

Uniform Boundedness and Global Existence of Solutions to a Quasilinear Diffusion Equation with Nonlocal Fisher-KPP Type Reaction Term

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Abstract. This paper deals with the Cauchy problem and Neumann initial boundary value problem for a quasilinear diffusion equation with nonlocal Fisher-KPP type reaction terms. We establish the uniform boundedness and global existence of solutions to the problems by using multipliers technique and modified Moser's iteration argument for some ranges of parameters. Moreover, the ranges of parameters have similar structure to that of the classical critical Fujita exponent.

1. Introduction

We investigate the following Cauchy problem ($\tau = 0$) and Neumann initial boundary value problem ($\tau = 1$) of the quasilinear diffusion equation with nonlocal Fisher-KPP type reaction term

$$(1.1) \quad \begin{cases} u_t = \Delta u^m + u^\alpha (1 - \int_{\Omega} u^\beta dx), & x \in \Omega, t > 0, \\ \tau \frac{\partial u^m}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) \geq 0, & x \in \Omega, \end{cases}$$

where $\tau \in \{0, 1\}$, $m > 0$, and $\alpha, \beta \geq 1$. When $\tau = 0$, Ω is assumed to be \mathbb{R}^N , therefore (1.1) is a Cauchy problem; while in the case of $\tau = 1$, we suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, hence (1.1) turns to a Neumann initial boundary value problem, ν is the outward unit normal vector on $\partial\Omega$. The nonnegative initial data u_0 is not identical to zero.

Nonlinear diffusion equations like (1.1) appear in various applications, which describes the diffusion of the concentration of Newtonian flow in a porous medium or the temperature of some combustible substances, see [6, 32]. In particular, equation (1.1) is a possible model for the diffusion system of some biological species, where $u(x, t)$ represents the density of the species at position x and time t , Δu^m portrays the mutation, which

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we view as a spreading of the characteristic, and the reaction term $u^\alpha(1 - \int_\Omega u^\beta dx)$ is considered as the rate of the reproduction. Due to the effect of spatial inhomogeneity, the occurrence of nonlocal term $\int_\Omega u^\beta dx$ denotes that the evolution of the species at a point in space depends not only on the density of species in partial region but also on the total region, refer to [2, 14, 18]. Moreover, as appeared in many literatures, e.g., [27, 28, 36], the nonlocal type reaction term $u^\alpha(1 - \int_\Omega u^\beta dx)$ can also describe Darwinian evolution of a structured population density or the behaviors of cancer cells with therapy. In the nonlinear diffusion theory, there exist obvious differences among the situations of slow ($m > 1$), fast ($0 < m < 1$), and linear ($m = 1$) diffusion. For example, there is a finite speed propagation in the slow and linear diffusion situations, whereas an infinite speed propagation exists in the case of fast diffusion.

During the past decades, there have been many works to deal with diffusion equations with local or nonlocal reaction terms, see [6, 22, 32, 34, 39]. Roughly, those works contain some focal topics such as existence and nonexistence of global solutions, blow-up phenomena, extinction phenomena, asymptotic behavior of the solutions as well as the critical exponent theory, etc. In this paper, we are particularly interested in the issues of global existence of solutions to the following quasilinear diffusion equation with nonlocal Fisher-KPP type reaction term

$$(1.2) \quad u_t = \Delta u^m + F(t, u, I(u)), \quad x \in \Omega, t > 0,$$

where $I(u) = \int_\Omega u(y, t) dy$. To the best of our knowledge, compared with the local reaction problems, few results are available for such nonlocal models. When $m = 1$ in (1.2). Bebernes [5, 7], Pao [33] and Liu et al. [26] considered the thermal explosion model of compressible gas with the nonlocal reaction term

$$F = e^u + aI(u_t) \quad \text{or} \quad F = e^u + aI(e^u),$$

and they obtained the solvability, asymptotic properties and blow-up phenomena of solutions under null Dirichlet boundary condition. Wang et al. [37] investigated the semilinear diffusion model with the reaction term

$$F = I(u^\alpha) - ku^\gamma, \quad \alpha, \gamma \geq 1,$$

and they derived sufficient condition for which the solutions exist globally and blow up in finite time by virtue of ODE analysis. Budd et al. [13], Hu and Yin [23] studied the partial differential equation with the following special nonlocal reaction term

$$F = u^\alpha - \frac{1}{|\Omega|} I(u^\alpha)$$

under null Neumann boundary condition, where $\alpha > 1$. The solutions of these problems have a conservation property and, based on the convexity argument, they proved the

nonexistence of global solutions under large initial energy. In addition, one can refer to [15,20,31] to review some latest researches on the nonlocal problems with fully nonlinear reaction terms, nonlocal semilinear parabolic equation with small positive initial energy and fourth-order thin-film equations. In [4], Anguiano et al. considered the diffusion equation with the reaction term

$$F = f(u)I(u)(1 - I(u))$$

under null Dirichlet boundary condition, and they obtained the existence of global attractor. Recently, for the Cauchy problem and the Neumann initial boundary value problem with the nonlocal reaction term

$$F = u^\alpha(1 - I(u^\beta)),$$

Bian and Chen studied the existence of global solutions and derived the critical exponents of the Fujita type (cf. [8,9]). Afterwards, Bian et al. [10] investigated the global existence and asymptotic behavior of solutions to Neumann initial boundary value problem with the nonlocal term

$$F = u^\alpha(1 - I(u)).$$

Besides, one can refer to the literature [12] to see studies on Fisher-KPP type equation with convolution operator.

When $m \neq 1$ in (1.2). Wang and Wo [38] investigated a fast diffusion equation with the nonlocal term

$$F = u^m - I(u^m)$$

under null Neumann boundary condition, and they proved a convergence of global solutions to some steady states in one-dimensional space. Fang et al. [16] studied the slow diffusion equation with the nonlocal term

$$F = u^\alpha I(u^\beta) - u^\gamma$$

under nonlocal boundary condition, and they established a new comparison principle and gave some sufficient conditions for which the solutions exist globally and blow up in finite time. Afterwards, Xu et al. [40] considered a fast diffusion equation with the nonlocal reaction term in [16] under a homogeneous Dirichlet boundary condition, and they derived some sufficient conditions for the extinction of nonnegative nontrivial weak solutions and the corresponding decay estimates by virtue of integral estimate method and ODE technique. In addition, for studies on the travelling fronts, entire solutions, and large-time behavior of solutions to local Fisher-KPP type diffusion problems, we refer to [21, 24, 25] and the references therein.

In view of the works mentioned above, much less effort has been devoted to the existence of uniformly bounded global solutions to quasilinear diffusion model (1.1). At a glance, our main difficulties lie in finding how the competitive relationship between the nonlinear diffusion term Δu^m and the nonlinear nonlocal term $\int_{\Omega} u^{\beta} dx$ affect the global existence of solutions. In particular, the method used in the nonlocal semilinear diffusion problems in the aforementioned works of literatures (see [8, 9]) is no longer directly applicable to our nonlocal quasilinear model, therefore, we need more delicate analysis to obtain the global existence. Motivated by these observations, applying multipliers technique and modified Moser's iteration argument, we will show that the solutions to (1.1) are uniformly bounded and exist globally under appropriate conditions. Indeed, our results improve and generalize the Theorem 1 in [9] and [8], respectively.

Note that for $m > 0$, the diffusion term $\Delta u^m(x, t)$ in problem (1.1) may be singular or degenerate on the set $\{(x, t) \mid u(x, t) = 0\}$, which leads to the nonexistence of classical solutions to problem (1.1), therefore, we consider the weak solutions in the distribution sense. It is well-known that the existence of local weak solutions in time to problem (1.1) can be obtained by the fixed point theorems or standard parabolic regularity theory that can be applied to get suitable estimates in the standard limiting process (cf. [3, 35, 39]), and hence, we omit the details here. It can be seen that there exists $T_{\max} > 0$ such that the unique weak solution u to problem (1.1) satisfies

$$u \in C([0, T_{\max}); L^{\infty}(\Omega)),$$

and either $T_{\max} = \infty$, or T_{\max} is finite with $\limsup_{t \rightarrow T_{\max}} \|u(\cdot, t)\|_{L^{\infty}(\Omega)} = +\infty$, which is similar to the results in [29, 30].

Now we are ready to give our main results. For convenience, let 2^* denote the critical exponent of Sobolev embedding, that is,

$$2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N \geq 3, \\ p \in (2, \infty) & \text{if } N = 2, \\ \infty & \text{if } N = 1. \end{cases}$$

We state the following existence of uniformly bounded global solutions in whole space \mathbb{R}^N as well as in bounded domain, respectively.

Theorem 1.1. *Let $\tau = 0$. Suppose that $N \geq 1$, $m > \max\{0, 1 - 2/N\}$, $\beta \geq 1$ and that α satisfies the inequalities*

$$\max\{1, m - \beta\} \leq \alpha < m + \left(1 - \frac{2}{2^*}\right)\beta.$$

If the nonnegative initial data u_0 is in $L^{\beta}(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$, then the Cauchy problem (1.1) has a unique nonnegative global solution, which is uniformly bounded.

Theorem 1.2. *Let $\tau = 1$. Suppose that $N \geq 1$, $m > 0$, $\beta \geq 1$ and that α satisfies the inequalities*

$$\max\{1, m - \beta\} \leq \alpha < m + \frac{2}{N}\beta.$$

If the nonnegative initial data u_0 is in $L^\infty(\Omega)$, then the Neumann initial boundary value problem (1.1) has a unique nonnegative global solution, which is uniformly bounded.

Remark 1.3. From Theorems 1.1 and 1.2 we see that, for the existence of global solution to (1.1), Cauchy problem and Neumann initial boundary value problem share exactly the same critical exponent $\alpha_c = m + \frac{2}{N}\beta$ ($N \geq 3$). In fact, the structure of the critical exponent α_c is similar to that of the well-known classical critical Fujita exponent to the local porous medium equation with power like source term $u_t = \Delta u^m + u^\alpha$. As we all know, for the local problem, when $\alpha < \alpha_c$, the solutions blow up in finite time for any initial data (cf. [17, 19, 29]). However, Theorems 1.1 and 1.2 show the opposite result, i.e., global solution exists, which indicates that the nonlocal term has a huge influence on the properties of solutions.

The present work is organized as follows. In Section 2, we give a proof for Theorem 1.1 on global boundedness for the solution to the Cauchy problem ($\tau = 0$). In Section 3, a proof for Theorem 1.2 on global boundedness for the solution to the Neumann initial boundary value problem ($\tau = 1$) is presented.

For simplicity, the variable of integral will be omitted without ambiguity, e.g., the integral $\int_\Omega f(x) dx$ is written as $\int_\Omega f(x)$.

2. Global boundedness for $\tau = 0$

In this section, we consider global boundedness of the solution to the Cauchy problem (1.1). To begin with, we review Lemma 2 in [9].

Lemma 2.1. [9] *If $N \geq 1$, $1 \leq r < q < 2^*$, and $\frac{q}{r} < \frac{2}{r} + 1 - \frac{2}{2^*}$, then for any $v \in H^1(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$, the following inequalities*

$$(2.1) \quad \begin{aligned} \|v\|_{L^q(\mathbb{R}^N)}^q &\leq \varepsilon \|\nabla v\|_{L^2(\mathbb{R}^N)}^2 + C\varepsilon^{-\frac{\lambda q}{2-\lambda q}} \|v\|_{L^r(\mathbb{R}^N)}^\mu, & N \geq 3, \\ \|v\|_{L^q(\mathbb{R}^N)}^q &\leq \varepsilon_1 \|\nabla v\|_{L^2(\mathbb{R}^N)}^2 + \varepsilon_2 \|v\|_{L^2(\mathbb{R}^N)}^2 \\ &\quad + C\left(\varepsilon_1^{-\frac{\lambda q}{2-\lambda q}} + \varepsilon_2^{-\frac{\lambda q}{2-\lambda q}}\right) \|v\|_{L^r(\mathbb{R}^N)}^\mu, & N = 1, 2 \end{aligned}$$

hold, where $C = C(N, q, r)$ and $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$ are arbitrary constants, and

$$\lambda = \frac{1/r - 1/q}{1/2 - 1/2^*} \in (0, 1), \quad \mu = \frac{2(1-\lambda)q}{2-\lambda q}.$$

We now introduce the following key proposition to prove Theorem 1.1.

Proposition 2.2. *Suppose that $N \geq 1$, $m > 0$, $\beta \geq 1$ and that α satisfies the inequalities*

$$\max\{1, m - \beta\} \leq \alpha < m + \left(1 - \frac{2}{2^*}\right)\beta.$$

For any $T \in (0, T_{max})$ and each $\beta \leq k < \infty$, if the nonnegative initial data u_0 is in $\bigcap_{\beta \leq k < \infty} L^k(\mathbb{R}^N)$, then any solution to the Cauchy problem (1.1) satisfies

$$\|u(\cdot, t)\|_{L^k(\mathbb{R}^N)} \leq C \quad \text{for all } t \in (0, T),$$

where $C = C(N, m, \alpha, \beta, k, \|u_0\|_{L^k(\mathbb{R}^N)})$.

Remark 2.3. Proposition 2.2 is a generalization of Proposition 2 in [9].

Proof of Proposition 2.2. We only give a proof for the case $N \geq 3$, since the cases of $N = 1, 2$ can be similarly proved. Multiplying the first equation in (1.1) by ku^{k-1} ($k > 1$) and integrating the result over \mathbb{R}^N , we obtain the equation

$$(2.2) \quad \frac{d}{dt} \int_{\mathbb{R}^N} u^k + \frac{4mk(k-1)}{(k+m-1)^2} \int_{\mathbb{R}^N} |\nabla u^{\frac{k+m-1}{2}}|^2 + k \int_{\mathbb{R}^N} u^{k+\alpha-1} \int_{\mathbb{R}^N} u^\beta = k \int_{\mathbb{R}^N} u^{k+\alpha-1}.$$

For each $k > \max\{1, \beta + 1 - \alpha\}$, we choose k' such that $k' > \beta$ and $\max\{1, \frac{k+m-1}{2}\} \leq k' < k + \alpha - 1$. Moreover, taking $v = u^{\frac{k+m-1}{2}}$, $q = \frac{2(k+\alpha-1)}{k+m-1}$, $r = \frac{2k'}{k+m-1}$, and $\varepsilon = \frac{2m(k-1)}{(k+m-1)^2}$ in (2.1), we have

$$(2.3) \quad \begin{aligned} \int_{\mathbb{R}^N} u^{k+\alpha-1} &= \|u^{\frac{k+m-1}{2}}\|_{L^q(\mathbb{R}^N)}^q \\ &\leq \frac{2m(k-1)}{(k+m-1)^2} \int_{\mathbb{R}^N} |\nabla u^{\frac{k+m-1}{2}}|^2 + C_1 \left(\frac{2m(k-1)}{(k+m-1)^2} \right)^{-\frac{\lambda}{2-\lambda q}} \|u\|_{L^{k'}(\mathbb{R}^N)}^a, \end{aligned}$$

where $C_1 = C_1(N, m, \alpha, k, k')$, $\lambda = \frac{\frac{1}{k'} - \frac{1}{k+\alpha-1}}{\frac{1}{k'} - \frac{1}{2^*(k+m-1)}} \in (0, 1)$, $a = \frac{(1-\lambda)(k+\alpha-1)}{1 - \frac{\lambda(k+\alpha-1)}{k+m-1}}$. Substituting

(2.3) into (2.2), there exists $C_2 = kC_1 \left(\frac{2m(k-1)}{(k+m-1)^2} \right)^{-\frac{\lambda q}{2-\lambda q}}$ such that

$$(2.4) \quad \frac{d}{dt} \int_{\mathbb{R}^N} u^k + \frac{2mk(k-1)}{(k+m-1)^2} \int_{\mathbb{R}^N} |\nabla u^{\frac{k+m-1}{2}}|^2 + k \int_{\mathbb{R}^N} u^{k+\alpha-1} \int_{\mathbb{R}^N} u^\beta \leq C_2 \|u\|_{L^{k'}(\mathbb{R}^N)}^a.$$

Since $\beta < k' < k + \alpha - 1$, the interpolation inequality yields that

$$(2.5) \quad \begin{aligned} \|u\|_{L^{k'}(\mathbb{R}^N)}^a &\leq \|u\|_{L^{k+\alpha-1}(\mathbb{R}^N)}^{a\theta} \|u\|_{L^\beta(\mathbb{R}^N)}^{a(1-\theta)} \\ &= \left(\|u\|_{L^{k+\alpha-1}(\mathbb{R}^N)}^{k+\alpha-1} \|u\|_{L^\beta(\mathbb{R}^N)}^\beta \right)^{\frac{a\theta}{k+\alpha-1}} \|u\|_{L^\beta(\mathbb{R}^N)}^{a(1-\theta) - \frac{\beta a \theta}{k+\alpha-1}}, \end{aligned}$$

where $\theta = \frac{\frac{1}{\beta} - \frac{1}{k'}}{\frac{1}{\beta} - \frac{1}{k+\alpha-1}} \in (0, 1)$. In addition, one can easily see the following equivalence

$$\frac{a\theta}{k+\alpha-1} < 1 \quad \iff \quad \alpha < m + \frac{2}{N}\beta.$$

Due to $\alpha + \beta \geq m$, we can choose $k' = \frac{k+\alpha+\beta-1}{2}$ in (2.5) and get $a(1-\theta) - \frac{\beta a \theta}{k+\alpha-1} = 0$. Applying Young's inequality to (2.5), we arrive at

$$(2.6) \quad \begin{aligned} C_2 \|u\|_{L^{k'}(\mathbb{R}^N)}^a &= C_2 (\|u\|_{L^{k+\alpha-1}(\mathbb{R}^N)}^{k+\alpha-1} \|u\|_{L^\beta(\mathbb{R}^N)}^\beta)^{\frac{a\theta}{k+\alpha-1}} \\ &\leq \frac{k}{2} \int_{\mathbb{R}^N} u^{k+\alpha-1} \int_{\mathbb{R}^N} u^\beta + C_3, \end{aligned}$$

where $C_3 = \frac{k+\alpha-1-a\theta}{k+\alpha-1} \left(\frac{2a\theta}{k(k+\alpha-1)}\right)^{\frac{a\theta}{k+\alpha-1-a\theta}} C_2^{\frac{k+\alpha-1}{k+\alpha-1-a\theta}}$. Substituting (2.6) into (2.4), we obtain the inequality

$$(2.7) \quad \frac{d}{dt} \int_{\mathbb{R}^N} u^k + \frac{2mk(k-1)}{(k+m-1)^2} \int_{\mathbb{R}^N} |\nabla u^{\frac{k+m-1}{2}}|^2 + \frac{k}{2} \int_{\mathbb{R}^N} u^{k+\alpha-1} \int_{\mathbb{R}^N} u^\beta \leq C_3$$

for each $k > \max\{1, \beta + 1 - \alpha\}$. It follows from Hölder's and Young's inequalities that

$$(2.8) \quad \int_{\mathbb{R}^N} u^{\frac{k+\alpha+\beta-1}{2}} \leq \frac{k}{2} \int_{\mathbb{R}^N} u^{k+\alpha-1} \int_{\mathbb{R}^N} u^\beta + \frac{1}{2k}.$$

In particular, picking $k = \alpha + \beta - 1$ in (2.7) and (2.8) entails

$$(2.9) \quad \frac{d}{dt} \int_{\mathbb{R}^N} u^{\alpha+\beta-1} + \int_{\mathbb{R}^N} u^{\alpha+\beta-1} \leq C_3 + \frac{1}{2(\alpha + \beta - 1)}.$$

Applying a differential inequality technique to (2.9), we obtain

$$\int_{\mathbb{R}^N} u^{\alpha+\beta-1} \leq \max \left\{ \int_{\mathbb{R}^N} u_0^{\alpha+\beta-1}, C_3 + \frac{1}{2(\alpha + \beta - 1)} \right\}.$$

Now, taking $k = \beta$ in (2.2), we get the differential inequality

$$\frac{d}{dt} \int_{\mathbb{R}^N} u^\beta \leq \beta \int_{\mathbb{R}^N} u^{\alpha+\beta-1} \left(1 - \int_{\mathbb{R}^N} u^\beta \right),$$

which results in $\int_{\mathbb{R}^N} u^\beta \leq \max \{ \int_{\mathbb{R}^N} u_0^\beta, 1 \}$. By the interpolation inequality, one can easily see that for all k such that $\beta \leq k \leq \alpha + \beta - 1$, the norm $\|u(\cdot, t)\|_{L^k(\mathbb{R}^N)}$ is uniform-in-time bounded.

When $\alpha + \beta - 1 < k < \infty$, taking $v = u^{\frac{k+m-1}{2}}$, $q = \tilde{q} := \frac{2k}{k+m-1}$, $r = \frac{k+\alpha+\beta-1}{k+m-1}$ and $\varepsilon = \frac{mk(k-1)}{(k+m-1)^2}$ in (2.1), and combining with (2.8), we know that

$$(2.10) \quad \begin{aligned} \int_{\mathbb{R}^N} u^k &= \|u^{\frac{k+m-1}{2}}\|_{L^{\tilde{q}}(\mathbb{R}^N)}^{\tilde{q}} \\ &\leq \frac{m(k-1)}{(k+m-1)^2} \int_{\mathbb{R}^N} |\nabla u^{\frac{k+m-1}{2}}|^2 + C_4 \left(\frac{mk(k-1)}{(k+m-1)^2} \right)^{-\frac{\tilde{\lambda}\tilde{q}}{2-\tilde{\lambda}\tilde{q}}} \|u\|_{L^{\frac{k+\alpha+\beta-1}{2}}(\mathbb{R}^N)}^{\tilde{a}} \\ &\leq \frac{m(k-1)}{(k+m-1)^2} \int_{\mathbb{R}^N} |\nabla u^{\frac{k+m-1}{2}}|^2 \\ &\quad + C_4 \left(\frac{mk(k-1)}{(k+m-1)^2} \right)^{-\frac{\tilde{\lambda}\tilde{q}}{2-\tilde{\lambda}\tilde{q}}} \left(\int_{\mathbb{R}^N} u^{k+\alpha-1} \int_{\mathbb{R}^N} u^\beta \right)^{\frac{\tilde{a}}{k+\alpha+\beta-1}}, \end{aligned}$$

where $C_4 = C_4(N, m, \alpha, \beta, k)$, $\tilde{\lambda} = \frac{\frac{1}{k+\alpha+\beta-1} - \frac{1}{2k}}{\frac{1}{k+\alpha+\beta-1} - \frac{1}{2^*(k+m-1)}} \in (0, 1)$, $\tilde{a} = \frac{k(1-\tilde{\lambda})}{1 - \frac{k\tilde{\lambda}}{k+m-1}}$. It is easy to see that $\frac{\tilde{a}}{k+\alpha+\beta-1} < 1$. We apply Young's inequality to (2.10) to see

$$(2.11) \quad \int_{\mathbb{R}^N} u^k \leq \frac{m(k-1)}{(k+m-1)^2} \int_{\mathbb{R}^N} |\nabla u^{\frac{k+m-1}{2}}|^2 + \frac{k}{2} \int_{\mathbb{R}^N} u^{k+\alpha-1} \int_{\mathbb{R}^N} u^\beta + C_5,$$

where $C_5 = C_5(N, m, \alpha, \beta, k)$. It follows from (2.7) and (2.11) that

$$\frac{d}{dt} \int_{\mathbb{R}^N} u^k + \int_{\mathbb{R}^N} u^k + \frac{mk(k-1)}{(k+m-1)^2} \int_{\mathbb{R}^N} |\nabla u^{\frac{k+m-1}{2}}|^2 \leq C_3 + C_5.$$

Therefore, for all k such that $k > \alpha + \beta - 1$, the norm $\|u(\cdot, t)\|_{L^k(\mathbb{R}^N)}$ is also uniform-in-time bounded.

The proof is complete. \square

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We also only need to prove the case $N \geq 3$. Let $q_k = 2^k + \alpha + \beta - 1$, where $k \in \mathbb{N}$. It is easy to see that $q_k \geq \beta$, and hence, the solution $u(\cdot, t)$ of the Cauchy problem is in $L^{q_k}(\mathbb{R}^N)$ for all $t \in (0, T)$ by Proposition 2.2. Taking $k = q_k$ in (2.2), we obtain the differential equation

$$(2.12) \quad \begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^N} u^{q_k} + \frac{4mq_k(q_k-1)}{(q_k+m-1)^2} \int_{\mathbb{R}^N} |\nabla u^{\frac{q_k+m-1}{2}}|^2 + q_k \int_{\mathbb{R}^N} u^{q_k+\alpha-1} \int_{\mathbb{R}^N} u^{q_k+\beta} \\ &= q_k \int_{\mathbb{R}^N} u^{q_k+\alpha-1}. \end{aligned}$$

The definition of q_k enables us to pick $C_m := 4m \min\{\frac{2}{(1+m)^2}, 1\}$ such that $\frac{4mq_k(q_k-1)}{(q_k+m-1)^2} \geq C_m > 0$. Then it follows from (2.12) that

$$(2.13) \quad \frac{d}{dt} \int_{\mathbb{R}^N} u^{q_k} + C_m \int_{\mathbb{R}^N} |\nabla u^{\frac{q_k+m-1}{2}}|^2 + q_k \int_{\mathbb{R}^N} u^{q_k+\alpha-1} \int_{\mathbb{R}^N} u^\beta \leq q_k \int_{\mathbb{R}^N} u^{q_k+\alpha-1}.$$

Taking $v = u^{\frac{q_k+m-1}{2}}$, $q = \bar{q} := \frac{2(q_k+\alpha-1)}{q_k+m-1}$, $r = \frac{2q_{k-1}}{q_k+m-1}$, and $\varepsilon = \frac{C_m}{2q_k}$ in (2.1), we have

$$(2.14) \quad \begin{aligned} & \int_{\mathbb{R}^N} u^{q_k+\alpha-1} = \|u^{\frac{q_k+m-1}{2}}\|_{L^{\bar{q}}(\mathbb{R}^N)}^{\bar{q}} \\ & \leq \frac{C_m}{2q_k} \int_{\mathbb{R}^N} |\nabla u^{\frac{q_k+m-1}{2}}|^2 + C_6 \left(\frac{C_m}{2q_k}\right)^{-\frac{\bar{\lambda}\bar{q}}{2-\bar{\lambda}\bar{q}}} \left(\int_{\mathbb{R}^N} u^{q_{k-1}}\right)^{a_1}, \end{aligned}$$

where $C_6 = C_6(N, m, \alpha, \beta, k)$, $\bar{\lambda} = \frac{\frac{1}{q_{k-1}} - \frac{1}{q_k+\alpha-1}}{\frac{1}{q_{k-1}} - \frac{1}{2^*(q_k+m-1)}} \in (0, 1)$, and $a_1 = 1 + \frac{q_k+\alpha-1-q_{k-1}}{q_{k-1} - \frac{N}{2}(\alpha-m)} < 2$.

Let

$$\delta = \frac{\bar{\lambda}\bar{q}}{2} = \frac{q_k - q_{k-1} + \alpha - 1}{q_k + m - 1 - \frac{2}{2^*}q_{k-1}} = g(2^{k-1}),$$

where $g(y) = \frac{y+\alpha-1}{(1+\frac{2}{N})y+\frac{2}{N}(\alpha+\beta-1)+m-1}$. It follows from $\alpha < m + \frac{2}{N}\beta$ that g is increasing on $[1, +\infty)$, and hence, $0 < g(1) \leq \delta \leq \frac{N}{N+2} < 1$. From (2.14), we arrive at

$$(2.15) \quad q_k \int_{\mathbb{R}^N} u^{q_k+\alpha-1} \leq \frac{C_m}{2} \int_{\mathbb{R}^N} |\nabla u^{\frac{q_k+m-1}{2}}|^2 + C_7 q_k^{\frac{\delta}{1-\delta}} \left(\int_{\mathbb{R}^N} u^{q_{k-1}} \right)^{a_1},$$

where $C_7 = C_7(N, m, \alpha, \beta)$. Similarly, taking $v = u^{\frac{q_k+m-1}{2}}$, $q = \frac{2q_k}{q_k+m-1}$, $r = \frac{2q_{k-1}}{q_k+m-1}$, and $\varepsilon = \frac{C_m}{2}$ in (2.1), it can be obtained that

$$(2.16) \quad \int_{\mathbb{R}^N} u^{q_k} \leq \frac{C_m}{2} \int_{\mathbb{R}^N} |\nabla u^{\frac{q_k+m-1}{2}}|^2 + C_8 \left(\int_{\mathbb{R}^N} u^{q_{k-1}} \right)^{a_2},$$

where $C_8 = C_8(N, m, \alpha, \beta)$ and $a_2 = 1 + \frac{q_k - q_{k-1}}{q_{k-1} - \frac{N}{2}(m-1)} < 2$. Adding (2.15) and (2.16) together, substituting the result into (2.13), we can find a constant $C_9 = \max\{C_7, C_8\}$ such that

$$(2.17) \quad \begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^N} u^{q_k} + \int_{\mathbb{R}^N} u^{q_k} &\leq C_7 q_k^{\frac{1}{1-\delta}} \left(\int_{\mathbb{R}^N} u^{q_{k-1}} \right)^{a_1} + C_8 \left(\int_{\mathbb{R}^N} u^{q_{k-1}} \right)^{a_2} \\ &\leq C_9 \left(q_k^{\frac{1}{1-\delta}} \left(\int_{\mathbb{R}^N} u^{q_{k-1}} \right)^{a_1} + \left(\int_{\mathbb{R}^N} u^{q_{k-1}} \right)^{a_2} \right) \\ &\leq 2C_9 q_k^{\frac{1}{1-\delta}} \max \left\{ 1, \left(\int_{\mathbb{R}^N} u^{q_{k-1}} \right)^2 \right\}. \end{aligned}$$

By virtue of (2.17) and the well-known Moser-Alikakos iteration procedure (cf. [1], [11, Lemma 4.1] or Appendix), we derive that

$$(2.18) \quad \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C \quad \text{for all } t \in (0, T),$$

where $C = C(N, m, \alpha, \beta, \|u_0\|_{L^\beta(\mathbb{R}^N)}, \|u_0\|_{L^\infty(\mathbb{R}^N)})$. Since $T \in (0, T_{max})$ is arbitrary, (2.18) yields that $T_{max} = \infty$, which completes the proof. \square

3. Global boundedness for $\tau = 1$

In this section, we consider global boundedness of the solution to the Neumann initial boundary value problem (1.1). To begin with, we review two useful inequalities below.

Lemma 3.1. *If $N \geq 1$, $p \geq 1$, $\gamma \in (0, p)$ and $w \in H^1(\Omega)$, then the Gagliardo-Nirenberg inequality*

$$(3.1) \quad \|w\|_{L^p(\Omega)} \leq c_{GN} \|w\|_{H^1(\Omega)}^\sigma \|w\|_{L^\gamma(\Omega)}^{1-\sigma}$$

holds, where $c_{GN} = c_{GN}(N, p, \gamma)$, $p(N-2) < 2N$, and $\sigma = \frac{1/\gamma - 1/p}{1/N - 1/2 + 1/\gamma} \in (0, 1)$. In addition, if $s > 0$, then the Poincaré inequality

$$(3.2) \quad \|w\|_{H^1(\Omega)} \leq c_P (\|\nabla w\|_{L^2(\Omega)} + \|w\|_{L^s(\Omega)})$$

holds, where $c_P = c_P(N, s)$.

We now introduce the following key proposition to prove Theorem 1.2.

Proposition 3.2. *Suppose that $N \geq 1$, $m > 0$, $\beta \geq 1$ and that α satisfies the inequalities*

$$\max\{1, m - \beta\} \leq \alpha < m + \frac{2}{N}\beta.$$

For any $T \in (0, T_{max})$ and each k such that $1 \leq k < \infty$, if the nonnegative initial data u_0 is in $\bigcap_{\beta \leq k < \infty} L^k(\Omega)$, then any nonnegative solution to the Neumann initial boundary value problem (1.1) satisfies the inequality

$$\|u(\cdot, t)\|_{L^k(\Omega)} \leq c \quad \text{for all } t \in (0, T),$$

where $c = c(N, m, \alpha, \beta, \|u_0\|_{L^k(\Omega)})$.

Remark 3.3. Proposition 3.2 is a generalization of Proposition 3 in [8].

Proof of Proposition 3.2. Multiplying the first equation in (1.1) by ku^{k-1} ($k > 1$) and integrating the result over Ω , we have

$$(3.3) \quad \frac{d}{dt} \int_{\Omega} u^k + \frac{4mk(k-1)}{(k+m-1)^2} \int_{\Omega} |\nabla u^{\frac{k+m-1}{2}}|^2 + k \int_{\Omega} u^{k+\alpha-1} \int_{\Omega} u^{\beta} = k \int_{\Omega} u^{k+\alpha-1}.$$

For each $k > \max\{1, \beta + 1 - \alpha\}$, taking $w = u^{\frac{k+m-1}{2}}$ and $p = \frac{2(k+\alpha-1)}{k+m-1}$ in (3.1), and using (3.2), it can be obtained that

$$(3.4) \quad \begin{aligned} & k \int_{\Omega} u^{k+\alpha-1} \\ & \leq kc_{GN}^p (c_P(\|\nabla u^{\frac{k+m-1}{2}}\|_{L^2(\Omega)}^2 + \|u^{\frac{k+m-1}{2}}\|_{L^s(\Omega)}^2))^{\frac{p\sigma}{2}} \|u^{\frac{k+m-1}{2}}\|_{L^{\gamma}(\Omega)}^{p(1-\sigma)} \\ & \leq c_1 \|\nabla u^{\frac{k+m-1}{2}}\|_{L^2(\Omega)}^{2\frac{p\sigma}{2}} \|u^{\frac{k+m-1}{2}}\|_{L^{\gamma}(\Omega)}^{\frac{2p(1-\sigma)}{2-p\sigma} \frac{2-p\sigma}{2}} + c_1 \|u^{\frac{k+m-1}{2}}\|_{L^s(\Omega)}^{p\sigma} \|u^{\frac{k+m-1}{2}}\|_{L^{\gamma}(\Omega)}^{p(1-\sigma)} \\ & \leq \frac{2mk(k-1)}{(k+m-1)^2} \int_{\Omega} |\nabla u^{\frac{k+m-1}{2}}|^2 + c_2 \|u^{\frac{k+m-1}{2}}\|_{L^{\gamma}(\Omega)}^{\frac{2p(1-\sigma)}{2-p\sigma}} \\ & \quad + c_1 \|u^{\frac{k+m-1}{2}}\|_{L^s(\Omega)}^{p\sigma} \|u^{\frac{k+m-1}{2}}\|_{L^{\gamma}(\Omega)}^{p(1-\sigma)}, \end{aligned}$$

where $(\frac{N(p-2)}{2})_+ < \gamma < p$, $s > 0$, $r = \frac{2k'}{k+m-1}$, $\sigma = \frac{1/\gamma-1/p}{1/N-1/2+1/\gamma} \in (0, 1)$, and c_1, c_2 depend on N, m, α, γ and k . Substituting (3.4) into (3.3), we get

$$(3.5) \quad \begin{aligned} & \frac{d}{dt} \int_{\Omega} u^k + \frac{2mk(k-1)}{(k+m-1)^2} \int_{\Omega} |\nabla u^{\frac{k+m-1}{2}}|^2 + k \int_{\Omega} u^{k+\alpha-1} \int_{\Omega} u^{\beta} \\ & \leq c_2 \|u^{\frac{k+m-1}{2}}\|_{L^{\gamma}(\Omega)}^{\frac{2p(1-\sigma)}{2-p\sigma}} + c_1 \|u^{\frac{k+m-1}{2}}\|_{L^s(\Omega)}^{p\sigma} \|u^{\frac{k+m-1}{2}}\|_{L^{\gamma}(\Omega)}^{p(1-\sigma)}. \end{aligned}$$

With $\gamma = \frac{k+\alpha+\beta-1}{k+m-1}$, Hölder's inequality yields

$$\|u^{\frac{k+m-1}{2}}\|_{L^{\gamma}(\Omega)} \leq \left(\int_{\Omega} u^{k+\alpha-1} \int_{\Omega} u^{\beta} \right)^{\frac{k+m-1}{2(k+\alpha+\beta-1)}}.$$

In addition, it can be shown that the following equivalence

$$\frac{k+m-1}{k+\alpha+\beta-1} \cdot \frac{p(1-\sigma)}{2-p\sigma} < 1 \iff \alpha < m + \frac{2}{N}\beta$$

holds, which yields that

$$(3.6) \quad \begin{aligned} c_2 \|u^{\frac{k+m-1}{2}}\|_{L^\gamma(\Omega)}^{\frac{2p(1-\sigma)}{2-p\sigma}} &\leq c_2 \left(\int_{\Omega} u^{k+\alpha-1} \int_{\Omega} u^\beta \right)^{\frac{k+m-1}{k+\alpha+\beta-1} \cdot \frac{p(1-\sigma)}{2-p\sigma}} \\ &\leq \frac{k}{4} \int_{\Omega} u^{k+\alpha-1} \int_{\Omega} u^\beta + c_3, \end{aligned}$$

and

$$(3.7) \quad \begin{aligned} c_1 \|u^{\frac{k+m-1}{2}}\|_{L^s(\Omega)}^{p\sigma} \|u^{\frac{k+m-1}{2}}\|_{L^\gamma(\Omega)}^{p(1-\sigma)} \\ \leq c_1 \|u^{\frac{k+m-1}{2}}\|_{L^s(\Omega)}^{p\sigma} \left(\int_{\Omega} u^{k+\alpha-1} \int_{\Omega} u^\beta \right)^{\frac{p(k+m-1)(1-\sigma)}{2(k+\alpha+\beta-1)}} \\ \leq \frac{k}{4} \int_{\Omega} u^{k+\alpha-1} \int_{\Omega} u^\beta + c_4 \|u^{\frac{k+m-1}{2}}\|_{L^s(\Omega)}^\chi, \end{aligned}$$

where $\chi = \frac{2p\sigma(k+\alpha+\beta-1)}{2(k+\alpha+\beta-1)-p(k+m-1)(1-\sigma)} > 0$, and c_3, c_4 depend on N, m, α, β and k . Substituting (3.6) and (3.7) into (3.5), we get the inequality

$$(3.8) \quad \begin{aligned} \frac{d}{dt} \int_{\Omega} u^k + \frac{2mk(k-1)}{(k+m-1)^2} \int_{\Omega} |\nabla u^{\frac{k+m-1}{2}}|^2 + \frac{k}{2} \int_{\Omega} u^{k+\alpha-1} \int_{\Omega} u^\beta \\ \leq c_3 + c_4 \|u^{\frac{k+m-1}{2}}\|_{L^s(\Omega)}^\chi. \end{aligned}$$

We utilize Hölder's inequality $(\int_{\Omega} u^\beta)^{\frac{k+\alpha-1}{\beta}} \leq |\Omega|^{\frac{k+\alpha-1-\beta}{\beta}} \int_{\Omega} u^{k+\alpha-1}$ to estimate

$$(3.9) \quad \left(\int_{\Omega} u^\beta \right)^{\frac{k+\alpha+\beta-1}{\beta}} \leq |\Omega|^{\frac{k+\alpha-1-\beta}{\beta}} \int_{\Omega} u^{k+\alpha-1} \int_{\Omega} u^\beta.$$

Taking $s = \frac{2\beta}{k+m-1}$, and therefore,

$$\frac{\beta\chi}{s(k+\alpha+\beta-1)} = \frac{p\sigma(k+m-1)}{2(k+\alpha+\beta-1)-p(k+m-1)(1-\sigma)} < 1.$$

It follows from (3.9) that

$$(3.10) \quad \begin{aligned} c_4 \|u^{\frac{k+m-1}{2}}\|_{L^s(\Omega)}^\chi &= c_4 \left(\int_{\Omega} u^\beta \right)^{\frac{k+\alpha+\beta-1}{\beta} \cdot \frac{\beta\chi}{s(k+\alpha+\beta-1)}} \\ &\leq \frac{k}{4} |\Omega|^{\frac{\beta+1-\alpha-k}{\beta}} \left(\int_{\Omega} u^\beta \right)^{\frac{k+\alpha+\beta-1}{\beta}} + c_5 \\ &\leq \frac{k}{4} \int_{\Omega} u^{k+\alpha-1} \int_{\Omega} u^\beta + c_5, \end{aligned}$$

where $c_5 = c_5(N, m, \alpha, \beta, k)$. Substituting (3.10) into (3.8), we have

$$(3.11) \quad \frac{d}{dt} \int_{\Omega} u^k + \frac{2mk(k-1)}{(k+m-1)^2} \int_{\Omega} |\nabla u^{\frac{k+m-1}{2}}|^2 + \frac{k}{4} \int_{\Omega} u^{k+\alpha-1} \int_{\Omega} u^{\beta} \leq c_3 + c_5.$$

In particular, taking $k = \beta$ in (3.11), from (3.9) we can choose M_1 and M_2 depending on N, m, α and β such that

$$(3.12) \quad \frac{d}{dt} \int_{\Omega} u^{\beta} + M_1 \left(\int_{\Omega} u^{\beta} \right)^{\frac{\alpha+2\beta-1}{\beta}} \leq M_2.$$

Differential inequality (3.12) results in

$$\int_{\Omega} u^{\beta} \leq \max \left\{ \int_{\Omega} u_0^{\beta}, \left(\frac{M_2}{M_1} \right)^{\frac{\beta}{\alpha+2\beta-1}} \right\}.$$

Therefore, we obtain the uniform boundedness for $\|u(\cdot, t)\|_{L^{\beta}(\Omega)}$. Thanks to Hölder's inequality, it can be seen that

$$\|u(\cdot, t)\|_{L^k(\Omega)} \leq c(k, N, m, \alpha, \beta, \|u_0\|_{L^{\beta}(\Omega)})$$

for all k such that $1 \leq k < \beta$ and $t \in (0, T)$.

For any $k > \max \left\{ \beta, m-1, \frac{N}{2}(1-m) \right\}$, taking $w = u^{\frac{k+m-1}{2}}$, $p = p_1 := \frac{2k}{k+m-1}$ in (3.1), we know that the inequality

$$(3.13) \quad \int_{\Omega} u^k = \|u^{\frac{k+m-1}{2}}\|_{L^{p_1}(\Omega)}^{p_1} \leq c_6 \|u^{\frac{k+m-1}{2}}\|_{H^1(\Omega)}^{p_1 \sigma_1} \|u^{\frac{k+m-1}{2}}\|_{L^{\gamma_1}(\Omega)}^{p_1(1-\sigma_1)}$$

holds for each $\gamma_1 \in (0, p_1)$, where $c_6 = c_6(N, m, \gamma_1, k)$ and $\sigma_1 = \frac{1/\gamma_1 - 1/p_1}{1/N - 1/2 + 1/\gamma_1} \in (0, 1)$. Taking $\gamma_1 = \frac{2\beta}{k+m-1}$, it is easy to see that $p_1 \sigma_1 < 2$. It then follows from the boundedness of $\|u(\cdot, t)\|_{L^{\beta}(\Omega)}$, (3.2) and (3.13) that

$$(3.14) \quad \begin{aligned} \int_{\Omega} u^k &\leq c_7 \|u^{\frac{k+m-1}{2}}\|_{H^1(\Omega)}^{p_1 \sigma_1} \\ &\leq c_8 \|\nabla u^{\frac{k+m-1}{2}}\|_{L^2(\Omega)}^{2 \cdot \frac{p_1 \sigma_1}{2}} + c_8 \|u^{\frac{k+m-1}{2}}\|_{L^{\gamma_1}(\Omega)}^{p_1 \sigma_1} \\ &\leq \frac{mk(k-1)}{(k+m-1)^2} \int_{\Omega} |\nabla u^{\frac{k+m-1}{2}}|^2 + c_9, \end{aligned}$$

where c_7, c_8, c_9 depend on k, N, m, α, β , and $\|u_0\|_{L^{\beta}(\Omega)}$. Substituting (3.14) into (3.11), we obtain the differential inequality

$$(3.15) \quad \frac{d}{dt} \int_{\Omega} u^k + \int_{\Omega} u^k \leq c_{10},$$

where $c_{10} = c_{10}(k, N, m, \alpha, \beta, \|u_0\|_{L^{\beta}(\Omega)})$, and hence, by an argument of differential inequality, (3.15) results in

$$\int_{\Omega} u^k \leq \max \left\{ \int_{\Omega} u_0^{\beta}, c_{10} \right\}$$

for any $k > \max \left\{ \beta, m - 1, \frac{N}{2}(1 - m) \right\}$.

When $\beta \leq k < \max \left\{ \beta, m - 1, \frac{N}{2}(1 - m) \right\}$, we apply the interpolation inequality to derive the uniform boundedness for $\|u(\cdot, t)\|_{L^k(\Omega)}$ and, from which, we can find a constant $c = c(k, N, m, \alpha, \beta, \|u_0\|_{L^k(\Omega)})$ such that

$$\|u(\cdot, t)\|_{L^k(\Omega)} \leq c \quad \text{for all } k \geq 1 \text{ and each } t \in (0, T).$$

The proof is completed. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Using a similar argument as the one used in the proof of Theorem 1.1 and applying the Moser–Alikakos technique [1], it can be shown that the solution $u(\cdot, t)$ to problem (1.1) satisfies

$$(3.16) \quad \|u(\cdot, t)\|_{L^\infty(\Omega)} \leq c \quad \text{for all } t \in (0, T),$$

where $c = c(N, m, \alpha, \beta, \|u_0\|_{L^\infty(\Omega)})$. Since $T \in (0, T_{\max})$ is arbitrary, (3.16) yields that $T_{\max} = \infty$, which completes the proof. □

A. Appendix

For completeness, we give a detailed proof for (2.18).

Proof of (2.18). Let $\rho = \frac{1}{1-\delta} = O(1)$. Then $\rho > 1$, thanks to (2.17), we can find a constant $C = C(N, m, \alpha, \beta) > 1$ such that

$$(A.1) \quad \frac{d}{dt} \int_{\mathbb{R}^N} u^{q_k} + \int_{\mathbb{R}^N} u^{q_k} \leq C q_k^\rho \max \left\{ 1, \left(\int_{\mathbb{R}^N} u^{q_{k-1}} \right)^2 \right\}.$$

Setting $y_k(t) = \int_{\mathbb{R}^N} u^{q_k}$, we have from (A.1) that

$$(A.2) \quad \begin{aligned} y_k(t) &\leq y_k(0)e^{-t} + C q_k^\rho \max \left\{ 1, \sup_{t \geq 0} y_{k-1}^2(t) \right\} (1 - e^{-t}) \\ &\leq y_k(0) + C q_k^\rho \max \left\{ 1, \sup_{t \geq 0} y_{k-1}^2(t) \right\}. \end{aligned}$$

Let $M := \max\{1, \|u_0\|_{L^\infty(\mathbb{R}^N)}, \|u_0\|_{L^\beta(\mathbb{R}^N)}\}$. The interpolation inequality yields that

$$(A.3) \quad y_k(0) = \int_{\mathbb{R}^N} u_0^{q_k} \leq \frac{\beta}{q_k} \|u_0\|_{L^\beta(\mathbb{R}^N)}^{q_k} + \frac{q_k - \beta}{q_k} \|u_0\|_{L^\infty(\mathbb{R}^N)}^{q_k} \leq M^{q_k}.$$

Substituting (A.3) into (A.2), there exist \bar{C} , \tilde{C} depend on N , α , β and M , such that

$$\begin{aligned}
 y_k(t) &\leq M^{q_k} + Cq_k^\rho \max \left\{ 1, \sup_{t \geq 1} y_{k-1}^2(t) \right\} \\
 &\leq 2Cq_k^\rho \max \left\{ M^{q_k}, \sup_{t \geq 0} y_{k-1}^2(t) \right\} \\
 &\leq \bar{C}2^{\rho k} \max \left\{ M^{q_k}, \sup_{t \geq 0} y_{k-2}^{2^2}(t) \right\} \\
 &\leq \tilde{C}\bar{C}^{1+2} 2^{\rho(k+2(k-1))} \max \left\{ M^{q_k}, \sup_{t \geq 0} y_{k-2}^{2^2}(t) \right\} \\
 &\leq \dots \\
 &\leq \tilde{C}\bar{C}^{\sum_{j=0}^{k-1} 2^j} 2^{\rho \sum_{j=0}^{k-1} 2^j(k-j+1)} \max \left\{ M^{q_k}, \sup_{t \geq 0} y_0^{2^k}(t) \right\} \\
 &= \tilde{C}\bar{C}^{2^k-1} 2^{\rho(2^{k+1}-k)} \max \left\{ M^{q_k}, \sup_{t \geq 0} y_0^{2^k}(t) \right\},
 \end{aligned}$$

here we have used the fact $q_k = 2^k + \alpha + \beta - 1 \leq (\alpha + \beta)2^k$. Therefore,

$$\int_{\mathbb{R}^N} u^{q_k} = y_k^{\frac{1}{q_k}} \leq C(N, \alpha, \beta, M).$$

Letting $k \rightarrow \infty$, we obtain the desired result (2.18). □

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