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Asymptotic Behavior of the Initial-boundary Value Problem of Landau-Lifshitz-Schrödinger Type

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Abstract. This paper is concerned with the asymptotic behavior of the classical solutions of a Landau-Lifshitz-Schrödinger-type problem with initial-boundary values when the parameter ε goes to zero. We establish several uniform estimates of u_ε by a conservation result and the standard parabolic method. Based on these results, we obtain parabolic behavior in the dissipative case and non-parabolic behavior of the semi-classical limits of those solutions respectively.

1. Introduction

Let $G \subset \mathbb{R}^2$ be a bounded and simply connected domain with smooth boundary ∂G, $B = \{x \in \mathbb{R}^2 \mid |x| < 1\}$, $S^2 = \{x = (x_1 + ix_2, x_3) \in \mathbb{C} \times \mathbb{R} \mid |x| = x_1^2 + x_2^2 + x_3^2 = 1\}$, $S^1 = \{x \in \mathbb{C} \times \mathbb{R} \mid x_1^2 + x_2^2 = 1, x_3 = 0\}$. Sometimes we write the vector value function $u = (u_1 + iu_2, u_3) = (u', u_3)$. Define $G_T = G \times (0,T]$ with $T \in (0,\infty)$. We are concerned with the limit behavior of the classical solution $u_\varepsilon \colon G_T \to S^2$ of the following problem of Landau-Lifshitz-Schrödinger type when the parameter $\varepsilon \to 0$

$$(1.1) \qquad \begin{cases} (a+ib)u'_t = \Delta u' + u'|\nabla u|^2 + \frac{1}{\varepsilon^2} u' u_3^2 & \text{on } G_T, \\ u_{3t} = \Delta u_3 + u_3|\nabla u|^2 + \frac{1}{\varepsilon^2} u_3(u_3^2 - 1) & \text{on } G_T, \\ u|_{\partial G \times \mathbb{R}^+} = g(x), & \\ u(x,0) = u_0(x), & x \in G, \end{cases}$$

where $a > 0$ and b are real constants, $g = (g', 0) \in C^\infty(\partial G, S^1)$, $u_0 = (u'_0, 0) \in C^\infty(\overline{G}, S^1)$, $u_0(x) = g(x)$ as $x \in \partial G \times \{t = 0\}$, and $\deg(g', \partial G) = 0$.

Equation (1.1) is related to the study of the Schrödinger operator $\partial_t - (a + bi)\Delta$ (cf. [7, 17]). When $a = 1$, $b = 0$ and S^2 is replaced by $\{x \in \mathbb{R}^3 \mid |x| = 1\}$, (1.1) can be rewritten as

$$(1.2) \qquad \begin{cases} u_t = \Delta u + u|\nabla u|^2 + \frac{1}{\varepsilon^2}(u u_3^2 - u_3 e_3) & \text{on } G_T, \\ u|_{\partial G \times \mathbb{R}^+} = g(x), & \\ u(x,0) = u_0(x), & x \in G, \end{cases}$$

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where $e_3 = (0, 0, 1)$. The system (1.2) arises in the study of high-energy physics (cf. [10, 16]). It controls the dynamics of planar ferromagnets and antiferromagnets. If the term $\frac{1}{\varepsilon^2}(uu_3^2 - u_3 e_3)$ is replaced by $\frac{1}{\varepsilon^2}u(1-|u|^2)$ and S^2 is replaced by $\mathbb{R}^2$, (1.2) becomes the Ginzburg-Landau system introduced in the theory of superconductors (see [2, 13, 18]). When $a = 0$ and $b = 1$, it is associated with the Gross-Pitaevskii-type equation (cf. [9, 18, 20]).

By virtue of $\deg(g', \partial G) = 0$, there exists $\theta_1(x) \in C^\infty(\partial G)$ and $\theta_2(x) \in C^\infty(\overline{G})$ such that $g' = e^{i\theta_1}$ on ∂G and $u_0' = e^{i\theta_2}$ in G, where $\theta_2|_{\partial G} = \theta_1$. Clearly, the following problem

$$\begin{cases} a\theta_t = \Delta\theta & \text{on } G_T, \\ \theta|_{\partial G\times\mathbb{R}^+} = \theta_1, & \\ \theta(x,0) = \theta_2(x), & x \in G \end{cases}$$

has a unique solution $\theta(x,t)$. Set $u_*' = e^{i\theta}$. Then u_*' is the unique solution to the problem (up to periods)

$$\begin{cases} aw_t = \Delta w + w|\nabla w|^2 & \text{on } G_T, \\ w|_{\partial G\times\mathbb{R}^+} = g'(x), & \\ w(x,0) = u_0'(x), & x \in G. \end{cases} \tag{1.3}$$

It is a heat flow of harmonic map.

In Section 2, we will establish a conservation of energy. Based on this result, u_ε converges to $(u_*', 0)$ when $\varepsilon \to 0$ as in [1,8], where u_*' is a heat flow of harmonic map. Thus, investigating the asymptotic behavior of the solution u_ε of the Landau-Lifshitz-Schrödinger problem is helpful to well understand the properties of the heat flow of harmonic maps. On the contrary, we may also understand the asymptotic properties of u_ε by means of the corresponding properties of u_*'.

Theorem 1.1. *Assume $u_\varepsilon\colon G_T \to S^2$ is a solution to* (1.1)*, where $T < \infty$ is independent of ε. Then as $\varepsilon \to 0$,*

$$\begin{aligned} u_\varepsilon &\rightharpoonup (u_*', 0) && \textit{weakly}^* \textit{ in } L^\infty(0,T; H^1(G,S^2)), \\ \frac{\partial}{\partial t}u_\varepsilon &\rightharpoonup \frac{\partial}{\partial t}(u_*', 0) && \textit{weakly in } L^2(0,T; L^2(G,S^2)), \\ u_\varepsilon &\to (u_*', 0) && \textit{in } L^2(0,T; L^2(G,S^2)), \\ u_{\varepsilon 3} &\to 0 && \textit{in } C^{\alpha,\alpha/2}(\overline{G_T}) \textit{ for some } 0 < \alpha < 1. \end{aligned}$$

Here, u_*' solves (1.3).

The first three results can be deduced directly by using energy estimate. The last result relies on the estimate of $\|u_{\varepsilon 3}\|_{W_p^{2,1}(\overline{G_T})}$. This type of estimate has already been obtained for evolution problems of the Ginzburg-Landau-type (cf. [6]) and the Landau-Lifshitz-type (cf. [11]). These problems are distinctly different. The Landau-Lifshitz equation is more difficult to handle since it only satisfies the natural growth condition (with respect to $|\nabla u|^2$), unlike the Ginzburg-Landau equation satisfying the controllable growth condition. The computations have been developed in the context of the harmonic maps and can be generalized to included the anisotropic perturbation and the evolution in time (cf. [4,5,15]). We will establish the analogous estimates for Landau-Lifshitz-Schrödinger problem (1.1) in Section 3.

Finally, in Section 4, we rescale u_ε in time as in [3,14]

$$v_\varepsilon(x,t) = u_\varepsilon(x, \varepsilon t).$$

Then the function $v_\varepsilon = (v'_\varepsilon, v_{\varepsilon 3})$ satisfies

$$\begin{cases} (a+ib)\frac{1}{\varepsilon}v'_t = \Delta v' + v'|\nabla v|^2 + \frac{1}{\varepsilon^2}v'v_3^2 & \text{on } G_T, \\ \frac{1}{\varepsilon}v_{3t} = \Delta v_3 + v_3|\nabla v|^2 + \frac{1}{\varepsilon^2}v_3(v_3^2-1) & \text{on } G_T, \\ v|_{\partial G\times\mathbb{R}^+} = g(x), \\ v(x,0) = u_0(x), & x \in G. \end{cases} \tag{1.4}$$

Let $b \neq 0$. Clearly, the following hyperbolic problem

$$\begin{cases} \theta_{tt} = \frac{2}{b^2}\Delta\theta & \text{on } G_T, \\ \theta|_{\partial G\times\mathbb{R}^+} = \theta_1, \\ \theta(x,0) = \theta_2(x), & x \in G \end{cases}$$

has a unique solution $\theta(x,t)$. Set $v'_* = e^{i\theta}$, then v'_* is the unique solution to the problem (up to periods)

$$\begin{cases} iw_t = -\frac{2w}{b^2}\int_0^t [\mathrm{Im}(\overline{w}\Delta w)]\,d\tau & \text{on } G_T, \\ w|_{\partial G\times\mathbb{R}^+} = g'(x), \\ w(x,0) = u'_0(x), & x \in G. \end{cases} \tag{1.5}$$

The following result shows the limit relation between v_ε and $(v'_*, 0)$, and the non-parabolic behavior of v'_*.

Theorem 1.2. *Assume $v_\varepsilon\colon G_T \to S^2$ is a solution to (1.4), where $T < \infty$ is independent of ε. If $a \neq a^2+b^2$,*

$$\begin{aligned} v_\varepsilon &\rightharpoonup (v'_*, 0) && \textit{weakly}^* \textit{ in } L^\infty(0,T;H^1(G,S^2)), \\ \frac{\partial}{\partial t}v_\varepsilon &\rightharpoonup \frac{\partial}{\partial t}(v'_*, 0) && \textit{weakly in } L^2(0,T;L^2(G,S^2)), \\ v_\varepsilon &\to (v'_*, 0) && \textit{in } L^2(0,T;L^2(G,S^2)), \end{aligned}$$

when $\varepsilon \to 0$. *Here,* v'_* *is a harmonic map on* G *with the boundary value* $g'(x)$. *On the other hand, when* $a = 0$, v'_* *must solve* (1.5) *as long as the convergence results above still hold true.*

2. Parabolic behavior

First we have a conservation result.

Proposition 2.1. *Assume* $u_\varepsilon\colon G \times \mathbb{R}^+ \to S^2$ *solves* (1.1). *Then we have*

$$\sup_{t>0}\left[\int_0^t \int_G \left(a\left|\frac{\partial}{\partial t}u'_\varepsilon\right|^2 + \left|\frac{\partial}{\partial t}u_{\varepsilon 3}\right|^2\right) dx d\tau + E_\varepsilon(u_\varepsilon(x,t))\right] = \frac{1}{2}\|\nabla u_0\|_2^2. \tag{2.1}$$

Here

$$E_\varepsilon(u) = \frac{1}{2}\int_G |\nabla u|^2\, dx + \frac{1}{2\varepsilon^2}\int_G |u_3|^2\, dx.$$

Proof. Taking the real part and imaginary part of the first equation in (1.1) and combining the second equation in (1.1), we have the following three equations

$$au_{1t} - bu_{2t} = \Delta u_1 + u_1|\nabla u|^2 + \frac{1}{\varepsilon^2}u_1u_3^2, \tag{2.2}$$

$$bu_{1t} + au_{2t} = \Delta u_2 + u_2|\nabla u|^2 + \frac{1}{\varepsilon^2}u_2u_3^2, \tag{2.3}$$

$$u_{3t} = \Delta u_3 + u_3|\nabla u|^2 + \frac{1}{\varepsilon^2}u_3(u_3^2 - 1). \tag{2.4}$$

Multiply (2.2), (2.3) and (2.4) with u_{1t}, u_{2t} and u_{3t}, respectively. Integrating over G, we get

$$\begin{aligned} &a\int_G |u_{1t}|^2\, dx - b\int_G u_{1t}u_{2t}\, dx \\ &= \int_G u_{1t}\Delta u_1\, dx + \frac{1}{2}\int_G (|u_1|^2)_t|\nabla u|^2\, dx + \frac{1}{2\varepsilon^2}\int_G (|u_1|^2)_t u_3^2\, dx, \end{aligned} \tag{2.5}$$

$$\begin{aligned} &b\int_G u_{1t}u_{2t}\, dx + a\int_G |u_{2t}|^2\, dx \\ &= \int_G u_{2t}\Delta u_2\, dx + \frac{1}{2}\int_G (|u_2|^2)_t|\nabla u|^2\, dx + \frac{1}{2\varepsilon^2}\int_G (|u_2|^2)_t u_3^2\, dx, \end{aligned} \tag{2.6}$$

$$\int_G |u_{3t}|^2\, dx = \int_G u_{3t}\Delta u_3\, dx + \frac{1}{2}\int_G (|u_3|^2)_t|\nabla u|^2\, dx + \frac{1}{2\varepsilon^2}\int_G (|u_3|^2)_t(u_3^2 - 1)\, dx. \tag{2.7}$$

Noting $u_t = g_t = 0$ on ∂G, we obtain by Green's theorem that

$$\int_G u_{it}\Delta u_i\, dx = -\frac{d}{2dt}\int_G |\nabla u_i|^2\, dx, \quad i = 1, 2, 3.$$

Insert into (2.5), (2.6) and (2.7), respectively. Noting $|u| = 1$, we have $(|u|^2)_t = 0$. Thus, combining the three results yields

$$\int_G a\big(|u_{1t}|^2 + |u_{2t}|^2\big)\,dx + \int_G |u_{3t}|^2\,dx = -\frac{d}{dt}E_\varepsilon(u(x,t)).$$

Integrating from 0 to t, we deduce that, for all $t > 0$,

$$\int_0^t \int_G \left(a\left|\frac{\partial}{\partial t}u'_\varepsilon\right|^2 + \left|\frac{\partial}{\partial t}u_{\varepsilon 3}\right|^2 \right) dx d\tau + E_\varepsilon(u_\varepsilon(x,t)) = E_\varepsilon(u(x,0)) = \frac{1}{2}\|\nabla u_0\|_2^2.$$

Proposition 2.1 is proved. $\square$

By the conservation result, we can see the parabolic behavior of u_ε when $\varepsilon \to 0$.

Proposition 2.2. *Assume* $u_\varepsilon \colon G_T \to S^2$ *is a solution to* (1.1), *where* $T < \infty$. *Then as* $\varepsilon \to 0$,

$$\begin{aligned} u_\varepsilon &\rightharpoonup (u'_*, 0) && \textit{weakly* in } L^\infty(0,T;H^1(G,S^2)),\\ \frac{\partial}{\partial t}u_\varepsilon &\rightharpoonup \frac{\partial}{\partial t}(u'_*, 0) && \textit{weakly in } L^2(0,T;L^2(G,S^2)),\\ u_\varepsilon &\to (u'_*, 0) && \textit{in } L^2(0,T;L^2(G,S^2)). \end{aligned}$$

Here, u'_* *solves* (1.3).

Proof. By Proposition 2.1, we can find a subsequence ε_k such that as $\varepsilon_k \to 0$,

$$\begin{cases} u_{\varepsilon_k} \rightharpoonup w & \text{weakly* in } L^\infty(0,T;H^1(G,S^2)),\\ \frac{\partial}{\partial t}u_{\varepsilon_k} \rightharpoonup \frac{\partial}{\partial t}w & \text{weakly in } L^2(0,T;L^2(G,S^2)),\\ u_{\varepsilon_k} \to w & \text{in } L^2(0,T;L^2(G,S^2)). \end{cases} \tag{2.8}$$

We claim that $w = (u'_*, 0)$, where u'_* is the unique solution to (1.3). When $\varepsilon \to 0$, by (2.1) we know $\sup_{t>0}\int_G u_{\varepsilon_k 3}^2\,dx \to 0$. This result shows $w_3 = 0$ a.e. on G_T. Since $u_{\varepsilon_k} = u = (u', u_3)$ solves (1.1), for all $\phi \in C_0^\infty(G_T)$, we have

$$\begin{aligned} \int_{G_T}(au_{1t} - bu_{2t})\phi\,dxdt &= \int_{G_T}\Delta u_1\phi\,dxdt + \int_{G_T}u_1|\nabla u|^2\phi\,dxdt + \frac{1}{\varepsilon^2}\int_{G_T}u_1u_3^2\phi\,dxdt,\\ \int_{G_T}(bu_{1t} + au_{2t})\phi\,dxdt &= \int_{G_T}\Delta u_2\phi\,dxdt + \int_{G_T}u_2|\nabla u|^2\phi\,dxdt + \frac{1}{\varepsilon^2}\int_{G_T}u_2u_3^2\phi\,dxdt. \end{aligned}$$

Take $\phi = u_2\zeta$ and $\phi = u_1\zeta$ in the first and second integral equations respectively. Here

$\zeta \in C_0^\infty(G_T)$. Then,

$$\begin{aligned}
&\int_{G_T} (au_1 - bu_2)(u_2\zeta)_t \, dxdt \\
&= \int_{G_T} \nabla u_1 \nabla (u_2\zeta) \, dxdt - \int_{G_T} u_1 |\nabla u|^2 u_2 \zeta \, dxdt - \frac{1}{\varepsilon^2} \int_{G_T} u_1 u_3^2 u_2 \zeta \, dxdt, \\
&\int_{G_T} (bu_1 + au_2)(u_1\zeta)_t \, dxdt \\
&= \int_{G_T} \nabla u_2 \nabla (u_1\zeta) \, dxdt - \int_{G_T} u_2 |\nabla u|^2 u_1 \zeta \, dxdt - \frac{1}{\varepsilon^2} \int_{G_T} u_2 u_3^2 u_1 \zeta \, dxdt.
\end{aligned}$$

Subtracting one from the other yields

$$\begin{aligned}
&\int_{G_T} a\zeta (u_1 u_{2t} - u_2 u_{1t}) \, dxdt - \int_{G_T} b\zeta (u_1 u_{1t} + u_2 u_{2t}) \, dxdt - \int_{G_T} b\zeta_t (u_1^2 + u_2^2) \, dxdt \\
&= \int_{G_T} \nabla \zeta (u_2 \nabla u_1 - u_1 \nabla u_2) \, dxdt.
\end{aligned}$$

Letting $\varepsilon \to 0$ and using (2.8), we obtain

$$a \int_{G_T} \zeta (w' \wedge w'_t) \, dxdt = \int_{G_T} \nabla \zeta (\nabla w' \wedge w') \, dxdt.$$

Integrating by parts, we get that for all $\zeta \in C_0^\infty(G_T)$,

$$a \int_{G_T} \zeta (w' \wedge w'_t) \, dxdt = \int_{G_T} \zeta \, \mathrm{div}(w' \wedge \nabla w') \, dxdt. \tag{2.9}$$

Let $w' = e^{i\theta}$, then $w'_t = ie^{i\theta}\theta_t$, $\nabla w' = ie^{i\theta}\nabla\theta$, and

$$a(w' \wedge w'_t) = a\theta_t, \quad \mathrm{div}(w' \wedge \Delta w') = \Delta\theta.$$

Thus, (2.9) leads to

$$a\theta_t = \Delta\theta.$$

Since the limit $(u'_*, 0)$ is unique, the convergence above can be generalized to all ε instead of the subsequence ε_k. This implies that u'_* satisfies (1.3). Proposition 2.2 is proved. □

Next, we will show that $|u'_\varepsilon|$ is positive for sufficiently small ε. We first need a Lipschitz continuity result which can be deduced by (2.1). In fact, this is predictable because the $W^{1,\infty}$-function is Lipschitz continuous (cf. Exercise 8 of Chapter 6 in [12]).

Proposition 2.3. *Assume $u = u_\varepsilon \colon \overline{G_T} \to S^2$ is a solution to (1.1). Then there exists a constant $C > 0$ independent of ε such that for any $x_1, x_2 \in G$ and $t \in [0, T]$,*

$$|u(x_1, t) - u(x_2, t)| \le C|x_1 - x_2|. \tag{2.10}$$

Proof. We give a complementary definition of $u \equiv 0$ on $\mathbb{R}^2 \setminus \overline{G_T}$. Thus, u is defined on $\mathbb{R}^2$.

We first consider inner estimate. Let J_η be the mollification operator, and write

$$u_\eta(x,t) = J_\eta u(x,t) = \int_0^T \int_{\mathbb{R}^2} j_\eta(x-y,t-\tau)u(y,\tau)\,dyd\tau,$$

where $0 < \eta < t < T-\eta$. Denote u_ε by u. For any $x_1, x_2 \in G^\lambda := \{x \in G \mid \operatorname{dist}(x, \partial G) \geq \lambda\}$, there holds

$$\begin{aligned}
&u_\eta(x_1,t) - u_\eta(x_2,t) \\
&= \int_0^T \int_{\mathbb{R}^2} \int_0^1 \frac{d}{ds} j_\eta(sx_1 + (1-s)x_2 - y, t-\tau)u(y,\tau)\,dsdyd\tau \\
&= (x_1 - x_2) \int_0^T \int_{\mathbb{R}^2} \int_0^1 \nabla_x j_\eta(sx_1 + (1-s)x_2 - y, t-\tau)u(y,\tau)\,dsdyd\tau \\
&= -(x_1 - x_2) \int_0^1 \int_{\mathbb{R}^2} \int_0^T \nabla_y j_\eta(sx_1 + (1-s)x_2 - y, t-\tau)u(y,\tau)\,d\tau dyds \\
&= (x_1 - x_2) \int_0^1 \int_{\mathbb{R}^2} \int_0^T j_\eta(sx_1 + (1-s)x_2 - y, t-\tau)\nabla_y u(y,\tau)\,d\tau dyds.
\end{aligned}$$

Applying (2.1) we obtain

$$\begin{aligned}
&|u_\eta(x_1,t) - u_\eta(x_2,t)| \\
&\leq |x_1 - x_2| \int_0^1 \int_{\mathbb{R}^2} \int_0^T |j_\eta(sx_1 + (1-s)x_2 - y, t-\tau)||\nabla_y u(y,\tau)|\,d\tau dyds \\
&\leq C|x_1 - x_2|.
\end{aligned}$$

Here $C > 0$ is independent of η and ε. Letting $\eta \to 0$, we can derive (2.10) for $x_1, x_2 \in G^\lambda$.

Next, we give the estimate near the boundary. Let $x_0 \in \partial G$. Without loss of generality, we assume $G \cap B_{2R}(x_0) = \{(x^1, x^2) \mid x^2 > 0\} \cap B_{2R}(x_0)$. Let J_η^+ be the mollification operator

$$u_\eta^+(x,t) = J_\eta^+ u(x,t) = \int_0^T \int_{\mathbb{R}^2} j_\eta(x^1 - y^1) j_\eta(x^2 - y^2 + 2\varepsilon) j_\eta(t-\tau) u(y,\tau)\,dyd\tau,$$

where $0 < t \leq T$. For any $x_1 = (x_1^1, x_1^2)$, $x_2 = (x_2^1, x_2^2) \in G \cap B_{2R}(x_0)$, using the same argument above, we can also deduce (2.10) near the boundary ∂G. Similarly, we also get (2.10) near $t = 0$. Proposition 2.3 is complete. □

Proposition 2.4. *Assume $u_\varepsilon \colon \overline{G_T} \to S^2$ is a solution to* (1.1). *Then $|u_\varepsilon'| \geq 1/2$ in $\overline{G_T}$ as long as ε is sufficiently small.*

Proof. We will prove that for each given $t \in (0, T]$,

$$|u_\varepsilon'(x,t)| \geq \frac{1}{2}, \quad \forall\, x \in G \tag{2.11}$$

as long as ε is sufficiently small. Otherwise, for some fixed $t_0 \in (0, T]$, we can find $x_\varepsilon \in G$ satisfying

$$|u'_\varepsilon(x_\varepsilon, t_0)| < \frac{1}{2}.$$

According to Proposition 2.3, we have

$$\begin{aligned} |u'_\varepsilon(x, t_0)| &\le |u'_\varepsilon(x, t_0) - u'_\varepsilon(x_\varepsilon, t_0)| + |u'_\varepsilon(x_\varepsilon, t_0)| \\ &\le C|x - x_\varepsilon| + \frac{1}{2} < \frac{3}{4} \quad \text{for } |x - x_\varepsilon| < \frac{1}{4C}. \end{aligned} \tag{2.12}$$

On the other hand, (2.10) implies $|u'_\varepsilon(x, t_0)|_{C^{0+1}(\overline{G})} \le C$. By means of the compact embedding theorem, we have

$$u'_\varepsilon(x, t_0) \to u'_*(x, t_0) \quad \text{in } C^\alpha(\overline{G}) \quad \text{as } \varepsilon \to 0,$$

where $\alpha \in (0, 1)$. Proposition 2.2 shows that the limit u'_* satisfies $|u'_*(x, t_0)| = 1$. This contradicts (2.12). Thus (2.11) holds true for any $t \in (0, T]$. In view of the initial-boundary condition, the proof of Proposition 2.4 is completed. □

3. Uniform estimates

The main result of this section is the Hölder convergence of $u_{\varepsilon 3}$ when $\varepsilon \to 0$. First, we establish a uniform estimate on the boundary.

Proposition 3.1. *Assume that $u_\varepsilon \colon G_T \to S^2$ is a solution to* (1.1). *Then, there exists a constant $C > 0$ which is independent of ε such that*

$$\int_0^T \int_{\partial G} \left| \frac{\partial u_\varepsilon}{\partial \nu} \right|^2 ds dt \le C, \tag{3.1}$$

where ν is the unit outward normal vector on ∂G.

Proof. Let $\mathbf{n} \in C^\infty(G, \partial B_1(0))$ such that $\mathbf{n} = \nu$ on ∂G, where ν is the unit outward normal vector. Denote u_ε by u. Multiply (2.2), (2.3) and (2.4) with $\mathbf{n} \cdot \nabla u_1$, $\mathbf{n} \cdot \nabla u_2$ and $\mathbf{n} \cdot \nabla u_3$ respectively and integrating on G_T. Since

$$\int_G \Delta u_i (\mathbf{n} \cdot \nabla u_i)\, dx = \int_{\partial G} \left| \frac{\partial u_i}{\partial \nu} \right|^2 ds - \int_G \nabla u_i \cdot \nabla(\mathbf{n} \cdot \nabla u_i)\, dx, \quad i = 1, 2, 3,$$

we have

$$\begin{aligned}\int_0^T\int_{\partial G}\left|\frac{\partial u}{\partial \nu}\right|^2 dsdt &= \int_{G_T} a(u'_t\cdot(\mathbf{n}\cdot\nabla u'))\,dxdt + \int_{G_T} b\det(u'_t(\mathbf{n}\cdot\nabla u'))\,dxdt\\ &\quad+\int_{G_T} u_{3t}(\mathbf{n}\cdot\nabla u_3)\,dxdt + \sum_{l=1}^{3}\int_{G_T}\nabla u_l\cdot\nabla(\mathbf{n}\cdot\nabla u_l)\,dxdt\\ &\quad-\int_{G_T}(|\nabla u|^2(u\cdot(\mathbf{n}\cdot\nabla u)))\,dxdt - \frac{1}{\varepsilon^2}\int_{G_T} u_3^2(u\cdot(\mathbf{n}\cdot\nabla u))\,dxdt\\ &\quad+\frac{1}{\varepsilon^2}\int_{G_T} u_3(\mathbf{n}\cdot\nabla u_3)\,dxdt.\end{aligned}$$

Here $\det(u',v') = u_1v_2 - u_2v_1$. Noting the smoothness of $\mathbf{n}$, from (2.1) and the Cauchy inequality, we can deduce

$$\int_{G_T}\big[|au_{1t}+bu_{2t}||\mathbf{n}\cdot\nabla u_1| + |au_{2t}+bu_{1t}||\mathbf{n}\cdot\nabla u_2| + |u_{3t}||\mathbf{n}\cdot\nabla u_3|\big]\,dxdt \le C.$$

In addition, using (2.1) we also have

$$\begin{aligned}\sum_{l=1}^{3}\int_G \nabla u_l\cdot\nabla(\mathbf{n}\cdot\nabla u_l)\,dx &\le C\int_G |\nabla u|^2\,dx + \frac12\left|\int_G \mathbf{n}\cdot\nabla(|\nabla u|^2)\,dx\right|\\ &\le C + \frac12\int_{\partial G}|\nabla u|^2\,ds.\end{aligned}$$

Noting $|u|=1$, we get $\frac12\nabla(|u|^2)=0$. Therefore,

$$-\int_G |\nabla u|^2(u\cdot(\mathbf{n}\cdot\nabla u))\,dx = -\frac12\int_G |\nabla u|^2\big(\mathbf{n}\cdot\nabla(|u|^2)\big)\,dx = 0.$$

In view of $u_3 = 0$ on ∂G, using (2.1) we can obtain that

$$\begin{aligned}&-\frac{1}{\varepsilon^2}\int_G u_3^2(u\cdot(\mathbf{n}\cdot\nabla u))\,dx + \frac{1}{\varepsilon^2}\int_G u_3(\mathbf{n}\cdot\nabla u_3)\,dx\\ &= -\frac{1}{2\varepsilon^2}\int_G u_3^2(\mathbf{n}\cdot\nabla(|u|^2))\,dx + \frac{1}{2\varepsilon^2}\int_G(\mathbf{n}\cdot\nabla u_3^2)\,dx\\ &= \frac{1}{2\varepsilon^2}\int_{\partial G} u_3^2\,ds - \frac{1}{2\varepsilon^2}\int_G u_3^2(\operatorname{div}\mathbf{n})\,dx \le C.\end{aligned}$$

Thus, by the boundary value condition in (1.1), we get

$$\int_0^T\int_{\partial G}\left|\frac{\partial u}{\partial\nu}\right|^2 dsdt \le C + \frac12\int_0^T\int_{\partial G}\left|\frac{\partial g}{\partial\tau}\right|^2 dsdt \le C,$$

where τ is the unit tangent vector on ∂G. Here $C>0$ is independent of ε. Proposition 3.1 is proved. □

Next, we will establish the uniform $W_2^{2,1}$-estimate of u_ε. Noting (2.1), we need the following result.

Proposition 3.2. *Assume that $u_\varepsilon\colon G_T \to S^2$ is a solution to (1.1). Let $(x_0,t_0) \in \overline{G_T}$, $Q_R = \overline{G_T} \cap (B_R(x_0) \times [t_0, t_0+R^2])$. Then, for suitably small $R>0$, there exists a constant $C = C(R,T) > 0$ which is independent of ε, such that*

$$\|D^2 u_\varepsilon\|_{L^2(Q_R)} \le C. \tag{3.2}$$

Proof. Differentiate (2.2), (2.3), and (2.4) with respect to x_j, then

$$au_{1x_jt} - bu_{2x_jt} = u_{1x_ix_ix_j} + (u_1|\nabla u|^2)_{x_j} + \frac{1}{\varepsilon^2}(u_1u_3^2)_{x_j}, \tag{3.3}$$

$$bu_{1x_jt} + au_{2x_jt} = u_{2x_ix_ix_j} + (u_2|\nabla u|^2)_{x_j} + \frac{1}{\varepsilon^2}(u_2u_3^2)_{x_j}, \tag{3.4}$$

$$u_{3x_jt} = u_{3x_ix_ix_j} + (u_3|\nabla u|^2)_{x_j} - \frac{1}{\varepsilon^2}(u_3|u'|^2)_{x_j}. \tag{3.5}$$

First we give the inner estimate. Let $\zeta = \zeta(x) \in C_0^\infty(\overline{B_{2R(x_0)}}, [0,1])$ satisfy $\zeta = 1$ in B_R, $|\nabla\zeta| \le CR^{-1}$. Multiply (3.3), (3.4) and (3.5) by $\zeta^2 u_{1x_j}$, $\zeta^2 u_{2x_j}$ and $\zeta^2 u_{3x_j}$, respectively. Then integrating over Q_{2R}, we get

$$\begin{aligned}
&\frac{a}{2}\int_{Q_{2R}} \zeta^2(|u_{1x_j}|^2)_t\,dxdt - b\int_{Q_{2R}} \zeta^2 u_{2x_jt}u_{1x_j}\,dxdt\\
&= \int_{Q_{2R}} \zeta^2 u_{1x_ix_ix_j}u_{1x_j}\,dxdt + \int_{Q_{2R}} \zeta^2(u_1|\nabla u|^2)_{x_j}u_{1x_j}\,dxdt + \int_{Q_{2R}} \frac{\zeta^2}{\varepsilon^2}(u_1u_3^2)_{x_j}u_{1x_j}\,dxdt,\\
&b\int_{Q_{2R}} \zeta^2 u_{1x_jt}u_{2x_j}\,dxdt + \frac{a}{2}\int_{Q_{2R}} \zeta^2(|u_{2x_j}|^2)_t\,dxdt\\
&= \int_{Q_{2R}} \zeta^2 u_{2x_ix_ix_j}u_{2x_j}\,dxdt + \int_{Q_{2R}} \zeta^2(u_2|\nabla u|^2)_{x_j}u_{2x_j}\,dxdt + \int_{Q_{2R}} \frac{\zeta^2}{\varepsilon^2}(u_2u_3^2)_{x_j}u_{2x_j}\,dxdt,
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2}\int_{Q_{2R}} \zeta^2(|u_{3x_j|^2})_t\,dxdt &= \int_{Q_{2R}} \zeta^2 u_{3x_ix_ix_j}u_{3x_j}\,dxdt + \int_{Q_{2R}} \zeta^2(u_3|\nabla u|^2)_{x_j}u_{3x_j}\,dxdt\\
&\quad - \int_{Q_{2R}} \frac{\zeta^2}{\varepsilon^2}(u_3|u'|^2)_{x_j}u_{3x_j}\,dxdt.
\end{aligned}$$

Therefore, using Green's theorem and noting $(|u|^2)_{x_j} = 0$, we obtain

$$\begin{aligned}
&\frac{a}{2}\int_{B_{2R}} \zeta^2|u'_{x_j}|^2(x,t_0+4R^2)\,dx + \frac{1}{2}\int_{B_{2R}} \zeta^2|u_{3x_j}|^2(x,t_0+4R^2)\,dx\\
&+ \int_{Q_{2R}} \zeta^2|u_{x_ix_j}|^2\,dxdt + \frac{1}{\varepsilon^2}\int_{Q_{2R}} \zeta^2|u_{3x_j}|^2\,dxdt\\
&= \frac{a}{2}\int_{B_{2R}} \zeta^2|u'_{x_j}|^2(x,t_0)\,dx + \frac{1}{2}\int_{B_{2R}} \zeta^2|u_{3x_j}|^2(x,t_0)\,dx
\end{aligned} \tag{3.6}$$

$$
\begin{aligned}
&+ b\int_{Q_{2R}} \zeta^2 \det(u'_t, u'_{x_jx_j})\,dxdt + 2b\int_{Q_{2R}} \zeta\zeta_{x_j}\det(u'_t, u'_{x_j})\,dxdt \\
&- 2\int_{Q_{2R}} \zeta\zeta_{x_j}(u_{x_j}\cdot u_{x_ix_j})\,dxdt + \int_{Q_{2R}} \zeta^2|\nabla u|^2|u_{x_j}|^2\,dxdt \\
&+ \frac{1}{\varepsilon^2}\int_{Q_{2R}} \zeta^2 u_3^2|u_{x_j}|^2\,dxdt.
\end{aligned}
$$

Here $B_{2R} = \overline{G}\cap B_{2R}(x_0)$. Using Cauchy's inequality and (2.1) to estimate the right-hand side of (3.6), for any $\delta\in(0,1)$, we obtain

$$
\begin{aligned}
\left|\int_{Q_{2R}} \zeta^2\det(u'_t, u'_{x_jx_j})\,dxdt\right| &\le \delta\int_{Q_{2R}} \zeta^2|u_{x_jx_j}|^2\,dxdt + C_\delta\int_{Q_{2R}} \zeta^2|u'_t|^2\,dxdt,\\
\left|\int_{Q_{2R}} \zeta\zeta_{x_j}\det(u'_t, u'_{x_j})\,dxdt\right| &\le \delta\int_{Q_{2R}} \zeta^2|\nabla u|^2\,dxdt + C_\delta\int_{Q_{2R}} |\nabla\zeta|^2|u'_t|^2\,dxdt,\\
\left|\int_{Q_{2R}} \zeta\zeta_{x_j}(u_{x_j}\cdot u_{x_ix_j})\,dxdt\right| &\le \delta\int_{Q_{2R}} \zeta^2|u_{x_jx_j}|^2\,dxdt + C_\delta\int_{Q_{2R}} |\nabla\zeta|^2|\nabla u|^2\,dxdt,\\
\left|\int_{Q_{2R}} \zeta^2|\nabla u|^2|u_{x_j}|^2\,dxdt\right| &\le C\int_{Q_{2R}} \zeta^2|\nabla u|^4\,dxdt.
\end{aligned}
$$

To estimate the last term, we use the first equation of (1.1) and Proposition 2.4 to deduce that

$$
\frac{1}{\varepsilon^2}u_3^2 = \frac{1}{|u'|^2}\left|\overline{u}'\left[(a+bi)u'_t - \Delta u' - u'|\nabla u|^2\right]\right| \le |u'|^{-1}\left[\sqrt{a^2+b^2}|u'_t| + |\Delta u'|\right].
$$

Therefore, using Proposition 2.4 and Cauchy's inequality, we obtain that for any $\delta\in(0,1)$,

$$
\begin{aligned}
\frac{1}{\varepsilon^2}\int_{Q_{2R}} \zeta^2 u_3^2|u_{x_j}|^2\,dxdt &\le \int_{Q_{2R}} \zeta^2|u_{x_j}|^2|u'|^{-1}\left[\sqrt{a^2+b^2}|u'_t| + |\Delta u'|\right]\,dxdt\\
&\le \delta\int_{Q_{2R}} \zeta^2|u'_t|^2\,dxdt + \delta\int_{Q_{2R}} \zeta^2|\Delta u|^2\,dxdt\\
&\quad + C_\delta\int_{Q_{2R}} \zeta^2|\nabla u|^4\,dxdt.
\end{aligned}
$$

Substituting these estimates (with δ sufficiently small) into (3.6), and using (2.1), we can deduce that

$$
\int_{Q_{2R}} \zeta^2|D^2u|^2\,dxdt \le C'\left(1 + \int_{Q_{2R}} \zeta^2|\nabla u|^4\,dxdt\right). \tag{3.7}
$$

Next we estimate the term $\int_{Q_{2R}} \zeta^2|\nabla u|^4\,dxdt$. By taking $\phi = \zeta|\nabla u|^2$ in the embedding inequality

$$
\left(\int_G \phi^2\,dx\right)^{1/2} \le C\int_G (|\nabla\phi| + |\phi|)\,dx, \quad \forall\,\phi\in W^{1,1}(G),
$$

and using (2.1), we have

$$\begin{aligned}\int_{Q_{2R}} \zeta^2|\nabla u|^4\,dxdt &\le C\left[\int_{Q_{2R}} \left(|\nabla\zeta||\nabla u|^2 + 2\zeta|\nabla u||D^2u| + \zeta|\nabla u|^2\right) dxdt\right]^2 \\ &\le C + C''\int_{Q_{2R}} |\nabla u|^2\,dxdt\cdot\int_{Q_{2R}}\zeta^2|D^2u|^2\,dxdt.\end{aligned}$$

Since $\sup_t \int_G |\nabla u|^2\,dx \le C$ (cf. (2.1)), we see $C''\int_{Q_{2R}}|\nabla u|^2\,dxdt \le 1/(4C')$ if R suitably small. Then,

$$\int_{Q_{2R}} \zeta^2|\nabla u|^4\,dxdt \le C + \frac{1}{4C'}\int_{Q_{2R}}\zeta^2|D^2u|^2\,dxdt.$$

Inserting this into (3.7), we get

$$\int_{Q_{2R}} \zeta^2|D^2u|^2\,dxdt \le C.$$

Noting that $\zeta = 1$ on $B_R(x_0)$, we can see the inner estimate.

Next we give the estimate near the boundary. Let $x_0 \in \partial G$. Without loss of generality, we assume $G\cap B_{2R}(x_0) = \{(x_1,x_2) \mid x_2 > 0\}\cap B_{2R}(x_0)$. Choose the cut-off function $\zeta(x)$ as above. Then (3.6) can be rewritten as

$$\begin{aligned}&\frac{a}{2}\int_{B_{2R}}\zeta^2|u'_{x_j}|^2(x,t_0+4R^2)\,dx + \frac{1}{2}\int_{B_{2R}}\zeta^2|u_{3x_j}|^2(x,t_0+4R^2)\,dx \\ &+\int_{Q_{2R}}\zeta^2|u_{x_ix_j}|^2\,dxdt + \frac{1}{\varepsilon^2}\int_{Q_{2R}}\zeta^2|u_{3x_j}|^2\,dxdt \\ =\;&\frac{a}{2}\int_{B_{2R}}\zeta^2|u'_{x_j}|^2(x,t_0)\,dx + \frac{1}{2}\int_{B_{2R}}\zeta^2|u_{3x_j}|^2(x,t_0)\,dx \\ &+ b\int_{Q_{2R}}\zeta^2\det(u'_t,u'_{2x_jx_j})\,dxdt + 2b\int_{Q_{2R}}\zeta\zeta_{x_j}\det(u'_t,u'_{x_j})\,dxdt \\ &-2\int_{Q_{2R}}\zeta\zeta_{x_j}(u_{x_j}\cdot u_{x_ix_j})\,dxdt + \int_{t_0}^{t_0+4R^2}\int_{B_{2R}\cap\{x_2=0\}}\zeta^2(u_{x_j}\cdot u_{x_jx_2})\,dsdt \\ &+\int_{Q_{2R}}\zeta^2|\nabla u|^2|u_{x_j}|^2\,dxdt + \frac{1}{\varepsilon^2}\int_{Q_{2R}}\zeta^2u_3^2|u_{x_j}|^2\,dxdt.\end{aligned} \tag{3.8}$$

Except for the tenth, eleventh and twelfth terms on the right-hand side of (3.8), the others can be handled as in the argument of the inner estimate. The tenth term is

$$\begin{aligned}&\int_{t_0}^{t_0+4R^2}\int_{B_{2R}\cap\{x_2=0\}}\zeta^2\sum_{j=1}^{2}u_{1x_j}u_{1x_jx_2}\,dsdt \\ =\;&\int_{t_0}^{t_0+4R^2}\int_{B_{2R}\cap\{x_2=0\}}\zeta^2\left[u_{1x_1}u_{1x_1x_2} + u_{1x_2}(\Delta u_1 - u_{1x_1x_1})\right]dsdt.\end{aligned}$$

Integrating by parts and noting $\zeta = 0$ at the two end points of the line segment $B_{2R} \cap \{x_2 = 0\}$, we get

$$\int_{B_{2R}\cap\{x_2=0\}} \zeta^2 u_{1x_1} u_{1x_1x_2}\, ds = \frac{1}{2}\int_{B_{2R}\cap\{x_2=0\}} \zeta^2 (u_{1x_1}^2)_{x_2}\, ds$$
$$= -\frac{1}{2}\int_{B_{2R}\cap\{x_2=0\}} (\zeta^2)_{x_2} u_{1x_1}^2\, ds.$$

Since (2.2) implies $\Delta u_1 = a u_{1t} - b u_{2t} - u_1|\nabla u|^2 - \frac{1}{\varepsilon^2} u_1 u_3^2 = -u_1|\nabla u|^2$ on ∂G, we have

$$\int_{B_{2R}\cap\{x_2=0\}} \zeta^2 u_{1x_2} \Delta u_1\, ds = -\int_{B_{2R}\cap\{x_2=0\}} \zeta^2 u_{1x_2} u_1 |\nabla u|^2\, ds$$
$$= -\frac{1}{2}\int_{B_{2R}\cap\{x_2=0\}} \zeta^2 (u_1^2)_{x_2} |\nabla u|^2\, ds.$$

Similarly, we have the same results for the eleventh and twelfth terms. Noting $|u| = 1$, and using (3.1) and Cauchy's inequality, we know that these terms are

$$-\int_{t_0}^{t_0+4R^2}\int_{B_{2R}\cap\{x_2=0\}} \left[\frac{1}{2}(\zeta^2)_{x_2}(u_{x_1})^2 + \frac{1}{2}\zeta^2(|u|^2)_{x_2}|\nabla u|^2 + \zeta^2(u_{x_2}\cdot u_{x_1x_1})\right] ds d\tau$$
$$\leq -\frac{1}{2}\int_{t_0}^{t_0+4R^2}\int_{B_{2R}\cap\{x_2=0\}} (\zeta^2)_{x_2}(g_{x_1})^2\, ds d\tau$$
$$+ C\int_{t_0}^{t_0+4R^2}\int_{B_{2R}\cap\{x_2=0\}} \zeta^2\left[(u_{x_2})^2 + (g_{x_1x_1})^2\right] ds d\tau \leq C.$$

Inserting this result into (3.8), we can also deduce the estimate of the second-order terms near the boundary. Thus, (3.2) is proved and hence Proposition 3.2 is complete. □

Finally, we gave a uniform $W_p^{2,1}$-estimate for some $p > 2$. Although it seems difficult to do for u_ε, we can handle $u_{\varepsilon 3}$ since the second equation of (1.1) is parabolic.

Proposition 3.3. *Assume that $u_\varepsilon\colon G_T \to S^2$ is a solution to (1.1). Let $(x_0, t_0) \in \overline{G_T}$, $Q_{R,r} = \overline{G_T} \cap (B_R(x_0) \times [t_0, t_0 + r])$. Then we can find $p > 2$ and $C > 0$ (which is independent of ε) such that*

$$\|u_{\varepsilon 3}\|_{W_p^{2,1}(Q_{R/2,r/2})} \leq C. \tag{3.9}$$

Proof. The second equation in (1.1) is

$$u_{3t} = \Delta u_3 + u_3|\nabla u|^2 - \frac{1}{\varepsilon^2} u_3 |u'|^2. \tag{3.10}$$

Set $\psi = \frac{1}{\varepsilon^2} u_3(x,t)$. Then

$$\psi |u'|^2 + \varepsilon^2 \psi_t = \varepsilon^2 \Delta\psi + \varepsilon^2 \psi |\nabla u|^2. \tag{3.11}$$

Let the cut-off function $\zeta \in C_0^\infty(Q_{R,r})$ satisfy $\zeta = 1$ on $Q_{R/2,r/2}$. Multiplying (3.11) with $\psi^{p-1}\zeta^p$ $(p>2)$ and integrating on $Q_{R,r}$, we get

$$\begin{aligned}&\int_{Q_{R,r}} \psi^p\zeta^p|u'|^2\,dxdt + \varepsilon^2\int_{Q_{R,r}} \psi_t\psi^{p-1}\zeta^p\,dxdt\\ &= \varepsilon^2\int_{Q_{R,r}} (\Delta\psi)\psi^{p-1}\zeta^p\,dxdt + \varepsilon^2\int_{Q_{R,r}} \psi^p\zeta^p|\nabla u|^2\,dxdt.\end{aligned} \tag{3.12}$$

Integrating by parts and noting $\psi = 0$ on ∂G_T, we have

$$\varepsilon^2\int_{Q_{R,r}} \psi_t\psi^{p-1}\zeta^p\,dxdt = -\varepsilon^2\int_{Q_{R,r}} \psi^p\zeta^{p-1}\zeta_t\,dxdt, \tag{3.13}$$

and

$$\begin{aligned}\varepsilon^2\int_{Q_{R,r}} (\Delta\psi)\psi^{p-1}\zeta^p\,dxdt &= -\varepsilon^2\int_{Q_{R,r}} \nabla\psi\nabla(\psi^{p-1}\zeta^p)\,dxdt\\ &= -\varepsilon^2(p-1)\int_{Q_{R,r}} \psi^{p-2}\zeta^p|\nabla\psi|^2\,dxdt\\ &\quad - \varepsilon^2 p\int_{Q_{R,r}} \psi^{p-1}\zeta^{p-1}\nabla\psi\nabla\zeta\,dxdt.\end{aligned} \tag{3.14}$$

Inserting (3.13) and (3.14) into (3.12), and applying Young's inequality, we can obtain that for any $\delta \in (0,1)$,

$$\begin{aligned}&\int_{Q_{R,r}} \psi^p\zeta^p|u'|^2\,dxdt + \varepsilon^2(p-1)\int_{Q_{R,r}} \psi^{p-2}\zeta^p|\nabla\psi|^2\,dxdt\\ &= \varepsilon^2\int_{Q_{R,r}} \left[\psi^p\zeta^{p-1}\zeta_t - p\psi^{p-1}\zeta^{p-1}\nabla\psi\nabla\zeta + \psi^p\zeta^p|\nabla u|^2\right]dxdt\\ &\le \delta\int_{Q_{R,r}} \psi^p\zeta^p\,dxdt + C(\delta)\int_{Q_{R,r}} \varepsilon^{2p}\left(|\psi|^p|\zeta_t|^p + p^p|\nabla\psi|^p|\nabla\zeta|^p + |\psi|^p\zeta^p|\nabla u|^{2p}\right)dxdt.\end{aligned}$$

Obviously, $\varepsilon^2\psi = u_3$, $\varepsilon^2\nabla\psi = \nabla u_3$. Noting Proposition 2.4 and choosing δ sufficiently small, we get

$$\int_{Q_{R,r}} \psi^p\zeta^p\,dxdt \le C\int_{Q_{R,r}} \left(|u_3|^p|\zeta_t|^p + |\nabla u_3|^p|\nabla\zeta|^p + |u_3|^p\zeta^p|\nabla u|^{2p}\right)dxdt. \tag{3.15}$$

Using (2.1) and (3.2), we have $\|u\|_{W_2^{2,1}(Q_{R,r})} \le C$. This implies that

$$\|u\|_{L^2(t_0,t_0+r^2;H^2(G\cap B_R))} + \|u_t\|_{L^2(t_0,t_0+r^2;L^2(G\cap B_R))} \le C. \tag{3.16}$$

Clearly, for all $q>4$, $H^2(G\cap B_R) \subset W^{1,q}(G\cap B_R) \subset L^2(G\cap B_R)$ and the imbedding map from $H^2(G\cap B_R)$ to $L^2(G\cap B_R)$ is compact. According to Corollary 4 in [19], by (3.16) we obtain that

$$\|u\|_{C(t_0,t_0+r^2;W^{1,q}(G\cap B_R))} \le C \quad \text{for all } q>4.$$

Here $C > 0$ is independent of ε. This implies

$$\|\nabla u\|_{L^q(Q_{R,r})} \le C \quad \text{for all } q > 4. \tag{3.17}$$

Noting $\zeta = 1$ on $Q_{R/2,r/2}$, and using (3.15) and (3.17), we obtain that for some $p > 2$, there exists a positive constant C which is independent of ε,

$$\int_{Q_{R/2,r/2}} \psi^p \, dxdt \le C. \tag{3.18}$$

Set $F_\varepsilon(x,t) = \left[u_3|\nabla u|^2 - \frac{1}{\varepsilon^2} u_3 |u'|^2\right](x,t)$. Then (3.10) becomes

$$u_{3t} = \Delta u_3 + F_\varepsilon. \tag{3.19}$$

Using (3.17) and (3.18), we get $\|F_\varepsilon\|_{L^p(G_T)} \le C$ for some $p > 2$, where C is independent of ε. Therefore, applying L^p theory for the parabolic equation (3.19), we know that (3.9) is true. Proposition 3.3 is proved. □

Proof of Theorem 1.1. According to Proposition 2.2, the first three results of Theorem 1.1 are proved.

By Proposition 3.3 and the t-anisotropy embedding inequality, we have

$$|u_{\varepsilon 3}|_{\gamma,\gamma/2;\overline{G_T}} \le C, \quad 0 < \gamma < \min\{1, 2 - 4/p\},$$

where $C > 0$ is independent of ε. This and Proposition 2.2 imply that there exists a subsequence $u_{\varepsilon_k 3}$ of $u_{\varepsilon 3}$ such that

$$u_{\varepsilon_k 3} \to 0 \quad \text{in } C^{\alpha,\alpha/2}(\overline{G_T})$$

for all $\alpha \in (0,\gamma)$ when $\varepsilon_k \to 0$. Since all the subsequences converge to the same limit 0, the convergence above still hold for $u_{\varepsilon 3}$. Theorem 1.1 is complete. □

Remark 3.4. Noting $|u'_\varepsilon| = \sqrt{1 - u_{\varepsilon 3}^2}$, we get

$$|u'_\varepsilon| \to 1 \quad \text{in } C^{\alpha,\alpha/2}(\overline{G_T}).$$

4. Non-parabolic behavior

Scale the solution u_ε to (1.1) in time

$$v_\varepsilon(x,t) = u_\varepsilon(x, \varepsilon t).$$

Then $v_\varepsilon = (v'_\varepsilon, v_{\varepsilon 3})$ satisfies

$$\begin{cases} (a+ib)\frac{1}{\varepsilon}v'_t = \Delta v' + v'|\nabla v|^2 + \frac{1}{\varepsilon^2}v'v_3^2 & \text{on } G_T, \\ \frac{1}{\varepsilon}v_{3t} = \Delta v_3 + v_3|\nabla v|^2 + \frac{1}{\varepsilon^2}v_3(v_3^2-1) & \text{on } G_T, \\ v|_{\partial G\times\mathbb{R}^+} = g(x), & \\ v(x,0) = u_0(x), & x \in G. \end{cases} \tag{4.1}$$

Similar to Proposition 2.1, we also have a conservation result.

Proposition 4.1. *Assume $v_\varepsilon\colon G\times\mathbb{R}^+ \to S^2$ satisfies* (4.1). *Then*

$$\sup_{t>0}\left[\int_0^t\int_G\left(\frac{a}{\varepsilon}\left|\frac{\partial}{\partial t}v'_\varepsilon\right|^2 + \frac{1}{\varepsilon}\left|\frac{\partial}{\partial t}v_{\varepsilon 3}\right|^2\right)dx\,d\tau + E_\varepsilon(v_\varepsilon(x,t))\right] = \frac{1}{2}\|\nabla u_0\|_2^2. \tag{4.2}$$

Here

$$E_\varepsilon(v) = \frac{1}{2}\int_G |\nabla v|^2\,dx + \frac{1}{2\varepsilon^2}\int_G |v_3|^2\,dx.$$

Proof. Similar to the derivation of (2.2)–(2.4), we have the following three equations

$$\frac{a}{\varepsilon}v_{1t} - \frac{b}{\varepsilon}v_{2t} = \Delta v_1 + v_1|\nabla v|^2 + \frac{1}{\varepsilon^2}v_1v_3^2, \tag{4.3}$$

$$\frac{b}{\varepsilon}v_{1t} + \frac{a}{\varepsilon}v_{2t} = \Delta v_2 + v_2|\nabla v|^2 + \frac{1}{\varepsilon^2}v_2v_3^2, \tag{4.4}$$

$$\frac{1}{\varepsilon}v_{3t} = \Delta v_3 + v_3|\nabla v|^2 + \frac{1}{\varepsilon^2}v_3(v_3^2-1). \tag{4.5}$$

Multiply (4.3), (4.4) and (4.5) with v_{1t}, v_{2t} and v_{3t}, respectively. Integrating over G, we have

$$\begin{aligned} &\frac{a}{\varepsilon}\int_G |v_{1t}|^2\,dx - \frac{b}{\varepsilon}\int_G v_{1t}v_{2t}\,dx \\ &= \int_G v_{1t}\Delta v_1\,dx + \frac{1}{2}\int_G (|v_1|^2)_t|\nabla v|^2\,dx + \frac{1}{2\varepsilon^2}\int_G (|v_1|^2)_t v_3^2\,dx, \end{aligned} \tag{4.6}$$

$$\begin{aligned} &\frac{b}{\varepsilon}\int_G v_{1t}v_{2t}\,dx + \frac{a}{\varepsilon}\int_G |v_{2t}|^2\,dx \\ &= \int_G v_{2t}\Delta v_2\,dx + \frac{1}{2}\int_G (|v_2|^2)_t|\nabla v|^2\,dx + \frac{1}{2\varepsilon^2}\int_G (|v_2|^2)_t v_3^2\,dx, \end{aligned} \tag{4.7}$$

$$\frac{1}{\varepsilon}\int_G |v_{3t}|^2\,dx = \int_G v_{3t}\Delta v_3\,dx + \frac{1}{2}\int_G (|v_3|^2)_t|\nabla v|^2\,dx + \frac{1}{2\varepsilon^2}\int_G (|v_3|^2)_t(v_3^2-1)\,dx. \tag{4.8}$$

Noting $v_t = 0$ on ∂G, we obtain by Green's theorem that

$$\int_G v_{it}\Delta v_i\,dx = -\frac{d}{2dt}\int_G |\nabla v_i|^2\,dx, \quad i = 1,2,3.$$

Inserting these into (4.6), (4.7) and (4.8), respectively. Noting $|v|^2 = 1$, we have

$$\int_G \frac{a}{\varepsilon}\big(|v_{1t}|^2 + |v_{2t}|^2\big)\,dx + \int_G \frac{1}{\varepsilon}|v_{3t}|^2\,dx = -\frac{d}{dt}E_\varepsilon(v(x,t)).$$

Integrating from 0 to t, we deduce that, for all $t > 0$,

$$\int_0^t \int_G \left(\frac{a}{\varepsilon} \left| \frac{\partial}{\partial t} v'_\varepsilon \right|^2 + \frac{1}{\varepsilon} \left| \frac{\partial}{\partial t} v_{\varepsilon 3} \right|^2 \right) dx d\tau + E_\varepsilon(v_\varepsilon(x,t)) = E_\varepsilon(v(x,0)) = \frac{1}{2}\|\nabla u_0\|_2^2.$$

Proposition 4.1 is proved. □

Proof of Theorem 1.2. By Proposition 4.1, we can find a positive constant C (independent of ε) such that

$$\|v_\varepsilon\|_{L^\infty(0,T;H^1(G,S^2))} \le C \quad \text{and} \quad \left\| \frac{\partial v_\varepsilon}{\partial t} \right\|_{L^2(0,T;L^2(G,S^2))} \le C.$$

Hence, there is a subsequence ε_k such that as $\varepsilon_k \to 0$,

$$\begin{aligned} v_{\varepsilon_k} &\rightharpoonup h && \text{weakly}^* \text{ in } L^\infty(0,T;H^1(G,S^2)),\\ \frac{\partial}{\partial t} v_{\varepsilon_k} &\rightharpoonup \frac{\partial}{\partial t} h && \text{weakly in } L^2(0,T;L^2(G,S^2)),\\ v_{\varepsilon_k} &\to h && \text{in } L^2(0,T;L^2(G,S^2)). \end{aligned}$$

Next, we prove that $h = (v'_*, 0)$ satisfies the conditions in Theorem 1.2. When $\varepsilon \to 0$, by (4.2) we get $\sup_{t>0} \int_G v_3^2 \, dx \to 0$. This result shows

$$h_3 = 0 \quad \text{a.e. on } G_T.$$

Since $v_{\varepsilon_k} = v = (v', v_3)$ satisfies (4.1), multiply the first equation in (4.1) by $\overline{v}'$. Taking the real part and imaginary part, we get

$$\frac{a}{2}\left(\frac{|v'|^2}{\varepsilon}\right)_t + \frac{b}{\varepsilon}(v_{1t}v_2 - v_{2t}v_1) = v_1 \Delta v_1 + v_2 \Delta v_2 + |v'|^2 |\nabla v|^2 + \frac{1}{\varepsilon^2}|v'|^2 v_3^2, \tag{4.9}$$

$$\frac{b}{2}\left(\frac{|v'|^2}{\varepsilon}\right)_t + \frac{a}{\varepsilon}(v_{2t}v_1 - v_{1t}v_2) = v_1 \Delta v_2 - v_2 \Delta v_1. \tag{4.10}$$

Multiplying the second equation in (4.1) by v_3, we have

$$\frac{1}{2}\left(\frac{v_3^2}{\varepsilon}\right)_t = v_3 \Delta v_3 + v_3^2 |\nabla v|^2 - \frac{1}{\varepsilon^2}|v'|^2 v_3^2. \tag{4.11}$$

Calculating by $a \times (4.9) + b \times (4.10) + a \times (4.11)$, we obtain

$$\frac{a - a^2 - b^2}{2}\left(\frac{v_3^2}{\varepsilon}\right)_t = a(v_1 \Delta v_1 + v_2 \Delta v_2 + v_3 \Delta v_3) + b(v_1 \Delta v_2 - v_2 \Delta v_1) + a|\nabla v|^2.$$

In view of $a \neq a^2 + b^2$, it follows that

$$\left(\frac{v_3^2}{\varepsilon}\right)_t = \frac{2}{a - a^2 - b^2}\left[a \operatorname{Re}(\overline{v}' \Delta v') + a v_3 \Delta v_3 + b \operatorname{Im}(\overline{v}' \Delta v') + a|\nabla v|^2\right].$$

Integrating from 0 to t, we get

$$\frac{v_3^2}{\varepsilon} = \frac{2}{a-a^2-b^2}\int_0^t \left[a\,\mathrm{Re}(\overline{v}'\Delta v') + av_3\Delta v_3 + b\,\mathrm{Im}(\overline{v}'\Delta v') + a|\nabla v|^2\right] d\tau. \tag{4.12}$$

By Proposition 4.1, $v_3^2/\varepsilon \rightharpoonup w$ in $\mathcal{D}'$ for some nonnegative function w. By (4.12), we know that

$$w = \frac{2}{a-a^2-b^2}\int_0^t \left[a\,\mathrm{Re}(\overline{v}'_*\Delta v'_*) + b\,\mathrm{Im}(\overline{v}'_*\Delta v'_*) + a|\nabla v_*|^2\right] d\tau.$$

Multiplying the first equation in (4.1) by ε and letting $\varepsilon \to 0$, we get

$$(a+ib)v'_{*t} = \frac{2v'_*}{a-a^2-b^2}\int_0^t \left[a\,\mathrm{Re}(\overline{v}'_*\Delta v'_*) + b\,\mathrm{Im}(\overline{v}'_*\Delta v'_*) + a|\nabla v_*|^2\right] d\tau.$$

Moreover, if we write $v_* = (e^{i\theta}, 0)$, the result above implies

$$(ai-b)\theta_t = \frac{2b}{a-a^2-b^2}\int_0^t \Delta\theta\, d\tau.$$

So we have the following non-parabolic results

$$a\theta_t = 0, \quad b\theta_t = \frac{2b}{a^2+b^2-a}\int_0^t \Delta\theta\, d\tau.$$

If $a > 0$, then $\theta_t = 0$, and we know θ only depends on x. Moveover, when $b \neq 0$, we have $\Delta\theta = 0$, i.e., v'_* is a harmonic map. If $a = 0$, then $b \neq 0$ and hence

$$\theta_{tt} = \frac{2}{b^2}\Delta\theta,$$

which implies that $(v'_*, 0)$ satisfies (1.5).

Since the limit $(v'_*, 0)$ is unique, the convergence above can be generalized to all ε instead of the subsequence ε_k. Theorem 1.2 is proved. $\square$

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