:,1,2) = \begin{pmatrix} 2\mathbf{i}+\mathbf{j} & -\mathbf{i} \\ \mathbf{j}+\mathbf{k} & \mathbf{i} \end{pmatrix},$$

$$\mathcal{A}_2(:,:,2,1) = \begin{pmatrix} -2\mathbf{i}+\mathbf{k} & \mathbf{i}-\mathbf{j} \\ 1+\mathbf{i} & \mathbf{i}+\mathbf{k} \end{pmatrix}, \quad \mathcal{A}_2(:,:,2,2) = \begin{pmatrix} \mathbf{i}-\mathbf{j} & 1+\mathbf{k} \\ \mathbf{j}-\mathbf{k} & \mathbf{i}+\mathbf{j} \end{pmatrix}, \quad \mathcal{B}_1(:,:,1,1) = \begin{pmatrix} \mathbf{j}-\mathbf{k} & 0 \\ 0 & \mathbf{i}-\mathbf{k} \end{pmatrix},$$

$$\mathcal{B}_1(:,:,1,2) = \begin{pmatrix} 1-\mathbf{i} & \mathbf{j} \\ \mathbf{i}+\mathbf{k} & -\mathbf{k} \end{pmatrix}, \quad \mathcal{B}_1(:,:,2,1) = \begin{pmatrix} \mathbf{j}-\mathbf{k} & -\mathbf{i} \\ \mathbf{i}+\mathbf{k} & -\mathbf{j} \end{pmatrix}, \quad \mathcal{B}_1(:,:,2,2) = \begin{pmatrix} -\mathbf{i} & 0 \\ \mathbf{j} & 0 \end{pmatrix},$$

$$\mathcal{B}_2(:,:,1,1) = \begin{pmatrix} \mathbf{i}-\mathbf{k} & \mathbf{i}+\mathbf{k} \\ \mathbf{j} & 1-\mathbf{i} \end{pmatrix}, \quad \mathcal{B}_2(:,:,1,2) = \begin{pmatrix} \mathbf{i}+\mathbf{j} & -\mathbf{i}+\mathbf{k} \\ 1+\mathbf{j} & -\mathbf{i} \end{pmatrix}, \quad \mathcal{B}_2(:,:,2,1) = \begin{pmatrix} \mathbf{i} & \mathbf{i}-\mathbf{j} \\ \mathbf{i} & \mathbf{i}-\mathbf{k} \end{pmatrix},$$

$$\mathcal{B}_2(:,:,2,2) = \begin{pmatrix} \mathbf{j}-\mathbf{k} & 0 \\ \mathbf{i}+\mathbf{j} & 2\mathbf{j} \end{pmatrix}, \quad \mathcal{C}_1(:,:,1,1) = \begin{pmatrix} 1 & 1+\mathbf{k} \\ \mathbf{i}-\mathbf{k} & \mathbf{j} \end{pmatrix}, \quad \mathcal{C}_1(:,:,1,2) = \begin{pmatrix} \mathbf{i}-\mathbf{j} & \mathbf{i}+\mathbf{k} \\ -\mathbf{j} & \mathbf{j} \end{pmatrix},$$

$$\mathcal{C}_1(:,:,2,1) = \begin{pmatrix} -\mathbf{j}+\mathbf{k} & \mathbf{i}-\mathbf{k} \\ \mathbf{j} & 1+\mathbf{j} \end{pmatrix}, \quad \mathcal{C}_1(:,:,2,2) = \begin{pmatrix} \mathbf{i}-\mathbf{k} & -\mathbf{j} \\ \mathbf{j}+\mathbf{k} & \mathbf{i}-\mathbf{k} \end{pmatrix}, \quad \mathcal{C}_2(:,:,1,1) = \begin{pmatrix} \mathbf{i}+\mathbf{j} & 1+\mathbf{i} \\ \mathbf{i}-\mathbf{k} & 1-\mathbf{j} \end{pmatrix},$$

$$\mathcal{C}_2(:,:,1,2) = \begin{pmatrix} \mathbf{i}-\mathbf{k} & -1+\mathbf{j} \\ -\mathbf{k} & 1+\mathbf{i}-\mathbf{j} \end{pmatrix}, \quad \mathcal{C}_2(:,:,2,1) = \begin{pmatrix} 1-\mathbf{j} & 1+\mathbf{k} \\ 1+\mathbf{j} & \mathbf{i}+\mathbf{k} \end{pmatrix}, \quad \mathcal{C}_2(:,:,2,2) = \begin{pmatrix} 1+\mathbf{i} & 1-\mathbf{i} \\ -\mathbf{i}-\mathbf{k} & \mathbf{j}-\mathbf{k} \end{pmatrix},$$

$$\mathcal{D}_1(:,:,1,1) = \begin{pmatrix} \mathbf{i}+\mathbf{j} & 1 \\ 1+\mathbf{i} & \mathbf{i} \end{pmatrix}, \quad \mathcal{D}_1(:,:,1,2) = \begin{pmatrix} \mathbf{i}-\mathbf{k} & \mathbf{j} \\ \mathbf{j} & \mathbf{i}+\mathbf{k} \end{pmatrix}, \quad \mathcal{D}_1(:,:,2,1) = \begin{pmatrix} 1+\mathbf{i} & \mathbf{k} \\ \mathbf{i}-\mathbf{k} & \mathbf{j} \end{pmatrix},$$

$$\mathcal{D}_1(:,:,2,2) = \begin{pmatrix} \mathbf{i}+\mathbf{k} & \mathbf{i} \\ \mathbf{j} & -\mathbf{k} \end{pmatrix}, \quad \mathcal{D}_2(:,:,1,1) = \begin{pmatrix} \mathbf{i}+\mathbf{j} & \mathbf{i}-\mathbf{k} \\ -1+\mathbf{j} & 1+2\mathbf{j} \end{pmatrix}, \quad \mathcal{D}_2(:,:,1,2) = \begin{pmatrix} \mathbf{i}-\mathbf{k} & \mathbf{j}+\mathbf{k} \\ \mathbf{k} & \mathbf{i}-\mathbf{j} \end{pmatrix},$$

$$\mathcal{D}_2(:,:,2,1) = \begin{pmatrix} -1-\mathbf{i} & \mathbf{k} \\ -\mathbf{i}+\mathbf{k} & 1+\mathbf{k} \end{pmatrix}, \quad \mathcal{D}_2(:,:,2,2) = \begin{pmatrix} \mathbf{i}+2\mathbf{j} & \mathbf{j}+\mathbf{k} \\ 1+\mathbf{k} & \mathbf{j}-2\mathbf{k} \end{pmatrix},$$

$$\mathcal{E}_1(:,:,1,1) = \begin{pmatrix} -2\mathbf{i}+\mathbf{k} & -3+4\mathbf{j}-7\mathbf{k} \\ -1+3\mathbf{i}+4\mathbf{j}-4\mathbf{k} & 7-4\mathbf{i}+3\mathbf{j}-6\mathbf{k} \end{pmatrix}, \quad \mathcal{E}_1(:,:,1,2) = \begin{pmatrix} -3-14\mathbf{i}-2\mathbf{j}+4\mathbf{k} & -3+3\mathbf{i}-5\mathbf{j}-6\mathbf{k} \\ -7-2\mathbf{i}-\mathbf{k} & 9+3\mathbf{i}+\mathbf{j} \end{pmatrix},$$

$$\mathcal{E}_1(:,:,2,1) = \begin{pmatrix} -9+4\mathbf{i}+5\mathbf{j}-6\mathbf{k} & 4-8\mathbf{i}+2\mathbf{j}-8\mathbf{k} \\ 10\mathbf{i}-6\mathbf{j}-4\mathbf{k} & -1-\mathbf{i}-10\mathbf{j}-\mathbf{k} \end{pmatrix}, \quad \mathcal{E}_1(:,:,2,2) = \begin{pmatrix} 1-10\mathbf{i}-\mathbf{j}+3\mathbf{k} & 4-2\mathbf{i}+8\mathbf{j}+6\mathbf{k} \\ -7+2\mathbf{i}-4\mathbf{j}-\mathbf{k} & -5\mathbf{i}-\mathbf{j}+11\mathbf{k} \end{pmatrix},$$

$$\mathcal{E}_2(:,:,1,1) = \begin{pmatrix} -20-4\mathbf{i}-12\mathbf{j}+8\mathbf{k} & 3-2\mathbf{i}+16\mathbf{j}-5\mathbf{k} \\ -1+10\mathbf{i}-7\mathbf{j}-4\mathbf{k} & -16-2\mathbf{i}-8\mathbf{j}-6\mathbf{k} \end{pmatrix}, \quad \mathcal{E}_2(:,:,1,2) = \begin{pmatrix} 10-13\mathbf{j}+4\mathbf{k} & -14-5\mathbf{i}+6\mathbf{j}+11\mathbf{k} \\ -4+14\mathbf{i}+7\mathbf{j}+7\mathbf{k} & -11+13\mathbf{i}-13\mathbf{j} \end{pmatrix},$$

$$\mathcal{E}_2(:,:,2,1) = \begin{pmatrix} 19+6\mathbf{i}-12\mathbf{j}-11\mathbf{k} & -4-5\mathbf{i}+5\mathbf{j} \\ -2-13\mathbf{i}+5\mathbf{j}-9\mathbf{k} & 13-9\mathbf{i}+14\mathbf{j}-\mathbf{k} \end{pmatrix}, \quad \mathcal{E}_2(:,:,2,2) = \begin{pmatrix} -3-2\mathbf{i}-12\mathbf{j}-\mathbf{k} & 5\mathbf{j}+9\mathbf{k} \\ -5+6\mathbf{i}+2\mathbf{j}+15\mathbf{k} & -6-3\mathbf{i}-9\mathbf{j}+6\mathbf{k} \end{pmatrix}.$$

Direct computation yields

$$\begin{aligned} \mathcal{R}_{\mathcal{A}_1} *_N \mathcal{E}_1 *_M \mathcal{L}_{\mathcal{D}_1} &= 0, & \mathcal{R}_{\mathcal{C}_1} *_N \mathcal{E}_1 *_M \mathcal{L}_{\mathcal{B}_1} &= 0, & \mathcal{R}_{\mathcal{M}_1} *_N \mathcal{R}_{\mathcal{A}_1} *_N \mathcal{E}_1 &= 0, \\ \mathcal{E}_1 *_M \mathcal{L}_{\mathcal{B}_1} *_M \mathcal{L}_{\mathcal{N}_1} &= 0, & \mathcal{R}_{\mathcal{A}_2} *_N \mathcal{E}_2 *_M \mathcal{L}_{\mathcal{D}_2} &= 0, & \mathcal{R}_{\mathcal{C}_2} *_N \mathcal{E}_2 *_M \mathcal{L}_{\mathcal{B}_2} &= 0, \\ \mathcal{R}_{\mathcal{M}_2} *_N \mathcal{R}_{\mathcal{A}_2} *_N \mathcal{E}_2 &= 0, & \mathcal{E}_2 *_M \mathcal{L}_{\mathcal{B}_2} *_M \mathcal{L}_{\mathcal{N}_2} &= 0, & \mathcal{R}_{\mathcal{M}} *_N \mathcal{R}_{\mathcal{A}} *_N \mathcal{E} &= 0, \\ \mathcal{E} *_M \mathcal{L}_{\mathcal{B}} *_M \mathcal{L}_{\mathcal{N}} &= 0, & \mathcal{R}_{\mathcal{A}} *_N \mathcal{E} *_M \mathcal{L}_{\mathcal{D}} &= 0, & \mathcal{R}_{\mathcal{C}} *_N \mathcal{E} *_M \mathcal{L}_{\mathcal{B}} &= 0. \end{aligned}$$

Therefore, all conditions in Theorem 3.1 are met, the system (1.2) is consistent. As a consequence, it is easy to verify that $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{Y}_1$ below satisfy the system (1.2):

$$\mathcal{X}_1(:,:,1,1) = \begin{pmatrix} \mathbf{i}-\mathbf{k} & \mathbf{j}+\mathbf{k} \\ -\mathbf{k} & 1-\mathbf{i} \end{pmatrix}, \quad \mathcal{X}_1(:,:,1,2) = \begin{pmatrix} -\mathbf{j}+\mathbf{k} & 1+\mathbf{i} \\ \mathbf{j} & 1-\mathbf{i} \end{pmatrix}, \quad \mathcal{X}_1(:,:,2,1) = \begin{pmatrix} 1+\mathbf{i} & \mathbf{i}-\mathbf{j} \\ 1-\mathbf{i} & \mathbf{i}+\mathbf{j} \end{pmatrix},$$

$$\mathcal{X}_1(:,:,2,2) = \begin{pmatrix} \mathbf{i} & \mathbf{i}-\mathbf{k} \\ \mathbf{i}+\mathbf{j} & \mathbf{j} \end{pmatrix}, \quad \mathcal{X}_2(:,:,1,1) = \begin{pmatrix} \mathbf{i}-\mathbf{k} & \mathbf{i}-\mathbf{j} \\ \mathbf{j}-\mathbf{k} & 1-\mathbf{j} \end{pmatrix}, \quad \mathcal{X}_2(:,:,1,2) = \begin{pmatrix} 1+\mathbf{j} & \mathbf{i}-\mathbf{k} \\ \mathbf{i}+\mathbf{k} & 1-\mathbf{k} \end{pmatrix},$$

$$\mathcal{X}_2(:,:,2,1)=\begin{pmatrix}\mathbf{i+k} & \mathbf{j-k}\\ \mathbf{i+j} & -1+\mathbf{j}\end{pmatrix},\quad \mathcal{X}_2(:,:,2,2)=\begin{pmatrix}-\mathbf{j} & \mathbf{i+j}\\ \mathbf{j-k} & 2\mathbf{j}\end{pmatrix},\quad \mathcal{Y}_1(:,:,1,1)=\begin{pmatrix}1-\mathbf{i} & \mathbf{i-k}\\ -1-2\mathbf{j} & \mathbf{i+k}\end{pmatrix},$$
$$\mathcal{Y}_1(:,:,1,2)=\begin{pmatrix}-\mathbf{i+j} & \mathbf{i}\\ \mathbf{i-k} & \mathbf{i+k}\end{pmatrix},\quad \mathcal{Y}_1(:,:,2,1)=\begin{pmatrix}1-\mathbf{j} & -\mathbf{k}\\ 1-\mathbf{k} & \mathbf{i}\end{pmatrix},\quad \mathcal{Y}_1(:,:,2,2)=\begin{pmatrix}\mathbf{i} & \mathbf{j-k}\\ \mathbf{j} & 1-\mathbf{k}\end{pmatrix}.$$

4. The η-Hermitian solutions to the tensor equation (1.1) and the system (1.3)

In this section, we investigate the η-Hermitian solutions to the equation (1.1) and the system (1.3) based on Theorem 3.1.

We first consider the equation (1.1) where $\mathcal{X}$, $\mathcal{Y}$ are unknown quaternion tensors with $\mathcal{Y}=\mathcal{Y}^{\eta^*}$, and $\mathcal{A}\in\mathbb{H}^{I_1\times\cdots\times I_N\times J_1\times\cdots\times J_N}$, $\mathcal{B}\in\mathbb{H}^{K_1\times\cdots\times K_M\times R_1\times\cdots\times R_M}$, $\mathcal{C}\in\mathbb{H}^{I_1\times\cdots\times I_N\times P_1\times\cdots\times P_N}$, $\mathcal{D}\in\mathbb{H}^{P_1\times\cdots\times P_N\times R_1\times\cdots\times R_M}$, $\mathcal{E}\in\mathbb{H}^{I_1\times\cdots\times I_N\times R_1\times\cdots\times R_M}$ are given. For simplicity, we put

$$\mathcal{M}_1=\mathcal{R}_\mathcal{A}*_N\mathcal{C},\quad \mathcal{N}_1=\mathcal{D}*_M\mathcal{L}_\mathcal{B},\quad \mathcal{S}_1=\mathcal{C}*_N\mathcal{L}_{\mathcal{M}_1},\quad \mathcal{S}_2=(\mathcal{R}_{\mathcal{N}_1}*_N\mathcal{D})^{\eta*},$$
$$\mathcal{A}_{11}=\begin{pmatrix}\mathcal{L}_{\mathcal{M}_1}*_N\mathcal{L}_{\mathcal{S}_1} & (\mathcal{R}_{\mathcal{N}_1})^{\eta*}*_N\mathcal{L}_{\mathcal{S}_2}\end{pmatrix},\quad \mathcal{B}_{11}=\begin{pmatrix}\mathcal{R}_\mathcal{D}\\ (\mathcal{L}_\mathcal{C})^{\eta*}\end{pmatrix},$$
$$\begin{aligned}\mathcal{E}_{11}=&(\mathcal{N}_1^\dagger)^{\eta*}*_M\mathcal{E}^{\eta*}*_N(\mathcal{C}^\dagger)^{\eta*}+\mathcal{S}_2^\dagger*_M\mathcal{S}_2*_N(\mathcal{D}^\dagger)^{\eta*}*_M\mathcal{E}^{\eta*}*_N(\mathcal{M}_1^\dagger)^{\eta*}\\&-\mathcal{M}_1^\dagger*_N\mathcal{E}*_M\mathcal{D}^\dagger-\mathcal{S}_1^\dagger*_N\mathcal{S}_1*_N\mathcal{C}^\dagger*_N\mathcal{E}*_M\mathcal{N}_1^\dagger,\end{aligned}$$
$$\mathcal{A}_1=\mathcal{R}_{\mathcal{A}_{11}}*_N\mathcal{L}_{\mathcal{M}_1},\ \mathcal{B}_1=\mathcal{R}_{\mathcal{N}_1}*_N\mathcal{L}_{\mathcal{B}_{11}},\ \mathcal{C}_1=\mathcal{R}_{\mathcal{A}_{11}}*_N(\mathcal{R}_{\mathcal{N}_1})^{\eta*},\ \mathcal{D}_1=(\mathcal{L}_{\mathcal{M}_1})^{\eta*}*_N\mathcal{L}_{\mathcal{B}_{11}},$$
$$\mathcal{E}_1=\mathcal{R}_{\mathcal{A}_{11}}*_N\mathcal{E}_{11}*_N\mathcal{L}_{\mathcal{B}_{11}},\quad \mathcal{M}=\mathcal{R}_{\mathcal{A}_1}*_N\mathcal{C}_1,\quad \mathcal{N}=\mathcal{D}_1*_N\mathcal{L}_{\mathcal{B}_1},\quad \mathcal{S}=\mathcal{C}_1*_N\mathcal{L}_\mathcal{M}.$$

Theorem 4.1. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be given quaternion tensors mentioned above. The equation* (1.1) *is consistent if and only if*

$$\mathcal{R}_\mathcal{A}*_N\mathcal{E}*_M\mathcal{L}_\mathcal{D}=0,\quad \mathcal{R}_\mathcal{C}*_N\mathcal{E}*_M\mathcal{L}_\mathcal{B}=0,\quad \mathcal{R}_{\mathcal{S}_1}*_N\mathcal{R}_\mathcal{A}*_N\mathcal{E}=0,$$
$$\mathcal{E}*_M\mathcal{L}_\mathcal{B}*_M\mathcal{L}_{\mathcal{N}_1}=0,\quad \mathcal{R}_{\mathcal{S}_2}*_M(\mathcal{L}_\mathcal{B})^{\eta*}*_M\mathcal{E}^{\eta*}=0,$$
$$\mathcal{E}^{\eta*}*_N(\mathcal{L}_\mathcal{A})^{\eta*}*_N(\mathcal{L}_{\mathcal{N}_1})^{\eta*}=0,\quad \mathcal{R}_\mathcal{M}*_N\mathcal{R}_{\mathcal{A}_1}*_N\mathcal{E}_1=0,$$
$$\mathcal{E}_1*_N\mathcal{L}_{\mathcal{B}_1}*_N\mathcal{L}_\mathcal{N}=0,\quad \mathcal{R}_{\mathcal{A}_1}*_N\mathcal{E}_1*_N\mathcal{L}_{\mathcal{D}_1}=0,\quad \mathcal{R}_{\mathcal{C}_1}*_N\mathcal{E}_1*_N\mathcal{L}_{\mathcal{B}_1}=0.$$

In this case, the η-Hermitian solution to the equation (1.1) *can be expressed as*

$$\mathcal{X}=\frac{\mathcal{X}_1+\mathcal{X}_2^{\eta*}}{2},\quad \mathcal{Y}=\frac{\mathcal{Y}_1+\mathcal{Y}_1^{\eta*}}{2},$$

where

$$\begin{aligned}\mathcal{X}_1=&\mathcal{A}^\dagger*_N\mathcal{E}*_M\mathcal{B}^\dagger-\mathcal{A}^\dagger*_N\mathcal{S}_1*_N\mathcal{C}^\dagger*_N\mathcal{E}*_M\mathcal{N}_1^\dagger*_N\mathcal{D}*_M\mathcal{B}^\dagger\\&-\mathcal{A}^\dagger*_N\mathcal{C}*_N\mathcal{M}_1^\dagger*_N\mathcal{E}*_M\mathcal{B}^\dagger-\mathcal{A}^\dagger*_N\mathcal{S}_1*_N\mathcal{V}_2*_M\mathcal{R}_{\mathcal{N}_1}*_N\mathcal{D}*_M\mathcal{B}^\dagger\\&+\mathcal{L}_\mathcal{A}*_N\mathcal{V}_4+\mathcal{V}_5*_M\mathcal{R}_\mathcal{B},\\ \mathcal{X}_2=&(\mathcal{B}^\dagger)^{\eta*}*_M\mathcal{E}^{\eta*}*_N(\mathcal{A}^\dagger)^{\eta*}-(\mathcal{B}^\dagger)^{\eta*}*_M\mathcal{D}^{\eta*}*_N(\mathcal{M}_1^\dagger)^{\eta*}*_N\mathcal{E}^{\eta*}*_N(\mathcal{A}^\dagger)^{\eta*}\end{aligned}$$

$$
\begin{aligned}
&- (\mathcal{B}^\dagger)^{\eta*} *_M \mathcal{S}_2 *_N (\mathcal{D}^\dagger)^{\eta*} *_M \mathcal{E}^{\eta*} *_N (\mathcal{N}_1^\dagger)^{\eta*} *_M \mathcal{C}^{\eta*} *_N (\mathcal{A}^\dagger)^{\eta*} + (\mathcal{R}_{\mathcal{B}})^{\eta*} *_M \mathcal{T}_4 \\
&+ \mathcal{T}_5 *_N (\mathcal{L}_{\mathcal{A}})^{\eta*} - (\mathcal{B}^\dagger)^{\eta*} *_M \mathcal{S}_2 *_N \mathcal{T}_2 *_M (\mathcal{L}_{\mathcal{N}_1})^{\eta*} *_M \mathcal{C}^{\eta*} *_N (\mathcal{A}^\dagger)^{\eta*}, \\
\mathcal{Y}_1 &= \mathcal{M}_1^\dagger *_N \mathcal{E} *_M \mathcal{D}^\dagger + \mathcal{S}_1^\dagger *_N \mathcal{S}_1 *_N \mathcal{C}^\dagger *_N \mathcal{E} *_M \mathcal{N}_1^\dagger + \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{L}_{\mathcal{S}_1} *_N \mathcal{V}_1 \\
&+ \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_N \mathcal{R}_{\mathcal{N}_1} + \mathcal{V}_3 *_N \mathcal{R}_{\mathcal{D}},
\end{aligned}
$$

or

$$
\begin{aligned}
\mathcal{Y}_1 &= (\mathcal{N}_1^\dagger)^{\eta*} *_M \mathcal{E}^{\eta*} *_N (\mathcal{C}^\dagger)^{\eta*} + \mathcal{S}_2^\dagger *_M \mathcal{S}_2 *_N (\mathcal{D}^\dagger)^{\eta*} *_N \mathcal{E}^{\eta*} *_N (\mathcal{M}_1^\dagger)^{\eta*} \\
&- (\mathcal{R}_{\mathcal{N}_1})^{\eta*} *_N \mathcal{L}_{\mathcal{S}_2} *_N \mathcal{T}_1 - (\mathcal{R}_{\mathcal{N}_1})^{\eta*} *_N \mathcal{T}_2 *_N (\mathcal{L}_{\mathcal{M}_1})^{\eta*} - \mathcal{T}_3 *_N (\mathcal{L}_{\mathcal{C}})^{\eta*},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{V}_1 &= (\mathcal{I}\ \ 0) *_N (\mathcal{A}_{11}^\dagger *_N (\mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_N \mathcal{R}_{\mathcal{N}_1} - (\mathcal{R}_{\mathcal{N}_1})^{\eta*} *_N \mathcal{T}_2 *_N (\mathcal{L}_{\mathcal{M}_1})^{\eta*}) \\
&+ \mathcal{W}_1 *_N \mathcal{B}_{11} + \mathcal{L}_{\mathcal{A}_{11}} *_N \mathcal{W}_2), \\
\mathcal{V}_2 &= \mathcal{A}_1^\dagger *_N \mathcal{E}_1 *_M \mathcal{B}_1^\dagger - \mathcal{A}_1^\dagger *_N \mathcal{S} *_N \mathcal{C}_1^\dagger *_N \mathcal{E}_1 *_N \mathcal{N}^\dagger *_N \mathcal{D}_1 *_N \mathcal{B}_1^\dagger \\
&- \mathcal{A}_1^\dagger *_N \mathcal{C}_1 *_N \mathcal{M}^\dagger *_N \mathcal{E}_1 *_N \mathcal{B}_1^\dagger - \mathcal{A}_1^\dagger *_N \mathcal{S} *_N \mathcal{W}_4 *_N \mathcal{R}_{\mathcal{N}} *_N \mathcal{D}_1 *_N \mathcal{B}_1^\dagger \\
&+ \mathcal{L}_{\mathcal{A}_1} *_N \mathcal{W}_5 + \mathcal{W}_6 *_N \mathcal{R}_{\mathcal{B}_1}, \\
\mathcal{V}_3 &= (\mathcal{R}_{\mathcal{A}_{11}} *_N (\mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_N \mathcal{R}_{\mathcal{N}_1} - (\mathcal{R}_{\mathcal{N}_1})^{\eta*} *_N \mathcal{T}_2 *_N (\mathcal{L}_{\mathcal{M}_1})^{\eta*}) *_N \mathcal{B}_{11}^\dagger \\
&- \mathcal{A}_{11} *_N \mathcal{W}_1 - \mathcal{W}_3 *_N \mathcal{R}_{\mathcal{B}_{11}}) *_N \begin{pmatrix} \mathcal{I} \\ 0 \end{pmatrix}, \\
\mathcal{T}_1 &= (0\ \ \mathcal{I}) *_N (\mathcal{A}_{11}^\dagger *_N (\mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_N \mathcal{R}_{\mathcal{N}_1} - (\mathcal{R}_{\mathcal{N}_1})^{\eta*} *_N \mathcal{T}_2 *_N (\mathcal{L}_{\mathcal{M}_1})^{\eta*}) \\
&+ \mathcal{W}_1 *_N \mathcal{B}_{11} + \mathcal{L}_{\mathcal{A}_{11}} *_N \mathcal{W}_2), \\
\mathcal{T}_2 &= \mathcal{M}^\dagger *_N \mathcal{E}_1 *_M \mathcal{D}_1^\dagger + \mathcal{S} *_N \mathcal{S}^\dagger *_N \mathcal{C}_1^\dagger *_N \mathcal{E}_1 *_N \mathcal{N}^\dagger + \mathcal{L}_{\mathcal{M}} *_N \mathcal{L}_{\mathcal{S}} *_N \mathcal{W}_7 \\
&+ \mathcal{L}_{\mathcal{M}} *_N \mathcal{W}_4 *_N \mathcal{R}_{\mathcal{N}} + \mathcal{W}_8 *_N \mathcal{R}_{\mathcal{D}_1}, \\
\mathcal{T}_3 &= (\mathcal{R}_{\mathcal{A}_{11}} *_N (\mathcal{E}_{11} - \mathcal{L}_{\mathcal{M}_1} *_N \mathcal{V}_2 *_N \mathcal{R}_{\mathcal{N}_1} - (\mathcal{R}_{\mathcal{N}_1})^{\eta*} *_N \mathcal{T}_2 *_N (\mathcal{L}_{\mathcal{M}_1})^{\eta*}) *_N \mathcal{B}_{11}^\dagger \\
&- \mathcal{A}_{11} *_N \mathcal{W}_1 - \mathcal{W}_3 *_N \mathcal{R}_{\mathcal{B}_{11}}) *_N \begin{pmatrix} 0 \\ \mathcal{I} \end{pmatrix},
\end{aligned}
$$

and $\mathcal{V}_4$, $\mathcal{V}_5$, $\mathcal{T}_4$, $\mathcal{T}_5$, $\mathcal{W}_1, \ldots, \mathcal{W}_8$ *are arbitrary tensors over* $\mathbb{H}$ *with appropriate sizes.*

Proof. We prove that the equation (1.1) has a solution with $\mathcal{Y}$ being η-Hermitian if and only if the following system

$$
\begin{aligned}
\mathcal{A} *_N \mathcal{X}_1 *_M \mathcal{B} + \mathcal{C} *_N \mathcal{Y}_1 *_N \mathcal{D} &= \mathcal{E}, \\
\mathcal{B}^{\eta*} *_M \mathcal{X}_2 *_N \mathcal{A}^{\eta*} + \mathcal{D}^{\eta*} *_N \mathcal{Y}_1 *_N \mathcal{C}^{\eta*} &= \mathcal{E}^{\eta*}
\end{aligned}
\tag{4.1}
$$

has a solution.

Firstly, if the equation (1.1) has a solution, denoted as $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}})$, then it is obvious that the system (4.1) has a solution shaped like $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1) = (\widetilde{\mathcal{X}}, \widetilde{\mathcal{X}}^{\eta *}, \widetilde{\mathcal{Y}})$.

Next we show that if the system (4.1) has a solution, then the equation (1.1) is solvable. Assume that the system (4.1) has a solution $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1)$. We will prove that

$$(\mathcal{X}, \mathcal{Y}) = \left(\frac{\mathcal{X}_1 + \mathcal{X}_2^{\eta *}}{2}, \frac{\mathcal{Y}_1 + \mathcal{Y}_1^{\eta *}}{2} \right) \tag{4.2}$$

is a solution to the equation (1.1).

Substituting (4.2) into the equation (1.1) yields

$$\begin{aligned}
&\mathcal{A} *_N \left(\frac{\mathcal{X}_1 + \mathcal{X}_2^{\eta *}}{2} \right) *_M \mathcal{B} + \mathcal{C} *_N \left(\frac{\mathcal{Y}_1 + \mathcal{Y}_1^{\eta *}}{2} \right) *_N \mathcal{D} \\
&= \frac{1}{2}\mathcal{A} *_N \mathcal{X}_1 *_M \mathcal{B} + \frac{1}{2}\mathcal{C} *_N \mathcal{Y}_1 *_N \mathcal{D} + \frac{1}{2}\mathcal{A} *_N \mathcal{X}_2^{\eta *} *_M \mathcal{B} + \frac{1}{2}\mathcal{C} *_N \mathcal{Y}_1^{\eta *} *_M \mathcal{D} \\
&= \frac{1}{2}(\mathcal{A} *_N \mathcal{X}_1 *_M \mathcal{B} + \mathcal{C} *_N \mathcal{Y}_1 *_N \mathcal{D}) + \frac{1}{2}(\mathcal{B}^{\eta *} *_M \mathcal{X}_2 *_N \mathcal{A}^{\eta *} + \mathcal{D}^{\eta *} *_N \mathcal{Y}_1 *_N \mathcal{C}^{\eta *})^{\eta *} \\
&= \mathcal{E}.
\end{aligned}$$

This implies that (4.2) is a solution to the equation (1.1). We can get the solvability conditions and the expression of the general η-Hermitian solution to the equation (1.1) by Theorem 3.1. $\square$

Remark 4.2. We can also present the solvability conditions and an expression of a solution with $\mathcal{X}$ being η-Hermitian to the tensor equation over $\mathbb{H}$:

$$\mathcal{A} *_N \mathcal{X} *_N \mathcal{B} + \mathcal{C} *_N \mathcal{Y} *_M \mathcal{D} = \mathcal{E},$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ are given, $\mathcal{X}$ and $\mathcal{Y}$ are unknown.

Now we turn our attention to consider the solvability conditions and the general η-Hermitian solution to the system (1.3) where $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{Y}_1$ are unknowns and $\mathcal{A}_1 \in \mathbb{H}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_N}$, $\mathcal{A}_2 \in \mathbb{H}^{I_1 \times \cdots \times I_N \times S_1 \times \cdots \times S_N}$, $\mathcal{B}_1 \in \mathbb{H}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_N}$, $\mathcal{B}_2 \in \mathbb{H}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_N}$, $\mathcal{E}_i \in \mathbb{H}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ are given quaternion tensors, and $\mathcal{E}_i$ $(i = 1, 2)$ are η-Hermitian. For simplicity, we set

$$\mathcal{M}_1 = \mathcal{R}_{\mathcal{A}_1} *_N \mathcal{B}_1, \quad \mathcal{M}_2 = \mathcal{R}_{\mathcal{A}_2} *_N \mathcal{B}_2, \quad \mathcal{S}_1 = \mathcal{B}_1 *_N \mathcal{L}_{\mathcal{M}_1}, \quad \mathcal{S}_2 = \mathcal{B}_2 *_N \mathcal{L}_{\mathcal{M}_2},$$

$$\mathcal{A}_{11} = (\mathcal{L}_{\mathcal{M}_1} *_N \mathcal{L}_{\mathcal{S}_1} \ \ \mathcal{L}_{\mathcal{M}_2} *_N \mathcal{L}_{\mathcal{S}_2}), \quad \mathcal{B}_{11} = \begin{pmatrix} (\mathcal{L}_{\mathcal{B}_1})^{\eta *} \\ (\mathcal{L}_{\mathcal{B}_2})^{\eta *} \end{pmatrix},$$

$$\begin{aligned}
\mathcal{E}_{11} = {} & \mathcal{M}_2^{\dagger} *_N \mathcal{E}_2 *_N (\mathcal{B}_2^{\dagger})^{\eta *} + \mathcal{S}_2^{\dagger} *_N \mathcal{S}_2 *_N \mathcal{B}_2^{\dagger} *_N \mathcal{E}_2 *_N (\mathcal{M}_2^{\dagger})^{\eta *} \\
& - \mathcal{M}_1^{\dagger} *_N \mathcal{E}_1 *_N (\mathcal{B}_1^{\dagger})^{\eta *} - \mathcal{S}_1^{\dagger} *_N \mathcal{S}_1 *_N \mathcal{B}_1^{\dagger} *_N \mathcal{E}_1 *_N (\mathcal{M}_1^{\dagger})^{\eta *},
\end{aligned}$$

$$\mathcal{A} = \mathcal{R}_{\mathcal{A}_{11}} *_N \mathcal{L}_{\mathcal{M}_1}, \ \mathcal{B} = (\mathcal{L}_{\mathcal{M}_1})^{\eta *} *_N \mathcal{L}_{\mathcal{B}_{11}}, \ \mathcal{C} = \mathcal{R}_{\mathcal{A}_{11}} *_N \mathcal{L}_{\mathcal{M}_2}, \ \mathcal{D} = (\mathcal{L}_{\mathcal{M}_2})^{\eta *} *_N \mathcal{L}_{\mathcal{B}_{11}},$$

$$\mathcal{E} = \mathcal{R}_{\mathcal{A}_{11}} *_N \mathcal{E}_{11} *_N \mathcal{L}_{\mathcal{B}_{11}}, \quad \mathcal{M} = \mathcal{R}_{\mathcal{A}} *_N \mathcal{C}, \quad \mathcal{N} = \mathcal{D} *_N \mathcal{L}_{\mathcal{B}}, \quad \mathcal{S} = \mathcal{C} *_N \mathcal{L}_{\mathcal{M}}.$$

Theorem 4.3. *Let $\mathcal{A}_i$, $\mathcal{B}_i$, $\mathcal{E}_i$ $(i = 1, 2)$ be given quaternion tensors. Then the system* (1.3) *is consistent if and only if*

$$\mathcal{R}_{\mathcal{A}_1} *_N \mathcal{E}_1 *_N (\mathcal{R}_{\mathcal{B}_1})^{\eta *} = 0, \quad \mathcal{R}_{\mathcal{M}_1} *_N \mathcal{R}_{\mathcal{A}_1} *_N \mathcal{E}_1 = 0,$$
$$\mathcal{R}_{\mathcal{A}_2} *_N \mathcal{E}_2 *_N (\mathcal{R}_{\mathcal{B}_2})^{\eta *} = 0, \quad \mathcal{R}_{\mathcal{M}_2} *_N \mathcal{R}_{\mathcal{A}_2} *_N \mathcal{E}_2 = 0, \quad \mathcal{R}_{\mathcal{M}} *_N \mathcal{R}_{\mathcal{A}} *_N \mathcal{E} = 0,$$
$$\mathcal{E} *_N \mathcal{L}_{\mathcal{B}} *_N \mathcal{L}_{\mathcal{N}} = 0, \quad \mathcal{R}_{\mathcal{A}} *_N \mathcal{E} *_N \mathcal{L}_{\mathcal{D}} = 0, \quad \mathcal{R}_{\mathcal{C}} *_N \mathcal{E} *_N \mathcal{L}_{\mathcal{B}} = 0.$$

In this case, the expression of η-Hermitian solution $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1)$ can be deduced from (3.8)–(3.11) *and* (3.12a)–(3.12f) *in Theorem* 3.1.

Proof. We show that the system (1.3) has an η-Hermitian solution if and only if the system of quaternion tensor equations

$$\begin{aligned} \mathcal{A}_1 *_N \mathcal{X} *_N \mathcal{A}_1^{\eta^*} + \mathcal{B}_1 *_N \mathcal{Y} *_N \mathcal{B}_1^{\eta^*} &= \mathcal{E}_1, \\ \mathcal{A}_2 *_N \mathcal{Z} *_N \mathcal{A}_2^{\eta^*} + \mathcal{B}_2 *_N \mathcal{Y} *_N \mathcal{B}_2^{\eta^*} &= \mathcal{E}_2 \end{aligned} \tag{4.3}$$

has a solution, where $\mathcal{E}_i$ $(i = 1, 2)$ are η-Hermitian quaternion tensors. If the system (1.3) has an η-Hermitian solution, i.e., $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1)$, then the system (4.3) obviously has a solution $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) = (\mathcal{X}_1, \mathcal{Y}_1, \mathcal{X}_2)$.

Conversely, if the system (4.3) has a solution $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$, then

$$(\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1) = \left(\frac{\mathcal{X} + \mathcal{X}^{\eta *}}{2}, \frac{\mathcal{Z} + \mathcal{Z}^{\eta *}}{2}, \frac{\mathcal{Y} + \mathcal{Y}^{\eta *}}{2} \right)$$

is an η-Hermitian solution to the system (1.3). Therefore we can get the solvability conditions by Theorem 3.1. And the expression of the η-Hermitian solution to the system (1.3) can be obtained from (3.8)–(3.11) and (3.12a)–(3.12f) when the conditions are met. □

5. Conclusion

In this paper, we have established some necessary and sufficient conditions for the existence of the general solution to the system (1.2) and constructed an expression of the general solution to the system when it is solvable in Theorem 3.1. We have provided an example to illustrate our main results. As applications of the system (1.2), we have given some necessary and sufficient conditions for the existence of an η-Hermitian solution to the two-sided Sylvester-type tensor equation (1.1) and the system (1.3). Moreover, we have presented expressions of such solutions when the solvability conditions are met.

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