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Gradient Estimates for the Nonlinear Parabolic Equation with Two Exponents on Riemannian Manifolds

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Abstract. In this paper, we study the nonlinear parabolic equation with two exponents on complete noncompact Riemannian maniflods. The special types of such equation include the Fisher-KPP equation, the parabolic Allen-Cahn equation and the Newell-Whitehead equation. We get the Souplet-Zhang's gradient estimates for the positive solutions to such equation. We also obtain the Liouville theorem for positive ancient solutions. Our results extend those of Souplet-Zhang (Bull. London. Math. Soc. 38 (2006), 1045–1053) and Zhu (Acta Math. Sci. Ser. B 36 (2016), no. 2, 514–526).

1. Introduction

Let M be a complete noncompact Riemannian manifold. In this paper, we consider the following nonlinear parabolic equation

$$\frac{\partial u}{\partial t} = \Delta u(x,t) + \lambda(x,t)u^p + \eta(x,t)u^q \tag{1.1}$$

on M, where the functions λ and η are C^1 in x and C^0 in t, p and q are positive constants with $p \geq 1$, $q \geq 1$. If $\lambda = -\eta = c$, $p = 1$ and $q = 2$, where c is a positive constant, then the equation (1.1) becomes

$$\frac{\partial u}{\partial t} = \Delta u + cu(1-u) \tag{1.2}$$

which is called the Fisher-KPP equation [6, 11]. It describes the propagation of an evolutionarily advantageous gene in a population and has many applications. Cao, Liu, Pendleton and Ward [4] derived some differential Harnack estimates for positive solutions to (1.2) on Riemannian manifolds. Geng and the author [7] extended the result of [4]. If $\lambda = 1$, $\eta = -1$, $p = 1$ and $q = 3$, then (1.1) becomes

$$\frac{\partial u}{\partial t} = \Delta u - (u^3 - u)$$

which is called the parabolic Allen-Cahn equation. A Harnack inequality for this equation was studied in [1]. The gradient estimates for the elliptic Allen-Cahn equation on

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Riemannian manifolds were obtained by the author in [9]. The special type of (1.1) also includes the Newell-Whitehead equation [16]

$$\frac{\partial u}{\partial t} = \Delta u + au - bu^3$$

where a and b are positive constants. It is used to model the change of concentration of a substance. The reader may refer to [2] for the recent results for such equation.

The gradient estimate is an important method in study on parabolic and elliptic equations. It was first proved by Yau [19] and Cheng-Yau [5], and was further developed by Li-Yau [13], Li [12], Hamilton [8], Negrin [15], Souplet and Zhang [17], Ma [14], Yang [18], etc. In [17], Souplelt and Zhang considered the heat equation

$$\frac{\partial u}{\partial t} = \Delta u \tag{1.3}$$

and proved the following result.

Theorem 1.1. *Let M be an n-dimensional Riemannian manifold with $n \geq 2$ and $\mathrm{Ricci}(M) \geq -k$, $k \geq 0$. If u is any positive solution to (1.3) in $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, +\infty)$ and $u \leq N$ in $Q_{R,T}$, then there holds*

$$\frac{|\nabla u(x,t)|}{u(x,t)} \leq c\left(\frac{1}{R} + \frac{1}{T^{1/2}} + \sqrt{k}\right)\left(1 + \log\frac{N}{u(x,t)}\right)$$

in $Q_{R/2,T/2}$, where $c = c(n)$.

Later, using the method of Souplet and Zhang, Zhu [20] studied the equation

$$\left(\Delta - \frac{\partial}{\partial t}\right)u(x,t) + h(x,t)u^p(x,t) = 0, \quad p > 1 \tag{1.4}$$

on compete noncomapct Riemannian manfolds, where the function $h(x,t)$ is assumed to be C^1 in the first variable and C^0 in the second variable. He proved the following result.

Theorem 1.2. *Let M be an n-dimensional Riemannian manifold with $n \geq 2$ and $\mathrm{Ricci}(M) \geq -k$, $k \geq 0$. If u is any positive solution to (1.4) in $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, +\infty)$ and $u \leq N$ in $Q_{R,T}$, then for any $\beta \in (0,2)$, there exists a constant $c = c(n, p, \beta)$ such that*

$$\frac{|\nabla u(x,t)|^2}{u(x,t)^\beta} \leq cN^{2-\beta}\left(\frac{1}{R^2} + \frac{1}{T} + k + N^{p-1}\|h^+\|_{L^\infty(Q_{R,T})} + N^{\frac{2}{3}(p-1)}\|\nabla h\|^{2/3}_{L^\infty(Q_{R,T})}\right) \tag{1.5}$$

in $Q_{R/2,T/2}$, where $h^+ = \max\{h, 0\}$.

The same method was also used by Huang and Ma [10] to obtain gradient estimates for the equations

$$\frac{\partial u}{\partial t} = \Delta u + \lambda u^{\alpha} \quad \text{and} \quad \frac{\partial u}{\partial t} = \Delta u + au\log u + bu$$

under the Ricci flow, where λ, α, a and b are constants.

In this paper, we get the following result.

Theorem 1.3. *Let M be an n-dimensional Riemannian manifold with $n \geq 2$ and* $\mathrm{Ricci}(M) \geq -k$, $k \geq 0$. *Suppose that $\lambda(x,t)$ and $\eta(x,t)$ are C^1 in x and C^0 in t, p and q are positive constants with $p \geq 1$, $q \geq 1$. If u is any positive solution to* (1.1) *in $Q_{R,T} \equiv B(x_0,R) \times [t_0 - T, t_0] \subset M \times (-\infty,+\infty)$ and $u \leq N$ in $Q_{R,T}$, then there exists a constant $c = c(n,p,q)$ such that*

$$\begin{aligned}\frac{|\nabla u(x,t)|}{u(x,t)} \leq c\bigg(&\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + N^{(p-1)/2}\|\lambda^+\|^{1/2}_{L^\infty(Q_{R,T})} + N^{(q-1)/2}\|\eta^+\|^{1/2}_{L^\infty(Q_{R,T})} \\ &+ N^{\frac{1}{3}(p-1)}\|\nabla\lambda\|^{1/3}_{L^\infty(Q_{R,T})} + N^{\frac{1}{3}(q-1)}\|\nabla\eta\|^{1/3}_{L^\infty(Q_{R,T})}\bigg)\left(1 + \log\frac{N}{u}\right)\end{aligned} \tag{1.6}$$

in $Q_{R/2,T/2}$, where $\lambda^+ = \max\{\lambda, 0\}$, $\eta^+ = \max\{\eta, 0\}$.

Note that the estimate (1.5) is equivalent to

$$\begin{aligned}&\frac{|\nabla u(x,t)|}{u(x,t)} \\ &\leq c\left(\frac{N}{u}\right)^{1-\beta/2}\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + N^{(p-1)/2}\|h^+\|^{1/2}_{L^\infty(Q_{R,T})} + N^{\frac{1}{3}(p-1)}\|\nabla h\|^{1/3}_{L^\infty(Q_{R,T})}\right).\end{aligned}$$

Applying Theorem 1.3 to (1.4) yields

$$\begin{aligned}&\frac{|\nabla u(x,t)|}{u(x,t)} \\ &\leq c\left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + N^{(p-1)/2}\|h^+\|^{1/2}_{L^\infty(Q_{R,T})} + N^{\frac{1}{3}(p-1)}\|\nabla h\|^{1/3}_{L^\infty(Q_{R,T})}\right)\left(1 + \log\frac{N}{u}\right).\end{aligned}$$

Since $\lim_{x\to+\infty} \frac{\log x}{x^{1-\beta/2}} = 0$, if N/u is large enough, then we have

$$1 + \log\frac{N}{u} \leq \left(\frac{N}{u}\right)^{1-\beta/2}.$$

So in this sense, the estimate (1.6) improves (1.5).

We also get the Liouville type theorem.

Theorem 1.4. *Let M be an n-dimensional Riemannian manifold with nonnegative Ricci curvature. Suppose that λ, η are nonpositive constants and one of them is negative, then* (1.1) *does not admit any positive ancient solution with $u(x,t) = e^{o(d(x)+\sqrt{|t|})}$ near infinity.*

The method of the proofs of main theorems comes from [10, 17, 20].

2. Proofs of main theorems

2.1. Proof of Theorem 1.3

Let $\widetilde{u} = u/N$. Then $\widetilde{u}$ satisfies

$$\frac{\partial \widetilde{u}}{\partial t} = \Delta \widetilde{u} + \widetilde{\lambda}\widetilde{u}^p + \widetilde{\eta}\widetilde{u}^q \tag{2.1}$$

where $\widetilde{\lambda} = \lambda N^{p-1}$, $\widetilde{\eta} = \eta N^{q-1}$. Noting $\widetilde{u} \le 1$, we let

$$f = \log \widetilde{u}, \quad \omega = |\nabla \ln(1-f)|^2. \tag{2.2}$$

In view of (2.1), we have

$$\Delta f + |\nabla f|^2 + \widetilde{\lambda} e^{(p-1)f} + \widetilde{\eta} e^{(q-1)f} - f_t = 0. \tag{2.3}$$

By (2.2) and (2.3), we have

$$\begin{aligned}
\omega_t &= \frac{2f_i(f_t)_i}{(1-f)^2} + \frac{2f_j^2 f_t}{(1-f)^3}\\
&= \frac{2f_i\big(f_{jji} + 2f_j f_{ji} + \widetilde{\lambda}_i e^{(p-1)f} + \widetilde{\lambda}(p-1)e^{(p-1)f} f_i + \widetilde{\eta}_i e^{(q-1)f} + \widetilde{\eta}(q-1)e^{(q-1)f} f_i\big)}{(1-f)^2}\\
&\quad + \frac{2f_j^2\big(f_{ii} + f_i^2 + \widetilde{\lambda} e^{(p-1)f} + \widetilde{\eta} e^{(q-1)f}\big)}{(1-f)^3}.
\end{aligned}$$

It follows from the similar calculation that

$$\begin{aligned}
\Delta\omega &= \frac{2f_{ij}^2 + 2f_j f_{jii}}{(1-f)^2} + \frac{8f_i f_{ij} f_j + 2f_j^2 f_{ii}}{(1-f)^3} + \frac{6f_i^2 f_j^2}{(1-f)^4}\\
&= \frac{2f_{ij}^2 + 2f_j f_{iij} + 2R_{ij} f_i f_j}{(1-f)^2} + \frac{8f_i f_{ij} f_j + 2f_j^2 f_{ii}}{(1-f)^3} + \frac{6f_i^2 f_j^2}{(1-f)^4}
\end{aligned}$$

where Bochner's identity is used. Noting that $R_{ij} f_i f_j \ge -k f_i^2$, we have

$$\begin{aligned}
\Delta\omega - \omega_t \ge\ & \frac{2f_{ij}^2 - 4f_i f_j f_{ij} - 2e^{(p-1)f} f_i \widetilde{\lambda}_i - 2\widetilde{\lambda}(p-1)e^{(p-1)f} f_i^2}{(1-f)^2}\\
& - \frac{2e^{(q-1)f} f_i \widetilde{\eta}_i + 2\widetilde{\eta}(q-1)e^{(q-1)f} f_i^2 + 2k f_i^2}{(1-f)^2}\\
& + \frac{8f_i f_{ij} f_j - 2f_j^2 f_i^2 - 2\widetilde{\lambda} e^{(p-1)f} f_j^2 - 2\widetilde{\eta} e^{(q-1)f} f_j^2}{(1-f)^3} + \frac{6f_i^2 f_j^2}{(1-f)^4}.
\end{aligned} \tag{2.4}$$

From (2.2), we deduce that

$$-\frac{2f}{1-f}\nabla f \nabla\omega = \frac{4f_i f_{ij} f_j}{(1-f)^2} + \frac{4f_i^2 f_j^2 - 4f_i f_{ij} f_j}{(1-f)^3} - \frac{4f_i^2 f_j^2}{(1-f)^4}. \tag{2.5}$$

Combining (2.4) and (2.5), we have

$$\Delta\omega - \omega_t - \frac{2f}{1-f}\nabla f\nabla\omega \geq \frac{2f_{ij}^2 - 2e^{(p-1)f}f_i\widetilde{\lambda}_i - 2\widetilde{\lambda}(p-1)e^{(p-1)f}f_i^2}{(1-f)^2} - \frac{2e^{(q-1)f}f_i\widetilde{\eta}_i + 2\widetilde{\eta}(q-1)e^{(q-1)f}f_i^2 + 2kf_i^2}{(1-f)^2} + \frac{4f_if_{ij}f_j + 2f_j^2f_i^2 - 2\widetilde{\lambda}e^{(p-1)f}f_j^2 - 2\widetilde{\eta}e^{(q-1)f}f_j^2}{(1-f)^3} + \frac{2f_i^2f_j^2}{(1-f)^4}.$$

Hölder's inequality implies that

$$\left|\frac{4f_if_{ij}f_j}{(1-f)^3}\right| \leq \frac{2f_{ij}^2}{(1-f)^2} + \frac{2f_i^2f_j^2}{(1-f)^4}.$$

Thus we have

$$\begin{aligned}
&\Delta\omega - \omega_t - \frac{2f}{1-f}\nabla f\nabla\omega \\
&\geq -\frac{2e^{(p-1)f}f_i\widetilde{\lambda}_i + 2e^{(q-1)f}f_i\widetilde{\eta}_i}{(1-f)^2} - \frac{2\widetilde{\lambda}(p-1)e^{(p-1)f}f_i^2 + 2\widetilde{\eta}(q-1)e^{(q-1)f}f_i^2 + 2kf_i^2}{(1-f)^2} \\
&\quad + \frac{2f_i^2f_j^2 - 2\widetilde{\lambda}e^{(p-1)f}f_j^2 - 2\widetilde{\eta}e^{(q-1)f}f_j^2}{(1-f)^3} \qquad (2.6) \\
&= 2(1-f)\omega^2 - 2\widetilde{\lambda}\left(p-1+\frac{1}{1-f}\right)e^{(p-1)f}\omega - 2\widetilde{\eta}\left(q-1+\frac{1}{1-f}\right)e^{(q-1)f}\omega \\
&\quad - \frac{2e^{(p-1)f}f_i\widetilde{\lambda}_i + 2e^{(q-1)f}f_i\widetilde{\eta}_i}{(1-f)^2} - 2k\omega.
\end{aligned}$$

Now we choose a smooth cut-off function $\psi = \psi(x,t)$ with compact support in $Q_{R,T}$ such that

(1) $\psi = \psi(r,t)$, $0 \leq \psi \leq 1$ with $\psi = 1$ in $Q_{R/2,T/2}$, where $r = d(x,x_0)$;

(2) ψ is decreasing with respect to r;

(3) for any $0 < \alpha < 1$, $|\partial_r\psi|/\psi^\alpha \leq C_\alpha/R$, $|\partial_r^2\psi|/\psi^\alpha \leq C_\alpha/R^2$;

(4) $|\partial_t\psi|/\psi^{1/2} \leq C/T$.

Using (2.6), we get

$$\begin{aligned}
&\Delta(\psi\omega) - 2\frac{\nabla\psi}{\psi}\cdot\nabla(\psi\omega) - (\psi\omega)_t \\
&\geq 2(1-f)\psi\omega^2 - 2\widetilde{\lambda}\left(p-1+\frac{1}{1-f}\right)e^{(p-1)f}\psi\omega - 2\widetilde{\eta}\left(q-1+\frac{1}{1-f}\right)e^{(q-1)f}\psi\omega \\
&\quad - 2k\psi\omega - \frac{2e^{(p-1)f}f_i\widetilde{\lambda}_i + 2e^{(q-1)f}f_i\widetilde{\eta}_i}{(1-f)^2}\psi + \frac{2f}{1-f}\nabla f\nabla(\psi\omega) - \frac{2f\omega}{1-f}\nabla f\nabla\psi
\end{aligned}$$

$$-\frac{2|\nabla\psi|^2}{\psi}\omega+(\Delta\psi)\omega-\psi_t\omega.$$

Suppose that $\psi\omega$ attains the maximum at (x_1, t_1). The argument in [3] implies that we can assume x_1 is not in the cut-locus of M. Then we have $\Delta(\psi\omega)\le 0$, $(\psi\omega)_t\ge 0$ and $\nabla(\psi\omega)=0$ at (x_1,t_1). It follows that

$$\begin{aligned}2(1-f)\psi\omega^2 \le\; & 2\widetilde{\lambda}\left(p-1+\frac{1}{1-f}\right)e^{(p-1)f}\psi\omega+2\widetilde{\eta}\left(q-1+\frac{1}{1-f}\right)e^{(q-1)f}\psi\omega\\ &+2k\psi\omega+\frac{2e^{(p-1)f}f_i\widetilde{\lambda}_i+2e^{(q-1)f}f_i\widetilde{\eta}_i}{(1-f)^2}\psi\\ &+\frac{2f\omega}{1-f}\nabla f\nabla\psi+\frac{2|\nabla\psi|^2}{\psi}\omega-(\Delta\psi)\omega+\psi_t\omega.\end{aligned}\tag{2.7}$$

In view of $p\ge 1$, $q\ge 1$ and $f\le 0$, we have

$$\begin{aligned}&2\widetilde{\lambda}\left(p-1+\frac{1}{1-f}\right)e^{(p-1)f}\psi\omega+2\widetilde{\eta}\left(q-1+\frac{1}{1-f}\right)e^{(q-1)f}\psi\omega\\ &\le 2\widetilde{\lambda}^+p\psi\omega+2\widetilde{\eta}^+q\psi\omega\\ &\le \frac{1}{16}\psi\omega^2+16\psi(\widetilde{\lambda}^+p)^2+\frac{1}{16}\psi\omega^2+16\psi(\widetilde{\eta}^+q)^2\\ &\le \frac{1}{8}\psi\omega^2+16(\widetilde{\lambda}^+p)^2+16(\widetilde{\eta}^+q)^2\end{aligned}\tag{2.8}$$

where $\widetilde{\lambda}^+=\max\{\widetilde{\lambda},0\}$, $\widetilde{\eta}^+=\max\{\widetilde{\eta},0\}$. Straightforward calculations show

$$\begin{aligned}&\frac{2e^{(p-1)f}f_i\widetilde{\lambda}_i+2e^{(q-1)f}f_i\widetilde{\eta}_i}{(1-f)^2}\psi\\ &\le \frac{f_i^4}{2(1-f)^4}\psi+\frac{3|\nabla\widetilde{\lambda}|^{4/3}}{2(1-f)^{4/3}}\psi+\frac{f_i^4}{2(1-f)^4}\psi+\frac{3|\nabla\widetilde{\eta}|^{4/3}}{2(1-f)^{4/3}}\psi\\ &\le \frac{f_i^4}{(1-f)^4}\psi+\frac{3}{2}\big(|\nabla\widetilde{\lambda}|^{4/3}+|\nabla\widetilde{\eta}|^{4/3}\big)\\ &\le (1-f)\psi\omega^2+\frac{3}{2}\big(|\nabla\widetilde{\lambda}|^{4/3}+|\nabla\widetilde{\eta}|^{4/3}\big),\end{aligned}\tag{2.9}$$

$$\begin{aligned}\left|\frac{2f\omega}{1-f}\nabla f\nabla\psi\right| &\le 2\omega^{3/2}|f||\nabla\psi|=2[\psi(1-f)\omega^2]^{3/4}\frac{|f||\nabla\psi|}{[\psi(1-f)]^{3/4}}\\ &\le \frac{1}{8}(1-f)\psi\omega^2+c\frac{(f|\nabla\psi|)^4}{[\psi(1-f)]^3}\\ &\le \frac{1}{8}(1-f)\psi\omega^2+c\frac{f^4}{R^4(1-f)^3},\end{aligned}\tag{2.10}$$

$$2k\psi\omega\le\frac{1}{8}(1-f)\psi\omega^2+ck^2.\tag{2.11}$$

By the estimates of Souplet and Zhang [17], we have

$$\frac{|\nabla\psi|^2}{\psi}\omega \le \frac{1}{8}\psi\omega^2 + c\frac{1}{R^4} \le \frac{1}{8}(1-f)\psi\omega^2 + c\frac{1}{R^4}, \tag{2.12}$$

$$-(\Delta\psi)\omega \le \frac{1}{8}\psi\omega^2 + c\frac{1}{R^4} + ck\frac{1}{R^2} \le \frac{1}{8}(1-f)\psi\omega^2 + c\frac{1}{R^4} + ck\frac{1}{R^2}, \tag{2.13}$$

$$|\psi_t|\omega \le \frac{1}{8}\psi\omega^2 + c\frac{1}{T^2} \le \frac{1}{8}(1-f)\psi\omega^2 + c\frac{1}{T^2}. \tag{2.14}$$

Combining (2.7)–(2.14), we obtain

$$\begin{aligned}\frac{1}{8}(1-f)\psi\omega^2 \le 16(\widetilde{\lambda}^+p)^2 + 16(\widetilde{\eta}^+q)^2 + \frac{3}{2}\left(|\nabla\widetilde{\lambda}|^{4/3} + |\nabla\widetilde{\eta}|^{4/3}\right)\\ + c\frac{f^4}{R^4(1-f)^3} + ck^2 + c\frac{1}{R^4} + ck\frac{1}{R^2} + c\frac{1}{T^2}.\end{aligned}$$

Hence

$$\begin{aligned}\psi\omega^2(x_1,t_1) \le cN^{2p-2}\|\lambda^+\|^2_{L^\infty(Q_{R,T})} + cN^{2q-2}\|\eta^+\|^2_{L^\infty(Q_{R,T})} + cN^{\frac{4}{3}(p-1)}\|\nabla\lambda\|^{4/3}_{L^\infty(Q_{R,T})}\\ + cN^{\frac{4}{3}(q-1)}\|\nabla\eta\|^{4/3}_{L^\infty(Q_{R,T})} + c\frac{f^4}{R^4(1-f)^4} + ck^2 + c\frac{1}{R^4} + c\frac{1}{T^2}.\end{aligned}$$

By above estimate, there holds for all (x,t) in $Q_{R,T}$,

$$\begin{aligned}\psi^2\omega^2(x,t) \le cN^{2p-2}\|\lambda^+\|^2_{L^\infty(Q_{R,T})} + cN^{2q-2}\|\eta^+\|^2_{L^\infty(Q_{R,T})} + cN^{\frac{4}{3}(p-1)}\|\nabla\lambda\|^{4/3}_{L^\infty(Q_{R,T})}\\ + cN^{\frac{4}{3}(q-1)}\|\nabla\eta\|^{4/3}_{L^\infty(Q_{R,T})} + c\frac{1}{R^4} + c\frac{1}{T^2} + ck^2.\end{aligned}$$

Noting that $\psi(x,t) = 1$ in $Q_{R/2,T/2}$, we get

$$\begin{aligned}\frac{|\nabla f(x,t)|}{1-f(x,t)} \le \frac{c}{R} + \frac{c}{\sqrt{T}} + c\sqrt{k} + cN^{(p-1)/2}\|\lambda^+\|^{1/2}_{L^\infty(Q_{R,T})} + cN^{(q-1)/2}\|\eta^+\|^{1/2}_{L^\infty(Q_{R,T})}\\ + cN^{\frac{1}{3}(p-1)}\|\nabla\lambda\|^{1/3}_{L^\infty(Q_{R,T})} + cN^{\frac{1}{3}(q-1)}\|\nabla\eta\|^{1/3}_{L^\infty(Q_{R,T})}.\end{aligned}$$

Finally we have

$$\begin{aligned}\frac{|\nabla u(x,t)|}{u(x,t)} \le c\Bigg(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} + N^{(p-1)/2}\|\lambda^+\|^{1/2}_{L^\infty(Q_{R,T})} + N^{(q-1)/2}\|\eta^+\|^{1/2}_{L^\infty(Q_{R,T})}\\ + N^{\frac{1}{3}(p-1)}\|\nabla\lambda\|^{1/3}_{L^\infty(Q_{R,T})} + N^{\frac{1}{3}(q-1)}\|\nabla\eta\|^{1/3}_{L^\infty(Q_{R,T})}\Bigg)\left(1+\log\frac{N}{u}\right).\end{aligned}$$

2.2. Proof of Theorem 1.4

We prove it by contradiction. Suppose that u is a positive solution to (1.1). Noting that λ and η are nonpositive constants, it follows from Theorem 1.3 that

$$\frac{|\nabla u(x,t)|}{u(x,t)} \le c\left(\frac{1}{R} + \frac{1}{\sqrt{T}}\right)\left(1+\log\frac{N}{u}\right). \tag{2.15}$$

By the same argument as in the proof of Theorem 1.2 in [17] and Theorem 1.8 in [20], fixing (x_0, t_0) and applying (2.15) to u on $B(x_0, R) \times [t_0 - R^2, t_0]$, we get

$$\frac{|\nabla u(x_0, t_0)|}{u(x_0, t_0)} \leq \frac{C}{R}[1 + o(R)].$$

It follows that $|\nabla u(x_0, t_0)| = 0$ by letting $R \to \infty$. Noting (x_0, t_0) is arbitrary, we have $u(x, t) = u(t)$. Then by (1.1), we get $\frac{du}{dt} = \lambda u^p + \eta u^q$. Without loss of generality, we assume that $\lambda < 0$.

If $p > 1$, integrating $\frac{du}{dt}$ on $[t, 0]$ with $t < 0$ implies that

$$\frac{1}{1-p}(u^{1-p}(0) - u^{1-p}(t)) \leq -\lambda t.$$

Then

$$u^{p-1}(t) \leq u^{p-1}(0) + (1-p)\lambda t.$$

This yields that if t is large enough, $u^{p-1}(t) < 0$ which contradicts that u is positive.

If $p = 1$, we get for $t < 0$

$$\log u(0) - \log u(t) \leq -\lambda t.$$

Hence $u(t) \geq u(0)e^{\lambda t}$, which contradicts $u(x, t) = e^{o(d(x)+\sqrt{|t|})}$ near infinity. We finish the proof.

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