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Stratifying Lie Strata of Hilbert Modular Varieties

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Abstract. In this survey we explain a stratification of a Hilbert modular variety $\mathscr{M}_E$ in characteristic $p > 0$ attached to a totally real number field E. This stratification refines the stratification of $\mathscr{M}_E$ by Lie type, and has the property that many strata are central leaves in $\mathscr{M}_E$, called *distinguished central leaves*.

In the case when the totally real field E is unramified above p, this stratification reduces to the stratification of $\mathscr{M}_E$ by α-type first introduced by Goren and Oort and studied by Yu, and coincides with the EO stratification on $\mathscr{M}_E$. Moreover it is known that every non-supersingular α-stratum of $\mathscr{M}_E$ is irreducible. To treat the general case where E may be ramified above p, a key ingredient is the notion of *congruity*, a p-adic numerical invariant for abelian varieties with real multiplication by $\mathcal{O}_E$ in characteristic p. For every Lie stratum $\mathcal{N}_{\underline{e}}$ on $\mathscr{M}_E$, this new invariant defines a finite number of locally closed subsets $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$, and $\mathcal{N}_{\underline{e}}$ is the disjoint union of these *Lie-congruity strata* $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ in $\mathcal{N}_{\underline{e}}$.

The incidence relation between the Lie-congruity strata enables one to show that the prime-to-p Hecke correspondences operate transitively on the set of all irreducible components of any distinguished central leaf in $\mathscr{M}_E$, see Theorems 7.1, 8.1 and 9.1. The Hecke transitivity implies, according to the method of prime-to-p monodromy of Hecke invariant subvarieties, that every non-supersingular *distinguished* central leaf in a Hilbert modular variety $\mathscr{M}_E$ is irreducible. The last irreducibility result is a key ingredient of the proof the Hecke orbit conjecture for Siegel modular varieties.

1. Introduction

Moduli spaces in characteristic $p > 0$ have natural stratifications coming from p-adic invariants of the geometric objects they classify. For instance, the moduli space $\mathscr{A}_{g,1,n}$ over $\overline{\mathbb{F}_p}$ which classifies g-dimensional principally polarized abelian varieties in characteristic p with symplectic level-n structure has several stratifications; e.g., the stratifications by the Newton polygon (respectively the p-rank, respectively the a-number) of an abelian variety. These stratifications are helpful in understanding the geometry and arithmetic of moduli spaces.

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This article is a survey of the *congruity stratification* of Lie strata in a Hilbert modular variety $\mathscr{M}_E$ over $\overline{\mathbb{F}_p}$ attached to a totally real number field E. We give complete definitions and statements of results but no proofs, so "summary of results" is perhaps more befitting than "survey". Detailed proofs can be found in [9, Chapter 4]. We have included a long introduction, together with preliminary materials in Sections 2–4, to make this article more accessible. The actual summary itself in Sections 5–9, which inevitably is quite technical, occupies about a third of the total number of pages. The last Section 10, consists of a list of related questions.

Throughout this paper p is a fixed prime number, E is a totally real number field.

1.1. Motivation: the Hecke orbit conjecture for Siegel modular varieties

This paper was motivated by the Hecke orbit problem for the Siegel modular variety $\mathscr{A}_{g,1,n}$ over $\overline{\mathbb{F}_p}$, with $n \geq 3$ and $\gcd(p,n) = 1$. For every $\overline{\mathbb{F}_p}$-point x of $\mathscr{A}_{g,1,n}$ which corresponds to a principally polarized abelian variety with symplectic level-n structure (A_x, λ_x, η_x), the *central leaf* $\mathcal{C}_{\mathscr{A}_{g,1,n}}(x)$ is the locally closed smooth subscheme of $\mathscr{A}_{g,1,n}$ over $\overline{\mathbb{F}_p}$, characterized by

$$\mathcal{C}_{\mathscr{A}_{g,1,n}}(x)(\overline{\mathbb{F}_p}) = \big\{ y \in \mathscr{A}_{g,1,n}(\overline{\mathbb{F}_p}) \mid (A_y, \lambda_y)[p^\infty] \cong (A_x, \lambda_x)[p^\infty] \big\},$$

see [30]. In other words for every geometric point $y \in \mathscr{A}_{g,1,n}(\overline{\mathbb{F}_p})$, y lies in the central leaf $\mathcal{C}_{\mathscr{A}_{g,1,n}}(x)$ if and only if the principally polarized p-divisible group $(A_y, \lambda_y)[p^\infty]$ attached to y is isomorphic to $(A_x, \lambda_x)[p^\infty]$.

On the other hand, the prime-to-p Hecke orbit of x is the countable subset $\mathcal{H}_{\mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})}(x)$ of $\mathscr{A}_{g,1,n}(\overline{\mathbb{F}_p})$, consisting of all points $[(A_z, \lambda_z, \eta_z)] \in \mathscr{A}_{g,1,n}(\overline{\mathbb{F}_p})$ such that there exists a prime-to-p quasi-isogeny $\alpha\colon A_z \dashrightarrow A_x$ such that $\alpha^*(\lambda_x) = \lambda_z$ and $\alpha^*(\eta_z) = \eta_x$.

The Hecke orbit conjecture for $\mathscr{A}_{g,1,n}$ predicts that

the prime-to-p Hecke orbit $\mathcal{H}_{\mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})}(x)$ *is Zariski dense in the central leaf* $\mathcal{C}_{\mathscr{A}_{g,1,n}}(x)$.

A complete proof will appear in [9, Chapter 8]; see also [5], [9, 8.1], [11] for outlines of the proof. Below is a brief description of how Hilbert modular varieties enter the proof.

A starting point of the proof of the Hecke orbit conjecture for Siegel modular varieties is Tate's theorem that every abelian variety over a finite field admits sufficiently many complex multiplications. This implies the existence of a Hecke-equivariant correspondence from a product of Hilbert modular varieties to the Siegel modular variety, called the "Hilbert trick".

> *For every* $x \in \mathscr{A}_{g,1,n}(\overline{\mathbb{F}_p})$, *there exists a product of totally real number fields* $E_1, \ldots, E_r$ *with* $\sum_{i=1}^r [E_i : \mathbb{Q}] = g$, *central leaves* $\mathcal{C}_{\mathscr{M}_{E_i}}(y_i)$ *in Hilbert mod-*

ular varieties $\mathscr{M}_{E_i}$, $i = 1, \dots, r$, and a finite-to-one Hecke-equivariant correspondence $f\colon \mathcal{C}_{\mathscr{M}_{E_1}}(y_1) \times \cdots \times \mathcal{C}_{\mathscr{M}_{E_r}}(y_r) \dashrightarrow \mathscr{A}_{g,1,n}$, such that the image under f of an irreducible component of $\mathcal{C}_{\mathscr{M}_{E_1}}(y_1) \times \cdots \times \mathcal{C}_{\mathscr{M}_{E_r}}(y_r)$ is in finite isogeny correspondence with a locally closed subvariety of the Zariski closure of $\mathcal{H}_{\mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})}(x)$ in $\mathcal{C}_{\mathscr{A}_{g,1,n}}(x)$.

In the above statement we have used the fact that the Zariski closure in $\mathcal{C}_{\mathscr{M}_{E_i}}(y_i)$ of the prime-to-p Hecke orbit of y_i, a countable subset of $\mathcal{C}_{\mathscr{M}_{E_i}}(y_i)$ is both open and closed in the central leaf $\mathcal{C}_{\mathscr{M}_{E_i}}(y_i)$. See Subsections 1.2 and 1.3 for the definition of Hilbert modular varieties $\mathscr{M}_{E_i}$ and central leaves in them.

With the help of the action of the local stabilizer subgroup at a supersingular point of the Zariski closure of the Hecke orbit $\mathcal{H}_{\mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})}(x)$ in $\mathscr{A}_{g,1,n}$, we may further assume that there is only one prime above p in $\mathcal{O}_{E_i}$ for each $i = 1, \dots, r$.

Changing the y_i's by E_i-isogeny, we may also assume that each of the central leaves $\mathcal{C}_{\mathscr{M}_{E_i}}(y_i)$ in $\mathscr{M}_{E_i}$ is *distinguished*; see Subsection 1.7. The critical information which comes from the congruity stratification of the Lie strata is Corollary 9.4:

Every non-supersingular distinguished central leaf in $\mathscr{M}_{E_i}$ is irreducible.

It is known that every central leaf in $\mathscr{M}_{E_i}$ contains a *robust hypersymmetric point* which corresponds to an abelian variety B_i over $\overline{\mathbb{F}}_p$ with at most two slopes, such that the natural map $\mathrm{End}(B_i) \otimes_{\mathbb{Z}} \mathbb{Q}_p \to \mathrm{End}(B_i[p^\infty]) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an isomorphism. So we conclude that the Zariski closure in $\mathcal{C}_{\mathscr{A}_{g,1,n}}(x)$ of the prime-to-p Hecke orbit $\mathcal{H}_{\mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})}(x)$ contains a robust hypersymmetric point; cf. [9, Chapter 8] and [10].

With a hypersymmetric point at hand, we are in a position to apply the local rigidity of p-divisible groups [7], and the density of $\mathcal{H}_{\mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})}(x)$ in $\mathcal{C}_{\mathscr{A}_{g,1,n}}(x)$ follows.

To recapitulate: the irreducibility of non-supersingular distinguished central leaves in a Hilbert modular variety $\mathscr{M}_E$ implies the existence of a robust hypersymmetric point in the Zariski closure of a Hecke orbit $\mathcal{H}_{\mathrm{Sp}_{2g}(\mathbb{A}_f^{(p)})}(x)$ in the central leaf $\mathcal{C}_{\mathscr{A}_{g,1,n}}(x)$. This irreducibility result is non-trivial—we don't know of any proof by "pure thought".

In Subsections 1.2–1.8 below we give a very brief introduction to Lie strata and Lie-congruity strata on Hilbert modular varieties, as a tour map for Sections 2–9.

1.2. Hilbert modular varieties

Given an element $[\widetilde{\mathscr{L}}]$ of the strict class group $\widetilde{\mathrm{Cl}}_E$ of E, a positive integer $n \geq 3$ prime to p, and a generator $\overline{\delta}$ of the free rank-one $(\mathcal{O}_E/n\mathcal{O}_E)$-module

$$\big(\mathscr{L}^\vee \otimes_{\mathcal{O}_E} \mathcal{D}_{E/\mathbb{Q}}^{-1}/n\mathscr{L}^\vee \otimes_{\mathcal{O}_E} \mathcal{D}_{E/\mathbb{Q}}^{-1}\big) \otimes_{(\mathbb{Z}/n\mathbb{Z})} \mathbb{\mu}_n(\overline{\mathbb{F}}_p),$$

we have an associated Hilbert modular variety $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$ over $\overline{\mathbb{F}_p}$, which classifies principally $\widetilde{\mathscr{L}}$-polarized abelian schemes of relative dimension g over $\overline{\mathbb{F}_p}$-schemes, with real multiplication by $\mathcal{O}_E$ with symplectic level-n structures. See Subsection 2.1, Definitions 2.2, 2.3 and 2.4 for the precise definitions.

The Hilbert modular variety $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$ is a reduced irreducible normal g-dimensional scheme over $\overline{\mathbb{F}_p}$, and is a local complete intersection. It is smooth over $\overline{\mathbb{F}_p}$ if and only if E is unramified above p. When E is ramified above p, the locus in $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$ consisting of singular points is a closed subset of $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$ all of whose irreducible components have codimension 2 in $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$.

Since geometric properties of the modular variety $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$ are all independent of the parameters $(\widetilde{\mathscr{L}}, n, \overline{\delta})$, we will shorten $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$ to $\mathscr{M}_E$ if no confusion is likely.

Remark. Strictly speaking the Hilbert modular variety $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$ are not a special cases of PEL type modular varieties as defined in [23] unless the element $[\widetilde{\mathscr{L}}]$ is 0 in the strict class group $\widetilde{\mathrm{Cl}}_E$. Still, the modular varieties $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$ should be regarded as analogs of the Siegel moduli space $\mathscr{A}_{g,1,n}$ of g-dimensional *principally polarized* abelian varieties with level-n structures; cf. Remark 2.2.3.

We will *not* discuss those moduli spaces which classify g-dimensional $\mathcal{O}_E$-linear abelian varieties in characteristic p together with $\mathcal{O}_E$-linear polarizations of some fixed degree which is divisible by p.

1.3. Central leaves in $\mathscr{M}_E$

For an $\overline{\mathbb{F}_p}$-point x of a Hilbert modular variety $\mathscr{M}_E$ which corresponds to an abelian variety with real multiplication by $\mathcal{O}_E$ with level-n-structure $(A_x, \lambda_x, \iota_x, \eta_x)$, the central leaf in $\mathscr{M}_E$ passing through x is, by definition, the locally closed smooth subvariety of $\mathscr{M}_E$ over $\overline{\mathbb{F}_p}$ such that

$$\mathcal{C}_{\mathscr{M}_E}(x)(\overline{\mathbb{F}_p}) = \big\{ y \in \mathscr{M}_E(\overline{\mathbb{F}_p}) \mid (A_y, \lambda_y, \iota_y)[p^\infty] \cong (A_x, \lambda_x, \iota_x)[p^\infty] \big\}.$$

Certain central leaves in Hilbert modular varieties $\mathscr{M}_E$, called *distinguished central leaves*, are irreducible. Every distinguished central leaf in $\mathscr{M}_E$ is a Lie-congruity stratum of $\mathscr{M}_E$. See Subsection 1.6 for a short description of Lie-congruity strata, and Definitions 6.2.1 and 6.3.5 for the definition of distinguished central leaves. Every central leaf in $\mathscr{M}_E$ is in a finite isogeny correspondence with a distinguished central leaf. As already mentioned, the proof of the Hecke orbit conjecture in [9, Chapter 8] depends heavily on the fact that every non-supersingular distinguished central leaf in $\mathscr{M}_E$ is irreducible.

1.4. The Lie stratification of $\mathscr{M}_E$

Deligne and Pappas [13] introduced a stratification of $\mathscr{M}_E$ by the isomorphism classes of $\mathrm{Lie}(A_x)$ as vector spaces with actions by $\mathcal{O}_E/p\mathcal{O}_E$, where x runs through all geometric points of $\mathscr{M}_E$, and $\mathrm{Lie}(A_x)$ is the Lie algebra of the fiber at x of the universal abelian scheme with real multiplication by $\mathcal{O}_E$ over $\mathscr{M}_E$. Under this stratification by *Lie type*, or the *Lie stratification* of $\mathscr{M}_E$, the set underlying $\mathscr{M}_E$ is the disjoint union of *Lie strata*: $\mathscr{M}_E = \bigsqcup_{\underline{e}\in\mathcal{T}_E^{\mathrm{sd}}} \mathcal{N}_{\underline{e}}$. Here $\mathcal{N}_{\underline{e}}$ is the locally closed subset of $\mathscr{M}_E$ such that a point x of $\mathscr{M}_E$ is in $\mathcal{N}_{\underline{e}}$ if and only if the isomorphism class of the $(\mathcal{O}_E/p\mathcal{O}_E)$-module $\mathrm{Lie}(A_x)$ corresponds to the element $\underline{e}$ of the indexing set $\mathcal{T}_E^{\mathrm{sd}}$. Moreover each Lie stratum $\mathcal{N}_{\underline{e}}$ with the reduced scheme structure is a *smooth* locally closed subscheme of $\mathscr{M}_E$, and the Zariski closure of $\mathcal{N}_{\underline{e}}$ is a local complete intersection and is normal.

The indexing set $\mathcal{T}_E^{\mathrm{sd}}$ for the Lie stratification of $\mathscr{M}_E$ has a natural structure as a finite poset (partially ordered set). The poset $\mathcal{T}_E^{\mathrm{sd}}$ is *ranked*, in the sense that all maximal chains between any two elements of $\mathcal{T}_E^{\mathrm{sd}}$ have the same length. Moreover it has a unique maximal element and also a unique minimal element. This partial ordering on $\mathcal{T}_E^{\mathrm{sd}}$ is compatible with the incidence relation of Lie strata, in the following sense: for any two elements $\underline{e}_1, \underline{e}_2 \in \mathcal{T}_E^{\mathrm{sd}}$, the Lie stratum $\mathcal{N}_{\underline{e}_1}$ is contained in the Zariski closure of $\mathcal{N}_{\underline{e}_2}$ if and only if $\underline{e}_1 \preceq \underline{e}_2$. The Lie stratum corresponding to the maximal element in $\mathcal{T}_E^{\mathrm{sd}}$ is the smooth locus $\mathscr{M}_E^{\mathrm{sm}}$, consisting of all smooth points of $\mathscr{M}_E$. The dimension of every irreducible component of the minimal Lie stratum $\mathcal{N}_{\min}$, indexed by the minimal element $\underline{e}_{E,\min}^{\mathrm{sd}}$ of $\mathcal{T}_E^{\mathrm{sd}}$, is $\sum_{e_v \text{ odd}} f_v$. The sum here runs through all places v of E above p whose ramification index e_v is odd, and $f_v := [\kappa_v : \mathbb{F}_p]$, the degree of the residue field κ_v of $\mathcal{O}_{E_v}$.

1.5. The α-stratification of $\mathscr{M}_E$ when E is unramified above p

In this subsection we assume that the totally real field E is unramified above p. In this case the poset $\mathcal{T}_E^{\mathrm{sd}}$ is a singleton, the Hilbert modular variety $\mathscr{M}_E$ over $\overline{\mathbb{F}_p}$ is smooth, and the Lie stratification of $\mathscr{M}_E$ is trivial.

(i) The stratification of the Hilbert modular variety $\mathscr{M}_E$ by the isomorphism classes of the maximal α-subgroup scheme of $A_x[p]$ with action by $\mathcal{O}_E/p\mathcal{O}_E$ attached to geometric points $[(A_x, \lambda_x, \iota_x, \eta_x)]$ of $\mathscr{M}_E$ was studied in [14], called the α-stratification.

(ii) One of the main results of [14] says that if $k \supseteq \mathbb{F}_p$ is an algebraically closed field and x, y are k-points of $\mathscr{M}_E$ such that the maximal $(\mathcal{O}_E/p\mathcal{O}_E)$-linear α-subgroup schemes of $A_x[p]$ and $A_y[p]$ are isomorphic, then there exists an $(\mathcal{O}_E/p\mathcal{O}_E)$-linear isomorphism from $A_x[p]$ to $A_y[p]$ which respects the principal polarizations. In other words, the stratification of $\mathscr{M}_E$ by α-types is a natural generalization of the EO stratification of Siegel modular varieties $\mathscr{A}_{g,1,n}$ defined in [29].

(iii) The α-stratification of $\mathscr{M}_E$ has a unique dense open stratum, which coincides with the ordinary locus $\mathscr{M}_E^{\mathrm{ord}}$ in $\mathscr{M}_E$, i.e., the open subset $\mathscr{M}_E^{\mathrm{ord}}$ of $\mathscr{M}_E$ whose points correspond to ordinary abelian varieties. The complement of $\mathscr{M}_E^{\mathrm{ord}}$ in $\mathscr{M}_E$ is a smooth divisor D with normal crossings, and the α-stratification coincides with the stratification of $\mathscr{M}_E$ associated to the divisor D. Moreover, some of the α-strata are actually central leaves in $\mathscr{M}_E$.

The congruity stratification of Lie strata of Hilbert modular varieties $\mathscr{M}_E$ for possibly ramified totally real fields E, described in Subsections 1.6–1.8 below, extends the α-stratification for Hilbert modular varieties attached to totally real fields unramified above p considered in this subsection: if E is unramified above p, then $\mathscr{M}_E$ has a single Lie stratum, and the congruity stratification of this unique Lie stratum coincides with the α-stratification.

1.6. The congruity stratification of Lie strata of $\mathscr{M}_E$

A numerical invariant $\underline{c}(A_x)$ with values in a finite poset $\boldsymbol{\tau}_E$, for g-dimensional principally $\widetilde{\mathscr{L}}$-polarized abelian varieties $(A_x, \lambda_x, \iota_x)$ with real multiplication by $\mathcal{O}_E$ attached to points x of $\mathscr{M}_E$, was defined in [40, 2.4]. This invariant, called *congruity* in [9, Chapter 4], is a refinement of the α-type of A_x, the invariant which defines the α-stratification. The invariant $\underline{c}(A_x)$ gives rise to a stratification on every Lie stratum $\mathcal{N}_{\underline{e}}$ of $\mathscr{M}_E$: we have $\mathcal{N}_{\underline{e}} = \bigsqcup_{\underline{c}} \mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$, where $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ is the locally closed subset of $\mathcal{N}_{\underline{e}}$, consisting of points of $\mathcal{N}_{\underline{e}}$ whose congruity is $\underline{c}$. Below are some properties of the stratification of $\mathcal{N}_{\underline{e}}$ by congruity, for every $\underline{e} \in \mathcal{T}_E^{\mathrm{sd}}$.

(i) The finite poset $\boldsymbol{\tau}_E$ is ranked, and has a unique maximal element, denoted by $\underline{0}$.

(ii) Each stratum $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ is a *smooth* locally closed subscheme of $\mathcal{N}_{\underline{e}}$.

(iii) The Zariski closure in $\mathcal{N}_{\underline{e}}$ of $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$, is equal to $\mathcal{Q}_{\preceq \underline{c}}(\mathcal{N}_{\underline{e}}) := \bigsqcup_{\underline{c}' \preceq \underline{c}} \mathcal{Q}_{\underline{c}'}(\mathcal{N}_{\underline{e}})$, which is smooth over $\overline{\mathbb{F}}_p$, for every $\underline{c} \in \boldsymbol{\tau}_E$.

(iv) The stratum $\mathcal{Q}_{\underline{0}}(\mathcal{N}_{\underline{e}})$ is a dense open subscheme of $\mathcal{N}_{\underline{e}}$.

(v) The complement of $\mathcal{Q}_{\underline{0}}(\mathcal{N}_{\underline{e}})$ in $\mathcal{N}_{\underline{e}}$, if non-empty, is a divisor in $\mathcal{N}_{\underline{e}}$ with normal crossings.

We call a subset of $\mathscr{M}_E$ of the form $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ a *Lie-congruity stratum* of $\mathscr{M}_E$.

A warning is in order, that the incidence relation between Lie-congruity strata is not well understood yet. For instance we don't know whether the Zariski closure $\big(\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})\big)^{\mathrm{zar}}$ in $\mathscr{M}_E$ of $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ is a union of Lie-congruity strata, for every Lie-congruity stratum $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$; see Question 10.1.

The Lie type and the congruity of an $\widetilde{\mathscr{L}}$-polarized abelian variety $(A_x, \lambda_x, \iota_x)$ in characteristic p depend only on the polarized $\mathcal{O}_E$-linear p-divisible group $(A_x[p^\infty], \lambda_x[p^\infty], \iota_x[p^\infty])$ associated to $(A_x, \lambda_x, \iota_x)$. Since the isomorphism class of $(A_x[p^\infty], \lambda_x[p^\infty], \iota_x[p^\infty])$ does not change under prime-to-p Hecke correspondences associated to $\mathrm{SL}_2(\mathbb{A}_f^{(p)})$, every Lie-congruity stratum $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ is stable under all prime-to-p Hecke correspondences on $\mathscr{M}_E$. More information on prime-to-p Hecke symmetries on Siegel and Hilbert modular varieties in characteristic p can be found in [4, 42].

1.7. Distinguished Lie-congruity strata

We will define a subset of the set of all Lie-congruity strata of $\mathscr{M}_E$, called *distinguished* Lie-congruity strata; see Definition 5.6. They have two useful properties:

(a) For any algebraically closed field $k \supseteq \mathbb{F}_p$ and any two k-points x_1, x_2 on a *distinguished* Lie-congruity stratum, there exists an $\mathcal{O}_E$-linear isomorphism from $A_{x_1}[p^\infty]$ to $A_{x_2}[p^\infty]$ which respects the principal $\mathcal{O}_E$-linear polarizations induced by λ_{x_1} and λ_{x_2}.

(b) For any algebraically closed field $k \supseteq \mathbb{F}_p$ and any k-point $x \in \mathscr{M}_E(k)$, there exists a k-point y in a *distinguished* Lie-congruity stratum, and an E-linear quasi-isogeny from $A_y[p^\infty]$ to $A_x[p^\infty]$ which respects the principal $\mathcal{O}_E$-linear polarizations.

Property (a) says that each distinguished Lie-congruity stratum is a *central leaf* of the Hilbert modular variety $\mathscr{M}_E$, consisting of all points in $\mathscr{M}_E$ with a fixed geometric isomorphism class of principally polarized $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$-linear p-divisible groups; these central leaves are said to be *distinguished*. Property (b) implies that every central leaf on $\mathscr{M}_E$ is in isogeny correspondence with a distinguished central leaf.

1.8. Specialization from distinguished central leaves to E-minimal points

A key result on the geometry of distinguished central leaves on a Hilbert modular variety $\mathscr{M}_E$ is Theorem 9.1, which asserts that the Zariski closure of every irreducible component of a distinguished Lie-congruity stratum on $\mathscr{M}_E$ contains an *E-minimal point*, i.e., an $\overline{\mathbb{F}_p}$-point $x_0 \in \mathcal{N}_{\min}(\overline{\mathbb{F}_p})$ whose underlying abelian variety A_{x_0} is *superspecial*. In other words A_{x_0} is isomorphic to the product of $[E : \mathbb{Q}]$ supersingular elliptic curves over $\overline{\mathbb{F}_p}$.

Theorem 9.1 follows from the combination of Theorems 7.1 and 8.1. Theorem 7.1 says that the Zariski closure of every irreducible component of a *distinguished* Lie-congruity stratum which is *not* contained in the minimal Lie stratum $\mathcal{N}_{\min}$ contains an irreducible component of a distinguished congruity stratum in $\mathcal{N}_{\min}$. Its proof is intricate; see Subsection 7.4 for an impressionistic sketch. Theorem 8.1 says that the Zariski closure of every

irreducible component of a congruity stratum in the minimal Lie stratum $\mathcal{N}_{\min}$ contains an E-minimal point.

A consequence of Theorem 9.1 is the fact that the prime-to-p Hecke correspondences operate transitively on the set of all irreducible components of any distinguished central leaf $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ of $\mathscr{M}_E$; see Corollary 9.3. This Hecke transitivity statement and the method of prime-to-p monodromy [6, 4.4] imply the irreducibility of every non-supersingular distinguished central leaf in $\mathscr{M}_E$; see Corollary 9.4. The last irreducibility result in Corollary 9.4 is a crucial ingredient of the proof of the Hecke orbit conjecture for Siegel modular varieties in characteristic p; see Remark 9.6. This connection to the Hecke orbit problem was the main motivation of [40].

The congruity stratification of a Lie stratum of a Hilbert modular variety was first defined in [40, 2.4]; a succinct proof of Theorem 7.1 is in [40, 3.3]. The observation that certain Lie-congruity strata are actually central leaves in Hilbert modular varieties appeared in [12, Section 6], [40, 2.5] and [40, 2.8]. The readers are referred to [9, Chapter 4], which is an expanded version of [40], for more information and proofs.

1.9.

The length of this article is due partly to a long Section 3 on the definitions of type spaces attached to a totally real number field. Readers who are familiar with basic properties of Hilbert modular varieties are urged to go directly to Section 5, and consult Section 3 only when necessary.

The rest of this article is organized as follows. In Section 2 we recall the definition of Hilbert modular varieties. In Section 3 we define indexing posets for the Lie stratification of $\mathscr{M}_E$ and for the congruity stratification on Lie strata of $\mathscr{M}_E$. The stratification of a Hilbert modular variety $\mathscr{M}_E$ by Lie type and its basic properties are reviewed in Section 4. The congruity invariant is defined in Section 5. In Section 6 we explain some basic properties of the congruity stratification of a Lie stratum, including the definition of distinguished Lie congruity strata. The main results on distinguished central leaves, Theorems 7.1 and 8.1, are explained in Sections 7 and 8 respectively. In Section 9 we explain how these two theorems imply the irreducibility of non-supersingular central leaves in Hilbert modular varieties. In Section 10 we discuss some questions on stratifications of $\mathscr{M}_E$.

2. Hilbert modular varieties

We fix a totally real field E, a prime number p, and an integer $n \in \mathbb{Z}$ with $n \geq 3$ which is relatively prime to p. Let $\mathbb{Z}[\zeta_n] := \mathbb{Z}[T]/(\Phi_n(T))$, where $\Phi_n(T)$ is the n-th cyclotomic polynomial. Let $\mathscr{L}$ be an invertible $\mathcal{O}_E$-module, and let $\mathscr{L}^+_{\mathbb{R}}$ be a notion of positivity for

$\mathscr{L}$, i.e., a choice of a connected component of

$$(\mathscr{L}_{\mathbb{R}})^{\times} := \mathrm{Isom}_{\mathcal{O}_{E\otimes_{\mathbb{Q}}\mathbb{R}}}\big(\mathcal{O}_{E\otimes_{\mathbb{Q}}\mathbb{R}}, \mathscr{L}\otimes_{\mathbb{Q}}\mathbb{R}\big).$$

Let $\widetilde{\mathscr{L}} := (\mathscr{L}, \mathscr{L}^{+}_{\mathbb{R}})$. Let

$$\overline{\delta} \in \big(\mathscr{L}^{\vee}\otimes_{\mathcal{O}_E}\mathcal{D}^{-1}_{E/\mathbb{Q}}/n\mathscr{L}^{\vee}\otimes_{\mathcal{O}_E}\mathcal{D}^{-1}_{E/\mathbb{Q}}\big)\otimes_{(\mathbb{Z}/n\mathbb{Z})} \text{ıı}\mu_n(\mathbb{Z}[\zeta_n,1/n])$$

be a generator of the free $(\mathcal{O}_E/n\mathcal{O}_E)$-module

$$\big(\mathscr{L}^{\vee}\otimes_{\mathcal{O}_E}\mathcal{D}^{-1}_{E/\mathbb{Q}}/n\mathscr{L}^{\vee}\otimes_{\mathcal{O}_E}\mathcal{D}^{-1}_{E/\mathbb{Q}}\big)\otimes_{(\mathbb{Z}/n\mathbb{Z})} \text{ıı}\mu_n(\mathbb{Z}[\zeta_n,1/n]),$$

where $\mathscr{L}^{\vee} := \mathrm{Hom}_{\mathcal{O}_E}(\mathscr{L},\mathcal{O}_E)$, $\mathcal{D}^{-1}_{E/\mathbb{Q}}$ is the inverse different of $E/\mathbb{Q}$, and $\text{ıı}\mu_n(\mathbb{Z}[\zeta_n,1/n])$ is the cyclic subgroup of $\mathbb{Z}[\zeta_n,1/n]^{\times}$ of order n generated by the element $\zeta_n\in\mathbb{Z}[\zeta_n,1/n]^{\times}$.

In this section we recall the definition of a Hilbert moduli scheme $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta}}$ over a number ring $\mathbb{Z}[1/n,\zeta_n]$ which classifies principally $\widetilde{\mathscr{L}}$-polarized abelian schemes with real multiplication by $\mathcal{O}_E$ with $\overline{\delta}$-symplectic level-n structures. Abelian varieties with real multiplication by $\mathcal{O}_E$ over $\mathbb{C}$ are uniformized as quotients of $\mathcal{O}_E\otimes_{\mathbb{Q}}\mathbb{C}$ by discrete rank-two projective $\mathcal{O}_E$-submodules of $\mathcal{O}_E\otimes_{\mathbb{Q}}\mathbb{C}$; see Remark 2.2.1(c)(iii). This leads to the uniformization of $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta}}\times_{\mathrm{Spec}(\mathbb{Z}[n,\zeta_n])}\mathrm{Spec}(\mathbb{C})$ and the interpretation of Hilbert modular forms as sections of equivariant line bundles on Hilbert modular varieties. See [35, p. 11] for an informative discussion on the history of Hilbert's modular group and modular functions, including the first papers [2,3] on Hilbert modular forms by Blumenthal, and the correction [26] by Maass of an error in [2,3].

After this section, we will consider only Hilbert modular varieties over $\overline{\mathbb{F}_p}$, and the notation $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$ for a geometric fiber of $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta}}$ in characteristic p will be shortened to $\mathscr{M}_E$.

2.1. Abelian varieties with real multiplication

2.1.1. Let S be a scheme. An *abelian scheme with real multiplication by* $\mathcal{O}_E$ over S is a pair $(A\to S,\iota)$, where $A\to S$ is an abelian scheme of relative dimension $[E:\mathbb{Q}]$, and $\iota\colon\mathcal{O}_E\to\mathrm{End}_S(A)$ is a ring homomorphism. Note that ι is injective because $\mathrm{End}_S(A)$ is torsion-free.

2.1.2. Let (A,ι) be an abelian variety with real multiplication by $\mathcal{O}_E$ over a field K.

(a) Denote by $\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A,A^t)$ the group of all K-homomorphisms $h\colon A\to A^t$ from A to the dual abelian variety A^t such that $h^t=h$ under the canonical identification of A with $(A^t)^t$. The abelian group $\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A,A^t)$ has a natural structure as a projective $\mathcal{O}_E$-module of rank one.

(b) The $(E\otimes_{\mathbb{Q}}\mathbb{R})^{\times}$-torsor

$$\mathrm{Isom}_{E\otimes_{\mathbb{Q}}\mathbb{R}}\left(E\otimes_{\mathbb{Q}}\mathbb{R},\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A,A^t)\otimes_{\mathbb{Q}}\mathbb{R}\right)$$

has a connected component, denoted by $\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A,A^t)^+_{\mathbb{R}}$, which is uniquely determined by the property that every element of $\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A,A^t)^+_{\mathbb{R}}\cap\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A,A^t)$ is a polarization of the abelian variety A.

Remark. The pair $(\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A,A^t),\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A,A^t)^+_{\mathbb{R}})$ attached to an abelian variety (A,ι) with real multiplication by $\mathcal{O}_E$ is an *invertible $\mathcal{O}_E$-module with a notion of positivity.* The set of all isomorphism classes of invertible $\mathcal{O}_E$-modules with notions of positivity is the *strict class group* $\widetilde{\mathrm{Cl}}_E$ of E. Thus the image of the pair $(\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A,A^t),\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A,A^t)^+_{\mathbb{R}})$ in $\widetilde{\mathrm{Cl}}_E$ is a discrete invariant of (A,ι).

Definition 2.2. Recall that $\widetilde{\mathscr{L}}=(\mathscr{L},\mathscr{L}^+_{\mathbb{R}})$ is an invertible $\mathcal{O}_E$-module with a notion of positivity. Let $(A\to S,\iota)$ be an abelian scheme over S with real multiplication by $\mathcal{O}_E$.

(1) An *$\widetilde{\mathscr{L}}$-polarization* of (A,ι) is an $\mathcal{O}_E$-homomorphism $\lambda\colon\mathscr{L}\to\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E,S}(A,A^t)$ such that for every point $s\in S$, the map $\lambda_s\colon\mathscr{L}\to\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A_s,A^t_s)$ is an isomorphism and sends every element of $\mathscr{L}\cap\mathscr{L}^+_{\mathbb{R}}$ to an $\mathcal{O}_E$-linear polarization of A_s.

(2) An $\widetilde{\mathscr{L}}$-polarization $\lambda\colon\mathscr{L}\to\mathrm{Hom}^{\mathrm{sym}}_S(A,A^t)$ is said to be *principal* if the natural homomorphism

$$\mathscr{L}\otimes_{\mathcal{O}_E}A\to A^t$$

is an isomorphism.

Here $\mathscr{L}\otimes_{\mathcal{O}_E}A$ is the tensor product of the $\mathcal{O}_E$-linear fppf sheaf A over S with the projective $\mathcal{O}_E$-module $\mathscr{L}$, which is representable by an abelian scheme over S, again denoted by $\mathscr{L}\otimes_{\mathcal{O}_E}A$.

Remark 2.2.1. Let (A,ι) be an abelian variety A with real multiplication by $\mathcal{O}_E$ over a field K.

(a) The natural map

$$\alpha_A\colon\ \mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A,A^t)\otimes_{\mathcal{O}_E}A\to A^t$$

is an isogeny. This isogeny α_A is an isomorphism if and only if there exists a principal $(\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A,A^t),\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A,A^t)^+_{\mathbb{R}})$-polarization on (A,ι).

(b) If the $\mathcal{O}_E$-linear abelian variety (A,α) admits an $\mathcal{O}_E$-linear *principal* polarization $\mu\colon A\to A^t$, then the invariant $[(\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A,A^t),\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A,A^t)^+_{\mathbb{R}})]$ of (A,ι) is the zero element of the strict class group $\widetilde{\mathrm{Cl}}_E$. So the invariant $[(\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A,A^t),$ $\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A,A^t)^+_{\mathbb{R}})]$ is an obstruction for the existence of an $\mathcal{O}_E$-linear principal polarization on (A,ι).

(c) Suppose that K is algebraically closed.

(i) If $\mathrm{char}(K) = 0$ then the isogeny $\alpha_A\colon \mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A, A^t) \otimes_{\mathcal{O}_E} A \to A^t$ is an isomorphism.

(ii) If $K = \mathbb{C}$, then $\mathrm{H}_1(A(\mathbb{C}), \mathbb{Z})$ is a projective $\mathcal{O}_E$-module of rank two, and the invertible $\mathcal{O}_E$-module $\mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A, A^t)$ is isomorphic to

$$\mathrm{Hom}_{\mathcal{O}_E}\left(\bigwedge^2_{\mathcal{O}_E} \mathrm{H}_1(A(\mathbb{C}), \mathbb{Z}), \mathcal{O}_E\right) \otimes_{\mathcal{O}_E} \mathcal{D}^{-1}_{E/\mathbb{Q}},$$

where $\mathcal{D}^{-1}_{E/\mathbb{Q}}$ is the inverse different of $E/\mathbb{Q}$.

It is a fact that for every discrete rank-two $\mathcal{O}_E$-submodule Q of $E \otimes_{\mathbb{Q}} \mathbb{C}$, there exists an abelian variety B with real multiplication by $\mathcal{O}_E$ over $\mathbb{C}$ such that $B(\mathbb{C})$ is $\mathcal{O}_E$-linearly isomorphic to $(E \otimes_{\mathbb{Q}} \mathbb{C})/Q$.

(iii) If $\mathrm{char}(K) = p > 0$, then $\mathrm{Ker}(\alpha_A)$ is a finite group scheme over K whose order is a power of p. Moreover there exists an abelian variety (A_1, ι_1) with real multiplication by $\mathcal{O}_E$ over K, a principal $\mathcal{O}_E$-linear polarization $\mu\colon A_1 \to A_1^t$, and an $\mathcal{O}_E$-linear isogeny $(A, \iota) \to (A_1, \iota_1)$ over K.

Remark 2.2.2. There are examples (A, ι) of abelian varieties with real multiplication by $\mathcal{O}_E$ such that the isogeny α_A is not an isomorphism, and can be constructed as follows. Suppose that there exists a finite place v of $\mathcal{O}_E$ above p with residue field degree $f_v > 1$. Then there exists an algebraically closed field K of characteristic p and an abelian variety (A_1, ι_1) over K with real multiplication by $\mathcal{O}_E$ whose Lie type is *not* self-dual. It follows from Lemmas 4.2.2 and 4.2.3 that the isogeny $\alpha_A\colon \mathrm{Hom}^{\mathrm{sym}}_{\mathcal{O}_E}(A_1, A_1^t) \otimes_{\mathcal{O}_E} A_1 \to A_1^t$ is *not* an isomorphism.

Remark 2.2.3. Let (A, λ, ι) be a principally $\widetilde{\mathscr{L}}$-polarized abelian scheme with real multiplication over a scheme S. As explained in Remark 2.2.1(a), there exists an $\mathcal{O}_E$-linear principal polarization on A if and only if the image of $\widetilde{\mathscr{L}}$ in the strict class group $\widetilde{\mathrm{Cl}}_E$ is 0. If the element $[\widetilde{\mathscr{L}}] \in \widetilde{\mathrm{Cl}}_E$ of $\widetilde{\mathscr{L}}$ is nonzero, then there is no $\mathcal{O}_E$-linear principal polarization $A \to A^t$, but there exist (many) separable $\mathcal{O}_E$-linear polarizations on A: by weak approximation for $\widetilde{\mathscr{L}}$, every element of $\mathscr{L} \cap \mathscr{L}^+_{\mathbb{R}}$ whose image in $\mathscr{L}/p\mathscr{L}$ is a generator of the cyclic $(\mathcal{O}_E/p\mathcal{O}_E)$-module $\mathscr{L}/p\mathscr{L}$ corresponds to an $\mathcal{O}_E$-linear separable polarization of (A, ι).

Definition 2.3. (a) Recall that $\overline{\delta}$ is a free generator of the $(\mathcal{O}_E/n\mathcal{O}_E)$-module

$$\left(\mathscr{L}^\vee \otimes \mathcal{D}^{-1}_{E/\mathbb{Q}}/n\mathscr{L}^\vee \otimes \mathcal{D}^{-1}_{E/\mathbb{Q}}\right) \otimes_{(\mathbb{Z}/n\mathbb{Z})} \mathit{\mu}_n(\mathbb{Z}[\zeta_n, 1/n]).$$

For each element $l \in \mathscr{L}$, denote by

$$\langle\,\cdot\,,\cdot\,\rangle_{n,\overline{\delta},l}\colon (\mathcal{O}_E/n\mathcal{O}_E)_S^{\oplus 2} \to \boldsymbol{\mu}_{n,S}$$

the alternating pairing on the free module $(\mathcal{O}_E/n\mathcal{O}_E)_S^{\oplus 2}$ over the sheaf of rings $(\mathcal{O}_E/n\mathcal{O}_E)_S$ on S_{et} attached to the free $(\mathcal{O}_E/n\mathcal{O}_E)$-module $(\mathcal{O}_E/n\mathcal{O}_E)^{\oplus 2}$ with generators $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, which is equal to the composition of the canonical $(\mathcal{O}_E/n\mathcal{O}_E)_S$-linear alternating pairing

$$(\mathcal{O}_E/n\mathcal{O}_E)_S^{\oplus 2} \to \bigwedge_{(\mathcal{O}_E/n\mathcal{O}_E)}^{2} (\mathcal{O}_E/n\mathcal{O}_E)_S^{\oplus 2}$$

with the homomorphism of group schemes

$$\bigwedge_{(\mathcal{O}_E/n\mathcal{O}_E)}^{2} (\mathcal{O}_E/n\mathcal{O}_E)_S^{\oplus 2} \to \boldsymbol{\mu}_{n,S}, \quad a\cdot \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \mapsto (\mathrm{Tr}_{E/\mathbb{Q}} \otimes 1)(l\cdot \overline{\delta}\cdot a)$$

for all local sections a of $(\mathcal{O}_E/n\mathcal{O}_E)_S$, where

$$\mathrm{Tr}_{E/\mathbb{Q}} \otimes 1\colon (\mathcal{D}_{E/\mathbb{Q}}^{-1}/n\mathcal{D}_{E/\mathbb{Q}}^{-1}) \otimes_{(\mathbb{Z}/n\mathbb{Z})} \boldsymbol{\mu}_{n,S} \to \boldsymbol{\mu}_{n,S}$$

is the homomorphism of group schemes induced by the $(E/\mathbb{Q})$-trace.

(b) Let S be a scheme over $\mathbb{Z}[1/n,\zeta_n]$. Let $(A \to S, \iota)$ be an abelian scheme over S with real multiplication by $\mathcal{O}_E$, and let λ be a principal $\widetilde{\mathscr{L}}$-polarization of (A,ι).

A $\overline{\delta}$-*symplectic level-n* structure on (A,λ,ι) is an $\mathcal{O}_E$-linear isomorphism

$$(\mathcal{O}_E/n\mathcal{O}_E)_S^2 \xrightarrow[\simeq]{\eta} A[n]$$

such that for every $l \in \mathscr{L}$, the diagram

$$\begin{array}{ccc} (\mathcal{O}_E/n\mathcal{O}_E)_S^2 \times_S (\mathcal{O}_E/n\mathcal{O}_E)_S^2 & \xrightarrow{\eta\times\eta} & A[n]\times_S A[n] \\ \downarrow{\scriptstyle \langle\,\cdot\,,\cdot\,\rangle_{n,\overline{\delta},l}} & & \downarrow{\scriptstyle 1_A\times\lambda(l)} \\ \boldsymbol{\mu}_{n,S} & \xleftarrow{\langle\,\cdot\,,\cdot\,\rangle_{n,A}} & A[n]\times_S A^t[n] \end{array}$$

commutes, where

$$\langle\,\cdot\,,\cdot\,\rangle_{n,A}\colon A[n]\times_S A^t[n] \to \boldsymbol{\mu}_{n,S}$$

is the Weil pairing for $A[n]$.

Definition 2.4. (a) Denote by

$$\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta}}\colon (\mathbb{Z}[1/n,\zeta_n]\text{-Schemes}) \to (\text{Sets})$$

the functor from the category of $\mathbb{Z}[1/n,\zeta_n]$-schemes to the category of sets as follows. For every $\mathbb{Z}[1/n,\zeta_n]$-scheme S, $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta}}(S)$ is the set of all isomorphism classes of quadruples

$(A \to S, \lambda, \iota, \eta)$, where $(A \to S, \iota)$ is an abelian scheme with real multiplication by $\mathcal{O}_E$ over S, λ is a principal $\widetilde{\mathscr{L}}$-polarization of (A, ι), and η is a $\overline{\delta}$-symplectic level-n structure.

(b) Pick and fix a ring homomorphism from $\mathbb{Z}[1/n, \zeta_n]$ an algebraic closure of $\overline{\mathbb{F}_p}$. Denote by $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$ the restriction of the moduli functor $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta}}$ to the category of $\overline{\mathbb{F}_p}$-schemes.

Proposition 2.5. (a) *The functor $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta}}$ is representable by a scheme of finite type over $\mathbb{Z}[1/n, \zeta_n]$, again denoted by $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta}}$. The structural morphism $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta}} \to \mathrm{Spec}(\mathbb{Z}[1/n, \zeta_n])$ is faithfully flat, normal and is a relative complete intersection.*

(b) *The geometric fiber $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$ is normal and irreducible. It is smooth over $\overline{\mathbb{F}_p}$ if and only if $E/\mathbb{Q}$ is unramified above p.*

(c) *Suppose that $E/\mathbb{Q}$ is ramified above p. Then the smooth locus $\mathscr{M}^{\widetilde{\mathscr{L}},\mathrm{sm}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$ of $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$ is a dense open subscheme of $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$. The singular locus*

$$V_{\mathrm{sing}} := \mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}} \smallsetminus \mathscr{M}^{\widetilde{\mathscr{L}},\mathrm{sm}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$$

with the reduced structure is a local complete intersection, and every irreducible component of V_{sing} is of codimension 2 in $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta}}$.

Proposition 2.5 is a consequence of [33, Theorem 5.1] and [13, Theorem 3.3]. Here we explain the irreducibility statement (b). We have an arithmetic toroidal compactification $\mathscr{M}^{\widetilde{\mathscr{L}},\Sigma}_{E,n,\overline{\delta}}$ of $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta}}$ over $\mathbb{Z}[1/n, \zeta_n]$ which is again a complete intersection over $\mathbb{Z}[1/n, \zeta_n]$. It follows that the geometric fiber $\mathscr{M}^{\widetilde{\mathscr{L}},\Sigma}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$ is normal. Since the geometric generic fiber of $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta}}$ is geometrically irreducible by complex uniformization, so is the geometric fiber in characteristic p.

Remark 2.5.1. As a special case of Remark 2.2.3, the universal abelian scheme $\boldsymbol{A} \to \mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$ with real multiplication by $\mathcal{O}_E$ has many separable $\mathcal{O}_E$-linear polarizations, and it admits an $\mathcal{O}_E$-linear *principal* polarization if and only if the element $[\widetilde{\mathscr{L}}]$ in the strict class group of E is zero. If $[\widetilde{\mathscr{L}}]$ is non-zero, one can view the $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$ as an analog of an irreducible component of $\mathscr{A}_{g,d,n}$ with $\gcd(d, p) = 1$, where $\mathscr{A}_{g,d,n}$ is the moduli space over $\overline{\mathbb{F}_p}$ of g-dimensional polarized abelian varieties (A, λ, η) with $\deg \lambda = d^2$ and level-n structure.

2.5.2. In the rest of this article we will consider only Hilbert modular varieties $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$ over $\overline{\mathbb{F}_p}$. Often we will also suppress the parameters $\widetilde{\mathscr{L}}$, n and $\overline{\delta}$ from the notation, and shorten $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta},\overline{\mathbb{F}_p}}$ to $\mathscr{M}_E$, if there is no risk of confusion.

3. Type spaces

This section contains the definitions of two finite posets $\mathcal{T}_E^{\mathrm{sd}}$, $\boldsymbol{\tau}_E$ associated to the totally real number field E. They are natural indexing sets for the Lie stratification of the Hilbert modular variety $\mathscr{M}_E$, and for the congruity stratification of Lie strata of $\mathscr{M}_E$. The partial ordering of these posets are compatible with the incidence relation between strata in the respective stratifications.

We recommend readers to go directly to Definition 3.2.1 and skip the rest of this section, and return to this section only when necessary.

3.1. Product decomposition of the ring $\mathcal{O}_E \otimes_{\mathbb{Z}} \Lambda(\overline{\mathbb{F}_p})$

3.1.1. Notations related to the totally real field E and its p-adic completion $E \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

- Denote by $\Sigma_{E,p}$ the set of places of E above p.
- Let $v \in \Sigma_{E,p}$ be a place of E above p and let E_v the v-adic completion of E at v.
  - Let $\mathcal{O}_v = \mathcal{O}_{E_v}$ be the ring of integers in E_v, let $\wp_v$ be the maximal ideal of $\mathcal{O}_{E_v}$, and let $\kappa_v := \mathcal{O}_{E_v}/\wp_v$, the residue field of $\mathcal{O}_{E_v}$.
  - Let $g_v := [E_v : \mathbb{Q}_p]$, $f_v := [\kappa_v : \overline{\mathbb{F}_p}]$, and let $e_v = e(E_v/\mathbb{Q}_p)$ be the absolute ramification index of E_v.
  - Denote by π_v a generator of the maximal ideal $\wp_v$ of $\mathcal{O}_v$.
  - Let E_v^{ur} be the maximal subfield of E_v unramified over $\mathbb{Q}_p$, and let $\mathcal{O}_{E_v^{\mathrm{ur}}}$ be the ring of integers of E_v^{ur}.
  - Let Fr_{κ_v} be the arithmetic Frobenius map ($y \mapsto y^p$) on κ_v. We will identify the Galois group $\mathrm{Gal}(\kappa_v/\mathbb{F}_p)$ with $\mathbb{Z}/f_v\mathbb{Z}$, by the isomorphism which sends the element Fr_{κ_v} in $\mathrm{Gal}(\kappa_v/\mathbb{F}_p)$ to the element $1 + f_v\mathbb{Z}$ in $\mathbb{Z}/f_v\mathbb{Z}$.
  - Let $f_E := \mathrm{lcm}\{f_v \mid v \in \Sigma_{E,p}\}$, the least common multiple of all f_v's.
- Let κ be perfect field containing κ_E, where κ_E denotes a finite field with p^{f_E} elements.
  - Denote by $\Lambda(\kappa)$ the ring of all p-adic Witt vectors with entries in κ.
  - Let σ be the continuous ring automorphism of $\Lambda(\kappa)$, which sends every infinite Witt vector $(a_0, a_1, \ldots, a_n, \ldots)$ in $\Lambda(\kappa)$ to $(a_0^p, a_1^p, \ldots, a_n^p, \ldots)$.
  - For every $v \in \Sigma_{E,p}$, define a torsor $I_v = I_{v,\kappa}$ for the cyclic group $\mathrm{Gal}(\kappa_v/\mathbb{F}_p) = \mathbb{Z}/f_v\mathbb{Z}$ by

$$I_v = I_{v,\kappa} := \mathrm{Hom}_{\mathrm{ring,cont}}(\mathcal{O}_{E_v^{\mathrm{ur}}}, \Lambda(\kappa)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{ring}}(\kappa_v, \kappa),$$

    so that for every element $i \in I_{v,\kappa}$, the element 1 mod $f_v \in \mathbb{Z}/f_v\mathbb{Z}$ sends i to the element $\sigma \circ i \in I_{v,\kappa}$.

3.1.2. We have direct product decompositions of rings

$$\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p = \prod_{v\in\Sigma_{E,p}} \mathcal{O}_{E_v} \quad \text{and} \quad \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{F}_p = \prod_{v\in\Sigma_{E,p}} \mathcal{O}_{E_v}/\wp_v^{e_v}.$$

For every perfect field κ containing κ_E, we have direct sum decompositions of rings

$$\mathcal{O}_{E_v} \otimes_{\mathbb{Z}_p} \Lambda(\kappa) = \prod_{i\in I_{v,\kappa}} \mathcal{O}_{E_v} \otimes_{(\mathcal{O}_{E_v^{\mathrm{ur}}},i)} \Lambda(\kappa) \quad \text{and} \quad \mathcal{O}_{E_v} \otimes_{\mathbb{Z}_p} \kappa = \prod_{i\in I_{v,\kappa}} \mathcal{O}_{E_v} \otimes_{(\mathcal{O}_{E_v^{\mathrm{ur}}},i)} \kappa.$$

So we get a product decomposition

$$\mathcal{O}_E \otimes_{\mathbb{Z}} \Lambda(\kappa) = \prod_{v\in\Sigma_{E,p}} \prod_{i\in I_{v,\kappa}} \mathcal{O}_{E_v} \otimes_{(\mathcal{O}_{E_v^{\mathrm{ur}}},i)} \Lambda(\kappa) = \prod_{v\in\Sigma_{E,p}} \prod_{i\in I_{v,\kappa}} \Lambda(\kappa)_v^{(i)}, \tag{3.1}$$

where $\Lambda(\kappa)_v^{(i)}$ denotes the complete discrete valuation ring

$$\Lambda(\kappa)_v^{(i)} := \mathcal{O}_{E_v} \otimes_{(\mathcal{O}_{E_v^{\mathrm{ur}}},i)} \Lambda(\kappa)$$

with residue field κ and absolute ramification index e_v. The maximal ideal of $\Lambda^{(i)} = \Lambda(\kappa)^{(i)}$ is generated by the element $\pi_v \in \mathcal{O}_{E_v}$. From the product decomposition (3.1) of the ring $\mathcal{O}_E \otimes_{\mathbb{Z}} \Lambda(\kappa)$, we see that every finitely generated module over $\mathcal{O}_E \otimes_{\mathbb{Z}} \Lambda(\kappa)$ is a direct sum, over all pairs (v,i), of direct sums of cyclic modules of the discrete valuation rings $\Lambda_v^{(i)}$.

Similarly, for every field K containing κ_E, not necessarily perfect, we have a product decomposition

$$\mathcal{O}_E \otimes_{\mathbb{Z}} K = \prod_{v\in\Sigma_{E,p}} \prod_{i\in I_{v,K}} \mathcal{O}_{E_v} \otimes_{(\mathcal{O}_{E_v^{\mathrm{ur}}},i)} K = \prod_{v\in\Sigma_{E,p}} \prod_{i\in I_{v,K}} \big(\kappa_v[T]/(T^{e_v})\big) \otimes_{(\kappa_v,i)} K, \tag{3.2}$$

where

$$I_{v,K} = I_{v,\kappa_E} := \mathrm{Hom}_{\mathrm{ring}}(\kappa_v, \kappa_E) = \mathrm{Hom}_{\mathrm{ring}}(\kappa_v, K)$$

for every $v \in \Sigma_{E,p}$, and $\kappa_v[T]$ is the polynomial ring over κ_v in the variable T. It follows that every finitely generated module over $\mathcal{O}_E \otimes_{\mathbb{Z}} K$ is isomorphic to a direct sum, over all pairs (v,i), of direct sum of cyclic modules of the ring

$$\big(\kappa_v[T]/(T^{e_v})\big) \otimes_{(\kappa_v,i)} K \cong K[T]/T^{e_v}K[T].$$

3.2. Type spaces of finite $(\mathcal{O}_E \otimes_{\mathbb{Z}} K)$-modules

In this subsection, K is a field containing κ_E. The *type spaces* considered in this article are isomorphisms classes of modules over $\mathcal{O}_E \otimes K$ which can be generated by two elements, together with natural partial ordering compatible with specialization.

Definition 3.2.1. Let $K \supseteq \mathbb{F}_p$ be a field which contains a finite subfield κ_E with p^{f_E} elements.

(i) Denote by $\mathcal{T}_{E,K}$ to be the set of all isomorphism classes of modules over $\mathcal{O}_E \otimes K$ which can be generated by two elements.

We will identify $\mathcal{T}_{E,K}$ with the set consisting of all collections of sequences of pairs of the form

$$\begin{aligned} \underline{e} = (\underline{e}_v)_{v\in\Sigma_{E,p}} &= \Big(\big(e_{v,1}^{(i)}, e_{v,2}^{(i)}\big)_{i\in I_{v,K}} \Big)_{v\in\Sigma_{E,p}} \\ \text{s.t.} \quad e_{v,1}^{(i)}, e_{v,2}^{(i)} &\in \{0,1,\dots,e_v\}, \quad e_{v,1}^{(i)} \le e_{v,2}^{(i)} \end{aligned} \tag{3.3}$$

for all $v \in \Sigma_{E,p}$ and all $i \in I_{v,K}$. An element $\underline{e}$ as above corresponds to the isomorphism class of the $(\mathcal{O}_E \otimes_{\mathbb{Z}} K)$-module

$$\begin{aligned} &\bigoplus_{v\in\Sigma_{E,p}} \bigoplus_{i\in I_{v,K}} \Big(\mathcal{O}_{E_v} \otimes_{(\mathcal{O}_{E_v^{\mathrm{ur}}},i)} K \big/ \pi_v^{e_{v,1}^{(i)}} \mathcal{O}_{E_v} \otimes_{(\mathcal{O}_{E_v^{\mathrm{ur}}},i)} K \Big) \\ \oplus &\Big(\mathcal{O}_{E_v} \otimes_{(\mathcal{O}_{E_v^{\mathrm{ur}}},i)} K \big/ \pi_v^{e_{v,2}^{(i)}} \mathcal{O}_{E_v} \otimes_{(\mathcal{O}_{E_v^{\mathrm{ur}}},i)} K \Big). \end{aligned}$$

(ii) Let $\boldsymbol{K}(\mathcal{O}_E \otimes_{\mathbb{Z}} K)$ denote the Grothendieck group of finitely generated $(\mathcal{O}_E \otimes_{\mathbb{Z}} K)$-modules. Denote by $\mathcal{T}_{E,K}^{\mathrm{sd}} \subset \mathcal{T}_{E,K}$ the subset consisting of all isomorphism classes of modules over $\mathcal{O}_E \otimes_{\mathbb{Z}} K$ whose image in $\boldsymbol{K}(\mathcal{O}_E \otimes K)$ is equal to the image of $\mathcal{O}_E \otimes K$ in $\boldsymbol{K}(\mathcal{O}_E \otimes_{\mathbb{Z}} K)$.

Following the explicit form (3.3) of $\mathcal{T}_{E,K}$, we will identify the type space $\mathcal{T}_{E,K}^{\mathrm{sd}}$ with the set of all collections of sequences of pairs

$$\Big(\big(e_{v,1}^{(i)}, e_{v,2}^{(i)}\big)_{i\in I_{v,K}} \Big)_{v\in\Sigma_{E,p}} \quad \text{s.t.} \quad e_{v,1}^{(i)}, e_{v,2}^{(i)} \in \mathbb{N},\ e_{v,1}^{(i)} \le e_{v,2}^{(i)},\ e_{v,1}^{(i)} + e_{v,2}^{(i)} = e_v$$

for all $v \in \Sigma_{E,p}$ and all $i \in I_{v,K}$. Here $\mathbb{N}$ denotes the set of all non-negative integers.

(iii) Denote by $\boldsymbol{\tau}_{E,K}$ the set of all collections of sequences

$$\underline{c} = (\underline{c}_v)_{v\in\Sigma_{E,p}} = \big((c_v^{(i)})_{i\in I_{v,K}} \big)_{v\in\Sigma_{E,p}} \quad \text{s.t.} \quad 0 \le c_v^{(i)} \le e_v$$

for all $v \in \Sigma_{E,p}$ and all $i \in I_{v,K}$. We will identify $\boldsymbol{\tau}_{E,K}$ with the set of all isomorphism classes of cyclic $(\mathcal{O}_E \otimes_{\mathbb{Z}} K)$-modules.

Remark 3.2.2. (a) For any two fields K, K' containing κ_E, every ring homomorphism $h\colon K \to K'$ induces isomorphisms

$$I_{v,K} \xrightarrow{\sim} I_{v,K'}, \quad \mathcal{T}_{E,K} \xrightarrow{\sim} \mathcal{T}_{E,K'}, \quad \mathcal{T}_{E,K}^{\mathrm{sd}} \xrightarrow{\sim} \mathcal{T}_{E,K'}^{\mathrm{sd}}, \quad \boldsymbol{\tau}_{E,K} \xrightarrow{\sim} \boldsymbol{\tau}_{E,K'}.$$

Often we will suppress the subscript "K" and shorten the notations $I_{v,K}$, $\mathcal{T}_{E,K}$, $\mathcal{T}_{E,K}^{\mathrm{sd}}$, $\tau_{E,K}$ to I_v, $\mathcal{T}_E$, $\mathcal{T}_E^{\mathrm{sd}}$, τ_E respectively, when there is no risk of confusion.

(b) The type space $\mathcal{T}_E$ is a direct product of spaces $\mathcal{T}_{E_v}$, where v runs through all $v \in \Sigma_{E,p}$. The same holds for $\mathcal{T}_E^{\mathrm{sd}}$ and $\boldsymbol{\tau}_E$. Explicitly, $\mathcal{T}_{E,v}$ consists of all sequences of pairs

$$\big(e_{v,1}^{(i)}, e_{v,2}^{(i)}\big)_{i\in I_v}, \quad e_{v,1}^{(i)}, e_{v,2}^{(i)} \in \{0,1,\dots,e_v\}, \quad e_{v,1}^{(i)} \le e_{v,2}^{(i)}, \ \ \forall\, i \in I_v,$$

and similarly for $\mathcal{T}_E^{\mathrm{sd}}$ and $\boldsymbol{\tau}_E$.

(c) Elements of $\boldsymbol{T}_{E,K}^{\mathrm{sd}}$ and $\mathcal{T}_{E,K}^{\mathrm{sd}}$ will be called *self-dual* types. The Lie type of any principally $\widetilde{\mathscr{L}}$-polarized abelian variety with real multiplication by $\mathcal{O}_E$ over K is self-dual. Similarly the Lie type of any self-dual rank-two $\mathcal{O}_E$-linear p-divisible group over K, is self-dual. See Lemmas 4.2.2 and 4.2.3.

Definition 3.3. We define partial orderings on the type spaces $\mathcal{T}_E$, $\mathcal{T}_E^{sd}$ and $\boldsymbol{\tau}_E$ as follows.

(i) Let $\underline{e}_1$, $\underline{e}_2$ be two elements of $\mathcal{T}_E$, $\underline{e}_j = \big(\big(e_{j;v,1}^{(i)}, e_{j;v,2}^{(i)}\big)_{i\in I_{v,k}}\big)_{v\in\Sigma_{E,p}}$ for $j = 1, 2$. Define $\underline{e}_2 \preceq \underline{e}_1$ if

$$e_{1;v,1}^{(i)} \le e_{2;v,1}^{(i)} \quad \text{and} \quad e_{1;v,1}^{(i)} + e_{1;v,2}^{(i)} \le e_{2;v,1}^{(i)} + e_{2;v,2}^{(i)}, \quad \forall\, v \in \Sigma_{E,p},\ \forall\, i \in I_v$$

for all $v \in \Sigma_{E,p}$ and all $i \in I_v$.

(ii) The restriction of the partial ordering $\preceq$ on $\mathcal{T}_E$ induces a partial ordering on $\mathcal{T}_E^{\mathrm{sd}}$. Explicitly, if $\underline{e}_1$, $\underline{e}_2$ are two elements of $\mathcal{T}_E^{\mathrm{sd}}$ with $\underline{e}_j = \big(\big(e_{j;v,1}^{(i)}, e_{j;v,2}^{(i)}\big)_{i\in I_{v,k}}\big)_{v\in\Sigma_{E,p}}$ for $j = 1, 2$, then

$$\underline{e}_2 \preceq \underline{e}_1 \quad \Longleftrightarrow \quad e_{1;v,1}^{(i)} \le e_{2;v,1}^{(i)}, \quad \forall\, v, i.$$

Note that the last condition is equivalent to $e_{2;v,2}^{(i)} \le e_{1;v,2}^{(i)}$ for all $v \in \Sigma_{E,p}$ and all $i \in I_v$.

(iii) For any two elements $\underline{c}_j = \big(c_{j;v}^{(i)}\big)_{v\in\Sigma_{E,p},i\in I_v}$ in $\boldsymbol{\tau}_E$, $j = 1, 2$,

$$\underline{c}_2 \preceq \underline{c}_1 \quad \Longleftrightarrow \quad c_{1;v}^{(i)} \le c_{2;v}^{(i)}, \quad \forall\, v \in \Sigma_{E,p},\ \forall\, i \in I_v.$$

Remark 3.3.1. (a) The poset $\mathcal{T}_E^{\mathrm{sd}}$ is *ranked*, in the sense that any two maximal chain between two comparable elements have the same length.

(b) The poset $\mathcal{T}_E^{\mathrm{sd}}$ has a unique maximal element $\underline{e}_{\max}$ and a unique minimal element $\underline{e}_{\min}$. The (v,i)-component $(e_{\max,v,1}^{(i)}, e_{\max,v,2}^{(i)})$ of $\underline{e}_{\max}$ is $(0, e_v)$ for all $v \in \Sigma_{E,p}$ and all $i \in I_v$. The (v,i)-component $(e_{\min,v,1}^{(i)}, e_{\min,v,2}^{(i)})$ of $\underline{e}_{\min}$ is $\big(\lfloor e_v/2\rfloor, \lceil e_v/2\rceil\big)$ for $v \in \Sigma_{E,p}$ and all $i \in I_v$. Here $\lfloor e_v/2\rfloor$ is rounding $e_v/2$ *down* to the nearest integer, while $\lceil e_v/2\rceil$ is rounding $e_v/2$ *up* to the nearest integer.

3.3.2. Let S be a scheme over a field containing κ_E. Let M be a coherent $\mathcal{O}_S$ module, endowed with an action by $\mathcal{O}_E$, i.e., a ring homomorphism $\mathcal{O}_E \to \mathrm{End}_S(M)$. For every

point $s \in S$, the product decomposition (3.2) with K replaced by the residue field $\kappa(s)$ at s induces a decomposition of the fiber M_s as a module over $\mathcal{O}_E \otimes_{\mathbb{Z}} \kappa(s)$. Denote by $\underline{e}_{M,s}$ the element in the type space $\mathcal{T}_{E,\kappa(s)} \xleftarrow{\sim} \mathcal{T}_{E,K}$ determined by the $(\mathcal{O}_E \otimes_{\mathbb{Z}} \kappa(s))$-module M_s.

Lemma 3.3.3. *Let M be a coherent $\mathcal{O}_S$-module as in Subsubsection* 3.3.2. *Let s, s' be points of S. If s is a specialization of s', then $\underline{e}_{M,s} \preceq \underline{e}_{M,s'}$*

The proof of Lemma 3.3.3 is left as an exercise.

Definition 3.3.4. Define a function strdim: $\mathcal{T}^{\mathrm{sd}}_{E,k} \to \mathbb{N}$ by

$$\mathrm{strdim}(\underline{e}) = \sum_{v \in \Sigma_{E,p}} \sum_{i \in I_v} \big(e^{(i)}_{v,2} - e^{(i)}_{v,1}\big)$$

if $\underline{e} = \big(((e^{(i)}_{v,1}, e^{(i)}_{v,2}))_{i \in I_v}\big)_{v \in \Sigma_{E,p}}$.

Definition 3.4 (Local version of Definition 3.2.1)**.** Let v be a place of E above p and let K be a field which contains a subfield with p^{f_v} elements. Let $I_{v,K} = \mathrm{Hom}_{\mathrm{ring}}(\kappa_v, K)$, regarded as a torsor over $\mathrm{Gal}(\kappa_v/\mathbb{F}_p) \cong \mathbb{Z}/f_v\mathbb{Z}$. The ring $\mathcal{O}_{E_v} \otimes_{\mathbb{Z}_p} K$ admits a natural product decomposition

$$\mathcal{O}_{E_v} \otimes_{\mathbb{Z}_p} K = \prod_{i \in I_{v,K}} \big(\mathcal{O}_{E_v} \otimes_{\mathcal{O}_{E_v^{\mathrm{ur}},i}} K\big).$$

(i) Denote by $\mathcal{T}_{E_v,K}$ the set of all isomorphism classes of modules over $\mathcal{O}_{E_v} \otimes_{\mathbb{Z}_p} K$ which can be generated by two elements. We identify $\mathcal{T}_{E_v,K}$ with the set consisting of all sequences of pairs

$$\underline{e}_v = \big((b^{(i)}_1, b^{(i)}_2)\big)_{i \in I_{v,K}}, \quad 0 \le b^{(i)}_1 \le b^{(i)}_2 \le e_v, \quad \forall\, i \in I_{v,K}.$$

A sequence of pairs as above corresponds to the isomorphism class of the $(\mathcal{O}_{E_v} \otimes_{\mathbb{Z}_p} K)$-module

$$\bigoplus_{i \in I_{v,K}} \big(\mathcal{O}_{E_v} \otimes_{\mathcal{O}_{E_v^{\mathrm{ur}},i}} K\big)\big/\big(\pi_v^{b^{(i)}_1} \mathcal{O}_{E_v} \otimes_{\mathcal{O}_{E_v^{\mathrm{ur}},i}} K\big) \oplus \big(\mathcal{O}_{E_v} \otimes_{\mathcal{O}_{E_v^{\mathrm{ur}},i}} K\big)\big/\big(\pi_v^{b^{(i)}_2} \mathcal{O}_{E_v} \otimes_{\mathcal{O}_{E_v^{\mathrm{ur}},i}} K\big).$$

(ii) Denote by $\mathcal{T}^{\mathrm{sd}}_{E_v,K}$ the subset of $\mathcal{T}_{E_v,K}$ consisting of all elements $\big((b^{(i)}_1, b^{(i)}_2)\big)_{i \in I_{v,K}}$ in $\mathcal{T}_{E_v,K}$ such that $b^{(i)}_1 + b^{(i)}_2 = e_v$ for all $i \in I_{v,K}$.

(iii) Denote by $\boldsymbol{\tau}_{E_v,K}$ the set of all sequences

$$\underline{c}_v = (c^{(i)}_v)_{i \in I_{v,K}}, \quad c^{(i)}_v \in \{0, 1, \ldots, e_v\}, \quad \forall\, i \in I_{v,K}.$$

This set $\boldsymbol{\tau}_{E_v,K}$ is naturally identified with the set of isomorphism classes of cyclic $(\mathcal{O}_{E_v} \otimes_{\mathbb{Z}_p} K)$-modules.

Remark 3.4.1. (a) Each of the local type spaces $\mathcal{T}_{E_v,K}$, $\mathcal{T}^{\mathrm{sd}}_{E_v,K}$ and $\boldsymbol{\tau}_{E_v,K}$ has a natural partial ordering, obtained by suppressing the quantifier "$\forall\, v \in \Sigma_{E,p}$" in Definition 3.3.

(b) Suppose that K is a field which contains a subfield with p^{f_E} elements. Then the finite poset $\mathcal{T}_{E,K}$ (respectively $\mathcal{T}^{\mathrm{sd}}_{E,K}$, $\boldsymbol{\tau}_{E,K}$) is the product, over all $v \in \Sigma_{E,p}$, of the finite posets $\mathcal{T}_{E_v,K}$ (respectively $\mathcal{T}^{\mathrm{sd}}_{E_v,K}$, $\boldsymbol{\tau}_{E_v,K}$).

(c) We have a local version $\mathrm{strdim}_v \colon \mathcal{T}^{\mathrm{sd}}_{E_v,K} \to \mathbb{N}$ of Definition 3.3.4, given by

$$\mathrm{strdim}_v\left((b_1^{(i)}, b_2^{(i)})_{i\in I_{v,K}}\right) = \sum_{i\in I_{v,K}} (b_2^{(i)} - b_1^{(i)}).$$

Then $\mathrm{strdim}(\underline{e}) = \sum_{v\in\Sigma_{E,p}} \mathrm{strdim}_v(\underline{e}_v)$.

3.5. Newton polygons for abelian varieties with real multiplication

We define a finite poset $\mathrm{NP}^{\mathrm{sd}}_E$, which encodes Newton polygons of g-dimensional abelian varieties with real multiplication by $\mathcal{O}_E$.

Definition 3.5.1. (a) Denote by $\mathrm{NP}^{\mathrm{sd}}_E$ the set of all families of pairs $((s_{v,1}, s_{v,2}))_{v\in\Sigma_{E,p}}$ of non-negative rational numbers such that

- $0 \le s_{v,1} \le s_{v,2} \le 1$,
- $s_{v,1} + s_{v,2} = 1$,
- $[E_v : \mathbb{Q}_p] \cdot s_{v,1} \in \mathbb{N}$ and $[E_v : \mathbb{Q}_p] \cdot s_{v,2} \in \mathbb{N}$ whenever $s_{v,1} < s_{v,2}$,

for all $v \in \Sigma_{E,p}$.

Elements of $\mathrm{NP}^{\mathrm{sd}}_E$ will be called *self-dual Newton E-polygons*, often shortened to "Newton E-polygons" or "Newton polygons" if there is no risk of confusion.

(b) For each element $\underline{s} = ((s_{v,1}, s_{v,2}))_{v\in\Sigma_{E,p}}$ in $\mathrm{NP}^{\mathrm{sd}}_E$, denote by $\Gamma_{\underline{s}_v} \subseteq \mathbb{R}^2$ the union of the segment from $(0,0)$ to $([E_v : \mathbb{Q}_p], [E_v : \mathbb{Q}_p]\cdot s_{v,1})$ and the segment from $([E_v : \mathbb{Q}_p], [E_v : \mathbb{Q}_p]\cdot s_{v,1})$ to $(2[E_v : \mathbb{Q}_p], [E_v : \mathbb{Q}_p])$, for every $v \in \Sigma_{E,p}$. This family $(\Gamma_{\underline{s}_v})_{v\in\Sigma_{E,p}}$ of polygons in $\mathbb{R}^2$ is the more classical form of $\underline{s}$.

Definition 3.5.2. We define a partial ordering on $\mathrm{NP}^{\mathrm{sd}}_E$ such that any two elements

$$\underline{s} = ((s_{v,1}, s_{v,2}))_{v\in\Sigma_{E,p}}, \quad \underline{s}' = ((s'_{v,1}, s'_{v,2}))_{v\in\Sigma_{E,p}} \in \mathrm{NP}^{\mathrm{sd}}_E,$$

we have

$$\underline{s} \preceq \underline{s}' \quad \Longleftrightarrow \quad \big(s_{v,1} \ge s'_{v,1},\ \ \forall\, v \in \Sigma_{E,p}\big).$$

Equivalently,

$$\underline{s} \preceq \underline{s}' \quad \Longleftrightarrow \quad \big(\Gamma_{\underline{s}_v} \text{ lies above } \Gamma_{\underline{s}'_v},\ \ \forall\, v \in \Sigma_{E,p}\big).$$

Remark 3.5.3. (a) The finite poset $\mathrm{NP}^{\mathrm{sd}}_E$ is a product of finite posets $\mathrm{NP}^{\mathrm{sd}}_{E_v}$ for all $v \in \Sigma_{E,p}$.

(b) For each v, the finite poset $\mathrm{NP}^{\mathrm{sd}}_{E_v}$ is linearly ordered, and

$$\mathrm{card}(\mathrm{NP}^{\mathrm{sd}}_{E_v}) = 1 + \lceil \tfrac{[E_v:\mathbb{Q}_p]}{2} \rceil.$$

(c) Let A be a g-dimensional abelian variety with real multiplication by $\mathcal{O}_E$ over an algebraically closed field $k \supseteq \mathbb{F}_p$. For each place v of E above p, the rank-two $\mathcal{O}_{E_v}$-linear p-divisible group $A[\wp_v^\infty]$ is either isogenous to direct product of two isoclinic p-divisible groups of height $[E_v : \mathbb{Q}_p]$, with slopes $s_{v,1}$ and $s_{v,2}$, or is isoclinic with slope $1/2$. In the latter case we say that $A[\wp_v^\infty]$ is supersingular and let $s_{v,1} = s_{v,2} = 1/2$. The element $((s_{v,1}, s_{v,2}))_{v\in\Sigma_{E,p}} \in \mathrm{NP}^{\mathrm{sd}}_E$ defined this way is said to be the Newton polygon associated to A; cf. Definition 4.2.5.

(d) The partial ordering on $\mathrm{NP}^{\mathrm{sd}}_E$ is compatible with specialization; see Lemma 4.2.8.

(e) The minimum element in $\mathrm{NP}^{\mathrm{sd}}_E$ for the above partial ordering is the element $\underline{s}_{E,\sigma}$ whose v-component is $(1/2, 1/2)$ for every $v \in \Sigma_{E,p}$, called the *supersingular* Newton polygon.

The maximal element in $\mathrm{NP}^{\mathrm{sd}}_E$ for the above partial ordering is the element $\underline{s}_{E,\mathrm{ord}}$ whose v-component is $(0, 1)$ for every $v \in \Sigma_{E,p}$, called the *ordinary* Newton polygon.

4. Stratifying Hilbert modular varieties by Lie types

4.1. Review of Dieudonné theory

We use covariant Dieudonné theory for commutative finite group schemes and p-divisible groups over perfect fields of characteristic p.

Let $\kappa \supseteq \mathbb{F}_p$ be a perfect field. Let $\Lambda(\kappa)$ be the ring of p-adic Witt vectors with entries in κ. Let σ be the continuous automorphism of $\Lambda(\kappa)$ which sends every element $(a_0, a_1, \ldots, a_n, \ldots) \in \Lambda(\kappa)$ to $(a_0^p, a_1^p, \ldots, a_n^p, \ldots)$.

Let R_κ be the (non-commutative) ring generated by $\Lambda(\kappa)$ and elements $\mathcal{F}$, $\mathcal{V}$, with defining relations

$$\mathcal{F}a = {}^\sigma a\,\mathcal{F}, \quad a\mathcal{V} = \mathcal{V}\,{}^\sigma a, \quad \forall\, a \in \Lambda(\kappa) \quad \text{and} \quad \mathcal{F}\mathcal{V} = \mathcal{V}\mathcal{F} = p.$$

Let $\boldsymbol{CFGS}_\kappa$ be the category of commutative finite group schemes over κ, and let $\boldsymbol{DM}_{\kappa,\mathrm{f}}$ be the category of modules over R_κ which are of finite length as modules over $\Lambda(\kappa)$.

Denote by p-$\boldsymbol{Div}_\kappa$ the category of p-divisible groups over κ, and denote by $\boldsymbol{DM}_\kappa$ the category of modules over R_κ which are free of finite rank as modules over $\Lambda(\kappa)$.

4.1.1. We have an exact covariant additive functor

$$\mathbb{D}\colon \boldsymbol{CFGS}_\kappa \to \boldsymbol{DM}_{\kappa,\mathrm{f}}$$

which is an equivalence of abelian categories and satisfies the following properties.

(i) For every commutative finite group scheme G over κ, with relative Frobenius and Verschiebung homomorphisms

$$\mathrm{Fr}_{G/\kappa}\colon G \to G^{(p)} := G \times_{(\mathrm{Spec}(\kappa),\mathrm{Fr}_\kappa^*)} \mathrm{Spec}(\kappa) \quad \text{and} \quad \mathrm{Ver}_{G/\kappa}\colon G^{(p)} \to G,$$

the operator $\mathcal{V}_G$ on $\mathbb{D}(G)$ is equal to the composition

$$\mathbb{D}(G) \xrightarrow{\mathbb{D}(\mathrm{Fr}_{G/\kappa})} \mathbb{D}(G^{(p)}) \xrightarrow{=} \Lambda(\kappa) \otimes_{(\sigma,\Lambda(\kappa))} \mathbb{D}(G) \xrightarrow{\sigma^{-1}\otimes 1_{\mathbb{D}(G)}} \mathbb{D}(G),$$

and the operator $\mathcal{F}_G$ is equal to the composition

$$\mathbb{D}(G) \xrightarrow{x\mapsto 1\otimes x} \Lambda(\kappa) \otimes_{(\sigma,\Lambda(\kappa))} \mathbb{D}(G) \xrightarrow{=} \mathbb{D}(G^{(p)}) \xrightarrow{\mathbb{D}(\mathrm{Ver}_{G/\kappa})} \mathbb{D}(G).$$

(ii) A commutative finite group scheme G over κ is *etale* (respectively *multiplicative*) if and only if $\mathcal{V}_G$ (respectively $\mathcal{F}_G$) is bijective.

(iii) If the relative Frobenius homomorphism

$$\mathrm{Fr}_{G/\kappa}\colon G \to G^{(p)}$$

is 0, then we have a natural isomorphism

$$\mathbb{D}(G) = \mathrm{Lie}(G)^{(p)} := \mathrm{Lie}(G) \otimes_{(\kappa,\mathrm{Fr}_\kappa)} \kappa.$$

4.1.2. The Dieudonné functor $\mathbb{D}$ in Subsubsection 4.1.1 induces an exact additive functor

$$\mathbb{D}\colon p\text{-}\boldsymbol{Div}_\kappa \to \boldsymbol{DM}_\kappa$$

which is an equivalence of categories with the following properties.

(i) For every p-divisible group X over κ and for every $m \in \mathbb{N}$, we have a functorial isomorphism

$$p^{-m}\mathbb{D}(X)/\mathbb{D}(X) \xrightarrow{\sim} \mathbb{D}(X[p^m])$$

of R_κ-modules.

(ii) The operators $\mathcal{V}_X$, $\mathcal{F}_X$ on the Dieudonné module $\mathbb{D}(X)$ of a p-divisible group X over κ are equal to the compositions

$$\mathbb{D}(X) \xrightarrow{\mathbb{D}(\mathrm{Fr}_{X/\kappa})} \mathbb{D}(X^{(p)}) \xrightarrow{=} \Lambda(\kappa) \otimes_{(\sigma,\Lambda(\kappa))} \mathbb{D}(X) \xrightarrow{\sigma^{-1}\otimes 1_{\mathbb{D}(X)}} \mathbb{D}(X)$$

and

$$\mathbb{D}(X) \xrightarrow{x\mapsto 1\otimes x} \Lambda(\kappa) \otimes_{(\sigma,\Lambda(\kappa))} \mathbb{D}(X) \xrightarrow{=} \mathbb{D}(X^{(p)}) \xrightarrow{\mathbb{D}(\mathrm{Ver}_{X/\kappa})} \mathbb{D}(X)$$

respectively.

(iii) A p-divisible group X over κ is etale (respectively multiplicative) if and only if $\mathcal{V}_X$ (respectively $\mathcal{F}_X$) is bijective.

(iv) For every p-divisible group X over κ we have natural isomorphisms

$$\mathrm{Lie}(X) \xrightarrow{\sim} \mathbb{D}(X)/\mathcal{V}(\mathbb{D}(X)) \quad \text{and} \quad H_1^{\mathrm{dR}}(X/\kappa) \xrightarrow{\sim} \mathbb{D}(X)/p\mathbb{D}(X),$$

so that the short exact sequence

$$0 \to \mathcal{V}(\mathbb{D}(X))/p\mathbb{D}(X) \to \mathbb{D}(X)/p\mathbb{D}(X) \to \mathbb{D}(X)/\mathcal{V}(\mathbb{D}(X)) \to 0$$

is identified with the Hodge filtration of the first de Rham homology group $H_1^{\mathrm{dR}}(X/\kappa)$ of X.

The formulas in Lemma 4.1.3 below have already been stated in Subsubsections 4.1.1(iii) and 4.1.2(iv). We give an explicit isomorphism $\mathbb{D}(X[F]) \cong \mathrm{Lie}(X[F])^{(p)}$ as an illustration.

Lemma 4.1.3. *Let X be a p-divisible group over a perfect field $\kappa \supseteq \mathbb{F}_p$, and let $M := \mathbb{D}(X)$ be the Dieudonné module of X. Let $X[F] = \mathrm{Ker}(\mathrm{Fr}_{X/\kappa}\colon X \to X^{(p)})$ be the kernel of relative Frobenius homomorphism of X. We have natural κ-linear isomorphisms*

$$\mathrm{Lie}(X) \cong M/\mathcal{V}M$$

and

$$\mathbb{D}(X[F]) \xrightarrow[\cong]{} p^{-1}\mathcal{F}(M)/M \xrightarrow[\simeq]{\mathbb{D}(\mathrm{Ver}_X)} (M/\mathcal{V}(M))^{(p)} \xrightarrow[\cong]{} \mathrm{Lie}(X)^{(p)}.$$

Proof. The covariant Dieudonné theory we used is normalized so that the Lie algebra $\mathrm{Lie}(X)$ of X is naturally isomorphic to $\mathbb{D}(X)/\mathcal{V}(\mathbb{D}(X))$. The isomorphism

$$\mathbb{D}(X[F]) \xrightarrow{\cong} p^{-1}\mathcal{F}(M)/M$$

follows from Subsubsection 4.1.2(i) and the exactness of the Dieudonné functor for commutative finite group schemes over κ, because

$$p^{-1}\mathcal{F}(M)/M = \mathrm{Ker}\left(p^{-1}M/M \xrightarrow{\mathcal{V}} p^{-1}M/M\right).$$

Clearly the σ^{-1}-linear operator $\mathcal{V}$ on $p^{-1}M/M$ induces a κ-linear isomorphism

$$(p^{-1}\mathcal{F}(M)/M)^{(p^{-1})} \xrightarrow{\simeq} M/\mathcal{V}(M). \qquad \square$$

Remark 4.1.4. (a) There are (at least) two versions of covariant Dieudonné theory, which differ from each other by a Frobenius twist. The properties in Subsubsections 4.1.1(iii) and 4.1.2(iv) are "normalization conditions" which specify the Dieudonné functors $G \rightsquigarrow \mathbb{D}(G)$ and $X \rightsquigarrow \mathbb{D}(X)$ in Subsubsections 4.1.1 and 4.1.2. Note also that the statement of

Lemma 4.1.3 will need to be modified by a Frobenius twist if a different Dieudonné theory is used.

(b) The functor $G \rightsquigarrow \mathbb{D}(G)$ used here is the covariant crystalline Dieudonné theory, denoted by $G \rightsquigarrow \mathrm{E}_*(G)$ in [8, B.3].

We refer to [8, B.3] for the literature and further information about Dieudonné theory, including the formulas for basic differential invariants in [8, B.3.11] and the comparison with the classical Dieudonné theory in [8, B.3.10].

4.2. Rank-two $\mathcal{O}_E$-linear p-divisible groups and their Dieudonné modules

We apply Dieudonné theory to the p-divisible groups associated to abelian varieties with real multiplication by $\mathcal{O}_E$.

Definition 4.2.1. Let S be an $\mathbb{F}_p$-scheme.

(a) Let v be a place of E above p. A *rank-two $\mathcal{O}_{E_v}$-linear p-divisible group* over S is a pair $(X_v \to S, \iota_v)$, where $X_v \to S$ is a p-divisible group over S of height $2[E_v : \mathbb{Q}_p]$, and $\iota_v \colon \mathcal{O}_{E_v} \to \mathrm{End}_S(X_v)$ is a ring homomorphism.

(b) A *rank-two $\mathcal{O}_E$-linear p-divisible group* over S is a pair $(X \to S, \iota)$ where $X \to S$ is a p-divisible group and $\iota \colon \mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p \to \mathrm{End}_S(X)$ is a ring homomorphism, such that for every $v \in \Sigma_{E,p}$, the p-divisible group $X[\wp_v^\infty] \to S$ has height $2[E_v : \mathbb{Q}_p]$. In other words, there exist rank-two $\mathcal{O}_{E_v}$-linear p-divisible groups $(X_v \to S, \iota_v)$, $v \in \Sigma_{E,p}$, such that the fiber product over S of the (X_v, ι_v)'s is $(\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p)$-linearly isomorphic to (X, ι).

(c) A rank-two $\mathcal{O}_{E_v}$-linear p-divisible group $(X_v \to S, \iota_v)$ is said to be *self-dual* if its Serre dual $(X_v^t \to S, \iota^t)$ is $\mathcal{O}_{E_v}$-linearly isomorphic to itself. A rank-two $\mathcal{O}_E$-linear p-divisible group $(X \to S, \iota)$ is self-dual if its Serre dual $(X^t \to S, \iota^t)$ is $\mathcal{O}_E$-linearly isomorphic to itself.

(d) A rank-two $\mathcal{O}_{E_v}$-linear p-divisible group $(X_v \to S, \iota_v)$ is said to be *self-dual up to isogeny* if there exists an $\mathcal{O}_{E_v}$-linear S-isogeny from (X_v, ι_v) to (X_v^t, ι_v^t). A rank-two $\mathcal{O}_E$-linear p-divisible group $(X \to S, \iota)$ is self-dual up to isogeny if there exists an $\mathcal{O}_E$-linear isogeny over S from X to X^t.

Lemma 4.2.2. *Let (A, ι) be an abelian variety with real multiplication by $\mathcal{O}_E$ over a field $K \supseteq \mathbb{F}_p$.*

(a) *The p-divisible group $A[p^\infty]$ with action by $\mathcal{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is a rank-two $\mathcal{O}_E$-linear p-divisible group which is self-dual up to isogeny.*

(b) *If (A,ι) admits a principal $\widetilde{\mathscr{L}}$-polarization, then the rank-two $\mathcal{O}_E$-linear p-divisible group $(A[p^\infty],\iota[p^\infty])$ is self-dual.*

Lemma 4.2.3. *Let (X,ι) be a self-dual rank-two $\mathcal{O}_E$-linear p-divisible group over a field $K \supset \mathbb{F}_p$, then the isomorphism class in $\boldsymbol{T}_{E,K}$ of the Lie algebra $\mathrm{Lie}(X,\iota)$ of (X,ι) is self-dual; i.e., the class $[\mathrm{Lie}(X,\iota)]$ of $\mathrm{Lie}(X,\iota)$ in the K-group of finitely generated $((\mathcal{O}_E/p\mathcal{O}_E)\otimes_{\mathbb{F}_p} K)$-modules is equal to the class $[(\mathcal{O}_E/p\mathcal{O}_E)\otimes_{\mathbb{F}_p} K]$ of $(\mathcal{O}_E/p\mathcal{O}_E)\otimes_{\mathbb{F}_p} K$.*

Lemma 4.2.4. *Let (X,ι) be a rank-two $\mathcal{O}_E$-linear p-divisible group over a field $K \supseteq \mathbb{F}_p$ which is self-dual up to isogeny. For every $v \in \Sigma_{E,p}$, suppose that the p-divisible group $X[\wp_v^\infty]$ is not isoclinic of slope $1/2$. Then $X[\wp_v^\infty]$ is isogenous to the product of two isoclinic $\mathcal{O}_{E_v}$-linear p-divisible groups Y_1, Y_2 of height $[E_v:\mathbb{Q}_p]$, and the slopes $s_{v,1}$, $s_{v,2}$ of Y_1, Y_2 satisfy*

$$0 \le s_{v,1}, s_{v,2} \le 1, \quad s_{v,1}+s_{v,2}=1 \quad \textit{and} \quad [E_v:\mathbb{Q}_p]\cdot s_{v,i} \in \mathbb{N}.$$

Definition 4.2.5. (a) Let (X,ι) be a rank-two $\mathcal{O}_E$-linear p-divisible group over a field $K \supseteq \mathbb{F}_p$ which is self-dual up to isogeny. The *Newton polygon* of (X,ι) is the element $((s_{v,1},s_{v,2}))_{v\in\Sigma_{E,p}} \in \mathrm{NP}_E^{\mathrm{sd}}$ such that

$$(s_{v,1},s_{v,2}) = \begin{cases} (s_{X,v,1},s_{X,v,2}) & \text{if } X[\wp_v^\infty] \text{ has two distinct slopes } s_{X,v,1} < s_{X,v,2}, \\ (1/2,1/2) & \text{if } X[\wp_v^\infty] \text{ is isoclinic.} \end{cases}$$

(b) Let (A,ι) be an abelian variety with real multiplication by $\mathcal{O}_E$ over a field $K \supseteq \mathbb{F}_p$. The *Newton polygon* of (A,ι) is by definition the Newton polygon of $(A[p^\infty],\iota[p^\infty])$.

Definition 4.2.6. For each $\underline{s} \in \mathrm{NP}_E^{\mathrm{sd}}$, denote by $\mathcal{W}_{\underline{s}} = \mathcal{W}_{\underline{s}}(\mathscr{M}_E)$ the locally closed subset of $\mathscr{M}_E$ consisting of all points $x \in \mathscr{M}_E$ whose Newton polygon is equal to $\underline{s}$. Clearly $\mathscr{M}_E$ is the disjoint union of all Newton polygon strata $\mathcal{W}_{\underline{s}}$, where $\underline{s}$ runs through all elements of $\mathrm{NP}_E^{\mathrm{sd}}$.

Remark. The Newton polygon stratum $\mathcal{W}_{\underline{s}_{E,\sigma}}$ corresponding to the minimal element $\underline{s}_{E,\sigma}$ in $\mathrm{NP}_E^{\mathrm{sd}}$, called the supersingular locus in $\mathscr{M}_E$, is the closed subset of $\mathscr{M}_E$ consisting of points whose underlying abelian varieties are isoclinic of slope $1/2$.

Remark 4.2.7. There is another notion of Newton polygon strata attached to a Newton polygon $\underline{s}$ in the literature. What was defined in Definition 4.2.5(a) is sometimes called the *open Newton polygon stratum* attached to $\underline{s}$ and denoted by $\mathcal{W}^0_{\underline{s}}$.

Lemma 4.2.8. *Let S be a scheme over $\mathbb{F}_p$ and let $(X \to S,\iota)$ be a rank-two $\mathcal{O}_E$-linear p-divisible group with real multiplication by $\mathcal{O}_E$ which is self-dual up to isogeny. Let z, z' be points of S, and let $\underline{s}_z, \underline{s}_{z'} \in \mathrm{NP}_E^{\mathrm{sd}}$ be the Newton -polygons of X_z and $X_{z'}$ respectively. If z is a specialization of z', then $\underline{s}_z \preceq \underline{s}_{z'}$.*

Corollary 4.2.9. *Let $\underline{s}$ be an element of $\mathrm{NP}_E^{\mathrm{sd}}$. The Zariski closure $(\mathcal{W}_{\underline{s}})^{\mathrm{zar}}$ of the Newton polygon stratum $\mathcal{W}_{\underline{s}}$ in $\mathscr{M}_E$ is contained in the union of all Newton polygon strata $\mathcal{W}_{\underline{s}'}$ with $\underline{s}' \preceq \underline{s}$:*

$$(\mathcal{W}_{\underline{s}})^{\mathrm{zar}} \subseteq \bigsqcup_{\underline{s}' \preceq \underline{s}} \mathcal{W}_{\underline{s}'}.$$

Remark. Whether $(\mathcal{W}_{\underline{s}})^{\mathrm{zar}}$ is equal to $\bigsqcup_{\underline{s}' \preceq \underline{s}} \mathcal{W}_{\underline{s}'}$ is the content of the Grothendieck conjecture on the Newton polygon stratification of $\mathscr{M}_E$; cf. Question 10.4(d)–(e).

Definition 4.2.10. Let κ be a field which contains a subfield with p^{f_E} elements. Let $\kappa_1 = \kappa^{\mathrm{perf}}$ be the perfection of κ.

(a) Let v be a place of E above p and let (X_v, ι_v) be a rank-two $\mathcal{O}_{E_v}$-linear p-divisible group over κ. The *Lie type* of (X_v, ι_v) is the element $\underline{e}_{X_v} \in \mathcal{T}_{E_v}$ which corresponds to the isomorphism class of the $(\mathcal{O}_{E_v} \otimes_{\mathbb{Z}_p} \kappa)$-module $\mathrm{Lie}(X_v)$. Equivalently, $\underline{e}_{X_v}$ is the element of $\mathcal{T}_{E_v,\kappa}$ whose image in $\mathcal{T}_{E_v,\kappa_1}$ under the bijection $\mathcal{T}_{E_v,\kappa} \xrightarrow{\sim} \mathcal{T}_{E_v,\kappa_1}$ is the isomorphism class of $\mathbb{D}(X_v \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\kappa_1))/\mathcal{V} \cdot \mathbb{D}(X_v \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\kappa_1))$.

(b) Let (X, ι) be a rank-two $\mathcal{O}_E$-linear p-divisible group over κ. The *Lie type* of (X, ι) is the element $\underline{e}_X \in \mathcal{T}_E$ which corresponds to the isomorphism class of the $(\mathcal{O}_E \otimes_{\mathbb{Z}} \kappa)$-module $\mathrm{Lie}(X)$. In other words $\underline{e}_X = (\underline{e}_{X_v})_{v \in \Sigma_{E,p}}$, where $\underline{e}_{X_v}$ is the Lie type of the rank-two $\mathcal{O}_{E_v}$-linear p-divisible group $X_v = X[\wp_v^\infty]$, for every place v of E above p.

(c) Let (A, ι) be an abelian variety with real multiplication by $\mathcal{O}_E$. The *Lie type* $\underline{e}_A$ of (A, ι) is by definition the Lie type of the rank-two $\mathcal{O}_E$-linear p-divisible group $(A[p^\infty], \iota[p^\infty])$.

Lemma 4.2.11. *Let $k \supseteq \mathbb{F}_p$ be an algebraically closed field, and let (X, ι) be a rank-two $\mathcal{O}_E$-linear p-divisible group. Let $\underline{e}_X = \big(e_{X,v,1}^{(i)}, e_{X,v,2}^{(i)}\big)_{v \in \Sigma_{E,p}, i \in I_v} \in \mathcal{T}_E$ be the Lie type of (X, ι). Then (X, ι) is self-dual if and only if $\underline{e}_X \in \mathcal{T}_E^{\mathrm{sd}}$, i.e., $e_{X,v,1}^{(i)} + e_{X,v,2}^{(i)} = e_v$ for all $v \in \Sigma_{E,p}$ and all $i \in I_v$.*

The definition and properties of Lie stratification of Hilbert modular varieties, due to Deligne and Pappas in [13, Sections 3–4], is summarized in Theorem 4.3 below.

Theorem 4.3. *For each $\underline{e} \in \mathcal{T}_E^{\mathrm{sd}}$, there exists a locally closed subscheme*

$$\mathcal{N}_{\underline{e}}(\mathscr{M}_E) \subseteq \mathscr{M}_E = \mathscr{M}_{E,n,\overline{\delta}}^{\widetilde{\mathscr{L}}}$$

over $\overline{\mathbb{F}_p}$, characterized by the property that for every point $x \in \mathscr{M}_E$, we have

$$x \in \mathcal{N}_{\underline{e}}(\mathscr{M}_E) \quad \Longleftrightarrow \quad \underline{e}_{(A_x, \iota_x)} = \underline{e}.$$

Here (A_x, ι_x) denotes the fiber at x of the universal abelian scheme with real multiplication by $\mathcal{O}_E$ over $\mathscr{M}_E$. The subvarieties $\mathcal{N}_{\underline{e}} = \mathcal{N}_{\underline{e}}(\mathscr{M}_E)$ satisfy the following properties:

(1) *The modular variety $\mathscr{M}_E$ is the disjoint union of these Lie strata:*

$$\mathscr{M}_E = \bigsqcup_{\underline{e}\in\mathcal{T}_E^{\mathrm{sd}}} \mathcal{N}_{\underline{e}}(\mathscr{M}_E) = \bigsqcup_{\underline{e}\in\mathcal{T}_E^{\mathrm{sd}}} \mathcal{N}_{\underline{e}}.$$

(2) *Each Lie stratum $\mathcal{N}_{\underline{e}}$ is a locally closed* smooth *subscheme of $\mathscr{M}_E$.*

(3) *The dimension of the Lie stratum $\mathcal{N}_{\underline{e}}$ indexed by an element*

$$\underline{e} = \Big(\big(e_{v,1}^{(i)}, e_{v,2}^{(i)}\big)_{i\in I_v} \Big)_{v\in\Sigma_{E,p}} \in \mathcal{T}_E^{\mathrm{sd}}$$

is given by

$$\dim(\mathcal{N}_{\underline{e}}) = \mathrm{strdim}(\underline{e}) = \sum_v \sum_{i\in I_v} \big(e_{v,2}^{(i)} - e_{v,1}^{(i)}\big).$$

In particular the dimension of the minimal Lie stratum $\mathcal{N}_{\min} = \mathcal{N}_{\underline{e}_{\min}}(\mathscr{M}_E)$ is

$$\dim(\mathcal{N}_{\min}) = \sum_{e_v \text{ odd}} f_v,$$

where the sum runs through all places v of E above p with odd ramification index e_v, and $f_v = [\kappa_v : \mathbb{F}_p]$ is the degree over $\mathbb{F}_p$ of the residue field κ_v of $\mathcal{O}_{E_v}$.

(4) *The maximal Lie stratum $\mathcal{N}_{\max} = \mathcal{N}_{\underline{e}_{\max}}(\mathscr{M}_E)$ coincides with the smooth locus of $\mathscr{M}_E$, i.e., the largest open subset of $\mathscr{M}_E$ which is smooth over $\overline{\mathbb{F}_p}$.*

(5) *The Zariski closure $(\mathcal{N}_{\underline{e}})^{\mathrm{zar}}$ of a Lie stratum $\mathcal{N}_{\underline{e}}$ in $\mathscr{M}_E$ is the disjoint union of all strata $\mathcal{N}_{\underline{e}'}$, where $\underline{e}'$ runs through all elements in $\mathcal{T}_E^{\mathrm{sd}}$ such that $\underline{e}' \preceq \underline{e}$, i.e.,*

$$(\mathcal{N}_{\underline{e}})^{\mathrm{zar}} = \bigsqcup_{\underline{e}'\preceq\underline{e}} \mathcal{N}_{\underline{e}'}.$$

(6) *For each $\underline{e} \in \mathcal{T}_E^{\mathrm{sd}}$, $(\mathcal{N}_{\underline{e}})^{\mathrm{zar}}$ is a local complete intersection and is normal, and $\mathcal{N}_{\underline{e}}$ is the largest open subset of $(\mathcal{N}_{\underline{e}})^{\mathrm{zar}}$ which is smooth over $\overline{\mathbb{F}_p}$.*

(7) *Suppose that $\underline{e}_1 \precneqq \underline{e}_2$ are elements of $\mathcal{T}_E^{\mathrm{sd}}$, and there is no element $\underline{e} \in \mathcal{T}_E^{\mathrm{sd}}$ such that $\underline{e}_1 \precneqq \underline{e} \precneqq \underline{e}_2$, then $\dim(\mathcal{N}_{\underline{e}_1}) = \dim(\mathcal{N}_{\underline{e}_2}) - 2$.*

5. The congruity invariant

In this section we define a numerical invariant, called *congruity*, attached to every rank-two $\mathcal{O}_E$-linear p-divisible group (X, ι) over an algebraically closed field k of characteristic p. When the congruity invariant $c_{(X,\iota)}$ takes certain specific values, the isomorphism class of (X, ι) is uniquely determined by the invariant $c_{(X,\iota)}$, in which case (X, ι) is said to be

distinguished; see Definitions 5.3.2, 5.4.3 and 5.6. The definition of the congruity invariant is described in the next two paragraphs.

In Definition 5.1.1 we define an $\mathbb{N}$-valued invariant $c(M; N, N')$ associated to an unordered pair (N, N') of two lattices in a rank-two module M over a discrete valuation ring, which measures the extent the of the failure of these two lattices in M to be transversal up to rescaling inside M.

For a rank-two $\mathcal{O}_E$-linear p-divisible group (X, ι) over a sufficiently large perfect field κ_1 of characteristic p, the Dieudonné module $\mathbb{D}(X)$ is a projective rank-two module over $\mathcal{O}_E \otimes_{\mathbb{Z}} \Lambda(\kappa_1)$, which decomposes into a direct sum, indexed by the disjoint union of the sets I_v, of rank-two projective modules $\mathbb{D}(X)_v^{(i)}$ over discrete valuation rings $\mathcal{O}_{E_v} \otimes_{(\mathcal{O}_{E_v}^{\mathrm{ur}}, i)} \Lambda(\kappa_1)$. The same holds for $\mathcal{F}(\mathbb{D}(X))$ and $\mathcal{V}(\mathbb{D}(X))$. For each place v of E above p and each $i \in I_v$, the construction in Definition 5.1.1 applied to $\mathbb{D}(X)_v^{(i)}$, $(\mathcal{F}\mathbb{D}(X))_v^{(i)}$ and a "transport" to $\mathbb{D}(X)_v^{(i)}$ of $(\mathcal{V}\mathbb{D}(X))_v^{(i')}$ for a suitable element $i' \in I_v$, gives the (v, i)-component of the *congruity invariant* $\underline{c}_{(X,\iota)}$ of (X, ι) defined in Definition 5.2.

5.1. Lattices in a free rank-two module over a discrete valuation ring

In this subsection, D denotes a discrete valuation ring with maximal ideal $\wp$ and residue field $\kappa = D/\wp$. Let M be a free rank-two D-module. We will define a discrete invariant $c(M; N, N')$ attached to two lattices N, N' in M.

Recall that the elementary divisors (b_1, b_2) of a lattice N in M, $b_1 \leq b_2$, are the two natural numbers determined by $M/N \cong D/\wp^{b_1} \oplus D/\wp^{b_2}$, or equivalently there exists a D-basis $v_1, v_2 \in M$ such that $N = \wp^{b_1} \cdot v_1 + \wp^{b_2} \cdot v_2$. Note that $b_1 = \max\{b \mid N \subseteq \wp^b M\}$, and $b_2 = \min\{b \mid N \supseteq \wp^b M\}$.

Definition 5.1.1 (An invariant of two lattices in D^2)**.** Let N, N' be two lattices in M, with elementary divisors (b_1, b_2) and (b_1', b_2') respectively, $b_1 \leq b_2$, $b_1' \leq b_2$. Define an invariant $c(M; N, N')$ by

$$M/(\wp^{-b_1} N + \wp^{-b_1'} N') \cong D/\wp^{c(M;N,N')}.$$

Note that the elementary divisors of $\wp^{-b_1} N$ and $\wp^{-b_1'} N'$ are $(0, b_2 - b_1)$ and $(0, b_2' - b_1')$ respectively. So $M/(\wp^{-b_1} N + \wp^{-b_1'} N')$ is a cyclic D-module of finite length. Note also that the equality

$$0 \leq c(M; N, N') \leq \min\{b_2 - b_1, b_2' - b_1'\}$$

holds.

Remark 5.1.2. The congruity invariant can be computed as follows. Let (x_1, x_2) be a D-basis of M such that $N = \wp^{b_1} x_1 + \wp^{b_2} x_2$, and let (y_1, y_2) be a D-basis of M such that

$N' = \wp^{b_1'} y_1 + \wp^{b_2'} y_2$. Define an element $(a_{ij})_{i,j=1,2} \in \mathrm{GL}_2(D)$ by

$$y_j = \sum_{i=1,2} a_{ij} x_i, \quad j = 1, 2.$$

From

$$\begin{aligned} \wp^{-b_1} N + \wp^{-b_1'} N' &= Dx_1 + D \cdot a_{21} x_2 + \wp^{b_2 - b_1} x_2 + \wp^{b_2' - b_1'} y_2 \\ &= Dy_1 + D \cdot a_{21} y_2 + \wp^{b_2 - b_1} x_2 + \wp^{b_2' - b_1'} y_2, \end{aligned}$$

we see that

$$c(M; N, N') = \min\{\mathrm{ord}_\wp(a_{21}), b_2 - b_1, b_2' - b_1'\}.$$

Definition 5.1.3. Let $\underline{e}_X = (\underline{e}_v)_{v \in \Sigma_{E,p}} = \big((e_{v,1}^{(i)}, e_{v,2}^{(i)})_{i \in I_v}\big)_{v \in \Sigma_{E,p}}$ be an element of $\mathcal{T}_E^{\mathrm{sd}}$. For every $v \in \Sigma_{E,p}$, define a function $t_{\underline{e}_v} \colon I_v \to \mathbb{N} \cup \{-\infty\}$ by

$$t_{\underline{e}_v}(i) := \min\{n \in \mathbb{N} \mid e_1^{(i+n)} < e_2^{(i+n)}\}, \quad \forall\, i \in I_v.$$

Note that $t_{\underline{e}_v}(i) = -\infty$ if and only if the set $\big\{n \in \mathbb{N} \mid e_1^{(i+n)} < e_2^{(i+n)}\big\}$ is empty, in which case the function $t_{\underline{e}_v}$ on I_v has constant value $-\infty$. Note also that $t_{\underline{e}_v}(i) = 0$ if and only if $e_{v,1}^{(i)} < e_{v,2}^{(i)}$.

Definition 5.2 (Congruity)**.** Let κ be a field which contains a finite subfield with p^{f_E} elements. Let $\kappa_1 = \kappa^{\mathrm{perf}}$ be the perfection of κ.

(a) Let v be a place of E above p. Let (X, ι) be a rank-two $\mathcal{O}_E$-linear p-divisible group over κ. Let $M_v := \mathbb{D}(X_v \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\kappa_1))$ be the covariant Dieudonné module of $X_v \times_{\mathrm{Spec}(\kappa)} \mathrm{Spec}(\kappa_1)$. Let

$$M_v = \bigoplus_{i \in I_v} M_v^{(i)}$$

be the decomposition of M_v corresponding to the product decomposition of the ring $\mathcal{O}_{E_v} \otimes_{\mathbb{Z}_p} \Lambda(\kappa)$ in Subsubsection 3.1.2. Note that $M_v^{(i)}$ is a free $\Lambda(\kappa)^{(i)}$-module of rank two for every $i \in I_v$. Moreover we have

$$\mathcal{F}(M_v^{(i)}) \subseteq M_v^{(i+1)} \quad \text{and} \quad \mathcal{V}(M_v^{(i)}) \subseteq M_v^{(i-1)}, \quad \forall\, (v, i).$$

Let

$$\underline{e}_X = (\underline{e}_v)_{v \in \Sigma_{E,p}} = \Big(\big(e_{v,1}^{(i)}, e_{v,2}^{(i)}\big)_{i \in I_v}\Big)_{v \in \Sigma_{E,p}}$$

be the Lie type of X.

We will define an element $\underline{c}_{X_v} = \big(c_{X_v}^{(i)}\big)_{i \in I_v}$ in $\boldsymbol{\tau}_{E_v,\kappa} \xrightarrow{\sim} \boldsymbol{\tau}_{E_v,\kappa_1}$, called the *congruity* of the rank-two $\mathcal{O}_{E_v}$-linear p-divisible group (X_v, ι_v). For every $i \in I_v$, the i-component

$c_{X_v}^{(i)}$ of $\underline{c}_{X_v}$ is given by

$$c_{X_v}^{(i)} = \begin{cases} 0 & \text{if } t_{\underline{e}_v}(i) = -\infty, \\ c\big(M_v^{(i)}; \mathcal{F}(M_v^{(i-1)}), \mathcal{V}(M_v^{(i+1)})\big) & \text{if } e_{v,1}^{(i)} < e_{v,2}^{(i)}, \\ c\big(M_v^{(i)}; \mathcal{F}(M_v^{(i-1)}), \\ \quad \pi_v^{-\sum_{0\le n\le t_{\underline{e}_v}(i)-1} e_{v,1}^{(i+n)}} \mathcal{V}^{t_{\underline{e}_v}(i)+1} M_v^{(i+t_{\underline{e}_v}(i)+1)}\big) & \text{if } t_{\underline{e}_v}(i) > 0. \end{cases}$$

(b) Let (X, ι) be a rank-two $\mathcal{O}_E$-linear p-divisible group over κ. For every $v \in \Sigma_{E,p}$, let $X_v := X[\wp_v^\infty]$, an $\mathcal{O}_{E_v}$-linear p-divisible group over κ. The *congruity invariant* $\underline{c}_X$ of (X, ι) is the element

$$\underline{c}_X := (\underline{c}_{X_v})_{v\in\Sigma_{E,p}} = \Big(\big(c_{X_v}^{(i)}\big)_{i\in I_v}\Big)_{v\in\Sigma_{E,p}}$$

of $\boldsymbol{\tau}_E = \boldsymbol{\tau}_{E,\kappa}$.

Remark 5.2.1. (a) In the case $t_{\underline{e}_v}(i) > 0$, we have

$$(\mathcal{V}(M_v))^{(i+n)} = \mathcal{V}(M_v^{(i+n+1)}) = \pi_v^{e_{v,1}^{(i+n)}} M_v^{(i+n)} \quad \text{for } n = 0, 1, \dots, t_{\underline{e}_v}(i) - 1,$$

and we use the σ^{-1}-linear isomorphisms

$$\pi_v^{-e_{v,1}^{(i+n)}} \mathcal{V} \colon M_v^{(i+n+1)} \xrightarrow{\sim} M_v^{(i+n)}, \quad n = 0, 1, \dots, t_{\underline{e}_v}(i) - 1$$

to transport the lattice $\mathcal{V}(M_v^{(i+t_{\underline{e}_v}(i)+1)})$ in $M_v^{(i+t_{\underline{e}_v}(i))}$ back to $M_v^{(i)}$. The (v, i)-component of $\underline{c}_X$ is defined to be the congruity of $\mathcal{F}(M_v^{(i-1)})$ and the result of the transport, two submodules of the free $\Lambda(\kappa)^{(i)}$-module $M_v^{(i)}$ of rank two.

(b) It is easy to see that the elementary divisors of the $\Lambda(\kappa)^{(i)}$-submodule $\mathcal{F}(M_v^{(i-1)}) \subseteq M_v^{(i)}$ are $\big(e_v - e_{v,2}^{(i-1)}, e_v - e_{v,1}^{(i-1)}\big)$ because $\mathcal{V} \circ \mathcal{F} = p = \mathcal{F} \circ \mathcal{V}$. Therefore,

$$0 \le c_{X,v}^{(i)} \le \min\{e_{v,2}^{(i-1)} - e_{v,1}^{(i-1)}, e_{v,2}^{(i+t)} - e_{v,1}^{(i+t)}\}, \quad \forall\, v \in \Sigma_{E,p},\ \forall\, i \in I_v,$$

where $t = t_{\underline{e}_v}(i)$.

Remark 5.2.2. The congruity invariant $\underline{c}_X$ and the Lie type of (X, ι) determine the α-type of (X, ι). The relations between the α-type and the Newton polygon were investigated in [39].

Proposition 5.3. *Let v be a place of E above p, and let (X_v, ι_v) be a rank-two $\mathcal{O}_{E_v}$-linear p-divisible group over an algebraically closed field $k \supseteq \mathbb{F}_p$. Let $\underline{e}_{X_v} = \big(e_{X_v,1}^{(i)}, e_{X_v,2}^{(i)}\big)_{i\in I_v}$, $\underline{c}_{X_v} = \big(c_{X_v}^{(i)}\big)_{i\in I_v}$ and $\underline{s}_{X_v} = (s_{X_v,1}, s_{X_v,2})$ be the Lie type, congruity and Newton polygon of (X_v, ι_v) respectively.*

(1) $[E_v:\mathbb{Q}_p]\cdot\sum_{i\in I_v} e^{(i)}_{X_v,1}\le s^{(i)}_{X_v,1}$.

(2) *The following statements are equivalent.*

(a) $[E_v:\mathbb{Q}_p]\cdot\sum_{i\in I_v} e^{(i)}_{X_v,1}= s^{(i)}_{X_v,1}$.

(b) $\underline{c}_{X_v}=\underline{0}$, *i.e.,* $c^{(i)}_{X_v}=0$ *for every* $i\in I_v$.

(c) *There exist*

- *isoclinic p-divisible groups* Y_1, Y_2 *over* k *of height* $[E_v:\mathbb{Q}_p]$ *of slopes* $s_{X_v,1}$ *and* $s_{X_v,2}$ *respectively, and*
- *ring homomorphisms* $\iota_j\colon \mathcal{O}_{E_v}\to \mathrm{End}(Y_j)$ *for* $j=1,2$, *and*
- *an* $\mathcal{O}_{E_v}$*-linear isomorphism*

$$(X,\iota)\xrightarrow{\sim}(Y_1,\iota_1)\times(Y_2,\iota_2).$$

See [9, Section 4.3] for a proof of Proposition 5.3.

Remark 5.3.1. For $j=1,2$, denote by $\underline{e}_{Y_j}$ the Lie type of the CM p-divisible group Y_j in Proposition 5.3(2), i.e., the element of $\boldsymbol{\tau}_{E_v}$ corresponding to the $(\mathcal{O}_{E_v}\otimes_{\mathbb{Z}_p} k)$-module $\mathbb{D}(Y_j)/\mathcal{V}(\mathbb{D}(Y_j))$. Then

$$\underline{e}_{Y_1}=(e^{(i)}_{X_v,1})_{i\in I_v},\quad \underline{e}_{Y_2}=(e^{(i)}_{X_v,2})_{i\in I_v}.$$

Since two $\mathcal{O}_{E_v}$-linear CM p-divisible groups over an algebraically closed field $k\supseteq\mathbb{F}_p$ with the same Lie type are $\mathcal{O}_{E_v}$-linearly isomorphic, the unordered pair $\{(Y_1,\iota_1),(Y_2,\iota_2)\}$ in Proposition 5.3 is uniquely determined by the congruity invariant $\underline{c}_{X_v}$ of (X_v,ι_v) up to isomorphism.

Definition 5.3.2 (Exquisite rank-two $\mathcal{O}_{E_v}$-linear p-divisible groups)**.** Let v be a place of E above p and let (X_v,ι_v) be a rank-two $\mathcal{O}_{E_v}$-linear p-divisible group over a field $K\supseteq\mathbb{F}_p$. We say that (X_v,ι_v) is *exquisite* if for some algebraically closed extension field k of K, or equivalently for every algebraically closed extension field k of K, the equivalent conditions (a)–(c) in Proposition 5.3(2) are satisfied.

Corollary 5.3.3. *Let* v *be a place of* E *above* p *and let* $k\supseteq\mathbb{F}_p$ *be an algebraically closed field. If* (X_1,ι_1) *and* (X_2,ι_2) *are two exquisite rank-two* $\mathcal{O}_{E_v}$*-linear p-divisible groups with the same Lie type, then there exists an* $\mathcal{O}_{E_v}$*-linear isomorphism from* (X_1,ι_1) *to* (X_2,ι_2).

5.4.

In this subsection we define the notion of *balanced self-dual rank-two p-divisible groups.* We begin with a preliminary statement in Lemma 5.4.1, whose proof is obvious.

Lemma 5.4.1. *Let K be a field which contains a subfield isomorphic to κ_v, where v is a place of E above p. Let (X_v, ι_v) be a self-dual rank-two $\mathcal{O}_{E_v}$-linear p-divisible group over K whose Lie type $\underline{e}_{X_v}$ is equal to the minimal element $\underline{e}^{\mathrm{sd}}_{E_v,\min}$ in $\mathcal{T}^{\mathrm{sd}}_{E_v}$.*

(a) *Suppose that the ramification index e_v of $E_v/\mathbb{Q}_p$ is even. Then the congruity $\underline{c}_{X_v}$ of (X_v, ι_v) is $\underline{0}$, i.e., $c^{(i)}_{X,v} = 0$ for all $i \in I_v$.*

(b) *Suppose that the ramification index e_v of $E_v/\mathbb{Q}_p$ is odd. Let $\underline{c}_{X_v} = \big(c^{(i)}_{X_v}\big)_{i\in I_v}$ be the congruity of (X_v, ι_v). Then $c^{(i)}_{X_v} \in \{0, 1\}$ for every $i \in I_v$.*

In the case of (b), the subset

$$\{i \in I_v \mid c^{(i)}_{X_v} = 1\}$$

will be called the *support* of $\underline{c}_{X_v}$.

Definition 5.4.2 (Balanced subsets of I_v)**.** (a) A non-empty subset J of the $(\mathbb{Z}/f_v\mathbb{Z})$-torsor I_v is an *interval* if $J \subsetneqq I_v$ and there exists an element $i_0 \in I$ and a natural number b with $0 \le b \le f_v - 2$ such that $J = \{i_0, i_0+1, \ldots, i_0+b\}$. Such an interval will be denoted by $[i_0, i_0 + b]$.

(b) A subset J of I_v is *connected* if $J = \emptyset$, or if $J = I_v$, or if J is an interval in I_v.

(c) A *connected component* of a subset I' of I_v is a maximal element in the family of connected subsets of I'.

(d) A subset I' of I is *balanced* if the cardinality of every connected component of I' is even. In particular the empty subset $\emptyset$ is balanced, and I_v is balanced if and only if f_v is even.

Definition 5.4.3 (Balanced rank-two $\mathcal{O}_{E_v}$-linear p-divisible groups)**.** Let v be a place of E above p. Let K be a field which contains a finite subfield isomorphic to κ_v. A self-dual rank-two $\mathcal{O}_{E_v}$-linear p-divisible group (X_v, ι_v) over K is said to be *balanced* if the following conditions are satisfied.

(a) The Lie type $\underline{e}_{X_v}$ of (X_v, ι_v) is equal to the minimal element

$$\underline{e}^{\mathrm{sd}}_{E_v,\min} = \big((\lfloor e_v/2\rfloor, \lceil e_v/2\rceil)\big)_{i\in I_v}$$

in the local type space $\mathcal{T}^{\mathrm{sd}}_{E_v}$.

(b) The support of $\underline{c}_{X_v}$ is a balanced subset of I_v.

Remark 5.4.4. Note that if the ramification index e_v of $E_v/\mathbb{Q}_p$ is even, then the condition in Definition 5.4.3(b) is empty, and the congruity $\underline{c}_{X_v}$ of every self-dual rank-two p-divisible group (X_v, ι_v) over a field K which satisfies Definition 5.4.3(a) is both exquisite and balanced. So the notion of balanced self-dual rank-two $\mathcal{O}_{E_v}$-linear p-divisible groups is of interest only when e_v is odd.

Proposition 5.4.5. *Let v be a place of E above p such that $e_v = 2d_v + 1$ is odd. Let $k \supseteq \mathbb{F}_p$ be an algebraically closed field.*

(a) *Let (X_v, ι_v) be a balanced rank-two $\mathcal{O}_{E_v}$-linear p-divisible group. Let I' be the support of $\underline{c}_{X_v} \neq \underline{0}$. Assume that $I' \neq \emptyset$, and let I_1 be a subset of I_v such that I' is the disjoint union of I_1 and $I_1 + 1$. There exist $\mathcal{O}_{E_v}$-linear p-divisible groups (Y_1, ι_1), (Y_2, ι_2) over k of height $[E_v : \mathbb{Q}_p]$ with the following properties:*

- *The Lie type $\underline{e}_{Y_1} = \left(e^{(i)}_{Y_1}\right)_{i \in I_v}$ of (Y_1, ι_1) is given by*

$$e^{(i)}_{Y_1} = \begin{cases} d_v + 1 & \text{if } i \in I_1, \\ d_v & \text{if } i \notin I_1. \end{cases}$$

- *The Lie type $\underline{e}_{Y_2} = \left(e^{(i)}_{Y_2}\right)_{i \in I_v}$ of (Y_2, ι_2) is given by*

$$e^{(i)}_{Y_2} = \begin{cases} d_v & \text{if } i \in I_1, \\ d_v + 1 & \text{if } i \notin I_1. \end{cases}$$

- *There exists an $\mathcal{O}_{E_v}$-linear isomorphism from (X_v, ι_v) to $(Y_1, \iota_1) \times (Y_2, \iota_2)$.*

(b) *Suppose that (X_v, ι_v) and (X'_v, ι'_v) are balanced rank-two $\mathcal{O}_{E_v}$-linear p-divisible groups over k with the same congruity invariant. Then (X_v, ι_v) is $\mathcal{O}_{E_v}$-linearly isomorphic to (X'_v, ι'_v).*

See [9, Section 4.4] for a proof of Proposition 5.4.5.

Definition 5.5 (E-minimal self-dual rank-two $\mathcal{O}_E$-linear p-divisible groups)**.** Let K be a field which contains a finite subfield with p^{f_E} elements.

(a) Let v place of E above p, and let (X_v, ι_v) be a self-dual rank-two $\mathcal{O}_{E_v}$-linear p-divisible group over K. We say that (X_v, ι_v) is E_v-minimal if its Lie type $\underline{e}_{X_v}$ is the minimal element $\underline{e}^{\mathrm{sd}}_{E_v,\min}$ in $\mathcal{T}^{\mathrm{sd}}_{E_v}$, and its congruity $\underline{c}_{X_v} = \left(c^{(i)}_{X_v}\right)_{i \in I_v}$ is given by

$$c^{(i)}_{X_v} = \begin{cases} 0 & \text{if } e_v \text{ is even}, \\ 1 & \text{if } e_v \text{ is odd} \end{cases}$$

for all $i \in I_v$.

(b) Let (X, ι) be a self-dual rank-two $\mathcal{O}_E$-linear p-divisible group over K. We say that (X, ι) is *E-minimal* if the v-component $(X[\wp_v^\infty], \iota[\wp_v^\infty])$ of (X, ι) is E_v-minimal for every place v of E above p.

Remark 5.5.1. (a) If (X_v, ι_v) is an E_v-minimal self-dual rank-two $\mathcal{O}_{E_v}$-linear p-divisible group over an algebraically closed field $k \supseteq \mathbb{F}_p$, then the p-divisible group X_v is superspecial in the sense that it is isomorphic to the product of $[E_v : \mathbb{Q}_p]$ copies of the p-divisible group attached to a supersingular elliptic curve over k.

(b) The terminology "E-minimal" comes from [31].

Lemma 5.5.2. *Let v be a place of E above p, and let $k \supseteq \overline{\mathbb{F}_p}$ be an algebraically closed field. Any two E_v-minimal self-dual rank-two $\mathcal{O}_{E_v}$-linear p-divisible groups are $\mathcal{O}_{E_v}$-linearly isomorphic.*

Definition 5.6 (Distinguished self-dual rank-two $\mathcal{O}_E$-linear p-divisible groups)**.** Let K be a field which contains a finite subfield with p^{f_E} elements. Let (X, ι) be a self-dual rank-two $\mathcal{O}_E$-linear p-divisible group over K. Write (X, ι) as a product of self-dual rank two $\mathcal{O}_{E_v}$-linear p-divisible groups over K:

$$(X, \iota) = \prod_{v \in \Sigma_{E,p}} (X_v, \iota_v),$$

where $(X_v, \iota_v) := (X[\wp_v^\infty], \iota[\wp_v^\infty])$ for each $v \in \Sigma_{E,p}$.

We say that (X, ι) is *distinguished* if for every $v \in \Sigma_{E,p}$, the self-dual rank-two $\mathcal{O}_{E_v}$-linear p-divisible group (X_v, ι_v) is exquisite, or is balanced, or is E_v-minimal.

Proposition 5.6.1. *Let (X, ι) and (X', ι') be distinguished self-dual rank-two $\mathcal{O}_E$-linear p-divisible groups over an algebraically closed field $k \supseteq \mathbb{F}_p$. If (X, ι) and (X', ι') have the same Lie type and the same congruity, then (X, ι) is $\mathcal{O}_E$-linearly isomorphic to (X', ι').*

Proof. This is the combination of Corollary 5.3.3, Proposition 5.4.5, and Lemma 5.5.2. □

6. The stratification of a Lie stratum by congruity

In this section $\mathscr{M}_E$ stands for a Hilbert modular variety $\mathscr{M}_{E,n,\overline{\delta}}^{\widetilde{\mathscr{L}}}$ over $\overline{\mathbb{F}_p}$, where $n \geq 3$ is a positive integer relatively prime to p, $\widetilde{\mathscr{L}} = (\mathscr{L}, \mathscr{L}_{\mathbb{R}}^+)$ is an invertible $\mathcal{O}_E$-module with a notion of positivity, and $\overline{\delta}$ is an $\mathcal{O}_E$-generator of $\mathscr{L}^\vee \otimes_{\mathcal{O}_E} \mathcal{D}_{E/\mathbb{Q}}^{-1} / n\mathscr{L}^\vee \otimes_{\mathcal{O}_E} \mathcal{D}_{E/\mathbb{Q}}^{-1}$. We will stratify each Lie stratum $\mathcal{N}_{\underline{e}}$ of $\mathscr{M}_E$ by the congruity invariant, and explore properties of the resulting congruity stratification of $\mathcal{N}_{\underline{e}}$.

Definition 6.1 (Congruity strata in a Lie stratum $\mathcal{N}_{\underline{e}}$)**.** Let

$$\underline{e} = (\underline{e}_v)_{v \in \Sigma_{E,p}} = \left(\left((e_{v,1}^{(i)}, e_{v,2}^{(i)}) \right)_{i \in I_v} \right)_{v \in \Sigma_{E,p}}$$

be an element of $\mathcal{T}_E^{\mathrm{sd}}$, and let $\mathcal{N}_{\underline{e}} = \mathcal{N}_{\underline{e}}(\mathscr{M}_E)$ be the Lie stratum in $\mathscr{M}_E$ consisting of points corresponding to abelian varieties with real multiplication by $\mathcal{O}_E$ whose Lie type is equal to $\underline{e}$.

For each $\underline{c} \in \boldsymbol{\tau}_E$, denote by $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ the reduced locally closed subscheme of $\mathcal{N}_{\underline{e}}$ such that for every algebraically closed extension field k of $\mathbb{F}_p$, the set $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})(k)$ of k-points of $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ consists of all points $x = [(A_x, \lambda_x, \iota_x, \eta_x)] \in \mathcal{N}_{\underline{e}}(k)$ such that the congruity invariant $\underline{c}_x$ of (A_x, ι_x) is equal to $\underline{c}$.

Clearly $\mathcal{N}_{\underline{e}}$ is the disjoint union of all congruity strata $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$, where $\underline{c}$ runs through all elements of $\boldsymbol{\tau}_E$. Note that there may exist pairs $(\underline{e}, \underline{c})$ such that $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ is empty.

Proposition 6.2. (1) *For every $\underline{c}$, the Zariski closure of $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ is the disjoint union of all $\mathcal{Q}_{\underline{c}'}(\mathcal{N}_{\underline{e}})$ with $\underline{c}' \preceq \underline{c}$:*

$$\big(\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})\big)^{\mathrm{zar}} = \bigsqcup_{\underline{c}' \preceq \underline{c}} \mathcal{Q}_{\underline{c}'}(\mathcal{N}_{\underline{e}}).$$

In particular $\mathcal{Q}_{\underline{0}}(\mathcal{N}_{\underline{e}})$ is a dense open subscheme of $\mathcal{N}_{\underline{e}}$.

(2) *If $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}}) \neq \emptyset$, then every irreducible component of $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}}) \neq \emptyset$ is a locally closed smooth subscheme of $\mathcal{N}_{\underline{e}}$ of codimension $|\underline{c}|$ in $\mathcal{N}_{\underline{e}}$, where*

$$|\underline{c}| := \sum_{v \in \Sigma_{E,p}} \sum_{i \in I_v} c_v^{(i)}.$$

(3) *For each $\underline{c}$ with $|\underline{c}| = 1$, the Zariski closure $\bigsqcup_{\underline{c}' \preceq \underline{c}} \mathcal{Q}_{\underline{c}'}(\mathcal{N}_{\underline{e}})$ of $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ is a smooth divisor on the smooth scheme $\mathcal{N}_{\underline{e}}$ over $\overline{\mathbb{F}}_p$. Together these smooth divisors on $\mathcal{N}_{\underline{e}}$ form a divisor with normal crossings, whose support is equal to $\mathcal{N}_{\underline{e}} \setminus \mathcal{Q}_{\underline{0}}(\mathcal{N}_{\underline{e}})$.*

Remark. Proposition 6.2 is proved using deformation theory of p-divisible groups and the theory of displays; see [9, Section 4.5].

Definition 6.2.1. A Lie-congruity stratum $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ in $\mathscr{M}_E$ is said to be *distinguished* if for every $v \in \Sigma_{E,p}$, either $\underline{c}_v = \underline{0}$, or $\underline{e}_v = \underline{e}^{\mathrm{sd}}_{E_v,\min}$ and $\underline{c}_v$ is balanced.

Remark. It follows from Propositions 5.3 and 5.4.5 that a Lie-congruity stratum $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ is distinguished if and only if for every geometric point x of $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$, the self-dual $\mathcal{O}_E$-linear p-divisible group attached to x is distinguished.

6.3. Central leaves in $\mathscr{M}_E$

Through every $\overline{\mathbb{F}}_p$-point $x \in \mathscr{M}_E$, we have a smooth locally closed subscheme $\mathcal{C}(x) = \mathcal{C}_{\mathscr{M}_E}(x) \subseteq \mathscr{M}_E$ satisfying the following properties.

(i) For every algebraically closed extension field k of $\overline{\mathbb{F}}_p$, we have

$$\mathcal{C}(x)(k) = \big\{y \in \mathscr{M}_E(x) \mid (A_y[p^\infty], \lambda_y[p^\infty], \iota_y[p^\infty]) \cong (A_x[p^\infty], \lambda_x[p^\infty], \iota_x[p^\infty])\big\}.$$

(ii) The restriction to $\mathcal{C}(x)$ of the universal self-dual rank-two $\mathcal{O}_E$-linear principally $\widetilde{\mathscr{L}}$-polarized p-divisible group $(A, \lambda, \iota)[p^\infty]$ over $\mathcal{C}(x)$ is sustained, i.e., for every positive integer m, the $\mathcal{C}(x)$-scheme

$$\boldsymbol{Isom}_{C(x)}\big((A_x, \lambda_x, \iota_x)[p^m], (A, \lambda, \iota)[p^m]\big) \to \mathcal{C}(x)$$

whose sections over every $\mathcal{C}(x)$-scheme S is

$$\mathrm{Isom}_S\big((A_x, \lambda_x, \iota_x)[p^m] \times_{\mathrm{Spec}(k)} S, (A, \lambda, \iota)[p^m] \times_{\mathcal{C}(x)} S\big),$$

is faithfully flat over $\mathcal{C}(x)$. See [9, Chapter 5] for the theory of sustained p-divisible groups.

(iii) For every positive integer m, there exist an integer j_0 such that the images of the restriction morphisms

$$\begin{aligned}&\boldsymbol{Isom}_{C(x)}\big((A_x, \lambda_x, \iota_x)[p^{m+j}], (A, \lambda, \iota)[p^{m+j}]\big)\\ \to\ &\boldsymbol{Isom}_{C(x)}\big((A_x, \lambda_x, \iota_x)[p^m], (A, \lambda, \iota)[p^m]\big)\end{aligned}$$

stabilize for all $j \geq j_0$ and defines a subscheme

$$\boldsymbol{Isom}^{\mathrm{stab}}_{C(x)}\big((A_x, \lambda_x, \iota_x)[p^m], (A, \lambda, \iota)[p^m]\big)$$

of $\boldsymbol{Isom}_{C(x)}\big((A_x, \lambda_x, \iota_x)[p^m], (A, \lambda, \iota)[p^m]\big)$, called the *stabilized Isom scheme* for $\mathcal{C}(x)$ at level m, which is finite locally free over $\mathcal{C}(x)$.

Remark 6.3.1. (a) The condition (i) uniquely determines $\mathcal{C}(x)$ as a subset of $\mathscr{M}_E$. The path adopted in [32], which may be called the "direct approach", is to first show that this subset is constructible. Then one shows, in a bootstrapping process, that the subset $\mathcal{C}(x)$ with reduced scheme structure is locally closed and smooth.

(b) The approach to central leaves in [9] via the notion of sustained p-divisible groups explained in [9, Chapter 5] has the advantage of conceptual clarity and helps revealing the local structures of central leaves. See Proposition 6.3.3 for an illustration.

Lemma 6.3.2. *Let x be an $\overline{\mathbb{F}}_p$-point of $\mathscr{M}_E$, let $\mathcal{C}(x)$ be the central leaf in $\mathscr{M}_E$ containing x, and let $\underline{s} = ((s_{v,1}, s_{v,2}))_{v\in\Sigma_{E,p}} \in \mathrm{NP}^{\mathrm{sd}}_E$ be the Newton polygon for (A_x, ι_x). Then*

$$\dim(\mathcal{C}(x)) = \sum_{v\in\Sigma_{E,p}} [E_v : \mathbb{Q}_p] \cdot (s_{v,2} - s_{v,1}).$$

Proposition 6.3.3. *For every $\overline{\mathbb{F}}_p$-point x of $\mathscr{M}_x$, the leaf $\mathcal{C}_{\mathscr{M}_E}(x)$ in $\mathscr{M}_E$ is quasi-affine.*

Proof. Consider the stabilized Isom scheme

$$\boldsymbol{Isom}_m^{\mathrm{stab}} = \boldsymbol{Isom}_{C(x)}^{\mathrm{stab}}\big((A_x, \lambda_x, \iota_x)[p^m], (A, \lambda, \iota)[p^m]\big)$$

for $\mathcal{C}(x)$ at level m, which is finite locally free over $\mathcal{C}(x)$. The pullback to $\boldsymbol{Isom}_m^{\mathrm{stab}}$ of the restriction to $\mathcal{C}(x)$ of the universal p-divisible group is constant. Therefore the pullback to $\boldsymbol{Isom}_m^{\mathrm{stab}}$ of the Hodge line bundle on $\mathscr{M}_E$ is trivial. As the Hodge bundle on $\mathscr{M}_E$ is ample, the proposition follows. □

Proposition 6.3.4. *A distinguished congruity stratum $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ in a Lie stratum $\mathcal{N}_{\underline{e}}$ is a central leaf.*

Proof. This is a rehash of Proposition 5.6.1. □

Definition 6.3.5. A central leaf in $\mathscr{M}_E$ which is equal to a distinguished congruity stratum in a Lie stratum $\mathcal{N}_{\underline{e}}$ is called a *distinguished central leaf* in $\mathscr{M}_E$; cf. Definition 6.2.1.

Lemma 6.3.6. *Newton polygons of distinguished central leaves in $\mathscr{M}_E$ exhausts all Newton polygons for E. In other words for every $\underline{s} \in \mathrm{NP}_E^{\mathrm{sd}}$, there exists a distinguished central leaf in $\mathscr{M}_E$ whose Newton polygon is $\underline{s}$.*

Proposition 6.4 below, on the incidence relation between distinguished congruity strata and the E-minimal locus follows from deformation theoretic considerations in [13, Sections 3–4] and the proof of Proposition 6.2; cf. [9, Section 4.5]. It is an important ingredient in the proof that prime-to-p Hecke correspondences operate transitively on the set of irreducible components of a distinguished central leaf in $\mathscr{M}_E$.

Proposition 6.4. *Let $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ be a distinguished Lie-congruity stratum on $\mathscr{M}_E$. Let $\mathcal{Q}'$ and $\mathcal{Q}''$ be irreducible components of $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$. If there is an E-minimal point $z \in \mathcal{N}_{\min}$ which is contained in the Zariski closure of $\mathcal{Q}'$ and also the Zariski closure of $\mathcal{Q}''$, then $\mathcal{Q}' = \mathcal{Q}''$.*

7. Distinguished leaves on non-minimal Lie strata

Theorem 7.1. *Let $\mathcal{N}_{\underline{e}}$ be a non-minimal Lie stratum in $\mathscr{M}_E$, i.e., $\underline{e} \neq \underline{e}_{E,\min}^{\mathrm{sd}}$. No irreducible component of a* distinguished *Lie-congruity stratum $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ is closed in $\mathscr{M}_E$. More precisely, for every* distinguished *Lie-congruity stratum $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ on $\mathcal{N}_{\underline{e}}$, the Zariski closure in $\mathscr{M}_E$ of every irreducible component of $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ contains an irreducible component of the* distinguished *Lie-congruity stratum $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\min})$ of the minimal Lie stratum $\mathcal{N}_{\min}$ in $\mathscr{M}_E$.*

Theorem 7.2 below is a slightly stronger version of Theorem 7.1, with the same proof.

Theorem 7.2. *Let $\underline{e} = (\underline{e}_v)_{v\in\Sigma_{E,p}}$ be an element of $\mathcal{T}_E^{\mathrm{sd}}$ and let $\underline{c} = (\underline{c}_v)_{v\in\Sigma_{E,p}}$ be an element of $\boldsymbol{\tau}_E$. Let $\mathcal{Q}$ be an irreducible component of a Lie-congruity stratum $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ in a Lie stratum $\mathcal{N}_{\underline{e}}$ of $\mathscr{M}_E$. Suppose that v_0 is a place of E above p such that $\underline{e}_{v_0} \neq \underline{e}^{\mathrm{sd}}_{E_{v_0},\min}$ and $\underline{c}_{v_0} = \underline{0}$. In other words the self-dual rank-two $\mathcal{O}_{E_{v_0}}$-linear p-divisible group $(A_x[\wp_{v_0}^\infty], \iota_x[\wp_{v_0}^\infty])$ attached to any geometric point x of $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ is distinguished and its Lie type is* not *equal to the minimal element $\underline{e}^{\mathrm{sd}}_{E_v,\min}$ in $\mathcal{T}^{\mathrm{sd}}_{E_v}$. Then the Zariski closure $\mathcal{Q}^{\mathrm{zar}}$ of $\mathcal{Q}$ contains an irreducible component of $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}'})$, where $\underline{e}' = (\underline{e}'_v)_{v\in\Sigma_{E,p}}$ is the element of $\mathcal{T}_E^{\mathrm{sd}}$ given by*

$$\underline{e}'_v = \begin{cases} \underline{e}^{\mathrm{sd}}_{E_{v_0},\min} & \text{if } v = v_0, \\ \underline{e}_v & \text{if } v \neq v_0. \end{cases}$$

Remark 7.3. The reason that we need a statement like Theorem 7.1 is that a priori, it is possible that a given distinguished Lie-congruity stratum $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ with $\mathcal{N}_{\underline{e}} \neq \mathcal{N}_{\min}$ may have an irreducible component $\mathcal{Q}$ which is equal to its own Zariski closure in $\mathscr{M}_E$. In the case when $\underline{c}$ is the maximal element $\underline{0}$ in $\boldsymbol{\tau}_E$ and $\underline{e}$ is *not* the minimal element $\underline{e}^{\mathrm{sd}}_{E,\min}$ in $\mathcal{T}_E^{\mathrm{sd}}$, the scenario that $\mathcal{Q}$ is closed in $\mathscr{M}_E$ is equivalent to the statement that $\mathcal{Q}$ is proper over $\overline{\mathbb{F}_p}$. Most people will consider this scenario to be very unlikely, but we don't know of any "easy proof".

Readers who are contemplating going through the details of this proof in [9, Chapter 4] undoubtedly will prefer a proof which is shorter and more appealing. Unfortunately such a proof has yet to be found.

7.4.

The main thrust of the proof of Theorem 7.1, described in [40, 2.5] and [9, Chapter 4], is an inductive procedure to construct, for any given irreducible component $\mathcal{Q}$ of a distinguished Lie-congruity stratum $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ with $\mathcal{N}_{\underline{e}} \neq \mathcal{N}_{\min}$, $\mathbb{P}^1$-families in the Zariski closure of $\mathcal{Q}$. Each of these $\mathbb{P}^1$ families of abelian varieties with real multiplication by $\mathcal{O}_E$ is $\mathcal{O}_E$-linearly isogenous to a constant abelian variety with real multiplication by $\mathcal{O}_E$, and there is a dense open subset U of $\mathbb{P}^1$ contained in $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$. The Lie types of points of $\mathbb{P}^1 \setminus U$ can be explicitly computed. In the majority of situations there is a point in $\mathbb{P}^1 \setminus U$ whose Lie type is strictly small than $\underline{e}$ in $\mathcal{T}_e^{\mathrm{sd}}$, and the inductive procedure stops if this is the case. If the Lie types of points of $\mathbb{P}^1 \setminus U$ are all equal to $\underline{e}$, one constructs another $\mathbb{P}^1$ family and then repeats the same operation. One shows that this procedure eventually produces a point of the minimal Lie stratum $\mathcal{N}_{\underline{e}}$ which lies in the Zariski closure of $\mathcal{Q}$.

8. Congruity strata on the minimal Lie stratum

Theorem 8.1. *Every irreducible component of a congruity stratum of the minimal Lie stratum $\mathcal{N}_{\min} = \mathcal{N}_{\underline{e}^{\mathrm{sd}}_{E,\min}}$ of $\mathscr{M}_E$ contains a minimal point, i.e., an irreducible component of the zero-dimensional congruity stratum in $\mathcal{N}_{\min}$.*

8.2.

The proof of Theorem 8.1 is a variant of the method called *Raynaud's trick* in [28, Section 4], and is based on the phenomenon, explained in Subsubsection 8.2.2, that the congruity stratification of the minimal Lie stratum $\mathcal{N}_{\min}(\mathscr{M}_E)$ bears prominent similarities to the α-type stratification of Hilbert modular varieties attached to a totally real field L which is *unramified* above p. See [9, Section 4.7] for details.

8.2.1. For a totally real field L unramified above p, there is only one Lie stratum, and the congruity invariant on $\mathscr{M}_L$ has the same information content as the type of the maximal α-subgroup schemes considered in [14]. It is shown in [14, Section 3] that for every α-type stratum W_τ of $\mathscr{M}_L$, there exists a positive integer $m > 0$ such that the restriction to W_τ of the m-th power of the Hodge line bundle on $\mathscr{M}_L$ is trivial, hence every α-stratum W_τ of $\mathscr{M}_L$ is quasi-affine.

8.2.2. Let $x \in \mathcal{N}_{\min}(\overline{\mathbb{F}}_p)$ be an $\overline{\mathbb{F}}_p$-point of the minimal Lie stratum of $\mathscr{M}_E$. Let $M_v = \mathbb{D}(A_x[\wp_v^\infty])$ be the Dieudonné module of the $\mathcal{O}_{E_v}$-linear p-divisible group $A_x[\wp_v^\infty]$. The assumption that $x \in \mathcal{N}_{\min}(\overline{\mathbb{F}}_p)$ implies that the operators $\mathcal{F}$, $\mathcal{V}$ on M_v are both divisible by $\pi_v^{\lfloor e_v/2 \rfloor}$. Let $\Phi_v := \pi_v^{-\lfloor e_v/2 \rfloor} \cdot \mathcal{F}|_{M_v}$, $\Psi_v := \pi_v^{-\lfloor e_v/2 \rfloor} \cdot \mathcal{V}|_{M_v}$. These two semi-linear operators on M_v commute with the action of $\mathcal{O}_v \otimes_{\mathbb{Z}_p} \Lambda(\overline{\mathbb{F}}_p)$, and

$$\Phi_v \circ \Psi_v = \Psi_v \circ \Phi_v = p \cdot \pi_v^{-2\lfloor e_v/2 \rfloor}.$$

(i) If e_v is even, then Φ_v is a σ-linear automorphism of M_v, Ψ_v is a σ^{-1}-linear automorphism, and such a triple (M_v, Φ_v, Ψ_v) is rigid.

(ii) On the other hand if e_v is odd, then $\pi'_v := p \cdot \pi_v^{-2\lfloor e_v/2 \rfloor}$ is a generator of $\wp_v$. For the decomposition $M_v = \bigoplus_{i \in I_{E_v}} M_v^{(i)}$ and the semi-linear operators Φ_v, Ψ_v, we have

$$\Phi_v(M_v^{(i)}) \subseteq M_v^{(i+1)}, \quad \Psi_v(M^{(i)}) \subseteq M_v^{(i-1)}, \quad \forall\, i \in I_{E_v}.$$

It follows that $\dim_{\overline{\mathbb{F}}_p} \big(M_v^{(i)}/\Phi_v(M_v^{(i-1)}) + \Psi_v(M_v^{(i+1)})\big) \leq 1$ for every $i \in I_{E_v}$. The congruity $\underline{c}_{x,v} = \big(c_{x,v}^{(i)}\big)_{i \in I_{E_v}}$ of $A_x[\wp_v^\infty]$ is given by

$$c_{x,v}^{(i)} = \begin{cases} 0 & \text{if } \Phi_v(M_v^{(i-1)}) + \Psi_v(M_v^{(i+1)}) = M_v^{(i)}, \\ 1 & \text{if } \Phi_v(M_v^{(i-1)}) + \Psi_v(M_v^{(i+1)}) \neq M_v^{(i)}. \end{cases}$$

This formula for $c^{(i)}_{x,v}$ is very similar to the formula for the (w,i)-component of the $\underline{a}$-invariant of an abelian variety with real multiplication by the ring of integers of a totally real field L unramified above p. Of course if E_v is unramified over $\mathbb{Q}_p$, then $e_v = 1$, and $c^{(i)}_{x,v} = 1$ if and only if $\mathcal{F}(M_v^{(i-1)}) + \mathcal{V}(M_v^{(i+1)}) \subsetneqq M_v^{(i)}$.

8.2.3. The similarity of the congruity stratification of the minimal Lie stratum $\mathcal{N}_{\underline{e}}$ on $\mathscr{M}_E$ with the α-type stratification of $\mathscr{M}_L$, as sketched in Subsubsectin 8.2.2, holds to such an extent that the argument in [14, Section 3], suitably interpreted, shows that the restriction of the Hodge line bundle on $\mathscr{M}_E$ to every congruity stratum $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\min})$ on $\mathcal{N}_{\min}$ is torsion. Therefore every congruity stratum $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\min})$ on $\mathcal{N}_{\min}$ is quasi-affine.

9. Irreducibility of non-supersingular distinguished leaves

Theorem 9.1. *The Zariski closure of every irreducible component of a distinguished Lie-congruity stratum in $\mathscr{M}_E$ contains an E-minimal point.*

Proof. Theorem 9.1 follows from Theorems 7.1 and 8.1. □

Replacing Theorem 7.1 by Theorem 7.2, we get a slightly stronger statement.

Theorem 9.2. *Let $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ be a Lie-congruity stratum in $\mathscr{M}_E$. Suppose that for every place v of E above p, either the v-component $\underline{e}_v$ of $\underline{e}$ is equal to $\underline{e}^{\mathrm{sd}}_{E_v,\min}$, or the v-component $\underline{c}_v$ of $\underline{c}$ is $\underline{0}$. Then every irreducible component of $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ contains an E-minimal point.*

Corollary 9.3. *The prime-to-p Hecke correspondences operate transitively on the set of all irreducible components of every non-supersingular distinguished Lie-congruity stratum of $\mathscr{M}_E$.*

Proof. This is a corollary of Theorem 9.1 and Proposition 6.4. □

Corollary 9.4. *Every non-supersingular distinguished Lie-congruity stratum in $\mathscr{M}_E$ is irreducible.*

Proof. This irreducibility statement is a consequence of Corollary 9.3 and the method in [6, 4.4] for proving irreducibility of Hecke-invariant subvarieties via prime-to-p monodromy. Note that [6, 4.4] was stated for Siegel modular varieties, but the proof therein shows that the statement also holds for Hilbert modular varieties; see [42, Section 6]. □

The same argument, using Theorem 9.2 instead of Theorem 9.1, gives a strengthened version of Corollary 9.4.

Corollary 9.5. *Let $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ be a Lie-congruity stratum in $\mathscr{M}_E$ which is* not *contained in the supersingular locus of $\mathscr{M}_E$. Suppose that for every place v of E above p, either the v-component $\underline{e}_v$ of $\underline{e}$ is equal to $\underline{e}^{\mathrm{sd}}_{E_v,\min}$, or the v-component $\underline{c}_v$ of $\underline{c}$ is $\underline{0}$. Then $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ is irreducible.*

Remark 9.6. Readers may consult [9, Chapter 8] for a proof of the Hecke orbit conjecture for Siegel modular varieties, and see Corollary 9.4 in action. See also Subsection 1.1 for the role played by the Hilbert trick and the irreducibility result in Corollary 9.4 in the proof of the Hecke orbit conjecture.

10. Some questions

Question 10.1 (Geometry of the Lie-congruity strata)**.** (a) Determine the Zariski closure $\big(\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})\big)^{\mathrm{zar}}$ of the Lie-congruity strata $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ in $\mathscr{M}_E$, for all $\underline{e} \in \mathcal{T}^{\mathrm{sd}}_E$ and all $\underline{c} \in \boldsymbol{\tau}_E$.

(b) Is $\big(\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})\big)^{\mathrm{zar}}$ normal, Cohen-Macaulay, or a local complete intersection?

(c) Is every Lie-congruity stratum $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ quasi-affine?

Remark 10.1.1. An optimist may ask whether there exists a stratification on $\mathscr{M}_E$ which refines the Lie stratification on $\mathscr{M}_E$ and induces the congruity stratification on every Lie stratum $\mathcal{N}_{\underline{e}}$. In other words:

> *Is the Zariski closure of a Lie-congruity stratum $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ a union of Lie-congruity strata, for every pair $(\underline{e},\underline{c}) \in \mathcal{T}^{\mathrm{sd}}_E \times \boldsymbol{\tau}_E$?*

Remark 10.1.2. Question 10.1 contains a preliminary part, namely determine all pairs $(\underline{e},\underline{c}) \in \mathcal{T}^{\mathrm{sd}}_E \times \boldsymbol{\tau}_E$ such that $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}}) \neq \emptyset$. The related problem on non-emptiness of EO strata and Newton strata of PEL-type Shimura varieties has been solved in [36].

Remark 10.1.3. We know that every distinguished Lie-congruity stratum is quasi-affine, and so is every congruity stratum on the minimal Lie stratum $\mathcal{N}_{\min}$. An optimist may ask whether the restriction of the Hodge line bundle to $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ is torsion for every pair $(\underline{e},\underline{c})$ in $\mathcal{T}^{\mathrm{sd}}_E \times \boldsymbol{\tau}_E$.

Question 10.2 (Irreducibility question for Lie-congruity strata)**.** Let $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ be a Lie-congruity stratum not contained in the supersingular locus of $\mathscr{M}_E$. Is $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$ irreducible?

Question 10.3 (Lie-congruity strata and the Newton polygon stratification)**.**

(a) Determine the subset

$$\big\{(\underline{s},\underline{e},\underline{c}) \in \mathrm{NP}^{\mathrm{sd}}_E \times \mathcal{T}^{\mathrm{sd}}_E \times \boldsymbol{\tau}_E \mid \mathcal{W}_{\underline{s}} \cap \mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}}) \neq \emptyset\big\}$$

of $\mathrm{NP}^{\mathrm{sd}}_E \times \mathcal{T}^{\mathrm{sd}}_E \times \boldsymbol{\tau}_E$.

(b) Given a Lie-congruity stratum $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$, determine the Newton polygon of the generic point of each irreducible component of $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$.

(c) Given a Newton polygon stratum $\mathcal{W}_{\underline{s}}$, determine the Lie type and the congruity of the generic point of each irreducible component of $\mathcal{W}_{\underline{s}}$.

Question 10.4 (Newton polygon strata and Grothendieck's conjecture)**.**

(a) For each Newton polygon $\underline{s} \in \mathrm{NP}_E^{\mathrm{sd}}$, determine the number and dimensions of irreducible components of $\mathcal{W}_{\underline{s}}$. Compare the set of irreducible components of $\mathcal{W}_{\underline{s}}$ and that of the special fiber the Rapoport–Zink space attached to $\mathcal{W}_{\underline{s}}$ modulo the twisted stabilizer group (denoted by $J(\mathbb{Q}_p)$ in [34, p. xiii]).

(b) Determine the singular locus of the Zariski closure $(\mathcal{W}_{\underline{s}})^{\mathrm{zar}}$ of $\mathcal{W}_{\underline{s}}$. Is every irreducible component of $(\mathcal{W}_{\underline{s}})^{\mathrm{zar}}$ normal (respectively Cohen-Macaulay)?

(c) Let $k \supseteq \mathbb{F}_p$ be an algebraically closed field. Let $\underline{e} = \big(\big(e_{v,1}^{(i)}, e_{v,2}^{(i)}\big)_{i\in I_v}\big)_{v\in\Sigma_{E,p}}$ be an element of $\mathcal{T}_E^{\mathrm{sd}}$. Let x be a k-point of the Lie stratum $\mathcal{N}_{\underline{e}}$ in $\mathscr{M}_E$. Let $\underline{s}_x$ be the Newton polygon of x. Let $\underline{s} = ((s_{v,1}, s_{v,2})) \in \mathrm{NP}_E^{\mathrm{sd}}$ be a Newton polygon such that $\underline{s}_x \preceq \underline{s}$ and $s_{v,1} \cdot [E_v : \mathbb{Q}_p] \geq \sum_{i\in I_v} e_{v,1}$ for all $v \in \Sigma_{E,p}$. Show that there exists a point y of $\mathcal{N}_{\underline{e}}$ such that x is contained in the Zariski closure of y and the Newton polygon of y is equal to $\underline{s}$.

(d) Let $k \supseteq \mathbb{F}_p$ be an algebraically closed field. Let x be a k-point of the Lie stratum $\mathcal{N}_{\underline{e}}$ in $\mathscr{M}_E$. Let $\underline{s}_x$ be the Newton polygon of x. Let $\underline{s} = ((s_{v,1}, s_{v,2})) \in \mathrm{NP}_E^{\mathrm{sd}}$ be a Newton polygon such that $\underline{s}_x \preceq \underline{s}$. Show that there exists a point y of $\mathscr{M}_E$ whose Newton polygon is equal to $\underline{s}$ and x is contained in the Zariski closure of y.

Remark. (i) It is not difficult to see that the statement (c) implies the statement (d). So the question (c) is in essence an approach to (d), which is Grothendieck's conjecture on the Newton polygon stratification for Hilbert modular varieties.

This conjecture of Grothendieck is known when p is unramified in E [18, 19], and also when every prime ideal of $\mathcal{O}_E$ containing $p\mathcal{O}_E$ has residue field $\mathbb{F}_p$ [38, Theorem 6.20].

(ii) For the question (a) when p is unramified in E and the residue field degree $f_v \leq 4$ for every place $v \in \Sigma_{E,p}$, the number of the irreducible components of the supersingular locus is computed by a formula [41, Theorem 4.12]; this formula involves the special value $\zeta_E(-1)$ of the Dedekind zeta function of E and certain constants $c(v)$ depending only on f_v, $v \in \Sigma_{E,p}$. Conjecturally [41, 4.13] the same formula for the number of irreducible components of the supersingular locus in $\mathscr{M}_E$ holds for all totally real field E, with the integer-valued constants $c(v)$ depending only on e_v and f_v, $v \in \Sigma_{E,p}$.

(iii) The sets of irreducible components of affine Deligne–Lusztig varieties modulo the action of the twisted stabilizer group are studied by Hamacher and Viehmann [20] and by Xiao and Zhu [37] very recently under a general good reduction setting. Their results give an explicit group-theoretic description of irreducible components of the corresponding Rapoport–Zink space in the question (a) when p is unramified in E.

(iv) For a general formulation of the notion of Newton invariant of isocrystals with additional structure, see [22, 24].

Question 10.5 (Lie-congruity vs. EKOR strata)**.** Compare the Lie-congruity stratification of $\mathscr{M}_E$ with the EKOR stratification in [21].

Remark. The Lie stratification on $\mathscr{M}_E$ coincides with the KR stratification introduced in [25, 27]. The Lie-congruity stratification and the EKOR stratification both refine the Lie stratification on $\mathscr{M}_E$. The geometry of KR strata of Siegel modular varieties is studied in [15–17].

Question 10.6 (Analog of the EO stratification on $\mathscr{M}_E$)**.** Find a good definition of the Ekedahl–Oort stratification for Hilbert modular varieties, so that the resulting EO stratification on $\mathscr{M}_E$ has properties similar to the EO stratification [29] on the Siegel moduli space $\mathscr{A}_{g,1,n}$ g-dimensional principally polarized abelian varieties over $\overline{\mathbb{F}_p}$ with symplectic level-n structures.

Remark 10.6.1. The EO stratification on $\mathscr{A}_{g,1,n}$ comes from geometric isomorphism classes of $(A[p], \lambda[p])$, where (A, λ) runs through g-dimensional principally polarized abelian varieties in characteristic p. The naive generalization to $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta}}$ involves understanding the classification of geometric isomorphism classes of triples $(A[p], \lambda[p], \iota[p])$, where $[(A, \lambda, \iota, \eta)]$ runs through geometric points of $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta}}$.

(a) When the totally real field E is unramified above p, this "naive notion" works nicely. Moreover we know from [14, Theorem 3.2.8] that the principally polarized $(\mathcal{O}_E/p\mathcal{O}_E)$-linear BT_1 group $(A_x[p], \lambda_x[p], \iota_x[p])$ attached to a geometric point x of $\mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta}}$ is determined by the α-type of $(A_x[p], \lambda_x[p], \iota_x[p])$.

(b) One difficulty when E is ramified above p is that the number of geometric isomorphism classes of $(A_x[p], \lambda_x[p], \iota_x[p])$ may not be finite; see [1, 38].

Remark 10.6.2. For any element $l \in \mathscr{L} \cap \mathscr{L}^+_{\mathbb{R}}$ such that $\mathscr{L} = \mathcal{O}_E \cdot l + p\mathscr{L}$, we have a *finite etale* morphism

$$f_l \colon \mathscr{M}^{\widetilde{\mathscr{L}}}_{E,n,\overline{\delta}} \to \mathscr{A}_{g,d,n},$$

where $d = \mathrm{card}(\mathscr{L}/\mathcal{O}_E \cdot l)$. A question related to Question 10.6 is to investigate basic geometric properties of the pull-back $f_l^* \mathcal{S}^{\mathrm{EO}}_\bullet$ of the EO stratification on $\mathscr{A}_{g,d,n}$, such as the

dimensions, irreducible components and smoothness of strata of $f_l^*\mathcal{S}_\bullet^{\mathrm{EO}}$ and the incidence relation between strata.

Some of the properties of the EO stratification on $\mathscr{A}_{g,d,n}$ holds for $f_l^*\mathcal{S}_\bullet^{\mathrm{EO}}$. For instance, every stratum in $f_l^*\mathcal{S}_\bullet^{\mathrm{EO}}$ is quasi-affine. But other geometric properties of strata in $f_l^*\mathcal{S}_\bullet^{\mathrm{EO}}$, such as dimension, singularity and the closure relation have not been determined. The relations between $f_l^*\mathcal{S}_\bullet^{\mathrm{EO}}$ and other stratifications of $\mathscr{M}_E$, such as the congruity stratification of Lie strata and the Newton polygon stratification, await further work.

Remark 10.6.3. Satisfactory answers to Question 10.1 may very well depend on future progress on Questions 10.1(a) and 10.5. For instance if it turns out that every Lie-congruity stratum is quasi-affine and every irreducible component $\mathcal{Q}$ of $\mathcal{Q}_{\underline{c}}(\mathcal{N}_{\underline{e}})$, the Zariski closure of $\mathcal{Q}$ is a union of irreducible components of Lie-congruity strata of $\mathscr{M}_E$, then one can made a good case that the Lie-congruity stratification is a good analog of the EO stratification for $\mathscr{M}_E$.

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