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Singular Limit Solutions for a 4-dimensional Semilinear Elliptic System of Liouville Type

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Abstract. We consider the existence of singular limit solutions for a nonlinear elliptic system of Liouville type with Navier boundary conditions. We use the nonlinear domain decomposition method and a Pohozaev type identity.

1. Introduction and statement of the results

Let $\Omega \subset \mathbb{R}^4$ be a regular bounded open domain in $\mathbb{R}^4$. We consider the following elliptic system

$$\begin{cases} \Delta^2 u_1 = \rho^4 e^{\gamma u_1 + (1-\gamma) u_2} & \text{in } \Omega, \\ \Delta^2 u_2 = \rho^4 e^{\xi u_2 + (1-\xi) u_1} & \text{in } \Omega, \\ u_i = \Delta u_i = 0, \ i = 1,2 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

here γ, ξ and ρ are constants. We assume that $\gamma, \xi \in (0,1)$ such that $\gamma + \xi > 1$. So in all the following, we have

$$\frac{1-\xi}{\gamma}, \frac{1-\gamma}{\xi} \in (0,1).$$

We are interested in the study of the existence of solutions with singular limits as the parameter ρ tends to 0.

The system (1.1) is a natural generalization of the equation

$$\Delta^2 u = 6e^{4u} \quad \text{in } \mathbb{R}^4. \tag{1.2}$$

Equation (1.2) is invariant under translation, rotation, dilatation in the Euclidean space and the Kelvin transforms. In [15], Lin proved the following important classification result of finite-mass solutions of equation (1.2).

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Theorem 1.1. [15] *Let u be a solution of* (1.2), *satisfying the finite-mass condition*

$$\int_{\mathbb{R}^4} e^{4u}\,dx < \infty \tag{1.3}$$

and $|u(x)| = o(|x|^2)$ at ∞. Then there exists some point $x^0 \in \mathbb{R}^4$ such that u is radially symmetric about x^0 and

$$u(x) = \ln\left(\frac{2\lambda}{1+\lambda^2|x-x^0|^2}\right).$$

This result is decisive for solving completely (1.2) under (1.3), because it reduces the problem to a simple ODE problem.

In [21], Wei and Ye constructed a non-radial solution of Liouville equation (1.2) under (1.3) with the following asymptotic behavior

$$u(x) = -\sum_{j=1}^{k} a_j(x_j - x_j^0)^2 - \alpha\ln|x| + c_0 + o(1), \quad |x| > 1 \quad \text{and} \quad \int_{\mathbb{R}^4} e^{4u(x)}\,dx = \frac{4\pi^2\alpha}{3}$$

for each fixed $x^0 \in \mathbb{R}^4$, $1 \le k \le 4$, $\alpha \in (1-k/4, 2)$ and $a_j > 0$ for $1 \le j \le k$.

We consider the corresponding Dirichlet problem on a bounded domain in $\mathbb{R}^2$,

$$-\Delta u = \rho^2 e^u \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega, \tag{1.4}$$

where the parameter ρ tends to 0. The study of this equation goes back to 1853 when Liouville derived a representation formula for all solutions of (1.4) that are defined in $\mathbb{R}^2$ [18].

It is well known that as the parameter ρ tends to 0, non-minimal solutions exist and they have singular limits. In [6], Baraket and Pacard proved

Theorem 1.2. [6] *Let Ω be a smooth open subset of $\mathbb{R}^2$ and $z^1, \dots, z^m \in \Omega$. Assume that $(z^1, \dots, z^m)$ is a nondegenerate critical point of the function*

$$F\colon (z^1, \dots, z^m) \in \mathbb{C}^m \to \sum_j h(z^j, z^j) + \sum_{j \ne l} g(z^j, z^l),$$

then there exist $\rho_0 > 0$ and $(u_\rho)_{\rho \in (0,\rho_0)}$ a one parameter family of solutions of (1.4) *such that*

$$\lim_{\rho \to 0} u_\rho = u^* := \sum_{j=1}^{m} g(\,\cdot\,, z^j) \quad \text{in } \mathcal{C}^{2,\alpha}_{\mathrm{loc}}(\Omega - \{z^1, \dots, z^m\}).$$

Here g is the Green's function defined as the solution of

$$\begin{cases} -\Delta_z g(z,z') = 8\pi\delta_{z=z'} & \text{in } \Omega, \\ g(z,z') = 0 & \text{on } \partial\Omega \end{cases}$$

and h is its smooth part defined by $h(z,z') = g(z,z') + 4\ln|z-z'|$. Some generalizations can be found in [3, 8, 14].

In dimension 4, other authors were motivated by similar problems, we refer the reader to [2, 4, 11, 12]. Wei in [20], has studied the behavior of solutions to the following nonlinear eigenvalue problem for the biharmonic operator Δ^2 in $\mathbb{R}^4$. More precisely, consider the following problem

$$\begin{cases} \Delta^2 u = \lambda f(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

When $f(u) = e^u$, we can see that (1.5) is issued from the conformal geometry by prescribing the so-called Q-curvature on 4-dimensional Riemannian manifolds. For more details and background material we refer to [1, 9, 13].

Before announcing the result of [20], we will introduce some notations. Let $G(x,x')$ defined over $\Omega\times\Omega$, be the Green function associated to the bi-laplacian operator with Navier boundary conditions, which is the solution of

$$\begin{cases} \Delta_x^2 G(x,x') = 64\pi^2 \delta_{x=x'} & \text{in } \Omega, \\ G(x,x') = \Delta_x G(x,x') = 0 & \text{on } \partial\Omega \end{cases}$$

and denote by $H(x,x') = G(x,x') + 8\ln|x-x'|$ its smooth part. Consider now the functional

$$E(x^1,\dots,x^m) = \sum_{j=1}^m H(x^j,x^j) + \sum_{j\neq l} G(x^j,x^l)$$

and denote by u^* the solution of

$$\begin{cases} \Delta^2 u^* = 64\pi^2 \sum_{j=1}^m \delta_{x^j} & \text{in } \Omega, \\ u^* = \Delta u^* = 0 & \text{on } \partial\Omega. \end{cases}$$

In [20], the author proved the following result.

Theorem 1.3. [20] *Let Ω be a smooth bounded domain in $\mathbb{R}^4$ and f a smooth nonnegative increasing function such that*

$$e^{-u} f(u) \text{ tends to } 1 \text{ as } u \to +\infty.$$

For u_λ solution of (1.5), denote by $\Sigma_\lambda = \lambda\int_\Omega f(u_\lambda)\,dx$. Then there are only three possibilities:

(i) *The $\{\Sigma_\lambda\}$ accumulate to 0. Then $\|u_\lambda\|_{L^\infty(\Omega)} \to 0$ as $\lambda \to 0$.*

(ii) *The $\{\Sigma_\lambda\}$ accumulate to $+\infty$. Then $u_\lambda \to +\infty$ as $\lambda \to 0$.*

(iii) *The* $\{\Sigma_\lambda\}$ *accumulate to* $64\pi^2 m$ *for some positive integer* m. *Then the limiting function* $u^* = \lim_{\lambda\to 0} u_\lambda$ *has* m *blow-up points,* $\{x^1, \ldots, x^m\}$*, where* $u_\lambda(x^i) \to +\infty$ *as* $\lambda \to 0$.

Moreover, $(x^1, \ldots, x^m)$ *is a critical point of* E.

In dimension 4, the authors in [5] considered the following problem

$$\begin{cases} \Delta^2 u_1 = \rho^4 e^{u_1+\gamma_1 u_2} & \text{in } \Omega, \\ \Delta^2 u_2 = \rho^4 e^{u_2+\gamma_2 u_1} & \text{in } \Omega, \\ u_i = \Delta u_i = 0, \ i = 1,2 & \text{on } \partial\Omega. \end{cases}$$

They proved the existence of singular limit solutions which blow-up on different points as ρ tends to 0 using the nonlinear domain decomposition method.

In dimension 2, the L-system has been interested by several authors [7, 10, 16, 17, 19]. Recently, the authors in [7] considered the following problem

$$\begin{cases} -\Delta u_1 = \rho^2 e^{\gamma u_1+(1-\gamma)u_2} & \text{in } \Omega, \\ -\Delta u_2 = \rho^2 e^{\xi u_2+(1-\xi)u_1} & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

They proved the existence of singular limit solutions with blow-up on common points as ρ tends to 0.

In this paper, we prove the following results.

Theorem 1.4. *Let* Ω *be a regular open subset of* $\mathbb{R}^4$ *and* $x^1, x^2, x^3 \in \Omega$ *be given disjoint points. Suppose that* (u_1^ρ, u_2^ρ) *is a one parameter family of solutions of* (1.1)*, such that*

$$\lim_{\rho\to 0} u_1^\rho = \frac{1}{\gamma} G(\,\cdot\,, x^1) + G(\,\cdot\,, x^2) = u_1^* \quad \textit{in} \ \ \mathcal{C}^{4,\alpha}_{\mathrm{loc}}(\Omega \setminus \{x^1, x^2\})$$

and

$$\lim_{\rho\to 0} u_2^\rho = \frac{1}{\xi} G(\,\cdot\,, x^3) + G(\,\cdot\,, x^2) = u_2^* \quad \textit{in} \ \ \mathcal{C}^{4,\alpha}_{\mathrm{loc}}(\Omega \setminus \{x^2, x^3\}).$$

Then (x^1, x^2, x^3) *is a critical point of the functional*

$$\begin{aligned} \mathcal{E}(x^1, x^2, x^3) = {} & \frac{1-\xi}{\gamma} H(x^1, x^1) + (2-\gamma-\xi) H(x^2, x^2) + \frac{1-\gamma}{\xi} H(x^3, x^3) \\ & + \frac{1-\gamma}{\xi}\frac{1-\xi}{\gamma} G(x^1, x^3) + \frac{1-\xi}{\gamma} G(x^1, x^2) + \frac{1-\gamma}{\xi} G(x^2, x^3). \end{aligned}$$

A natural question that arises: can one find a solution that concentrates in a common point x^2. Before giving a partial answer of this question, we define an auxiliary function which is a cut-off function in $C_0^\infty(\Omega)$ such that $\varphi \equiv 1$ in $B(x^1, r_0) \cup B(x^3, r_0)$ and $\varphi \equiv 0$ in $\Omega \setminus (B(x^1, r_0) \cup B(x^3, r_0))$, where $r_0 > 0$ and such that $B(x^i, 2r_0) \subset \Omega$ for $i = 1, 3$ and $B(x^1, 2r_0) \cap B(x^3, 2r_0) = \emptyset$.

Theorem 1.5. *Let Ω be a regular open subset of $\mathbb{R}^4$ and $x^1, x^2, x^3 \in \Omega$ be given disjoint points. Suppose that (x^1, x^2, x^3) is a nondegenerate critical point of the functional*

$$\begin{aligned}\mathcal{E}(x^1,x^2,x^3) = \frac{1-\xi}{\gamma}H(x^1,x^1) + (2-\gamma-\xi)H(x^2,x^2) + \frac{1-\gamma}{\xi}H(x^3,x^3) \\ + \frac{1-\gamma}{\xi}\frac{1-\xi}{\gamma}G(x^1,x^3) + \frac{1-\xi}{\gamma}G(x^1,x^2) + \frac{1-\gamma}{\xi}G(x^2,x^3).\end{aligned}$$

Then there exist γ_0 and ξ_0 in $(0,1)$ such that for all $\gamma \in (\gamma_0, 1)$ and $\xi \in (\xi_0, 1)$, there exist $\rho_0 > 0$ and $(u_1^\rho, u_2^\rho)_{\rho\le\rho_0}$ a one parameter family of solutions of (1.1), such that

$$\lim_{\rho\to 0}\varphi u_1^\rho = \frac{\varphi}{\gamma}G(\,\cdot\,,x^1) \ \text{ in } \ \mathcal{C}^{4,\alpha}_{\mathrm{loc}}(\Omega\setminus\{x^1\}), \quad \lim_{\rho\to 0}\varphi u_2^\rho = \frac{\varphi}{\xi}G(\,\cdot\,,x^3) \ \text{ in } \ \mathcal{C}^{4,\alpha}_{\mathrm{loc}}(\Omega\setminus\{x^3\})$$

and

$$\begin{aligned}&\lim_{\rho\to 0}\big((1-\xi)u_1^\rho + (1-\gamma)u_2^\rho\big) \\ &= \frac{1-\xi}{\gamma}G(\,\cdot\,,x^1) + \frac{1-\gamma}{\xi}G(\,\cdot\,,x^3) + (2-\gamma-\xi)G(\,\cdot\,,x^2) \ \text{ in } \ \mathcal{C}^{4,\alpha}_{\mathrm{loc}}(\Omega\setminus\{x^1,x^2,x^3\}).\end{aligned}$$

Unfortunately, we are not able to give the asymptotic behavior of u_1^ρ and u_2^ρ separately. But, under an additional assumption on the points set $\{x^1, x^2, x^3\}$, we give a positive answer.

Theorem 1.6. *Let Ω be a regular open subset of $\mathbb{R}^4$ and $x^1, x^2, x^3 \in \Omega$ be given disjoint points. Suppose that (x^1, x^2, x^3) is a nondegenerate critical point of the functional*

$$\begin{aligned}\mathcal{E}(x^1,x^2,x^3) = \frac{1-\xi}{\gamma}H(x^1,x^1) + (2-\gamma-\xi)H(x^2,x^2) + \frac{1-\gamma}{\xi}H(x^3,x^3) \\ + \frac{1-\gamma}{\xi}\frac{1-\xi}{\gamma}G(x^1,x^3) + \frac{1-\xi}{\gamma}G(x^1,x^2) + \frac{1-\gamma}{\xi}G(x^2,x^3)\end{aligned}$$

such that

$$(1.6) \qquad \frac{1}{\gamma}G(x^2,x^1) = \frac{1}{\xi}G(x^2,x^3) \quad \text{and} \quad \frac{1}{\gamma}\nabla G(\,\cdot\,,x^1)(x^2) = \frac{1}{\xi}\nabla G(\,\cdot\,,x^3)(x^2).$$

Then there exist γ_0 and ξ_0 in $(0,1)$ such that for all $\gamma \in (\gamma_0, 1)$ and $\xi \in (\xi_0, 1)$, there exist $\rho_0 > 0$ and $(u_1^\rho, u_2^\rho)_{\rho\le\rho_0}$ a one parameter family of solutions of (1.1), such that

$$\begin{aligned}\lim_{\rho\to 0}u_1^\rho &= \frac{1}{\gamma}G(\,\cdot\,,x^1) + G(\,\cdot\,,x^2) \qquad \text{in } \ \mathcal{C}^{4,\alpha}_{\mathrm{loc}}(\Omega\setminus\{x^1,x^2\}),\\ \lim_{\rho\to 0}u_2^\rho &= \frac{1}{\xi}G(\,\cdot\,,x^3) + G(\,\cdot\,,x^2) \qquad \text{in } \ \mathcal{C}^{4,\alpha}_{\mathrm{loc}}(\Omega\setminus\{x^2,x^3\}).\end{aligned}$$

Remark 1.7. Denote by S_i the blow up points set of u_i for $i = 1, 2$.

1. Theorem 1.4 can be extended, but the computation is more complicated. Indeed we replace $\mathcal{E}(x^1,x^2,x^3)$ by $\mathcal{F}(x^p\in S_1\setminus S_1\cap S_2, x^k\in S_1\cap S_2, x^l\in S_2\setminus S_1\cap S_2)$, given by
$$\begin{aligned}&\mathcal{F}(x^p\in S_1\setminus S_1\cap S_2, x^k\in S_1\cap S_2, x^l\in S_2\setminus S_1\cap S_2)\\ &=\frac{1-\gamma}{\xi}\frac{1-\xi}{\gamma}\sum_{\substack{x^l\in S_2\setminus S_1\cap S_2\\ x^p\in S_1\setminus S_1\cap S_2}} G(x^l,x^p)\\ &\quad+\frac{1-\xi}{\gamma}\Bigg(\sum_{x^p\in S_1\setminus S_1\cap S_2} H(x^p,x^p)+\sum_{\substack{x^k\in S_1\cap S_2\\ x^p\in S_1\setminus S_1\cap S_2}} G(x^k,x^p)+\sum_{\substack{x^p,x^{p'}\in S_1\setminus S_1\cap S_2\\ p\neq p'}} G(x^p,x^{p'})\Bigg)\\ &\quad+\frac{1-\gamma}{\xi}\Bigg(\sum_{x^l\in S_2\setminus S_1\cap S_2} H(x^l,x^l)+\sum_{\substack{x^l\in S_2\setminus S_1\cap S_2\\ x^k\in S_1\cap S_2}} G(x^l,x^k)+\sum_{\substack{x^l,x^{l'}\in S_2\setminus S_1\cap S_2\\ l\neq l'}} G(x^l,x^{l'})\Bigg)\\ &\quad+(2-\gamma-\xi)\Bigg(\sum_{x^k\in S_1\cap S_2} H(x^k,x^k)+\sum_{x^k,x^{k'}\in S_1\cap S_2} G(x^k,x^{k'})\Bigg).\end{aligned}$$

2. Theorem 1.5 can be also extended by assuming that $(x^p\in S_1\setminus S_1\cap S_2, x^k\in S_1\cap S_2, x^l\in S_2\setminus S_1\cap S_2)$ is a nondegenerate critical point of $\mathcal{F}$.

3. Theorem 1.6 is hold if $S_1\setminus S_1\cap S_2\neq\emptyset$ and $S_2\setminus S_1\cap S_2\neq\emptyset$ and the condition (1.6) will be replaced for all $x^k\in S_1\cap S_2$ by
$$\frac{1}{\gamma}\sum_{x^p\in S_1\setminus S_1\cap S_2} G(x^k,x^p)=\frac{1}{\xi}\sum_{x^l\in S_2\setminus S_1\cap S_2} G(x^k,x^l)$$
and
$$\frac{1}{\gamma}\sum_{x^p\in S_1\setminus S_1\cap S_2} \nabla G(\,\cdot\,,x^p)(x^k)=\frac{1}{\xi}\sum_{x^l\in S_2\setminus S_1\cap S_2} \nabla G(\,\cdot\,,x^l)(x^k).$$

4. To have a readable and clear proof, we considered the choice of $S_1=\{x^1,x^2\}$ and $S_2=\{x^2,x^3\}$, which is already supposed in Theorems 1.4, 1.5 and 1.6.

5. Our functional presents the same terms as those of the functional introduced by Baraket-Pacard [6], the only difference here is that these terms are weighted by certain weights not equals.

6. The conditions of Theorem 1.6 are certainly not valid on all domain Ω of $\mathbb{R}^4$. It is thought that a certain symmetry of the domain must be imposed for the condition (1.6) to be verified.

2. Proof of Theorem 1.4

We first give the green identity for the bilaplacian operator:
$$\int_\Omega(\Delta^2u)\cdot v-(\Delta^2v)\cdot u=\int_{\partial\Omega}\left(\frac{\partial\Delta u}{\partial\nu}\cdot v-\Delta u\frac{\partial v}{\partial\nu}+\frac{\partial u}{\partial\nu}\cdot\Delta v-u\cdot\frac{\partial\Delta v}{\partial\nu}\right)d\sigma.$$

2.1. Behavior of solution around x^2

We multiply the equation $\Delta^2 u_1 = \rho^4 e^{\gamma u_1+(1-\gamma)u_2}$ by $\nabla(\gamma u_1 + (1-\gamma)u_2)$ and then integrating over $B_2 = B(x^2,\eta)$ where η fixed small enough, we obtain a pohozaev type identity

$$\gamma \int_{B_2} (\Delta^2 u_1)\nabla u_1 + (1-\gamma)\int_{B_2}(\Delta^2 u_1)\nabla u_2 = \rho^4 \int_{\partial B_2}(e^{\gamma u_1+(1-\gamma)u_2} - 1)\nu \, d\sigma. \tag{2.1}$$

Using the Green's formula we obtain

$$\begin{aligned}\int_{B_2}(\Delta^2 u_1)\nabla u_1 &= -\int_{B_2}(\Delta u_1)\nabla(\Delta(u_1)) - \int_{B_2}\nabla(\nabla(\Delta u_1)\cdot\nabla u_1)\\ &\quad + \int_{\partial B_2}\nabla(\Delta u_1)\cdot\nu\nabla u_1\, d\sigma + \int_{\partial B_2}\nabla u_1\cdot\nu\nabla(\Delta u_1)\, d\sigma\\ &= -\frac{1}{2}\int_{\partial B_2}(\Delta u_1)^2\nu\, d\sigma - \int_{\partial B_2}(\nabla(\Delta u_1)\cdot\nabla u_1)\nu\, d\sigma\\ &\quad + \int_{\partial B_2}\nabla(\Delta u_1)\cdot\nu\nabla u_1\, d\sigma + \int_{\partial B_2}\nabla u_1\cdot\nu\nabla(\Delta u_1)\, d\sigma.\end{aligned}$$

Similarly we multiply the equation $\Delta^2 u_2 = \rho^4 e^{\xi u_2+(1-\xi)u_1}$ by $\nabla(\xi u_2 + (1-\xi)u_1)$ and then integrating over $B_2 = B(x^2,\eta)$ we obtain a pohozaev type identity

$$\xi \int_{B_2} (\Delta^2 u_2)\nabla u_2 + (1-\xi)\int_{B_2}(\Delta^2 u_2)\nabla u_1 = \rho^4 \int_{\partial B_2}(e^{\xi u_2+(1-\xi)u_1} - 1)\nu \, d\sigma. \tag{2.2}$$

Using the Green's formula we obtain

$$\begin{aligned}\int_{B_2}(\Delta^2 u_2)\nabla u_2 &= -\int_{B_2}(\Delta u_2)\nabla(\Delta(u_2)) - \int_{B_2}\nabla(\nabla(\Delta u_2)\cdot\nabla u_2)\\ &\quad + \int_{\partial B_2}\nabla(\Delta u_2)\cdot\nu\nabla u_2\, d\sigma + \int_{\partial B_2}\nabla u_2\cdot\nu\nabla(\Delta u_2)\, d\sigma\\ &= -\frac{1}{2}\int_{\partial B_2}(\Delta u_2)^2\nu\, d\sigma - \int_{\partial B_2}(\nabla(\Delta u_2)\cdot\nabla u_2)\nu\, d\sigma\\ &\quad + \int_{\partial B_2}\nabla(\Delta u_2)\cdot\nu\nabla u_2\, d\sigma + \int_{\partial B_2}\nabla u_2\cdot\nu\nabla(\Delta u_2)\, d\sigma.\end{aligned}$$

Making use of the identity

$$\begin{aligned}&\int_{B_2}\Delta^2 u_2\nabla u_1 + \int_{B_2}\Delta^2 u_1\nabla u_2\\ &= -\int_{B_2}\nabla(\Delta u_2\cdot\Delta u_1) + \int_{\partial B_2}\frac{\partial(\Delta u_2)}{\partial\nu}\nabla u_1\, d\sigma + \int_{\partial B_2}\frac{\partial(\Delta u_1)}{\partial\nu}\nabla u_2\, d\sigma\\ &= -\int_{\partial B_2}(\Delta u_2\cdot\Delta u_1)\nu\, d\sigma + \int_{\partial B_2}\frac{\partial(\Delta u_2)}{\partial\nu}\nabla u_1\, d\sigma + \int_{\partial B_2}\frac{\partial(\Delta u_1)}{\partial\nu}\nabla u_2\, d\sigma,\end{aligned}$$

then by combination of (2.1) and (2.2) we obtain

(2.3)
$$\begin{aligned}
&\gamma(1-\xi)\int_{\partial B_2}\left[\frac{-1}{2}(\Delta u_1)^2\nu-(\nabla(\Delta u_1)\cdot\nabla u_1)\nu+\nabla(\Delta u_1)\cdot\nu\nabla u_1+\nabla u_1\cdot\nu\nabla(\Delta u_1)\right]d\sigma\\
&+\xi(1-\gamma)\int_{\partial B_2}\left[\frac{-1}{2}(\Delta u_2)^2\nu-(\nabla(\Delta u_2)\cdot\nabla u_2)\nu+\nabla(\Delta u_2)\cdot\nu\nabla u_2+\nabla u_2\cdot\nu\nabla(\Delta u_2)\right]d\sigma\\
&+(1-\gamma)(1-\xi)\left[-\int_{\partial B_2}(\Delta u_2\cdot\Delta u_1)\nu\,d\sigma+\int_{\partial B_2}\frac{\partial(\Delta u_2)}{\partial\nu}\nabla u_1\,d\sigma+\int_{\partial B_2}\frac{\partial(\Delta u_1)}{\partial\nu}\nabla u_2\,d\sigma\right]\\
&=\rho^4(1-\xi)\int_{\partial B_2}(e^{\gamma u_1+(1-\gamma)u_2}-1)\nu\,d\sigma+\rho^4(1-\gamma)\int_{\partial B_2}(e^{\xi u_2+(1-\xi)u_1}-1)\nu\,d\sigma.
\end{aligned}$$

In the desire to construct solutions of the system that blow-up in the point x^2 this means that, if ρ tends to zero,

$$u_1\to u_1^*(x)=G(x,x^2)+\frac{1}{\gamma}G(x,x^1)\quad\text{and}\quad u_2\to u_2^*(x)=G(x,x^2)+\frac{1}{\xi}G(x,x^3).$$

Since we have $G(x,x^2)=-8\ln|x-x^2|+H(x,x^2)$, where H is a smooth function in Ω, then

$$\begin{aligned}
u_1^*(x)&=G(x,x^2)+\frac{1}{\gamma}G(x,x^1)=-8\ln|x-x^2|+H(x,x^2)+\frac{1}{\gamma}G(x,x^1)\\
&=-8\ln|x-x^2|+R(x,x^2)
\end{aligned}$$

and

$$\begin{aligned}
u_2^*(x)&=G(x,x^2)+\frac{1}{\xi}G(x,x^3)=-8\ln|x-x^2|+H(x,x^2)+\frac{1}{\xi}G(x,x^3)\\
&=-8\ln|x-x^2|+K(x,x^2).
\end{aligned}$$

Thanks to the fact that the solutions of the system (1.1) are regular on $\Omega\setminus\{x^1,x^2,x^3\}$ and by inserting the profile of the limits of the solutions in the identity (2.3) when $\rho\to0$ and η fixed small enough, we obtain

$$\lim_{\rho\to0}\rho^4(1-\xi)\int_{\partial B_2}(e^{\gamma u_1+(1-\gamma)u_2}-1)\nu\,d\sigma+\rho^4(1-\gamma)\int_{\partial B_2}(e^{\xi u_2+(1-\xi)u_1}-1)\nu\,d\sigma=0,$$

then

$$\begin{aligned}
&\gamma(1-\xi)\int_{\partial B_2}\left[\frac{-1}{2}(\Delta u_1^*)^2\nu-(\nabla(\Delta u_1^*)\cdot\nabla u_1^*)\nu+\nabla(\Delta u_1^*)\cdot\nu\nabla u_1^*+\nabla u_1^*\cdot\nu\nabla(\Delta u_1^*)\right]d\sigma\\
&+\xi(1-\gamma)\int_{\partial B_2}\left[\frac{-1}{2}(\Delta u_2^*)^2\nu-(\nabla(\Delta u_2^*)\cdot\nabla u_2^*)\nu+\nabla(\Delta u_2^*)\cdot\nu\nabla u_2^*+\nabla u_2^*\cdot\nu\nabla(\Delta u_2^*)\right]d\sigma\\
&+(1-\gamma)(1-\xi)\left[-\int_{\partial B_2}(\Delta u_2^*\cdot\Delta u_1^*)\nu\,d\sigma+\int_{\partial B_2}\frac{\partial(\Delta u_2^*)}{\partial\nu}\nabla u_1^*\,d\sigma+\int_{\partial B_2}\frac{\partial(\Delta u_1^*)}{\partial\nu}\nabla u_2^*\,d\sigma\right]=0.
\end{aligned}$$

We set

$$
\begin{aligned}
I_{lhs} &= \gamma(1-\xi)\int_{\partial B_2}\left[\frac{-1}{2}(\Delta u_1^*)^2\nu - (\nabla(\Delta u_1^*)\cdot\nabla u_1^*)\nu + \nabla(\Delta u_1^*)\cdot\nu\nabla u_1^* + \nabla u_1^*\cdot\nu\nabla(\Delta u_1^*)\right]d\sigma \\
&\quad + \xi(1-\gamma)\int_{\partial B_2}\left[\frac{-1}{2}(\Delta u_2^*)^2\nu - (\nabla(\Delta u_2^*)\cdot\nabla u_2^*)\nu + \nabla(\Delta u_2^*)\cdot\nu\nabla u_2^* + \nabla u_2^*\cdot\nu\nabla(\Delta u_2^*)\right]d\sigma \\
&\quad + (1-\gamma)(1-\xi)\left[-\int_{\partial B_2}(\Delta u_2^*\cdot\Delta u_1^*)\nu\,d\sigma + \int_{\partial B_2}\frac{\partial(\Delta u_2^*)}{\partial\nu}\nabla u_1^*\,d\sigma + \int_{\partial B_2}\frac{\partial(\Delta u_1^*)}{\partial\nu}\nabla u_2^*\,d\sigma\right],
\end{aligned}
$$

by computation, we prove that

$$
\begin{aligned}
I_{lhs} &= -\frac{8}{\eta}\gamma(1-\xi)\int_{\partial B_2}\nabla\Delta R(x,x^2)\,d\sigma - \frac{8}{\eta}\xi(1-\gamma)\int_{\partial B_2}\nabla\Delta K(x,x^2)\,d\sigma \\
&\quad + \frac{8}{\eta}(1-\xi)(1-\gamma)\int_{\partial B_2}((\nabla\Delta K(x,x^2)+\nabla\Delta R(x,x^2))\cdot\nu)\nu\,d\sigma \\
&\quad + \frac{16}{\eta^2}\left[(1-\xi)\int_{\partial B_2}\Delta R(x,x^2)\nu\,d\sigma + (1-\gamma)\int_{\partial B_2}\Delta K(x,x^2)\nu\,d\sigma\right] \\
&\quad + \frac{32}{\eta^3}\left[(1-\xi)\int_{\partial B_2}\nabla R(x,x^2)\,d\sigma + (1-\gamma)\int_{\partial B_2}\nabla K(x,x^2)\,d\sigma\right] + O(\eta).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&-8\eta\gamma(1-\xi)\int_{\partial B_2}\nabla\Delta R(x,x^2)\,d\sigma - 8\eta\xi(1-\gamma)\int_{\partial B_2}\nabla\Delta K(x,x^2)\,d\sigma \\
&+ 8\eta(1-\xi)(1-\gamma)\int_{\partial B_2}((\nabla\Delta K(x,x^2)+\nabla\Delta R(x,x^2))\cdot\nu)\nu\,d\sigma \\
&+ 16\left[(1-\xi)\int_{\partial B_2}\Delta R(x,x^2)\nu\,d\sigma + (1-\gamma)\int_{\partial B_2}\Delta K(x,x^2)\nu\,d\sigma\right] \\
&+ \frac{32}{\eta}\left[(1-\xi)\int_{\partial B_2}\nabla R(x,x^2)\,d\sigma + (1-\gamma)\int_{\partial B_2}\nabla K(x,x^2)\,d\sigma\right] = O(\eta^3).
\end{aligned}
$$

Writing $\nabla R(x,x^2) = \nabla R(x^2,x^2) + O(\eta)$ and $\nabla K(x,x^2) = \nabla K(x^2,x^2) + O(\eta)$ we obtain

$$
(1-\xi)\nabla R(x^2,x^2) + (1-\gamma)\nabla K(x^2,x^2) = O(\eta^3),
$$

which means that x^2 is a critical point of the functional

$$
\mathcal{E}_2\colon x \longmapsto \frac{1-\xi}{\gamma(2-\gamma-\xi)}G(x,x^1) + H(x,x^2) + \frac{1-\gamma}{\xi(2-\gamma-\xi)}G(x,x^3). \tag{2.4}
$$

2.2. Behavior of solution around x^1 and x^3

We multiply the equation $\Delta^2 u_1 = \rho^4 e^{\gamma u_1 + (1-\gamma)u_2}$ by $\nabla(\gamma u_1 + (1-\gamma)u_2)$ and then integrating over $B_1 = B(x^1,\eta)$ we obtain a Pohozaev type identity

$$
\gamma\int_{B_1}(\Delta^2 u_1)\nabla u_1 + (1-\gamma)\int_{B_1}(\Delta^2 u_1)\nabla u_2 = \rho^4\int_{\partial B_1}(e^{\gamma u_1+(1-\gamma)u_2}-1)\nu\,d\sigma. \tag{2.5}
$$

Using the Green's formula we obtain

$$\begin{aligned}\int_{B_1} (\Delta^2 u_1)\nabla u_1 &= -\int_{B_1} (\Delta u_1)\nabla(\Delta(u_1)) - \int_{B_1} \nabla(\nabla(\Delta u_1)\cdot\nabla u_1)\\ &\quad+\int_{\partial B_1} \nabla(\Delta u_1)\cdot\nu\nabla u_1\, d\sigma + \int_{\partial B_1} \nabla u_1\cdot\nu\nabla(\Delta u_1)\, d\sigma\\ &= -\frac{1}{2}\int_{\partial B_1} (\Delta u_1)^2\nu\, d\sigma - \int_{\partial B_1} (\nabla(\Delta u_1)\cdot\nabla u_1)\nu\, d\sigma\\ &\quad+\int_{\partial B_1} \nabla(\Delta u_1)\cdot\nu\nabla u_1\, d\sigma + \int_{\partial B_1} \nabla u_1\cdot\nu\nabla(\Delta u_1)\, d\sigma.\end{aligned}$$

Similarly we multiply the equation $\Delta^2 u_2 = \rho^4 e^{\xi u_2+(1-\xi)u_1}$ by $\nabla(\xi u_2 + (1-\xi)u_1)$ and then integrating over $B_1 = B(x^1,\eta)$ we obtain a Pohozaev type identity

$$\xi\int_{B_1} (\Delta^2 u_2)\nabla u_2 + (1-\xi)\int_{B_1} (\Delta^2 u_2)\nabla u_1 = \rho^4\int_{\partial B_1} (e^{\xi u_2+(1-\xi)u_1} - 1)\nu\, d\sigma. \tag{2.6}$$

Using the Green's formula we obtain

$$\begin{aligned}\int_{B_1} (\Delta^2 u_2)\nabla u_2 &= -\int_{B_1} (\Delta u_2)\nabla(\Delta(u_2)) - \int_{B_1} \nabla(\nabla(\Delta u_2)\cdot\nabla u_2)\\ &\quad+\int_{\partial B_1} \nabla(\Delta u_2)\cdot\nu\nabla u_2\, d\sigma + \int_{\partial B_1} \nabla u_2\cdot\nu\nabla(\Delta u_2)\, d\sigma\\ &= -\frac{1}{2}\int_{\partial B_1} (\Delta u_2)^2\nu\, d\sigma - \int_{\partial B_1} (\nabla(\Delta u_2)\cdot\nabla u_2)\nu\, d\sigma\\ &\quad+\int_{\partial B_1} \nabla(\Delta u_2)\cdot\nu\nabla u_2\, d\sigma + \int_{\partial B_1} \nabla u_2\cdot\nu\nabla(\Delta u_2)\, d\sigma.\end{aligned}$$

Making use of the identity

$$\begin{aligned}&\int_{B_1} \Delta^2 u_2\nabla u_1 + \int_{B_1} \Delta^2 u_1\nabla u_2\\ &= -\int_{B_1} \nabla(\Delta u_2\cdot\Delta u_1) + \int_{\partial B_1} \frac{\partial(\Delta u_2)}{\partial\nu}\nabla u_1\, d\sigma + \int_{\partial B_1} \frac{\partial(\Delta u_1)}{\partial\nu}\nabla u_2\, d\sigma\\ &= -\int_{\partial B_1} (\Delta u_2\cdot\Delta u_1)\nu\, d\sigma + \int_{\partial B_1} \frac{\partial(\Delta u_2)}{\partial\nu}\nabla u_1\, d\sigma + \int_{\partial B_1} \frac{\partial(\Delta u_1)}{\partial\nu}\nabla u_2\, d\sigma,\end{aligned}$$

then by combination of (2.5) and (2.6) we obtain

$$\begin{aligned}&\gamma(1-\xi)\int_{\partial B_1}\left[\frac{-1}{2}(\Delta u_1)^2\nu - (\nabla(\Delta u_1)\cdot\nabla u_1)\nu + \nabla(\Delta u_1)\cdot\nu\nabla u_1 + \nabla u_1\cdot\nu\nabla(\Delta u_1)\right] d\sigma\\ &+\xi(1-\gamma)\int_{\partial B_1}\left[\frac{-1}{2}(\Delta u_2)^2\nu - (\nabla(\Delta u_2)\cdot\nabla u_2)\nu + \nabla(\Delta u_2)\cdot\nu\nabla u_2 + \nabla u_2\cdot\nu\nabla(\Delta u_2)\right] d\sigma\\ &+(1-\gamma)(1-\xi)\left[-\int_{\partial B_1} (\Delta u_2\cdot\Delta u_1)\nu\, d\sigma + \int_{\partial B_1} \frac{\partial(\Delta u_2)}{\partial\nu}\nabla u_1\, d\sigma + \int_{\partial B_1} \frac{\partial(\Delta u_1)}{\partial\nu}\nabla u_2\, d\sigma\right]\\ &= \rho^4(1-\xi)\int_{\partial B_1} (e^{\gamma u_1+(1-\gamma)u_2} - 1)\nu\, d\sigma + \rho^4(1-\gamma)\int_{\partial B_1} (e^{\xi u_2+(1-\xi)u_1} - 1)\nu\, d\sigma.\end{aligned}$$

In the desire to construct solutions of the system that blow-up at the point x^1 this means that, if ρ tends to zero,

$$u_1 \to u_1^*(x) = G(x,x^2) + \frac{1}{\gamma}G(x,x^1) \quad \text{and} \quad u_2 \to u_2^*(x) = G(x,x^2) + \frac{1}{\xi}G(x,x^3).$$

Since we have $G(x,x^1) = -8\ln|x-x^1| + H(x,x^1)$, where H is a smooth function in Ω, then

$$\begin{aligned} u_1^*(x) &= G(x,x^2) + \frac{1}{\gamma}G(x,x^1) = -\frac{8}{\gamma}\ln|x-x^1| + \frac{1}{\gamma}H(x,x^1) + G(x,x^2) \\ &= -\frac{8}{\gamma}\ln|x-x^1| + S(x,x^1) \end{aligned}$$

and

$$u_2^*(x) = G(x,x^2) + \frac{1}{\xi}G(x,x^3) = T(x,x^2,x^3).$$

Thanks to the fact that the solutions of the system (1.1) are regular on $\Omega \setminus \{x^1,x^2,x^3\}$ and by inserting the profile of the limits of the solutions in the identity (2.3) when $\rho \to 0$ and η fixed small enough, we obtain

$$\lim_{\rho\to 0} \rho^4(1-\xi)\int_{\partial B_1}(e^{\gamma u_1+(1-\gamma)u_2}-1)\nu\,d\sigma + \rho^4(1-\gamma)\int_{\partial B_2}(e^{\xi u_2+(1-\xi)u_1}-1)\nu\,d\sigma = 0,$$

then

$$\begin{aligned} &\gamma(1-\xi)\int_{\partial B_1}\left[\frac{-1}{2}(\Delta u_1^*)^2\nu - (\nabla(\Delta u_1^*)\cdot\nabla u_1^*)\nu + \nabla(\Delta u_1^*)\cdot\nu\nabla u_1^* + \nabla u_1^*\cdot\nu\nabla(\Delta u_1^*)\right]d\sigma \\ &+ \xi(1-\gamma)\int_{\partial B_1}\left[\frac{-1}{2}(\Delta u_2^*)^2\nu - (\nabla(\Delta u_2^*)\cdot\nabla u_2^*)\nu + \nabla(\Delta u_2^*)\cdot\nu\nabla u_2^* + \nabla u_2^*\cdot\nu\nabla(\Delta u_2^*)\right]d\sigma \\ &+ (1-\gamma)(1-\xi)\left[-\int_{\partial B_1}(\Delta u_2^*\cdot\Delta u_1^*)\nu\,d\sigma + \int_{\partial B_1}\frac{\partial(\Delta u_2^*)}{\partial\nu}\nabla u_1^*\,d\sigma + \int_{\partial B_1}\frac{\partial(\Delta u_1^*)}{\partial\nu}\nabla u_2^*\,d\sigma\right] = 0. \end{aligned}$$

We set

$$\begin{aligned} I_{lhs} = &\gamma(1-\xi)\int_{\partial B_1}\left[\frac{-1}{2}(\Delta u_1^*)^2\nu - (\nabla(\Delta u_1^*)\cdot\nabla u_1^*)\nu + \nabla(\Delta u_1^*)\cdot\nu\nabla u_1^* + \nabla u_1^*\cdot\nu\nabla(\Delta u_1^*)\right]d\sigma \\ &+ \xi(1-\gamma)\int_{\partial B_1}\left[\frac{-1}{2}(\Delta u_2^*)^2\nu - (\nabla(\Delta u_2^*)\cdot\nabla u_2^*)\nu + \nabla(\Delta u_2^*)\cdot\nu\nabla u_2^* + \nabla u_2^*\cdot\nu\nabla(\Delta u_2^*)\right]d\sigma \\ &+ (1-\gamma)(1-\xi)\left[-\int_{\partial B_1}(\Delta u_2^*\cdot\Delta u_1^*)\nu\,d\sigma + \int_{\partial B_1}\frac{\partial(\Delta u_2^*)}{\partial\nu}\nabla u_1^*\,d\sigma + \int_{\partial B_1}\frac{\partial(\Delta u_1^*)}{\partial\nu}\nabla u_2^*\,d\sigma\right], \end{aligned}$$

by computation, we prove that

$$\begin{aligned} I_{lhs} = &-\frac{8}{\eta}(1-\xi)\int_{\partial B_1}\nabla\Delta S(x,x^1)\,d\sigma + \frac{8}{\gamma\eta}(1-\gamma)(1-\xi)\int_{\partial B_1}\left(\frac{\partial\Delta T(x,x^2,x^3)}{\partial\nu}\right)\nu\,d\sigma \\ &+ \frac{16}{\eta^2}\left[(1-\xi)\int_{\partial B_1}\Delta S(x,x^1)\nu\,d\sigma + \frac{(1-\gamma)(1-\xi)}{\gamma}\int_{\partial B_1}\Delta T(x,x^2,x^3)\nu\,d\sigma\right] \\ &+ \frac{32}{\eta^3}\left[(1-\xi)\int_{\partial B_1}\nabla S(x,x^1)\,d\sigma + \frac{(1-\gamma)(1-\xi)}{\gamma}\int_{\partial B_1}\nabla T(x,x^2,x^3)\,d\sigma\right] + O(\eta). \end{aligned}$$

Then we have

$$\begin{aligned}&-8\eta(1-\xi)\int_{\partial B_1}\nabla\Delta S(x,x^1)\,d\sigma+\frac{8\eta}{\gamma}(1-\gamma)(1-\xi)\int_{\partial B_1}\left(\frac{\partial\Delta T(x,x^2,x^3)}{\partial\nu}\right)\nu\,d\sigma\\&+16\left[(1-\xi)\int_{\partial B_1}\Delta S(x,x^1)\nu\,d\sigma+\frac{(1-\gamma)(1-\xi)}{\gamma}\int_{\partial B_1}\Delta T(x,x^2,x^3)\nu\,d\sigma\right]\\&+\frac{32}{\eta}\left[(1-\xi)\int_{\partial B_1}\nabla S(x,x^1)\,d\sigma+\frac{(1-\gamma)(1-\xi)}{\gamma}\int_{\partial B_1}\nabla T(x,x^2,x^3)\,d\sigma\right]=O(\eta^3).\end{aligned}$$

Writing $\nabla S(x,x^1)=\nabla S(x^1,x^1)+O(\eta)$ and $\nabla T(x,x^2,x^3)=\nabla T(x^1,x^2,x^3)+O(\eta)$ we obtain

$$(1-\xi)\nabla S(x^1,x^1)+\frac{(1-\gamma)(1-\xi)}{\gamma}\nabla T(x^1,x^2,x^3)=O(\eta^3),$$

which means that x^1 is a critical point of the functional

$$\mathcal{E}_1\colon x\longmapsto H(\,\cdot\,,x^1)+G(\,\cdot\,,x^2)+\frac{1-\gamma}{\xi}G(\,\cdot\,,x^3). \tag{2.7}$$

In $B_3=B(x^3,\eta)$, we proceed similarly as in $B_1=B(x^1,\eta)$ and respect the changes we obtain that x^3 is a critical point of the functional

$$\mathcal{E}_3\colon x\longmapsto H(\,\cdot\,,x^3)+G(\,\cdot\,,x^2)+\frac{1-\xi}{\gamma}G(\,\cdot\,,x^1). \tag{2.8}$$

Finally by combination of (2.4), (2.7) and (2.8) we conclude that the point (x^1,x^2,x^3) is a critical point of the functional $\mathcal{E}$ defined by

$$\begin{aligned}\mathcal{E}(x^1,x^2,x^3)=\frac{1-\xi}{\gamma}&H(x^1,x^1)+(2-\gamma-\xi)H(x^2,x^2)+\frac{1-\gamma}{\xi}H(x^3,x^3)\\&+\frac{1-\xi}{\gamma}G(x^1,x^2)+\frac{(1-\xi)(1-\gamma)}{\gamma\xi}G(x^1,x^3)+\frac{1-\gamma}{\xi}G(x^3,x^2).\end{aligned}$$

3. Proof of Theorem 1.5

3.1. Construction of the approximate solution

We denote by ε the smallest positive parameter satisfying

$$\rho^4=\frac{384\varepsilon^4}{(1+\varepsilon^2)^4}.$$

Let

$$u_\varepsilon(x):=4\ln(1+\varepsilon^2)-4\ln(\varepsilon^2+|x|^2),$$

which is a solution of

$$\Delta^2u=\rho^4e^u\quad\text{in }\mathbb{R}^4. \tag{3.1}$$

Hence for all $\tau > 0$ the function

$$u_{\varepsilon,\tau}(x) := 4\ln(1+\varepsilon^2) + 4\ln\tau - 4\ln(\varepsilon^2 + |\tau x|^2) \tag{3.2}$$

is also a solution to (3.1).

3.1.1. A linearized operator

First we introduce some definitions and notations.

Definition 3.1. Given $k \in \mathbb{N}$, $\alpha \in (0,1)$, $\mu \in \mathbb{R}$ and $|x| = r$, we define the Hölder weighted space $\mathcal{C}^{k,\alpha}_\mu(\mathbb{R}^4)$ as the space of functions $w \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R}^4)$ for which the following norm

$$\|u\|_{\mathcal{C}^{k,\alpha}_\mu(\mathbb{R}^4)} = \|u\|_{\mathcal{C}^{k,\alpha}(\overline{B}_1(0))} + \sup_{r\geq 1}\left((1+r^2)^{-\mu/2}\|u(r\,\cdot\,)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_1(0)-B_{1/2}(0))}\right)$$

is finite. Similarly, for given $\overline{r} \geq 1$, let $\mathcal{C}^{k,\alpha}_\mu(B_{\overline{r}}(0))$ be the space of functions in $\mathcal{C}^{k,\alpha}(B_{\overline{r}}(0))$ for which the following norm

$$\|u\|_{\mathcal{C}^{k,\alpha}_\mu(B_{\overline{r}}(0))} = \|u\|_{\mathcal{C}^{k,\alpha}(B_1(0))} + \sup_{1\leq r\leq \overline{r}}\left(r^{-\mu}\|u(r\,\cdot\,)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_1(0)-B_{1/2}(0))}\right)$$

is finite. Finally, set $B^*_r(x^i) = B_r(x^i) - \{x^i\}$, let $\mathcal{C}^{k,\alpha}_\mu(\overline{B}^*_1(0))$ be the space of functions in $\mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\overline{B}^*_1(0))$ for which the following norm

$$\|u\|_{\mathcal{C}^{k,\alpha}_\mu(\overline{B}^*_1(0))} = \sup_{r\leq 1/2}\left(r^{-\mu}\|u(r\,\cdot\,)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_2(0)-B_1(0))}\right)$$

is finite.

We define the linear elliptic operator $\mathbb{L}$ by

$$\mathbb{L} := \Delta^2 - \frac{384}{(1+r^2)^4},$$

which is the linearized operator of $\Delta^2 u - \rho^4 e^u = 0$ about the radial symmetric solution $u_{\varepsilon=1,\tau=1}$ defined by (3.2). When $k \geq 2$, we let $\left[\mathcal{C}^{k,\alpha}_\mu(\overline{\Omega})\right]_0$ to be the subspace of functions $w \in \mathcal{C}^{k,\alpha}_\mu(\overline{\Omega})$ satisfying $\Delta w = w = 0$ on $\partial\Omega$.

For all $\varepsilon, \tau_i > 0$, $i = 1,2,3$ and $\gamma, \xi \in (0,1)$, we define

$$r_\varepsilon := \max\left(\varepsilon^{1/2}, \varepsilon^{(\gamma+\xi-1)/\gamma}, \varepsilon^{(\gamma+\xi-1)/\xi}\right) \quad \text{and} \quad R^i_\varepsilon := \tau_i\frac{r_\varepsilon}{\varepsilon}.$$

Proposition 3.2. [4] *All bounded solutions of $\mathbb{L}w = 0$ on $\mathbb{R}^4$ are linear combination of*

$$\phi_0(x) = 4\frac{1-|x|^2}{1+|x|^2} \quad \textit{and} \quad \phi_i(x) = \frac{8x_i}{1+|x|^2} \quad \textit{for } i = 1,\dots,4.$$

Moreover, for $\mu > 1$, $\mu \notin \mathbb{Z}$, the operator $\mathbb{L}\colon \mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \to \mathcal{C}^{0,\alpha}_{\mu-4}(\mathbb{R}^4)$ is surjective.

In the following, we denote by $\mathcal{G}_\mu$ to be a right inverse of $\mathbb{L}$. Similarly, using the fact that any bounded bi-harmonic solution on $\mathbb{R}^4$ is constant, we claim

Proposition 3.3. *Let $\delta > 0$, $\delta \notin \mathbb{Z}$ then Δ^2 is surjective from $\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)$ to $\mathcal{C}^{0,\alpha}_{\delta-4}(\mathbb{R}^4)$.*

We denote by $\mathcal{K}_\delta \colon \mathcal{C}^{0,\alpha}_{\delta-4}(\mathbb{R}^4) \to \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)$ a right inverse of Δ^2 for $\delta > 0$, $\delta \notin \mathbb{Z}$.

Finally, we consider punctured domains. Given $\widetilde{x}^1$, $\widetilde{x}^2$, $\widetilde{x}^3$ three distinct points in Ω, we define $\widetilde{\boldsymbol{x}} := (\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3)$ and $\overline{\Omega}^*(\widetilde{\boldsymbol{x}}) := \overline{\Omega} - \{\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3\}$. Let $r_0 > 0$ be small such that $\overline{B}_{r_0}(\widetilde{x}^i)$ are disjoint and included in Ω. For all $r \in (0, r_0)$, we define

$$\overline{\Omega}_r(\widetilde{\boldsymbol{x}}) := \overline{\Omega} - \bigcup_{i=1}^{3} B_r(\widetilde{x}^i).$$

Definition 3.4. Let $k \in \mathbb{R}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, we introduce the Hölder weighted space $\mathcal{C}^{k,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))$ as the space of functions $w \in \mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))$ such that

$$\|w\|_{\mathcal{C}^{k,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))} := \|w\|_{\mathcal{C}^{k,\alpha}(\overline{\Omega}_{r_0/2}(\widetilde{\boldsymbol{x}}))} + \sum_{i=1}^{3} \sup_{0<r\leq r_0/2} \left(r^{-\nu}\|w(\widetilde{x}^i + r\,\cdot)\|_{\mathcal{C}^{k,\alpha}(\overline{B}_2(0)-B_1(0))}\right)$$

is finite. Furthermore, for $k \geq 2$, let $\left[\mathcal{C}^{k,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))\right]_0$ to be the set of $w \in \mathcal{C}^{k,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))$ satisfying $\Delta w = w = 0$ on $\partial\Omega$.

We recall the following result.

Proposition 3.5. [12] *Let $\nu < 0$, $\nu \notin \mathbb{Z}$ then Δ^2 is surjective from $\left[\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))\right]_0$ to $\mathcal{C}^{0,\alpha}_{\nu-4}(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))$.*

We denote by $\widetilde{\mathcal{G}}_\nu \colon \mathcal{C}^{0,\alpha}_{\nu-4}(\overline{\Omega}^*(\widetilde{\boldsymbol{x}})) \to \left[\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))\right]_0$ a right inverse of Δ^2 for $\nu < 0$, $\nu \notin \mathbb{Z}$.

3.1.2. Ansatz and first estimates

For all $\sigma \geq 1$, we denote by $\xi_{\mu,\sigma} \colon \mathcal{C}^{0,\alpha}_\mu(\overline{B}_\sigma(0)) \to \mathcal{C}^{0,\alpha}_\mu(\mathbb{R}^4)$ the extension operator defined by

$$\begin{cases} \xi_{\mu,\sigma}(f)(x) \equiv f(x) & \text{for } |x| \leq \sigma, \\ \xi_{\mu,\sigma}(f)(x) = \chi\left(\frac{|x|}{\sigma}\right) f\left(\sigma\frac{x}{|x|}\right) & \text{for } |x| \geq \sigma. \end{cases} \tag{3.3}$$

Here χ is a cut-off function over $\mathbb{R}_+$, which is equal to 1 for $t \leq 1$ and equal to 0 for $t \geq 2$.

It is easy to check that there exists a constant $c = \overline{c}(\mu) > 0$, independent of σ such that

$$\|\xi_{\mu,\sigma}(w)\|_{\mathcal{C}^{0,\alpha}_\mu(\mathbb{R}^4)} \leq \overline{c}\|w\|_{\mathcal{C}^{0,\alpha}_\mu(\overline{B}_\sigma(0))}. \tag{3.4}$$

Now we define an ansatz for solution of (1.1):

$$\widetilde{u}_1(x)=\begin{cases}\frac{1}{\gamma}u_{\varepsilon,\tau_1}(x-x^1)-\frac{1-\gamma}{\gamma}G(x,x^2)-\frac{1-\gamma}{\gamma\xi}G(x,x^3)-\frac{\ln\gamma}{\gamma}, & x\in B_{r_\varepsilon}(x^1),\\ u_{\varepsilon,\tau_2}(x-x^2), & x\in B_{r_\varepsilon}(x^2),\\ \frac{1}{\gamma}G(x,x^1)+G(x,x^2), & x\in\Omega\setminus\bigcup_{i=1}^2 B_{r_\varepsilon}(x^i)\end{cases}$$

and

$$\widetilde{u}_2(x)=\begin{cases}\frac{1}{\xi}u_{\varepsilon,\tau_3}(x-x^3)-\frac{1-\xi}{\xi}G(x,x^2)-\frac{1-\xi}{\gamma\xi}G(x,x^1)-\frac{\ln\xi}{\xi}, & x\in B_{r_\varepsilon}(x^3),\\ u_{\varepsilon,\tau_2}(x-x^2), & x\in B_{r_\varepsilon}(x^2),\\ \frac{1}{\xi}G(x,x^3)+G(x,x^2), & x\in\Omega\setminus\bigcup_{i=2}^3 B_{r_\varepsilon}(x^i).\end{cases}$$

Therefore, in $B_{r_\varepsilon}(x^1)$, there holds

$$\Delta^2\widetilde{u}_1-\rho^4e^{\gamma\widetilde{u}_1+(1-\gamma)\widetilde{u}_2}=0,$$

$$\Delta^2\widetilde{u}_2-\rho^4e^{\xi\widetilde{u}_2+(1-\xi)\widetilde{u}_1}=\frac{-384\varepsilon^4\tau_1^{4\frac{1-\xi}{\gamma}}(1+\varepsilon^2)^{4\frac{1-\gamma-\xi}{\gamma}}}{\gamma^{\frac{1-\xi}{\gamma}}(\varepsilon^2+\tau_1^2|x-x^1|^2)^{4\frac{1-\xi}{\gamma}}}e^{\frac{\gamma+\xi-1}{\gamma}G(x,x^2)+\frac{\gamma+\xi-1}{\gamma\xi}G(x,x^3)}.$$

Then, for $r=|x-x^1|$ and $0<\delta<(\gamma+\xi-1)/\gamma$, we have

$$\begin{aligned}\left\|\Delta^2\widetilde{u}_2-\rho^4e^{\xi\widetilde{u}_2+(1-\xi)\widetilde{u}_1}\right\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(B_{r_\varepsilon}(x^1))}&\le C\sup_{r<r_\varepsilon}\frac{\tau_1^{4\frac{1-\xi}{\gamma}}\varepsilon^4}{\gamma^{\frac{1-\xi}{\gamma}}(1+\varepsilon^2)^{4-4\frac{1-\xi}{\gamma}}}\frac{r^{4-\delta}}{\varepsilon^{8\frac{1-\xi}{\gamma}}\left(1+\left(\frac{\tau_1}{\varepsilon}r\right)^2\right)^{4\frac{1-\xi}{\gamma}}}\\ &\le C\sup_{r<R^1_\varepsilon}\frac{\varepsilon^{8-8\frac{1-\xi}{\gamma}-\delta}}{(1+\varepsilon^2)^{4-4\frac{1-\xi}{\gamma}}}\frac{r^{4-\delta}}{(1+r^2)^{4\frac{1-\xi}{\gamma}}}\\ &\le C\sup_{r<R^1_\varepsilon}\varepsilon^{8-8\frac{1-\xi}{\gamma}-\delta}S(r),\end{aligned}$$

where $S(r)=\frac{r^{4-\delta}}{(1+r^2)^{4(1-\xi)/\gamma}}$.

If $4-\delta-8(1-\xi)/\gamma\le 0$, then S is bounded on $\mathbb{R}_+$, hence

$$\left\|\Delta^2\widetilde{u}_2-\rho^4e^{\xi\widetilde{u}_2+(1-\xi)\widetilde{u}_1}\right\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(B_{r_\varepsilon}(x^1))}\le C\varepsilon^{8-8\frac{1-\xi}{\gamma}-\delta}\le Cr_\varepsilon^2.$$

If $4-\delta-8(1-\xi)/\gamma>0$, $\sup_{[0,r_\varepsilon/\varepsilon)}S(r)=S\left(\frac{r_\varepsilon}{\varepsilon}\right)$, then

$$\left\|\Delta^2\widetilde{u}_2-\rho^4e^{\xi\widetilde{u}_2+(1-\xi)\widetilde{u}_1}\right\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(B_{r_\varepsilon}(x^1))}\le Cr_\varepsilon^2\quad\text{as}\ \ \varepsilon^{8-8\frac{1-\xi}{\gamma}-\delta}S\left(\frac{r_\varepsilon}{\varepsilon}\right)\le Cr_\varepsilon^2.$$

Similarly in $B_{r_\varepsilon}(x^3)$, there holds

$$\Delta^2\widetilde{u}_1-\rho^4e^{\gamma\widetilde{u}_1+(1-\gamma)\widetilde{u}_2}=\frac{-384\varepsilon^4\tau_3^{4\frac{1-\gamma}{\xi}}(1+\varepsilon^2)^{4\frac{1-\gamma-\xi}{\xi}}}{\xi^{\frac{1-\gamma}{\xi}}(\varepsilon^2+\tau_3^2|x-x^3|^2)^{4\frac{1-\gamma}{\xi}}}e^{\frac{\gamma+\xi-1}{\xi}G(x,x^2)+\frac{\gamma+\xi-1}{\gamma\xi}G(x,x^1)},$$

$$\Delta^2\widetilde{u}_2-\rho^4e^{\xi\widetilde{u}_2+(1-\xi)\widetilde{u}_1}=0.$$

Then, for $r = |x - x^3|$ and $0 < \delta < (\gamma + \xi - 1)/\xi$, we have the same estimates

$$\left\|\Delta^2 \widetilde{u}_1 - \rho^4 e^{\gamma \widetilde{u}_1 + (1-\gamma)\widetilde{u}_2}\right\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(B_{r_\varepsilon}(x^3))} \leq C r_\varepsilon^2.$$

Finally, in $B_{r_\varepsilon}(x^2)$, we have an exact solution of the system.

3.1.3. Bi-harmonic extensions

Next, we will study the properties of interior and exterior bi-harmonic extensions.

Given $(\varphi, \psi), (\widetilde{\varphi}, \widetilde{\psi}) \in \mathcal{C}^{4,\alpha}(S^3) \times \mathcal{C}^{2,\alpha}(S^3)$, we define respectively $H^{\text{int}} = H^{\text{int}}(\varphi, \psi; \cdot\,) = H^{\text{int}}_{\varphi,\psi}$ and $H^{\text{ext}} = H^{\text{ext}}(\widetilde{\varphi}, \widetilde{\psi}; \cdot\,) = H^{\text{ext}}_{\widetilde{\varphi},\widetilde{\psi}}$ to be the solution of

$$\begin{cases} \Delta^2 H^{\text{int}} = 0 & \text{in } B_1(0), \\ H^{\text{int}} = \varphi & \text{on } \partial B_1(0), \\ \Delta H^{\text{int}} = \psi & \text{on } \partial B_1(0), \end{cases} \quad \text{and} \quad \begin{cases} \Delta^2 H^{\text{ext}} = 0 & \text{in } \mathbb{R}^4 - B_1(0), \\ H^{\text{ext}} = \widetilde{\varphi} & \text{on } \partial B_1(0), \\ \Delta H^{\text{ext}} = \widetilde{\psi} & \text{on } \partial B_1(0), \end{cases}$$

which decays at infinity. We will also use

Definition 3.6. Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, we define the space $\mathcal{C}^{k,\alpha}_\nu(\mathbb{R}^4 - B_1(0))$ as the space of functions $w \in \mathcal{C}^{k,\alpha}_{\text{loc}}(\mathbb{R}^4 - B_1(0))$ for which the following norm

$$\|w\|_{\mathcal{C}^{k,\alpha}_\nu(\mathbb{R}^4 - B_1(0))} = \sup_{r \geq 1} \left(r^{-\nu} \|w(r\,\cdot\,)\|_{\mathcal{C}^{k,\alpha}_\nu(\overline{B}_2(0) - B_1(0))} \right)$$

is finite.

We denote by $e_1, \ldots, e_4$ the coordinate functions on S^3.

Lemma 3.7. [2] *Assume that*

$$\int_{S^3} (8\varphi - \psi)\, dv_{S^3} = 0 \quad \textit{and} \quad \int_{S^3} (12\varphi - \psi) e_\ell\, dv_{S^3} = 0 \quad \textit{for } \ell = 1, \ldots, 4. \tag{3.5}$$

Then there exists $c > 0$ such that

$$\|H^{\text{int}}_{\varphi,\psi}\|_{\mathcal{C}^{4,\alpha}_2(\overline{B}^*_1(0))} \leq c\big(\|\varphi\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi\|_{\mathcal{C}^{2,\alpha}(S^3)}\big).$$

Similarly, there exists $c > 0$ such that if

$$\int_{S^3} \widetilde{\psi}\, dv_{S^3} = 0, \tag{3.6}$$

then

$$\|H^{\text{ext}}_{\widetilde{\varphi},\widetilde{\psi}}\|_{\mathcal{C}^{4,\alpha}_{-1}(\mathbb{R}^4 - B_1(0))} \leq c\big(\|\widetilde{\varphi}\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\widetilde{\psi}\|_{\mathcal{C}^{2,\alpha}(S^3)}\big).$$

If $F \subset L^2(S^3)$ be a subspace S^3, we denote $F^\perp$ to be the subspace of F which are $L^2(S^3)$-orthogonal to the functions $1, e_1, \ldots, e_4$. We will need the following result.

Lemma 3.8. [2] *The mapping*

$$\begin{array}{rcc} \mathcal{P}: \ \mathcal{C}^{4,\alpha}(S^3)^\perp \times \mathcal{C}^{2,\alpha}(S^3)^\perp & \longrightarrow & \mathcal{C}^{3,\alpha}(S^3)^\perp \times \mathcal{C}^{1,\alpha}(S^3)^\perp \\ (\varphi,\psi) & \longmapsto & \big(\partial_r(H^{\mathrm{int}}_{\varphi,\psi} - H^{\mathrm{ext}}_{\varphi,\psi}), \partial_r(\Delta H^{\mathrm{int}}_{\varphi,\psi} - \Delta H^{\mathrm{ext}}_{\varphi,\psi})\big) \end{array}$$

is an isomorphism.

3.2. The nonlinear interior problem

Here, we are interested to study the system

$$\Delta^2 u_1 = \rho^4 e^{\gamma u_1 + (1-\gamma)u_2}, \quad \Delta^2 u_2 = \rho^4 e^{\xi u_2 + (1-\xi)u_1}. \tag{3.7}$$

Using the following transformations

$$\begin{cases} v_1(x) = u_1\big(\frac{\varepsilon}{\tau_1}x\big) + \frac{8}{\gamma}\ln\varepsilon - \frac{4}{\gamma}\ln\big(\frac{\tau_1(1+\varepsilon^2)}{2}\big) & \text{in } B_{r_\varepsilon}(x^1), \\ v_2(x) = u_2\big(\frac{\varepsilon}{\tau_1}x\big) & \text{in } B_{r_\varepsilon}(x^1), \end{cases}$$

$$\begin{cases} v_1(x) = u_1\big(\frac{\varepsilon}{\tau_2}x\big) + 8\ln\varepsilon - 4\ln\big(\frac{\tau_2(1+\varepsilon^2)}{2}\big) & \text{in } B_{r_\varepsilon}(x^2), \\ v_2(x) = u_2\big(\frac{\varepsilon}{\tau_2}x\big) + 8\ln\varepsilon - 4\ln\big(\frac{\tau_2(1+\varepsilon^2)}{2}\big) & \text{in } B_{r_\varepsilon}(x^2) \end{cases}$$

and

$$\begin{cases} v_1(x) = u_1\big(\frac{\varepsilon}{\tau_3}x\big) & \text{in } B_{r_\varepsilon}(x^3), \\ v_2(x) = u_2\big(\frac{\varepsilon}{\tau_3}x\big) + \frac{8}{\xi}\ln\varepsilon - \frac{4}{\xi}\ln\big(\frac{\tau_3(1+\varepsilon^2)}{2}\big) & \text{in } B_{r_\varepsilon}(x^3). \end{cases}$$

So the previous systems can be written as

$$\begin{cases} \Delta^2 v_1 = 24 e^{\gamma v_1 + (1-\gamma)v_2} & \text{in } B_{R^1_\varepsilon}(x^1), \\ \Delta^2 v_2 = 24 C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}} \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}} e^{\xi v_2 + (1-\xi)v_1} & \text{in } B_{R^1_\varepsilon}(x^1), \end{cases} \tag{3.8}$$

$$\begin{cases} \Delta^2 v_1 = 24 e^{\gamma v_1 + (1-\gamma)v_2} & \text{in } B_{R^2_\varepsilon}(x^2), \\ \Delta^2 v_2 = 24 e^{\xi v_2 + (1-\xi)v_1} & \text{in } B_{R^2_\varepsilon}(x^2) \end{cases} \tag{3.9}$$

and

$$\begin{cases} \Delta^2 v_1 = 24 C_{3,\varepsilon}^{4\frac{\gamma+\xi-1}{\xi}} \varepsilon^{8\frac{\gamma+\xi-1}{\xi}} e^{\gamma v_1 + (1-\gamma)v_2} & \text{in } B_{R^3_\varepsilon}(x^3), \\ \Delta^2 v_2 = 24 e^{\xi v_2 + (1-\xi)v_1} & \text{in } B_{R^3_\varepsilon}(x^3), \end{cases} \tag{3.10}$$

where $C_{i,\varepsilon} = \frac{2}{\tau_i(1+\varepsilon^2)}$ for $i = 1, 3$. Here $\tau_i > 0$ is a constant which will be fixed later.

Given $\varphi^i := (\varphi^i_1, \varphi^i_2) \in (\mathcal{C}^{4,\alpha}(S^3))^2$ and $\psi^i := (\psi^i_1, \psi^i_2) \in (\mathcal{C}^{2,\alpha}(S^3))^2$ such that (φ^i_1, ψ^i_1) and (φ^i_2, ψ^i_2) satisfy (3.5). We denote by $\overline{u} = u_{\varepsilon=1,\tau_i=1}$, we write for $x \in B_{R^1_\varepsilon}(x^1)$ the following system

$$\begin{aligned} v_1(x) &= \frac{1}{\gamma}\overline{u}(x-x^1) - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) \\ &\quad - \frac{\ln\gamma}{\gamma} + H^{\mathrm{int}}\left(\varphi_1^1, \psi_1^1; \frac{x-x^1}{R_\varepsilon^1}\right) + h_1^1(x), \\ v_2(x) &= \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + H^{\mathrm{int}}\left(\varphi_2^1, \psi_2^1; \frac{x-x^1}{R_\varepsilon^1}\right) + h_2^1(x). \end{aligned}$$

Using the fact that H^{int} is bi-harmonic and that $e^{\overline{u}(x-x^1)} = \frac{16}{(1+|x-x^1|^2)^4}$, we see that this amounts to solve the system

$$\mathbb{L}h_1^1 = \frac{384}{\gamma(1+r^2)^4}\left(e^{\gamma\left(h_1^1+H^{\mathrm{int}}_{\varphi_1^1,\psi_1^1}\right)+(1-\gamma)\left(h_2^1+H^{\mathrm{int}}_{\varphi_2^1,\psi_2^1}\right)} - \gamma h_1^1 - 1\right),$$

$$\tag{3.11} \begin{aligned} \Delta^2 h_2^1 &= \frac{24 C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}} 16^{\frac{1-\xi}{\gamma}} \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}}}{\gamma^{\frac{1-\xi}{\gamma}}(1+r^2)^{4\frac{1-\xi}{\gamma}}} \\ &\quad \times e^{\frac{\gamma+\xi-1}{\gamma}G\left(\frac{\varepsilon x}{\tau_1},x^2\right)+\frac{\gamma+\xi-1}{\gamma\xi}G\left(\frac{\varepsilon x}{\tau_1},x^3\right)+\xi\left(h_2^1+H^{\mathrm{int}}_{\varphi_2^1,\psi_2^1}\right)+(1-\xi)\left(h_1^1+H^{\mathrm{int}}_{\varphi_1^1,\psi_1^1}\right)}. \end{aligned}$$

We denote by

$$\mathbb{L}h_1^1 = \mathcal{T}_1(h_1^1, h_2^1) \quad \text{and} \quad \Delta^2 h_2^1 = \mathcal{T}_2(h_1^1, h_2^1).$$

Fix $\mu \in (1,2)$ and $\delta \in \left(0, \min\{(\frac{\gamma+\xi-1}{\gamma}), (\frac{\gamma+\xi-1}{\xi})\}\right)$. To find a solution of (3.11), it is enough to find a fixed point (h_1^1, h_2^1) in a small ball of $\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)$ solutions of

$$\tag{3.12} \begin{aligned} h_1^1 &= \mathcal{G}_\mu \circ \xi_{\mu, R_\varepsilon^1} \circ \mathcal{T}_1(h_1^1, h_2^1) = \mathcal{N}_1(h_1^1, h_2^1), \\ h_2^1 &= \mathcal{K}_\delta \circ \xi_{\delta, R_\varepsilon^1} \circ \mathcal{T}_2(h_1^1, h_2^1) = \mathcal{M}_1(h_1^1, h_2^1). \end{aligned}$$

Here $\xi_{\mu,R_\varepsilon^1}$ is defined in (3.3), $\mathcal{G}_\mu$ and $\mathcal{K}_\delta$ are defined after Propositions 3.2 and 3.3, respectively.

Given $\kappa > 0$ (whose value will be fixed later on), we further assume that the functions φ_j^1 and ψ_j^1 satisfy

$$\tag{3.13} \|\varphi_j^1\|_{\mathcal{C}^{4,\alpha}(S^3)} \le \kappa r_\varepsilon^2 \quad \text{and} \quad \|\psi_j^1\|_{\mathcal{C}^{2,\alpha}(S^3)} \le \kappa r_\varepsilon^2 \quad \text{for } j = 1, 2.$$

Then we have the following result.

Lemma 3.9. *Let $\varphi^1 := (\varphi_1^1, \varphi_2^1) \in (\mathcal{C}^{4,\alpha}(S^3))^2$ and $\psi^1 := (\psi_1^1, \psi_2^1) \in (\mathcal{C}^{2,\alpha}(S^3))^2$ such that (φ_1^1, ψ_1^1) and (φ_2^1, ψ_2^1) satisfy* (3.5) *and* (3.13). *Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$, $c_\kappa > 0$ and $\gamma_0 \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\gamma \in (\gamma_0, 1)$, $\mu \in (1,2)$ and $\delta \in \left(0, \min\{(\frac{\gamma+\xi-1}{\gamma}), (\frac{\gamma+\xi-1}{\xi})\}\right)$. We have*

$$\|\mathcal{N}_1(0,0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2, \quad \|\mathcal{M}_1(0,0)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2,$$

$$\|\mathcal{N}_1(h_1^1, h_2^1) - \mathcal{N}_1(k_1^1, k_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2 \|h_1^1 - k_1^1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} + c_\kappa(1-\gamma)\|h_2^1 - k_2^1\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}$$

and

$$\|\mathcal{M}_1(h_1^1,h_2^1)-\mathcal{M}_1(k_1^1,k_2^1)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)}\le c_\kappa r_\varepsilon^2\|(h_1^1,h_2^1)-(k_1^1,k_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)},$$

provided $(h_1^1,h_2^1),(k_1^1,k_2^1)\in\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)$ *satisfying*

$$\|(h_1^1,h_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)}\le 2c_\kappa r_\varepsilon^2,\quad \|(k_1^1,k_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)}\le 2c_\kappa r_\varepsilon^2. \tag{3.14}$$

Proof. The proof of the first and the second estimates follows from the asymptotic behavior of H^{int} together with the assumption on the norms of φ_j^1 and ψ_j^1 given by (3.13) and it follows from the estimate of H^{int}, given by Lemma 3.7, that

$$\left\|H^{\text{int}}_{\varphi_j^1,\psi_j^1}\left(\frac{r}{R_\varepsilon^1}\cdot\right)\right\|_{\mathcal{C}^{4,\alpha}(\overline{B}_2(0)-B_1(0))}\le Cr^2(R_\varepsilon^1)^{-2}\big(\|\varphi_j^1\|_{\mathcal{C}^{4,\alpha}(S^3)}+\|\psi_j^1\|_{\mathcal{C}^{2,\alpha}(S^3)}\big)$$

for all $r\le R_\varepsilon^1/2$. Then by (3.13), we get

$$\left\|H^{\text{int}}_{\varphi_j^1,\psi_j^1}\left(\frac{r}{R_\varepsilon^1}\cdot\right)\right\|_{\mathcal{C}^{4,\alpha}(\overline{B}_2(0)-B_1(0))}\le c_\kappa\varepsilon^2r^2.$$

On the other hand,

$$\begin{aligned}\sup_{r\le R_\varepsilon^1} r^{4-\mu}|\mathcal{T}_1(0,0)|&\le\sup_{r\le R_\varepsilon^1}\frac{384r^{4-\mu}}{(1+r^2)^4}\frac{1}{\gamma}\left|e^{\gamma H^{\text{int}}_{\varphi_1^1,\psi_1^1}+(1-\gamma)H^{\text{int}}_{\varphi_2^1,\psi_2^1}}-1\right|\\&\le\sup_{r\le R_\varepsilon^1}\frac{384r^{4-\mu}}{(1+r^2)^4}\frac{1}{\gamma}\big(\gamma r^2\|H^{\text{int}}_{\varphi_1^1,\psi_1^1}\|_{\mathcal{C}_2^{4,\alpha}}+(1-\gamma)r^2\|H^{\text{int}}_{\varphi_2^1,\psi_2^1}\|_{\mathcal{C}_2^{4,\alpha}}\big).\end{aligned}$$

Making use of Proposition 3.2 together with (3.4), for $\mu\in(1,2)$, we get that there exists $c_\kappa>0$ such that

$$\|\mathcal{N}_1(0,0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}\le c_\kappa r_\varepsilon^2.$$

For the second estimate, we have

$$\begin{aligned}\sup_{r\le R_\varepsilon^1} r^{4-\delta}|\mathcal{T}_2(0,0)|&\le c\sup_{r\le R_\varepsilon^1}C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}}\varepsilon^{8\frac{\gamma+\xi-1}{\gamma}}r^{4-\delta}\left(\frac{16}{(1+r^2)^4}\right)^{\frac{1-\xi}{\gamma}}\\&\qquad\times e^{\frac{\gamma+\xi-1}{\gamma}G\left(\frac{\varepsilon x}{\tau_1},x^2\right)+\frac{\gamma+\xi-1}{\gamma\xi}G\left(\frac{\varepsilon x}{\tau_1},x^3\right)+\xi H^{\text{int}}_{\varphi_2^1,\psi_2^1}+(1-\xi)H^{\text{int}}_{\varphi_1^1,\psi_1^1}}\\&\le c\sup_{r\le R_\varepsilon^1}\varepsilon^{8\frac{\gamma+\xi-1}{\gamma}}r^{4-\delta}\left(\frac{16}{(1+r^2)^4}\right)^{\frac{1-\xi}{\gamma}}\\&\qquad\times\big(\xi r^2\|H^{\text{int}}_{\varphi_2^1,\psi_2^1}\|_{\mathcal{C}_2^{4,\alpha}}+(1-\xi)r^2\|H^{\text{int}}_{\varphi_1^1,\psi_1^1}\|_{\mathcal{C}_2^{4,\alpha}}+1\big).\end{aligned}$$

Using the same argument as above, we get $\|\mathcal{M}_1(0,0)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)}\le c_\kappa r_\varepsilon^2$.

To derive the third estimate, for (h_1^1, h_2^1), (k_1^1, k_2^1) verifying (3.14), we have

$$\begin{aligned}
&\sup_{r\le R_\varepsilon^1} r^{4-\mu}|\mathcal{T}_1(h_1^1,h_2^1)-\mathcal{T}_1(k_1^1,k_2^1)| \\
&\le \sup_{r\le R_\varepsilon^1} \frac{384r^{4-\mu}}{(1+r^2)^4}\frac{1}{\gamma}\Big|\big(e^{\gamma h_1^1+\gamma H^{\rm int}_{\varphi_1^1,\psi_1^1}+(1-\gamma)h_2^1+(1-\gamma)H^{\rm int}_{\varphi_2^1,\psi_2^1}}-\gamma h_1^1-1\big) \\
&\qquad\qquad -\big(e^{\gamma k_1^1+\gamma H^{\rm int}_{\varphi_1^1,\psi_1^1}+(1-\gamma)k_2^1+(1-\gamma)H^{\rm int}_{\varphi_2^1,\psi_2^1}}-\gamma k_1^1-1\big)\Big| \\
&\le c\sup_{r\le R_\varepsilon^1} \frac{384r^{4-\mu}}{(1+r^2)^4}\frac{1}{\gamma}\big[\gamma^2((h_1^1)^2-(k_1^1)^2)+(1-\gamma)|h_2^1-k_2^1|\big] \\
&\le c\sup_{r\le R_\varepsilon^1} \frac{384r^{4-\mu}}{(1+r^2)^4}\frac{1}{\gamma}\big[\gamma^2 r^{2\mu}\big(\|h_1^1\|_{\mathcal{C}^{4,\alpha}_\mu}+\|k_1^1\|_{\mathcal{C}^{4,\alpha}_\mu}\big)\|h_1^1-k_1^1\|_{\mathcal{C}^{4,\alpha}_\mu}+(1-\gamma)r^\delta\|h_2^1-k_2^1\|_{\mathcal{C}^{4,\alpha}_\delta}\big].
\end{aligned}$$

We conclude that

$$\|\mathcal{N}_1(h_1^1,h_2^1)-\mathcal{N}_1(k_1^1,k_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}\le c_\kappa r_\varepsilon^2\|h_1^1-k_1^1\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}+c_\kappa(1-\gamma)\|h_2^1-k_2^1\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}. \tag{3.15}$$

On the other hand, we have

$$\begin{aligned}
&\sup_{r\le R_\varepsilon^1} r^{4-\delta}|\mathcal{T}_2(h_1^1,h_2^1)-\mathcal{T}_2(k_1^1,k_2^1)| \\
&\le \sup_{r\le R_\varepsilon^1} 24C_{1,\varepsilon}^{4^{\frac{\gamma+\xi-1}{\gamma}}}\gamma^{-\frac{1-\xi}{\gamma}}\varepsilon^{8^{\frac{\gamma+\xi-1}{\gamma}}}\left(\frac{16}{(1+r^2)^4}\right)^{\frac{1-\xi}{\gamma}} r^{4-\delta}e^{\frac{\gamma+\xi-1}{\gamma}G\left(\frac{\varepsilon x}{\tau_1},x^2\right)+\frac{\gamma+\xi-1}{\gamma\xi}G\left(\frac{\varepsilon x}{\tau_1},x^3\right)} \\
&\qquad\times\Big|e^{\xi h_2^1+\xi H^{\rm int}_{\varphi_2^1,\psi_2^1}+(1-\xi)h_1^1+(1-\xi)H^{\rm int}_{\varphi_1^1,\psi_1^1}}-e^{\xi k_2^1+\xi H^{\rm int}_{\varphi_2^1,\psi_2^1}+(1-\xi)k_1^1+(1-\xi)H^{\rm int}_{\varphi_1^1,\psi_1^1}}\Big| \\
&\le c\sup_{r\le R_\varepsilon^1} 24C_{1,\varepsilon}^{4^{\frac{\gamma+\xi-1}{\gamma}}}\gamma^{-\frac{1-\xi}{\gamma}}\varepsilon^{8^{\frac{\gamma+\xi-1}{\gamma}}}\left(\frac{16}{(1+r^2)^4}\right)^{\frac{1-\xi}{\gamma}} r^{4-\delta}\big[\xi|h_2^1-k_2^1|+(1-\xi)|h_1^1-k_1^1|\big] \\
&\le c\sup_{r\le R_\varepsilon^1} 24C_{1,\varepsilon}^{4^{\frac{\gamma+\xi-1}{\gamma}}}\gamma^{-\frac{1-\xi}{\gamma}}\varepsilon^{8^{\frac{\gamma+\xi-1}{\gamma}}}\left(\frac{16}{(1+r^2)^4}\right)^{\frac{1-\xi}{\gamma}} \\
&\qquad\times r^{4-\delta}\big[\xi r^\delta\|h_2^1-k_2^1\|_{\mathcal{C}^{4,\alpha}_\delta}+(1-\xi)r^\mu\|h_1^1-k_1^1\|_{\mathcal{C}^{4,\alpha}_\mu}\big].
\end{aligned}$$

We conclude that

$$\|\mathcal{M}_1(h_1^1,h_2^1)-\mathcal{M}_1(k_1^1,k_2^1)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}\le c_\kappa r_\varepsilon^2\|(h_1^1,h_2^1)-(k_1^1,k_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}. \tag{3.16}$$ □

Reducing ε_κ if necessary, we can assume that $c_\kappa r_\varepsilon^2 < 1/2$ for all $\varepsilon\in(0,\varepsilon_\kappa)$. There exists also $\gamma_0\in(0,1)$ such that $c_\kappa(1-\gamma)\le 1/2$ for all $\gamma\in(\gamma_0,1)$. Therefore (3.15) and (3.16) are enough to show that

$$(h_1^1,h_2^1)\mapsto(\mathcal{N}_1(h_1^1,h_2^1),\mathcal{M}_1(h_1^1,h_2^1))$$

is a contraction from the ball

$$\big\{(h_1^1,h_2^1)\in\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4):\|(h_1^1,h_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)}\le 2c_\kappa r_\varepsilon^2\big\}$$

into itself. Then applying a contraction mapping argument, we obtain the following proposition.

Proposition 3.10. *Given $\kappa > 0$, $\mu \in (1,2)$ and $\delta \in \left(0, \min\{(\frac{\gamma+\xi-1}{\gamma}), (\frac{\gamma+\xi-1}{\xi})\}\right)$, there exist $\varepsilon_\kappa > 0$, $c_\kappa > 0$ and $\gamma_0 \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\gamma \in (\gamma_0, 1)$, for all τ_1 in some fixed compact subset of $[\tau_1^-, \tau_1^+] \subset (0,\infty)$ and for φ_j^1 and ψ_j^1 satisfying* (3.5) *and* (3.13)*, there exists a unique (h_1^1, h_2^1) $(:= (h_{1,\varepsilon,\tau_1,\varphi_1^1,\psi_1^1}, h_{2,\varepsilon,\tau_1,\varphi_2^1,\psi_2^1}))$ solution of* (3.12) *such that*

$$\|(h_1^1, h_2^1)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2.$$

Hence

$$\begin{aligned} v_1(x) &:= \frac{1}{\gamma}\overline{u}(x-x^1) - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) \\ &\quad - \frac{\ln\gamma}{\gamma} + h_1^1(x) + H^{\mathrm{int}}\left(\varphi_1^1, \psi_1^1; \frac{x-x^1}{R_\varepsilon^1}\right), \\ v_2(x) &:= \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + h_2^1(x) + H^{\mathrm{int}}\left(\varphi_2^1, \psi_2^1; \frac{x-x^1}{R_\varepsilon^1}\right) \end{aligned}$$

solves (3.8) *in $B_{R_\varepsilon^1}(x^1)$.*

In $B_{R_\varepsilon^3}(x^3)$, following the same arguments as the first case by reversing the roles of the functions u_1 and u_2 and by respecting the changes of the coefficients we can prove that there exists $(h_1^3, h_2^3) \in \mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)$ such that

$$\|(h_1^3, h_2^3)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)\times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2.$$

Furthermore (h_1^3, h_2^3) solves the equations

$$\begin{aligned} \Delta^2 h_1^3 &= \frac{24 C_{3,\varepsilon}^{4\frac{\gamma+\xi-1}{\xi}} 16^{\frac{1-\gamma}{\xi}} \varepsilon^{8\frac{\gamma+\xi-1}{\xi}}}{\xi^{\frac{1-\gamma}{\xi}} (1+r^2)^{4\frac{1-\gamma}{\xi}}} \\ &\quad \times e^{\frac{\gamma+\xi-1}{\xi} G\left(\frac{\varepsilon x}{\tau_3}, x^2\right) + \frac{\gamma+\xi-1}{\gamma\xi} G\left(\frac{\varepsilon x}{\tau_3}, x^1\right) + \gamma\left(h_1^3 + H^{\mathrm{int}}_{\varphi_1^3,\psi_1^3}\right) + (1-\gamma)\left(h_2^3 + H^{\mathrm{int}}_{\varphi_2^3,\psi_2^3}\right)}, \\ \mathbb{L} h_2^3 &= \frac{384}{\xi(1+r^2)^4}\left[e^{\xi\left(h_2^3 + H^{\mathrm{int}}_{\varphi_2^3,\psi_2^3}\right) + (1-\xi)\left(h_1^3 + H^{\mathrm{int}}_{\varphi_1^3,\psi_1^3}\right)} - \xi h_2^3 - 1\right]. \end{aligned}$$

Given $\kappa > 0$ (whose value will be fixed later on), we further assume that the functions φ_j^3 and ψ_j^3 satisfy

$$\|\varphi_j^3\|_{\mathcal{C}^{4,\alpha}(S^3)} \le \kappa r_\varepsilon^2 \quad \text{and} \quad \|\psi_j^3\|_{\mathcal{C}^{2,\alpha}(S^3)} \le \kappa r_\varepsilon^2 \quad \text{for } j = 1, 2. \tag{3.17}$$

Then we have the following proposition.

Proposition 3.11. *Given $\kappa > 0$, $\mu \in (1,2)$ and $\delta \in \left(0, \min\{(\frac{\gamma+\xi-1}{\gamma}), (\frac{\gamma+\xi-1}{\xi})\}\right)$, there exist $\varepsilon_\kappa > 0$, $c_\kappa > 0$ and $\xi_0 \in (0,1)$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\xi \in (\xi_0, 1)$, for all τ_3 in some fixed compact subset of $[\tau_3^-, \tau_3^+] \subset (0, \infty)$ and for φ_j^3 and ψ_j^3 satisfying* (3.5) *and* (3.17)*, there exists a unique (h_1^3, h_2^3) $(:= (h_{1,\varepsilon,\tau_3,\varphi_1^3,\psi_1^3}, h_{2,\varepsilon,\tau_3,\varphi_2^3,\psi_2^3}))$ solution of* (3.12) *such that*

$$\|(h_1^3, h_2^3)\|_{C_\delta^{4,\alpha}(\mathbb{R}^4) \times C_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_\varepsilon^2.$$

Hence

$$\begin{aligned} v_1(x) &:= \frac{1}{\gamma} G\left(\frac{\varepsilon x}{\tau_3}, x^1\right) + G\left(\frac{\varepsilon x}{\tau_3}, x^2\right) + h_1^3(x) + H^{\text{int}}\left(\varphi_1^3, \psi_1^3, \frac{x - x^3}{R_\varepsilon^3}\right), \\ v_2(x) &:= \frac{1}{\xi}\overline{u}(x - x^3) - \frac{1-\xi}{\xi} G\left(\frac{\varepsilon x}{\tau_3}, x^2\right) - \frac{1-\xi}{\gamma\xi} G\left(\frac{\varepsilon x}{\tau_3}, x^1\right) \\ &\quad - \frac{\ln \xi}{\xi} + h_2^3(x) + H^{\text{int}}\left(\varphi_2^3, \psi_2^3, \frac{x - x^3}{R_\varepsilon^3}\right) \end{aligned}$$

solves (3.10) *in $B_{R_\varepsilon^3}(x^3)$.*

In $B_{R_\varepsilon^2}(x^2)$, we look for a solution of (3.9) of the form

$$\begin{aligned} v_1(x) &= \overline{u}(x - x^2) + H^{\text{int}}\left(\varphi_1^2, \psi_1^2; \frac{x - x^2}{R_\varepsilon^2}\right) + h_1^2(x), \\ v_2(x) &= \overline{u}(x - x^2) + H^{\text{int}}\left(\varphi_2^2, \psi_2^2; \frac{x - x^2}{R_\varepsilon^2}\right) + h_2^2(x). \end{aligned}$$

This amounts to solve the equations

$$\begin{aligned} \mathbb{L}h_1^2 &= \frac{384}{(1+r^2)^4}\left[e^{\gamma\left(h_1^2 + H^{\text{int}}_{\varphi_1^2,\psi_1^2}\right) + (1-\gamma)\left(h_2^2 + H^{\text{int}}_{\varphi_2^2,\psi_2^2}\right)} - h_1^2 - 1\right], \\ \mathbb{L}h_2^2 &= \frac{384}{(1+r^2)^4}\left[e^{\xi\left(h_2^2 + H^{\text{int}}_{\varphi_2^2,\psi_2^2}\right) + (1-\xi)\left(h_1^2 + H^{\text{int}}_{\varphi_1^2,\psi_1^2}\right)} - h_2^2 - 1\right]. \end{aligned} \tag{3.18}$$

We denote by

$$\mathbb{L}h_1^2 = \mathcal{T}_3(h_1^2, h_2^2) \quad \text{and} \quad \mathbb{L}h_2^2 = \mathcal{T}_4(h_1^2, h_2^2).$$

To find a solution of (3.18), it is enough to find a fixed point (h_1^2, h_2^2) in a small ball of $\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)$, solutions of

$$\begin{aligned} h_1^2 &= \mathcal{G}_\mu \circ \xi_{\mu, R_\varepsilon^2} \circ \mathcal{T}_3(h_1^2, h_2^2) = \mathcal{N}_2(h_1^2, h_2^2), \\ h_2^2 &= \mathcal{G}_\mu \circ \xi_{\mu, R_\varepsilon^2} \circ \mathcal{T}_4(h_1^2, h_2^2) = \mathcal{M}_2(h_1^2, h_2^2). \end{aligned} \tag{3.19}$$

Given $\kappa > 0$ (whose value will be fixed later on), we further assume that the functions φ_j^2 and ψ_j^2 satisfy

$$\|\varphi_j^2\|_{\mathcal{C}^{4,\alpha}(S^3)} \leq \kappa r_\varepsilon^2 \quad \text{and} \quad \|\psi_j^2\|_{\mathcal{C}^{2,\alpha}(S^3)} \leq \kappa r_\varepsilon^2 \quad \text{for } j = 1, 2. \tag{3.20}$$

Then, we have the following result.

Lemma 3.12. *Let $\mu \in (1,2)$, γ_0 and $\xi_0 \in (0,1)$. Given $\kappa > 0$, there exist $\varepsilon_\kappa > 0$ and $c_\kappa > 0$ such that for all $\varepsilon \in (0,\varepsilon_\kappa)$, $\gamma \in (\gamma_0, 1)$ and $\xi \in (\xi_0, 1)$. We have*

$$\|\mathcal{N}_2(0,0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2, \quad \|\mathcal{M}_2(0,0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2,$$
$$\|\mathcal{N}_2(h_1^2,h_2^2) - \mathcal{N}_2(k_1^2,k_2^2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa(1-\gamma)\|(h_1^2,h_2^2) - (k_1^2,k_2^2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)},$$
$$\|\mathcal{M}_2(h_1^2,h_2^2) - \mathcal{M}_2(k_1^2,k_2^2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa(1-\xi)\|(h_1^2,h_2^2) - (k_1^2,k_2^2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)},$$

provided (h_1^2,h_2^2), (k_1^2,k_2^2) in $\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)$ satisfying

$$\|(h_1^2,h_2^2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2 \quad \text{and} \quad \|(k_1^2,k_2^2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2. \tag{3.21}$$

Proof. The proof of the first and the second estimates follows from the asymptotic behavior of H^{int} together with the assumption on the norms of φ_j^2 and ψ_j^2 given by (3.20) and it follows from the estimate of H^{int}, given by Lemma 3.7, that

$$\left\| H^{\text{int}}_{\varphi_j^2,\psi_j^2}\left(\frac{r}{R_\varepsilon^2}\,\cdot\right)\right\|_{\mathcal{C}^{4,\alpha}(\overline{B}_2(0)-B_1(0))} \le C r^2 (R_\varepsilon^2)^{-2}\big(\|\varphi_j^2\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi_j^2\|_{\mathcal{C}^{2,\alpha}(S^3)}\big)$$

for all $r \le R_\varepsilon^2/2$. Then by (3.20), we get

$$\left\| H^{\text{int}}_{\varphi_j^2,\psi_j^2}\left(\frac{r}{R_\varepsilon^2}\,\cdot\right)\right\|_{\mathcal{C}^{4,\alpha}(\overline{B}_2(0)-B_1(0))} \le c_\kappa \varepsilon^2 r^2.$$

On the other hand,

$$\begin{aligned}\sup_{r\le R_\varepsilon^2} r^{4-\mu}|\mathcal{T}_3(0,0)| &\le \sup_{r\le R_\varepsilon^2} \frac{384 r^{4-\mu}}{(1+r^2)^4}\left|e^{\gamma H^{\text{int}}_{\varphi_1^2,\psi_1^2}+(1-\gamma)H^{\text{int}}_{\varphi_2^2,\psi_2^2}} - 1\right| \\ &\le \sup_{r\le R_\varepsilon^2} \frac{384 r^{4-\mu}}{(1+r^2)^4}\big(\gamma r^2\|H^{\text{int}}_{\varphi_1^2,\psi_1^2}\|_{\mathcal{C}_2^{4,\alpha}} + (1-\gamma) r^2 \|H^{\text{int}}_{\varphi_2^2,\psi_2^2}\|_{\mathcal{C}_2^{4,\alpha}}\big).\end{aligned}$$

Making use of Proposition 3.2 together with (3.4), for $\mu \in (1,2)$, we get that there exists $c_\kappa > 0$ such that

$$\|\mathcal{N}_2(0,0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2.$$

For the second estimate, we use the same techniques to prove

$$\|\mathcal{M}_2(0,0)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2.$$

To derive the third estimate, for (h_1^2,h_2^2), (k_1^2,k_2^2) verifying (3.21), we have

$$\begin{aligned}&\sup_{r\le R_\varepsilon^2} r^{4-\mu}|\mathcal{T}_3(h_1^2,h_2^2) - \mathcal{T}_3(k_1^2,k_2^2)| \\ &\le \sup_{r\le R_\varepsilon^2} \frac{384 r^{4-\mu}}{(1+r^2)^4}\Big|\big(e^{\gamma h_1^2 + \gamma H^{\text{int}}_{\varphi_1^2,\psi_1^2} + (1-\gamma)h_2^2 + (1-\gamma)H^{\text{int}}_{\varphi_2^2,\psi_2^2}} - h_1^2\big)\end{aligned}$$

$$
\begin{aligned}
&\qquad - \left(e^{\gamma k_1^2+\gamma H^{\mathrm{int}}_{\varphi_1^2,\psi_1^2}+(1-\gamma)k_2^2+(1-\gamma)H^{\mathrm{int}}_{\varphi_2^2,\psi_2^2}} - k_1^2\right)\Big| \\
&\le c \sup_{r\le R_\varepsilon^2} \frac{384 r^{4-\mu}}{(1+r^2)^4}\big|(\gamma-1)(h_1^2-k_1^2)+(1-\gamma)(h_2^2-k_2^2)\big| \\
&\le c \sup_{r\le R_\varepsilon^2} \frac{384 r^{4-\mu}}{(1+r^2)^4}(1-\gamma)\big[r^\mu\|h_1^2-k_1^2\|_{\mathcal{C}^{4,\alpha}_\mu}+r^\mu\|h_2^2-k_2^2\|_{\mathcal{C}^{4,\alpha}_\mu}\big].
\end{aligned}
$$

We conclude that

$$
(3.22)\quad \|\mathcal{N}_2(h_1^2,h_2^2)-\mathcal{N}_2(k_1^2,k_2^2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa(1-\gamma)\|(h_1^2,h_2^2)-(k_1^2,k_2^2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}.
$$

Similarly, we get

$$
(3.23)\quad \|\mathcal{M}_2(h_1^2,h_2^2)-\mathcal{M}_2(k_1^2,k_2^2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le c_\kappa(1-\xi)\|(h_1^2,h_2^2)-(k_1^2,k_2^2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)}.
$$

□

Then there exist γ_0 and $\xi_0 \in (0,1)$ such that $c_\kappa(1-\gamma)\le 1/2$ and $c_\kappa(1-\xi)\le 1/2$ for all $\gamma\in(\gamma_0,1)$ and $\xi\in(\xi_0,1)$. Therefore (3.22) and (3.23) are enough to show that

$$
(h_1^2,h_2^2)\mapsto(\mathcal{N}_2(h_1^2,h_2^2),\mathcal{M}_2(h_1^2,h_2^2))
$$

is a contraction from the ball

$$
\left\{(h_1^2,h_2^2)\in\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) : \|(h_1^2,h_2^2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2\right\}
$$

into itself. Then applying a contraction mapping argument, we obtain the following proposition.

Proposition 3.13. *Given $\kappa>0$, $\mu\in(1,2)$, $\gamma_0\in(0,1)$ and $\xi_0\in(0,1)$, there exist $\varepsilon_\kappa>0$ and $c_\kappa>0$ such that for all $\varepsilon\in(0,\varepsilon_\kappa)$, $\gamma\in(\gamma_0,1)$ and $\xi\in(\xi_0,1)$, for all τ_2 in some fixed compact subset of $[\tau_2^-,\tau_2^+]\subset(0,\infty)$ and for φ_j^2 and ψ_j^2 satisfying* (3.5) *and* (3.20)*, there exists a unique (h_1^2,h_2^2) $(:=(h_{1,\varepsilon,\tau_2,\varphi_1^2,\psi_1^2},h_{2,\varepsilon,\tau_2,\varphi_2^2,\psi_2^2}))$ solution of* (3.19) *such that*

$$
\|(h_1^2,h_2^2)\|_{C^{4,\alpha}_\mu(\mathbb{R}^4)\times C^{4,\alpha}_\mu(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2.
$$

Hence

$$
\begin{aligned}
v_1(x) &:= \overline{u}(x-x^2)+h_1^2(x)+H^{\mathrm{int}}\left(\varphi_1^2,\psi_1^2;\frac{x-x^2}{R_\varepsilon^2}\right),\\
v_2(x) &:= \overline{u}(x-x^2)+h_2^2(x)+H^{\mathrm{int}}\left(\varphi_2^2,\psi_2^2;\frac{x-x^2}{R_\varepsilon^2}\right)
\end{aligned}
$$

solves (3.9) *in $B_{R_\varepsilon^2}(x^2)$.*

Remark also that the functions (h_1^i,h_2^i) $(:=(h^i_{1,\varepsilon,\tau_i,\varphi_1^i,\psi_1^i},h^i_{2,\varepsilon,\tau_i,\varphi_2^i,\psi_2^i}))$, for $i\in\{1,2,3\}$, depend continuously on the parameter τ_i.

3.3. The nonlinear exterior problem

Given $\widetilde{\boldsymbol{x}} := (\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3) \in \Omega^3$ close to $\boldsymbol{x} := (x^1, x^2, x^3)$, $\boldsymbol{\lambda} := (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ close to 0, $\widetilde{\boldsymbol{\varphi}}_1 := (\widetilde{\varphi}_1^1, \widetilde{\varphi}_1^2, \widetilde{\varphi}_1^3) \in (\mathcal{C}^{4,\alpha}(S^3))^3$, $\widetilde{\boldsymbol{\varphi}}_2 := (\widetilde{\varphi}_2^1, \widetilde{\varphi}_2^2, \widetilde{\varphi}_2^3) \in (\mathcal{C}^{4,\alpha}(S^3))^3$, $\widetilde{\boldsymbol{\psi}}_1 := (\widetilde{\psi}_1^1, \widetilde{\psi}_1^2, \widetilde{\psi}_1^3) \in (\mathcal{C}^{2,\alpha}(S^3))^3$ and $\widetilde{\boldsymbol{\psi}}_2 := (\widetilde{\psi}_2^1, \widetilde{\psi}_2^2, \widetilde{\psi}_2^3) \in (\mathcal{C}^{2,\alpha}(S^3))^3$ satisfying (3.6). Let $\widetilde{\boldsymbol{w}}_1$ and $\widetilde{\boldsymbol{w}}_2$ be defined by

$$\widetilde{\boldsymbol{w}}_1(x) := \frac{1+\lambda_1}{\gamma} G(x, \widetilde{x}^1) + (1+\lambda_2) G(x, \widetilde{x}^2) + \sum_{i=1}^{3} \chi_{r_0}(x - \widetilde{x}^i) H^{\mathrm{ext}} \left(\widetilde{\varphi}_1^i, \widetilde{\psi}_1^i; \frac{x - \widetilde{x}^i}{r_\varepsilon} \right)$$

and

$$\widetilde{\boldsymbol{w}}_2(x) := \frac{1+\lambda_3}{\xi} G(x, \widetilde{x}^3) + (1+\lambda_2) G(x, \widetilde{x}^2) + \sum_{i=1}^{3} \chi_{r_0}(x - \widetilde{x}^i) H^{\mathrm{ext}} \left(\widetilde{\varphi}_2^i, \widetilde{\psi}_2^i; \frac{x - \widetilde{x}^i}{r_\varepsilon} \right).$$

Here χ_{r_0} is a cut-off function identically equal to 1 in $B_{r_0/2}(0)$ and identically equal to 0 outside $B_{r_0}(0)$. We would like to find a solution of the system

$$\Delta^2 u_1 = \rho^4 e^{\gamma u_1 + (1-\gamma) u_2} \quad \text{and} \quad \Delta^2 u_2 = \rho^4 e^{\xi u_2 + (1-\xi) u_1} \tag{3.24}$$

in the domain $\overline{\Omega}_{r_\varepsilon}(\widetilde{x})$ with $u_k = \widetilde{\boldsymbol{w}}_k + \widetilde{v}_k$ a perturbation of $\widetilde{\boldsymbol{w}}_k$, $k = 1, 2$. This amounts to solve in $\overline{\Omega}_{r_\varepsilon}(\widetilde{x})$,

$$\Delta^2 \widetilde{v}_1 = \rho^4 e^{\gamma(\widetilde{\boldsymbol{w}}_1 + \widetilde{v}_1) + (1-\gamma)(\widetilde{\boldsymbol{w}}_2 + \widetilde{v}_2)} - \Delta^2 \widetilde{\boldsymbol{w}}_1 \quad \text{and} \quad \Delta^2 \widetilde{v}_2 = \rho^4 e^{\xi(\widetilde{\boldsymbol{w}}_2 + \widetilde{v}_2) + (1-\xi)(\widetilde{\boldsymbol{w}}_1 + \widetilde{v}_1)} - \Delta^2 \widetilde{\boldsymbol{w}}_2. \tag{3.25}$$

For all $\sigma \in (0, r_0/2)$ and all $\widetilde{\boldsymbol{x}} = (\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3) \in \Omega^3$ such that $\|\boldsymbol{x} - \widetilde{\boldsymbol{x}}\| \le r_0/2$, where $\boldsymbol{x} = (x^1, x^2, x^3)$, we denote by $\widetilde{\xi}_{\sigma,\widetilde{\boldsymbol{x}}} \colon \mathcal{C}_\nu^{0,\alpha}(\overline{\Omega}_\sigma(\widetilde{\boldsymbol{x}})) \to \mathcal{C}_\nu^{0,\alpha}(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))$ the extension operator defined by

$$\begin{cases} \widetilde{\xi}_{\sigma,\widetilde{\boldsymbol{x}}}(f) \equiv f & \text{in } \overline{\Omega}_\sigma(\widetilde{\boldsymbol{x}}), \\ \widetilde{\xi}_{\sigma,\widetilde{\boldsymbol{x}}}(f)(\widetilde{x}^j + x) = \widetilde{\chi}\big(\frac{|x|}{\sigma}\big) f\big(\widetilde{x}^j + \sigma \frac{x}{|x|}\big) & \text{in } B_\sigma(\widetilde{x}^j) - B_{\sigma/2}(\widetilde{x}^j), \ \forall\, 1 \le j \le 3, \\ \widetilde{\xi}_{\sigma,\widetilde{\boldsymbol{x}}}(f) \equiv 0 & \text{in } B_{\sigma/2}(\widetilde{x}^1) \cup B_{\sigma/2}(\widetilde{x}^2) \cup B_{\sigma/2}(\widetilde{x}^3). \end{cases}$$

Here $\widetilde{\chi}$ is a cut-off function over $\mathbb{R}_+$ which is equal to 1 for $t \ge 1$ and equal to 0 for $t \le 1/2$. Obviously, there exists a constant $\overline{c} = \overline{c}(\nu) > 0$ only depending on ν such that

$$\big\|\widetilde{\xi}_{\sigma,\widetilde{\boldsymbol{x}}}(w)\big\|_{\mathcal{C}_\nu^{0,\alpha}(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))} \le \overline{c} \|w\|_{\mathcal{C}_\nu^{0,\alpha}(\overline{\Omega}_\sigma(\widetilde{\boldsymbol{x}}))}. \tag{3.26}$$

We fix $\nu \in (-1, 0)$, to solve (3.25), it is enough to find $(\widetilde{v}_1, \widetilde{v}_2) \in (\mathcal{C}_\nu^{4,\alpha}(\overline{\Omega}^*(\widetilde{\boldsymbol{x}})))^2$ solution of

$$\widetilde{v}_1 = \widetilde{\mathcal{K}}_\nu \circ \widetilde{\xi}_{r_\varepsilon,\widetilde{\boldsymbol{x}}} \circ \widetilde{S}_1(\widetilde{v}_1, \widetilde{v}_2) \quad \text{and} \quad \widetilde{v}_2 = \widetilde{\mathcal{K}}_\nu \circ \widetilde{\xi}_{r_\varepsilon,\widetilde{\boldsymbol{x}}} \circ \widetilde{S}_2(\widetilde{v}_1, \widetilde{v}_2), \tag{3.27}$$

where

$$\widetilde{S}_1(\widetilde{v}_1,\widetilde{v}_2)=\rho^4 e^{\gamma(\widetilde{\boldsymbol{w}}_1+\widetilde{v}_1)+(1-\gamma)(\widetilde{\boldsymbol{w}}_2+\widetilde{v}_2)}-\Delta^2\widetilde{\boldsymbol{w}}_1$$

and

$$\widetilde{S}_2(\widetilde{v}_1,\widetilde{v}_2)=\rho^4 e^{\xi(\widetilde{\boldsymbol{w}}_2+\widetilde{v}_2)+(1-\xi)(\widetilde{\boldsymbol{w}}_1+\widetilde{v}_1)}-\Delta^2\widetilde{\boldsymbol{w}}_2.$$

We denote by

$$\widetilde{\mathcal{N}}(\widetilde{v}_1,\widetilde{v}_2)=\widetilde{\mathcal{K}}_\nu\circ\widetilde{\xi}_{r_\varepsilon,\widetilde{\boldsymbol{x}}}\circ\widetilde{S}_1(\widetilde{v}_1,\widetilde{v}_2)\quad\text{and}\quad\widetilde{\mathcal{M}}(\widetilde{v}_1,\widetilde{v}_2)=\widetilde{\mathcal{K}}_\nu\circ\widetilde{\xi}_{r_\varepsilon,\widetilde{\boldsymbol{x}}}\circ\widetilde{S}_2(\widetilde{v}_1,\widetilde{v}_2).$$

Given $\kappa>0$ (whose value will be fixed later on), we further assume that for $i\in\{1,2,3\}$ and $j\in\{1,2\}$ the functions $\widetilde{\varphi}^i_j$, $\widetilde{\psi}^i_j$, the parameters λ_i and the point $\widetilde{\boldsymbol{x}}=(\widetilde{x}^1,\widetilde{x}^2,\widetilde{x}^3)$ satisfy

$$\|\widetilde{\varphi}^i_j\|_{\mathcal{C}^{4,\alpha}(S^3)}\le\kappa r_\varepsilon^2,\quad\|\widetilde{\psi}^i_j\|_{\mathcal{C}^{2,\alpha}(S^3)}\le\kappa r_\varepsilon^2,\tag{3.28}$$

$$|\lambda_i|\le\kappa r_\varepsilon^2,\quad|\widetilde{x}^i-x^i|\le\kappa r_\varepsilon.\tag{3.29}$$

Then the following result holds.

Lemma 3.14. *Under the above assumptions, there exists a constant $c_\kappa>0$ such that*

$$\|\widetilde{\mathcal{N}}(0,0)\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))}\le c_\kappa r_\varepsilon^2,\quad\|\widetilde{\mathcal{M}}(0,0)\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))}\le c_\kappa r_\varepsilon^2,$$
$$\|\widetilde{\mathcal{N}}(\widetilde{v}_1,\widetilde{v}_2)-\widetilde{\mathcal{N}}(\widetilde{v}'_1,\widetilde{v}'_2)\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))}\le c_\kappa r_\varepsilon^2\|(\widetilde{v}_1,\widetilde{v}_2)-(\widetilde{v}'_1,\widetilde{v}'_2)\|_{(\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}})))^2}$$

and

$$\|\widetilde{\mathcal{M}}(\widetilde{v}_1,\widetilde{v}_2)-\widetilde{\mathcal{M}}(\widetilde{v}'_1,\widetilde{v}'_2)\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))}\le c_\kappa r_\varepsilon^2\|(\widetilde{v}_1,\widetilde{v}_2)-(\widetilde{v}'_1,\widetilde{v}'_2)\|_{(\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}})))^2},$$

provided $(\widetilde{v}_1,\widetilde{v}_2,\widetilde{v}'_1,\widetilde{v}'_2)\in(\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}})))^4$ satisfy

$$\|(\widetilde{v}_1,\widetilde{v}_2)\|_{(\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}})))^2}\le 2c_\kappa r_\varepsilon^2\quad\text{and}\quad\|(\widetilde{v}'_1,\widetilde{v}'_2)\|_{(\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}})))^2}\le 2c_\kappa r_\varepsilon^2.\tag{3.30}$$

Proof. As for the interior problem, the proof of the two first estimates follows from the asymptotic behavior of H^{ext} together with the assumption on the norm of boundary data $\widetilde{\varphi}^i_j$ and $\widetilde{\psi}^i_j$ given by (3.28). Indeed, let c_κ be a constant depending only on κ, by Lemma 3.7,

$$\left|H^{\text{ext}}\left(\widetilde{\varphi}^i_j,\widetilde{\psi}^i_j;\frac{x-\widetilde{x}^i}{r_\varepsilon}\right)\right|\le c_\kappa r_\varepsilon^3 r^{-1}.\tag{3.31}$$

On the other hand,

$$\widetilde{S}_1(0,0)=\rho^4e^{\gamma\widetilde{\boldsymbol{w}}_1+(1-\gamma)\widetilde{\boldsymbol{w}}_2}-\Delta^2\widetilde{\boldsymbol{w}}_1\quad\text{and}\quad\widetilde{S}_2(0,0)=\rho^4e^{\xi\widetilde{\boldsymbol{w}}_2+(1-\xi)\widetilde{\boldsymbol{w}}_1}-\Delta^2\widetilde{\boldsymbol{w}}_2.$$

We will estimate $\widetilde{S}_1(0,0)$ in different subregions of $\overline{\Omega}^*(\widetilde{\boldsymbol{x}})$.

• In $B_{r_0/2}(\widetilde{x}^1) - B_{r_\varepsilon}(\widetilde{x}^1)$, we have $\chi_{r_0}(x-\widetilde{x}^1) = 1$, $\chi_{r_0}(x-\widetilde{x}^2) = 0$, $\chi_{r_0}(x-\widetilde{x}^3) = 0$ and $\Delta^2 \widetilde{\boldsymbol{w}}_1 = 0$, so that $|\widetilde{S}_1(0,0)| = \rho^4 e^{\gamma \widetilde{w}_1 + (1-\gamma)\widetilde{w}_2}$. Then

$$|\widetilde{S}_1(0,0)| \le c_\kappa \varepsilon^4 |x-\widetilde{x}^1|^{-8(1+\lambda_1)} \le c_\kappa \varepsilon^4 r^{-8(1+\lambda_1)}.$$

Hence, for $\nu \in (-1,0)$ and λ_1 small enough, we get

$$\|\widetilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{r_0/2}(\widetilde{x}^1))} \le \sup_{r_\varepsilon \le r \le r_0/2} r^{4-\nu} |\widetilde{S}_1(0,0)| \le c_\kappa \varepsilon^4 r_\varepsilon^{-4}.$$

• In $B_{r_0}(\widetilde{x}^1) - B_{r_0/2}(\widetilde{x}^1)$, using the estimate (3.31), then we have

$$\begin{aligned} |\widetilde{S}_1(0,0)| &\le c_\kappa \varepsilon^4 r^{-8(1+\lambda_1)} + \left| [\Delta^2, \chi_{r_0}(x-\widetilde{x}^1)] H^{\text{ext}} \left(\widetilde{\varphi}_1^1, \widetilde{\psi}_1^1; \frac{x-\widetilde{x}^1}{r_\varepsilon} \right) \right| \\ &\le c_\kappa \big(\varepsilon^4 r^{-8(1+\lambda_1)} + r^{-1} r_\varepsilon^3 \big), \end{aligned}$$

where

$$\begin{aligned} [\Delta^2, \chi_{r_0}] w = {} & w \Delta^2 \chi_{r_0} + 2\Delta w \Delta \chi_{r_0} + 4\nabla(\Delta w) \cdot \nabla \chi_{r_0} \\ & + 4\nabla w \cdot \nabla(\Delta \chi_{r_0}) + 4 \sum_{i,j=1}^{4} \frac{\partial^2 \chi_{r_0}}{\partial x_i \partial x_j} \frac{\partial^2 w}{\partial x_i \partial x_j}. \end{aligned}$$

Hence, for $\nu \in (-1,0)$ and λ_1 small enough, we get

$$\|\widetilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{r_0}(\widetilde{x}^1) - B_{r_0/2}(\widetilde{x}^1))} \le \sup_{r_0/2 \le r \le r_0} r^{4-\nu} |\widetilde{S}_1(0,0)| \le c_\kappa r_\varepsilon^2.$$

• In $B_{r_0/2}(\widetilde{x}^2) - B_{r_\varepsilon}(\widetilde{x}^2)$, we have $\chi_{r_0}(x-\widetilde{x}^1) = 0$, $\chi_{r_0}(x-\widetilde{x}^2) = 1$, $\chi_{r_0}(x-\widetilde{x}^3) = 0$ and $\Delta^2 \widetilde{\boldsymbol{w}}_1 = 0$, so that $\widetilde{S}_1(0,0) = \rho^4 e^{\gamma \widetilde{\boldsymbol{w}}_1 + (1-\gamma)\widetilde{\boldsymbol{w}}_2}$. Then

$$|\widetilde{S}_1(0,0)| \le c_\kappa \varepsilon^4 |x-\widetilde{x}^2|^{-8(1+\lambda_2)} \le c_\kappa \varepsilon^4 r^{-8(1+\lambda_2)}.$$

Hence, for $\nu \in (-1,0)$ and λ_2 small enough, we get

$$\|\widetilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{r_0}(\widetilde{x}^2))} \le \sup_{r_\varepsilon \le r \le r_0/2} r^{4-\nu} |\widetilde{S}_1(0,0)| \le c_\kappa r_\varepsilon^2.$$

• In $B_{r_0}(\widetilde{x}^2) - B_{r_0/2}(\widetilde{x}^2)$, using the estimate (3.31), there holds

$$\begin{aligned} |\widetilde{S}_1(0,0)| &\le c_\kappa \varepsilon^4 r^{-8(1+\lambda_2)} + \left| [\Delta^2, \chi_{r_0}(x-\widetilde{x}^2)] H^{\text{ext}} \left(\widetilde{\varphi}_1^2, \widetilde{\psi}_1^2; \frac{x-\widetilde{x}^2}{r_\varepsilon} \right) \right| \\ &\le c_\kappa \big(\varepsilon^4 r^{-8(1+\lambda_2)} + r^{-1} r_\varepsilon^3 \big). \end{aligned}$$

Hence, for $\nu \in (-1,0)$ and λ_2 small enough, we get

$$\|\widetilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{r_0}(\widetilde{x}^2))} \le \sup_{r_0/2 \le r \le r_0} r^{4-\nu} |\widetilde{S}_1(0,0)| \le c_\kappa r_\varepsilon^2.$$

Similarly, for $\nu \in (-1,0)$ and λ_3 small enough, we can prove the same result for $\widetilde{x}^3$.

• In $\Omega - \big(B_{r_0}(\widetilde{x}^1) \cup B_{r_0}(\widetilde{x}^2) \cup B_{r_0}(\widetilde{x}^3)\big)$, we have $\chi_{r_0}(x-\widetilde{x}^1) = 0$, $\chi_{r_0}(x-\widetilde{x}^2) = 0$, $\chi_{r_0}(x-\widetilde{x}^3) = 0$ and $\Delta^2 \widetilde{\boldsymbol{w}}_1 = 0$. So for $\nu \in (-1,0)$, we have

$$\|\widetilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(\overline{\Omega} - \bigcup_{i=1}^3 B_{r_0}(\widetilde{x}^i))} \le \sup_{r \ge r_0} r^{4-\nu} |\widetilde{S}_1(0,0)| \le c_\kappa \varepsilon^4.$$

We conclude that

$$\|\widetilde{S}_1(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(\overline{\Omega}_{r_0}(\widetilde{\boldsymbol{x}}))} \le c_\kappa r_\varepsilon^4.$$

Now, we are interested in the second equation of the previous system.

• In $B_{r_0/2}(\widetilde{x}^1) - B_{r_\varepsilon}(\widetilde{x}^1)$, we have $\chi_{r_0}(x-\widetilde{x}^1) = 1$, $\chi_{r_0}(x-\widetilde{x}^2) = 0$, $\chi_{r_0}(x-\widetilde{x}^3) = 0$ and $\Delta^2 \widetilde{\boldsymbol{w}}_1 = 0$, so that $|\widetilde{S}_2(0,0)| = \rho^4 e^{\xi \widetilde{w}_2 + (1-\xi)\widetilde{w}_1}$. Then

$$|\widetilde{S}_2(0,0)| \le c_\kappa \varepsilon^4 |x-\widetilde{x}^1|^{-8\frac{(1-\xi)(1+\lambda_1)}{\gamma}} \le c_\kappa \varepsilon^4 r^{-8\frac{(1-\xi)(1+\lambda_1)}{\gamma}}.$$

Hence, for $\nu \in (-1,0)$ and λ_1 small enough, we get

$$\|\widetilde{S}_2(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{r_0/2}(\widetilde{x}^1))} \le \sup_{r_\varepsilon \le r \le r_0/2} r^{4-\nu} |\widetilde{S}_2(0,0)| \le c_\kappa r_\varepsilon^2.$$

• In $B_{r_0}(\widetilde{x}^1) - B_{r_0/2}(\widetilde{x}^1)$, using the estimate (3.31), then we have

$$\begin{aligned} |\widetilde{S}_2(0,0)| &\le c_\kappa \varepsilon^4 r^{-8\frac{(1-\xi)(1+\lambda_1)}{\gamma}} + \left| [\Delta^2, \chi_{r_0}(x-\widetilde{x}^1)] H^{\text{ext}}\left(\widetilde{\varphi}_2^1, \widetilde{\psi}_2^1; \frac{x-\widetilde{x}^1}{r_\varepsilon}\right) \right| \\ &\le c_\kappa \big(\varepsilon^4 r^{-8\frac{(1-\xi)(1+\lambda_1)}{\gamma}} + r^{-1} r_\varepsilon^3\big), \end{aligned}$$

where

$$\begin{aligned} [\Delta^2, \chi_{r_0}] w &= w \Delta^2 \chi_{r_0} + 2\Delta w \Delta \chi_{r_0} + 4\nabla(\Delta w) \cdot \nabla \chi_{r_0} \\ &\quad + 4\nabla w \cdot \nabla(\Delta \chi_{r_0}) + 4 \sum_{i,j=1}^4 \frac{\partial^2 \chi_{r_0}}{\partial x_i \partial x_j} \frac{\partial^2 w}{\partial x_i \partial x_j}. \end{aligned}$$

Hence, for $\nu \in (-1,0)$ and λ_1 small enough, we get

$$\|\widetilde{S}_2(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{r_0}(\widetilde{x}^1) - B_{r_0/2}(\widetilde{x}^1))} \le \sup_{r_0/2 \le r \le r_0} r^{4-\nu} |\widetilde{S}_2(0,0)| \le c_\kappa r_\varepsilon^2.$$

• In $B_{r_0/2}(\widetilde{x}^2) - B_{r_\varepsilon}(\widetilde{x}^2)$, we have $\chi_{r_0}(x-\widetilde{x}^1) = 0$, $\chi_{r_0}(x-\widetilde{x}^2) = 1$, $\chi_{r_0}(x-\widetilde{x}^3) = 0$ and $\Delta^2 \widetilde{\boldsymbol{w}}_2 = 0$, so that $\widetilde{S}_2(0,0) = \rho^4 e^{\xi \widetilde{\boldsymbol{w}}_2 + (1-\xi)\widetilde{\boldsymbol{w}}_1}$. Then

$$|\widetilde{S}_2(0,0)| \le c_\kappa \varepsilon^4 |x-\widetilde{x}^2|^{-8(1+\lambda_2)} \le c_\kappa \varepsilon^4 r^{-8(1+\lambda_2)}.$$

Hence, for $\nu \in (-1,0)$ and λ_2 small enough, we get

$$\|\widetilde{S}_2(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{r_0}(\widetilde{x}^2))} \le \sup_{r_\varepsilon \le r \le r_0/2} r^{4-\nu} |\widetilde{S}_2(0,0)| \le c_\kappa r_\varepsilon^2.$$

• In $B_{r_0}(\widetilde{x}^2) - B_{r_0/2}(\widetilde{x}^2)$, using the estimate (3.31), there holds

$$\begin{aligned}|\widetilde{S}_2(0,0)| &\le c_\kappa \varepsilon^4 r^{-8(1+\lambda_2)} + \left|[\Delta^2, \chi_{r_0}(x-\widetilde{x}^2)]H^{\text{ext}}\left(\widetilde{\varphi}_2^2, \widetilde{\psi}_2^2; \frac{x-\widetilde{x}^2}{r_\varepsilon}\right)\right| \\ &\le c_\kappa\big(\varepsilon^4 r^{-8(1+\lambda_2)} + r^{-1} r_\varepsilon^3\big).\end{aligned}$$

Hence, for $\nu \in (-1,0)$ and λ_2 small enough, we get

$$\|\widetilde{S}_2(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(B_{r_0}(\widetilde{x}^2))} \le \sup_{r_0/2 \le r \le r_0} r^{4-\nu}|\widetilde{S}_2(0,0)| \le c_\kappa r_\varepsilon^2.$$

Similarly, for $\nu \in (-1,0)$ and λ_3 small enough, we can prove the same result for $\widetilde{x}^3$.

• In $\Omega - \big(B_{r_0}(\widetilde{x}^1) \cup B_{r_0}(\widetilde{x}^2) \cup B_{r_0}(\widetilde{x}^3)\big)$, we have $\chi_{r_0}(x-\widetilde{x}^1) = 0$, $\chi_{r_0}(x-\widetilde{x}^2) = 0$, $\chi_{r_0}(x-\widetilde{x}^3) = 0$ and $\Delta^2 \widetilde{\boldsymbol{w}}_1 = 0$. So for $\nu \in (-1,0)$, we have

$$\|\widetilde{S}_2(0,0)\|_{\mathcal{C}^{0,\alpha}_{\nu-4}(\overline{\Omega} - \bigcup_{i=1}^3 B_{r_0}(\widetilde{x}^i))} \le \sup_{r \ge r_0} r^{4-\nu}|\widetilde{S}_2(0,0)| \le c_\kappa \varepsilon^4.$$

Making use of Proposition 3.5 together with (3.26) we conclude that

$$\|\widetilde{\mathcal{N}}(0,0)\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))} \le c_\kappa r_\varepsilon^2 \quad \text{and} \quad \|\widetilde{\mathcal{M}}(0,0)\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))} \le c_\kappa r_\varepsilon^2.$$

For the proof of the third estimate, let $\widetilde{v}_1$, $\widetilde{v}_2$, $\widetilde{v}_1'$ and $\widetilde{v}_2' \in \mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*)$ satisfy (3.30), we have

$$\begin{aligned}\big|\widetilde{S}_1(\widetilde{v}_1,\widetilde{v}_2) - \widetilde{S}_1(\widetilde{v}_1',\widetilde{v}_2')\big| &\le c_\kappa \varepsilon^4 e^{\gamma\widetilde{w}_1 + (1-\gamma)\widetilde{w}_2}\big|e^{\gamma\widetilde{v}_1+(1-\gamma)\widetilde{v}_2} - e^{\gamma\widetilde{v}_1'+(1-\gamma)\widetilde{v}_2'}\big| \\ &\le c_\kappa\varepsilon^4\big(\gamma|\widetilde{v}_1 - \widetilde{v}_1'| + (1-\gamma)|\widetilde{v}_2 - \widetilde{v}_2'|\big).\end{aligned}$$

So, for λ_i, $i = 1,2,3$, small enough and using the estimate (3.26), there exists c_κ (depending on κ) such that

$$\|\widetilde{\mathcal{N}}(\widetilde{v}_1,\widetilde{v}_2) - \widetilde{\mathcal{N}}(\widetilde{v}_1',\widetilde{v}_2')\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))} \le c_\kappa r_\varepsilon^2\big(\|\widetilde{v}_1 - \widetilde{v}_1'\|_{C^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))} + \|\widetilde{v}_2 - \widetilde{v}_2'\|_{C^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))}\big). \tag{3.32}$$

Similarly we can use the same arguments to prove

$$\|\widetilde{\mathcal{M}}(\widetilde{v}_1,\widetilde{v}_2) - \widetilde{\mathcal{M}}(\widetilde{v}_1',\widetilde{v}_2')\|_{\mathcal{C}^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))} \le c_\kappa r_\varepsilon^2\big(\|\widetilde{v}_1 - \widetilde{v}_1'\|_{C^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))} + \|\widetilde{v}_2 - \widetilde{v}_2'\|_{C^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))}\big). \tag{3.33}$$

□

Reducing ε_κ if necessary, we can assume that $c_\kappa r_\varepsilon^2 \le 1/2$ for all $\varepsilon \in (0, \varepsilon_\kappa)$. Then, (3.32) and (3.33) are enough to show that

$$(\widetilde{v}_1,\widetilde{v}_2) \mapsto \big(\widetilde{\mathcal{N}}(\widetilde{v}_1,\widetilde{v}_2), \widetilde{\mathcal{M}}(\widetilde{v}_1,\widetilde{v}_2)\big)$$

is a contraction from the ball

$$\big\{(\widetilde{v}_1,\widetilde{v}_2) \in (C^{4,\alpha}_\nu(\mathbb{R}^4))^2 : \|(\widetilde{v}_1,\widetilde{v}_2)\|_{(C^{4,\alpha}_\nu(\mathbb{R}^4))^2} \le 2c_\kappa r_\varepsilon^2\big\}$$

into itself. Hence there exists a unique fixed point $(\widetilde{v}_1,\widetilde{v}_2)$ in this set, which is a solution of (3.27). Applying a fixed point Theorem for contraction mappings, we conclude that

Proposition 3.15. *Given $\kappa > 0$, there exists $\varepsilon_\kappa > 0$ (depending on κ) such that for any $\varepsilon \in (0, \varepsilon_\kappa)$, λ_i and $\widetilde{x}^i$ satisfying* (3.29) *and functions $\widetilde{\varphi}^i_j$ and $\widetilde{\psi}^i_j$ satisfying* (3.6) *and* (3.28)*, there exists a unique $(\widetilde{v}_1, \widetilde{v}_2)$ $(:= (\widetilde{v}_{1,\varepsilon,\lambda_1,\lambda_2,\widetilde{\boldsymbol{x}},\widetilde{\varphi}^i_1,\widetilde{\psi}^i_1}, \widetilde{v}_{2,\varepsilon,\lambda_2,\lambda_3,\widetilde{\boldsymbol{x}},\widetilde{\varphi}^i_2,\widetilde{\psi}^i_2}))$ solution of* (3.27) *so that for v_k $(k = 1, 2)$ defined by*

$$v_1(x) := \frac{1+\lambda_1}{\gamma} G(x, \widetilde{x}^1) + (1+\lambda_2) G(x, \widetilde{x}^2) + \sum_{i=1}^{3} \chi_{r_0}(x - \widetilde{x}^i) H^{\text{ext}}\left(\widetilde{\varphi}^i_1, \widetilde{\psi}^i_1; \frac{x - \widetilde{x}^i}{r_\varepsilon}\right) + \widetilde{v}_1(x),$$

$$v_2(x) := \frac{1+\lambda_3}{\xi} G(x, \widetilde{x}^3) + (1+\lambda_2) G(x, \widetilde{x}^2) + \sum_{i=1}^{3} \chi_{r_0}(x - \widetilde{x}^i) H^{\text{ext}}\left(\widetilde{\varphi}^i_2, \widetilde{\psi}^i_2; \frac{x - \widetilde{x}^i}{r_\varepsilon}\right) + \widetilde{v}_2(x)$$

solves (3.24) *in $\overline{\Omega}_{r_\varepsilon}(\widetilde{\boldsymbol{x}})$. In addition, we have*

$$\|(\widetilde{v}_1, \widetilde{v}_2)\|_{C^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))} \le 2c_\kappa r_\varepsilon^2.$$

3.4. The nonlinear Cauchy-data matching

We will gather the results of the previous sections. Using the previous notations, assume that $\widetilde{\boldsymbol{x}} := (\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3) \in \Omega^3$ are given close to $\boldsymbol{x} := (x^1, x^2, x^3)$. Assume also that

$$\boldsymbol{\tau} := (\tau_1, \tau_2, \tau_3) \in [\tau_1^-, \tau_1^+] \times [\tau_2^-, \tau_2^+] \times [\tau_3^-, \tau_3^+] \subset (0, \infty)^3$$

are given (the values of τ_l^- and τ_l^+, for $l = 1, 2, 3$ will be fixed later). First, we consider some set of boundary data $\boldsymbol{\varphi}^i := (\varphi^i_1, \varphi^i_2) \in (\mathcal{C}^{4,\alpha}(S^3))^2$ and $\boldsymbol{\psi}^i := (\psi^i_1, \psi^i_2) \in (\mathcal{C}^{2,\alpha}(S^3))^2$. Let $\varepsilon \in (0, \varepsilon_\kappa)$ and according to the result of Propositions 3.10, 3.11 and 3.13, we can find, $u_{\text{int}} := (u_{\text{int},1}, u_{\text{int},2})$ a solution of (3.7) in $B_{r_\varepsilon}(\widetilde{x}^1) \cup B_{r_\varepsilon}(\widetilde{x}^2) \cup B_{r_\varepsilon}(\widetilde{x}^3)$, which can be decomposed as

$$u_{\text{int},1}(x) := \begin{cases} \frac{1}{\gamma} u_{\varepsilon,\tau_1}(x - \widetilde{x}^1) - \frac{1-\gamma}{\gamma} G(x, \widetilde{x}^2) - \frac{1-\gamma}{\gamma\xi} G(x, \widetilde{x}^3) \\ \quad - \frac{\ln\gamma}{\gamma} + H^{\text{int}}_{\varphi^1_1,\psi^1_1}\left(\frac{x-\widetilde{x}^1}{r_\varepsilon}\right) + h^1_1\left(\frac{R^1_\varepsilon(x-\widetilde{x}^1)}{r_\varepsilon}\right) & \text{in } B_{r_\varepsilon}(\widetilde{x}^1), \\ u_{\varepsilon,\tau_2}(x - \widetilde{x}^2) + H^{\text{int}}_{\varphi^2_1,\psi^2_1}\left(\frac{x-\widetilde{x}^2}{r_\varepsilon}\right) + h^2_1\left(\frac{R^2_\varepsilon(x-\widetilde{x}^2)}{r_\varepsilon}\right) & \text{in } B_{r_\varepsilon}(\widetilde{x}^2), \\ \frac{1}{\gamma} G(x, \widetilde{x}^1) + G(x, \widetilde{x}^2) + H^{\text{int}}_{\varphi^3_1,\psi^3_1}\left(\frac{x-\widetilde{x}^3}{r_\varepsilon}\right) + h^3_1\left(\frac{R^3_\varepsilon(x-\widetilde{x}^3)}{r_\varepsilon}\right) & \text{in } B_{r_\varepsilon}(\widetilde{x}^3) \end{cases}$$

and

$$u_{\text{int},2}(x) := \begin{cases} \frac{1}{\xi} G(x, \widetilde{x}^3) + G(x, \widetilde{x}^2) + H^{\text{int}}_{\varphi^1_2,\psi^1_2}\left(\frac{x-\widetilde{x}^1}{r_\varepsilon}\right) + h^1_2\left(\frac{R^1_\varepsilon(x-\widetilde{x}^1)}{r_\varepsilon}\right) & \text{in } B_{r_\varepsilon}(\widetilde{x}^1), \\ u_{\varepsilon,\tau_2}(x - \widetilde{x}^2) + H^{\text{int}}_{\varphi^2_2,\psi^2_2}\left(\frac{x-\widetilde{x}^2}{r_\varepsilon}\right) + h^2_2\left(\frac{R^2_\varepsilon(x-\widetilde{x}^2)}{r_\varepsilon}\right) & \text{in } B_{r_\varepsilon}(\widetilde{x}^2), \\ \frac{1}{\xi} u_{\varepsilon,\tau_3}(x - \widetilde{x}^3) - \frac{1-\xi}{\xi} G(x, \widetilde{x}^2) - \frac{1-\xi}{\gamma\xi} G(x, \widetilde{x}^1) \\ \quad - \frac{\ln\xi}{\xi} + H^{\text{int}}_{\varphi^3_2,\psi^3_2}\left(\frac{x-\widetilde{x}^3}{r_\varepsilon}\right) + h^3_2\left(\frac{R^3_\varepsilon(x-\widetilde{x}^3)}{r_\varepsilon}\right) & \text{in } B_{r_\varepsilon}(\widetilde{x}^3), \end{cases}$$

where for $i \in \{1, 2, 3\}$ and $j \in \{1, 2\}$, $R^i_\varepsilon = \tau_i \frac{r_\varepsilon}{\varepsilon}$ and the functions h^i_j satisfy

$$\|(h^1_1, h^1_2)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4) \times \mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2, \quad \|(h^2_1, h^2_2)\|_{(\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4))^2} \le 2c_\kappa r_\varepsilon^2$$

and

$$\|(h_1^3, h_2^3)\|_{\mathcal{C}_\delta^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \leq 2c_\kappa r_\varepsilon^2.$$

Similarly, given some boundary data $\widetilde{\varphi}_j^i \in C^{4,\alpha}(S^3)$, $\widetilde{\psi}_j^i \in C^{2,\alpha}(S^3)$ satisfying (3.6), $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$ satisfying (3.29), provided $\varepsilon \in (0, \varepsilon_\kappa)$, by Proposition 3.15, we find a solution $u_{\mathrm{ext}} := (u_{\mathrm{ext},1}, u_{\mathrm{ext},2})$ of (3.7) in $\overline{\Omega} \setminus (B_{r_\varepsilon}(\widetilde{x}^1) \cup B_{r_\varepsilon}(\widetilde{x}^2)) \cup B_{r_\varepsilon}(\widetilde{x}^3))$ which can be decomposed as

$$\begin{aligned} u_{\mathrm{ext},1}(x) &:= \frac{1+\lambda_1}{\gamma} G(x, \widetilde{x}^1) + (1+\lambda_2) G(x, \widetilde{x}^2) \\ &\quad + \sum_{i=1}^{3} \chi_{r_0}(x - \widetilde{x}^i) H^{\mathrm{ext}}\left(\widetilde{\varphi}_1^i, \widetilde{\psi}_1^i; \frac{x-\widetilde{x}^i}{r_\varepsilon}\right) + \widetilde{v}_1(x), \\ u_{\mathrm{ext},2}(x) &:= \frac{1+\lambda_3}{\xi} G(x, \widetilde{x}^3) + (1+\lambda_2) G(x, \widetilde{x}^2) \\ &\quad + \sum_{i=1}^{3} \chi_{r_0}(x - \widetilde{x}^i) H^{\mathrm{ext}}\left(\widetilde{\varphi}_2^i, \widetilde{\psi}_2^i; \frac{x-\widetilde{x}^i}{r_\varepsilon}\right) + \widetilde{v}_2(x), \end{aligned}$$

with $\widetilde{v}_1, \widetilde{v}_2 \in C_\nu^{4,\alpha}(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))$ satisfying

$$\|(\widetilde{v}_1, \widetilde{v}_2)\|_{(C_\nu^{4,\alpha}(\overline{\Omega}^*(\widetilde{\boldsymbol{x}})))^2} \leq 2c_\kappa r_\varepsilon^2.$$

It remains to determine the parameters and the boundary data in such a way that the function equal to u_{int} in $B_{r_\varepsilon}(\widetilde{x}^1) \cup B_{r_\varepsilon}(\widetilde{x}^2) \cup B_{r_\varepsilon}(\widetilde{x}^3)$ and equal to u_{ext} in $\overline{\Omega}_{r_\varepsilon}(\widetilde{\boldsymbol{x}})$ is a smooth function. This amounts to find the boundary data and the parameters so that, for each $j = 1, 2$,

$$\begin{aligned} u_{\mathrm{int},j} &= u_{\mathrm{ext},j}, & \partial_r u_{\mathrm{int},j} &= \partial_r u_{\mathrm{ext},j}, \\ \Delta u_{\mathrm{int},j} &= \Delta u_{\mathrm{ext},j}, & \partial_r \Delta u_{\mathrm{int},j} &= \partial_r \Delta u_{\mathrm{ext},j} \end{aligned} \tag{3.34}$$

on $\partial B_{r_\varepsilon}(\widetilde{x}^1)$, $\partial B_{r_\varepsilon}(\widetilde{x}^2)$ and $\partial B_{r_\varepsilon}(\widetilde{x}^3)$.

Suppose that (3.34) is verified, this provides that for each ε small enough $u_\varepsilon \in \mathcal{C}^{4,\alpha}$ (which is obtained by patching together the functions u_{int} and the function u_{ext}), a weak solution of our system and elliptic regularity theory implies that this solution is in fact smooth. That will complete the proof since, as ε tends to 0, the sequence of solutions we have obtain satisfies the required singular limit behaviors.

Before we proceed, the following remarks are due. First it will be convenient to observe that the function u_{ε,τ_i} can be expanded as

$$u_{\varepsilon,\tau_i}(x) = -4\ln\tau_i - 8\ln|x| + \mathcal{O}\left(\frac{\varepsilon^2\tau_i^{-2}}{|x|^2}\right) \quad \text{on } \partial B_{r_\varepsilon}(0).$$

• On $\partial B_{r_\varepsilon}(\widetilde{x}^1)$, we have

$$
\begin{aligned}
&(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(x) \\
&= -\frac{4}{\gamma}\ln\tau_1 + \frac{8\lambda_1}{\gamma}\ln|x-\widetilde{x}^1| - \frac{1-\gamma}{\gamma\xi}G(x,\widetilde{x}^3) - \frac{\ln\gamma}{\gamma} \\
&\quad + h_1^1\left(R_\varepsilon^1 \frac{x-\widetilde{x}^1}{r_\varepsilon}\right) + H^{\mathrm{int}}\left(\varphi_1^1,\psi_1^1;\frac{x-\widetilde{x}^1}{r_\varepsilon}\right) - H^{\mathrm{ext}}\left(\widetilde{\varphi}_1^1,\widetilde{\psi}_1^1;\frac{x-\widetilde{x}^1}{r_\varepsilon}\right) \\
&\quad - \frac{1+\lambda_1}{\gamma}H(x,\widetilde{x}^1) - \left(1+\lambda_2+\frac{1-\gamma}{\gamma}\right)G(x,\widetilde{x}^2) + \mathcal{O}\left(\frac{\varepsilon^2\tau_1^{-2}}{|x-\widetilde{x}^1|^2}\right) + \mathcal{O}(r_\varepsilon^2).
\end{aligned}
\tag{3.35}
$$

Next, even though all functions are defined on $\partial B_{r_\varepsilon}(\widetilde{x}^1)$ in (3.34), it will be more convenient to solve on S^3 the following set of equations

$$
\begin{aligned}
(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) &= 0, & \partial_r(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) &= 0, \\
\Delta(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) &= 0, & \partial_r\Delta(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) &= 0.
\end{aligned}
\tag{3.36}
$$

Since the boundary data are chosen to satisfy (3.5) or (3.6). We decompose

$$
\begin{aligned}
\varphi_1^1 &= \varphi_{1,0}^1 + \varphi_{1,1}^1 + \varphi_1^{1,\perp}, & \psi_1^1 &= 8\varphi_{1,0}^1 + 12\varphi_{1,1}^1 + \psi_1^{1,\perp}, \\
\widetilde{\varphi}_1^1 &= \widetilde{\varphi}_{1,0}^1 + \widetilde{\varphi}_{1,1}^1 + \widetilde{\varphi}_1^{1,\perp}, & \widetilde{\psi}_1^1 &= \widetilde{\psi}_{1,1}^1 + \widetilde{\psi}_1^{1,\perp},
\end{aligned}
$$

where $\varphi_{1,0}^1, \widetilde{\varphi}_{1,0}^1 \in \mathbb{E}_0 = \mathbb{R}$ are constant on S^3, $\varphi_{1,1}^1$, $\widetilde{\varphi}_{1,1}^1$, $\widetilde{\psi}_{1,1}^1$ belong to $\mathbb{E}_1 = \mathrm{Span}\{e_1, e_2, e_3, e_4\}$ and $\varphi_1^{1,\perp}$, $\widetilde{\varphi}_1^{1,\perp}$, $\psi_1^{1,\perp}$, $\widetilde{\psi}_1^{1,\perp}$ are $L^2(S^3)$ orthogonal to $\mathbb{E}_0$ and $\mathbb{E}_1$.

Using (3.35), we have for $x \in S^3$,

$$
\begin{aligned}
&(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{x}^1 + r_\varepsilon x) \\
&= -\frac{4}{\gamma}\ln\tau_1 + \frac{8\lambda_1}{\gamma}\ln(r_\varepsilon|x|) - \frac{1}{\gamma}\left(H(\widetilde{x}^1,\widetilde{x}^1) + G(\widetilde{x}^1,\widetilde{x}^2) + \frac{1-\gamma}{\xi}G(\widetilde{x}^1,\widetilde{x}^3)\right) \\
&\quad + H^{\mathrm{int}}(\varphi_1^1,\psi_1^1;x) - H^{\mathrm{ext}}(\widetilde{\varphi}_1^1,\widetilde{\psi}_1^1;x) - \frac{\ln\gamma}{\gamma} \\
&\quad - \frac{\lambda_1}{\gamma}H(\widetilde{x}^1,\widetilde{x}^1) - \lambda_2 G(\widetilde{x}^1,\widetilde{x}^2) + \mathcal{O}(r_\varepsilon^2).
\end{aligned}
$$

Then, the projection of the equations (3.36) over $\mathbb{E}_0$ will yield

$$
\begin{aligned}
-4\ln\tau_1 + 8\lambda_1\ln r_\varepsilon - \ln\gamma + \gamma\varphi_{1,0}^1 - \gamma\widetilde{\varphi}_{1,0}^1 - \mathcal{E}_1(\widetilde{x}^1,\widetilde{\boldsymbol{x}}) + \mathcal{O}(r_\varepsilon^2) &= 0, \\
8\lambda_1 + 2\gamma\varphi_{1,0}^1 + 2\gamma\widetilde{\varphi}_{1,0}^1 + \mathcal{O}(r_\varepsilon^2) &= 0, \\
16\lambda_1 + 8\gamma\varphi_{1,0}^1 + \mathcal{O}(r_\varepsilon^2) &= 0, \\
-32\lambda_1 + \mathcal{O}(r_\varepsilon^2) &= 0,
\end{aligned}
\tag{3.37}
$$

where

$$
\mathcal{E}_1(\,\cdot\,,\widetilde{\boldsymbol{x}}) := H(\,\cdot\,,\widetilde{x}^1) + G(\,\cdot\,,\widetilde{x}^2) + \frac{1-\gamma}{\xi}G(\,\cdot\,,\widetilde{x}^3).
$$

The system (3.37) can be simply written as

$$\lambda_1 = \mathcal{O}(r_\varepsilon^2), \quad \varphi_{1,0}^1 = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_{1,0}^1 = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \frac{1}{\ln r_\varepsilon}\big[4\ln\tau_1 + \ln\gamma + \mathcal{E}_1(\widetilde{x}^1, \widetilde{\boldsymbol{x}})\big] = \mathcal{O}(r_\varepsilon^2).$$

We are now in a position to define τ_1^- and τ_1^+. In fact, according to the above analysis, as ε tends to 0, we expect that $\widetilde{x}^i$ will converge to x^i for $i \in \{1,2,3\}$ and τ_1 will converge to τ_1^* satisfying

$$4\ln\tau_1^* = -\ln\gamma - \mathcal{E}_1(x^1, \boldsymbol{x}).$$

Hence it is enough to choose τ_1^- and τ_1^+ in such a way that

$$4\ln(\tau_1^-) < -\ln\gamma - \mathcal{E}_1(x^1, \boldsymbol{x}) < 4\ln(\tau_1^+).$$

Consider now the projection of (3.36) over $\mathbb{E}_1$. Given a smooth function f defined in Ω, we identify its gradient $\nabla f = (\partial_{x_1} f, \ldots, \partial_{x_4} f)$ with the element of $\mathbb{E}_1$,

$$\overline{\nabla} f = \sum_{i=1}^{4} \partial_{x_i} f e_i.$$

Keeping these notations in mind, we obtain the system of equations

$$\begin{aligned}
\varphi_{1,1}^1 - \widetilde{\varphi}_{1,1}^1 - \overline{\nabla}\mathcal{E}_1(\widetilde{x}^1, \widetilde{\boldsymbol{x}}) + \mathcal{O}(r_\varepsilon^2) &= 0,\\
3\varphi_{1,1}^1 + 3\widetilde{\varphi}_{1,1}^1 + \frac{1}{2}\widetilde{\psi}_{1,1}^1 - \overline{\nabla}\mathcal{E}_1(\widetilde{x}^1, \widetilde{\boldsymbol{x}}) + \mathcal{O}(r_\varepsilon^2) &= 0,\\
15\varphi_{1,1}^1 - 3\widetilde{\varphi}_{1,1}^1 - \widetilde{\psi}_{1,1}^1 + \mathcal{O}(r_\varepsilon^2) &= 0,\\
15\varphi_{1,1}^1 + 15\widetilde{\varphi}_{1,1}^1 + \frac{18}{4}\widetilde{\psi}_{1,1}^1 + \mathcal{O}(r_\varepsilon^2) &= 0,
\end{aligned}$$

which can be simplified as follows

$$\varphi_{1,1}^1 = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_{1,1}^1 = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\psi}_{1,1}^1 = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \overline{\nabla}\mathcal{E}_1(\widetilde{x}^1, \widetilde{\boldsymbol{x}}) = \mathcal{O}(r_\varepsilon^2).$$

Finally, we consider the projection onto $L^2(S^3)^\perp$. This yields the system

$$\begin{aligned}
&\varphi_1^{1,\perp} - \widetilde{\varphi}_1^{1,\perp} + \mathcal{O}(r_\varepsilon^2) = 0, &&\partial_r\big(H^{\text{int}}_{\varphi_1^{1,\perp},\psi_1^{1,\perp}} - H^{\text{ext}}_{\widetilde{\varphi}_1^{1,\perp},\widetilde{\psi}_1^{1,\perp}}\big) + \mathcal{O}(r_\varepsilon^2) = 0,\\
&\psi_1^{1,\perp} - \widetilde{\psi}_1^{1,\perp} + \mathcal{O}(r_\varepsilon^2) = 0, &&\partial_r\Delta\big(H^{\text{int}}_{\varphi_1^{1,\perp},\psi_1^{1,\perp}} - H^{\text{ext}}_{\widetilde{\varphi}_1^{1,\perp},\widetilde{\psi}_1^{1,\perp}}\big) + \mathcal{O}(r_\varepsilon^2) = 0.
\end{aligned}$$

Thanks to the result of Lemma 3.8, this last system can be rewritten as

$$\varphi_1^{1,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_1^{1,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \psi_1^{1,\perp} = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \widetilde{\psi}_1^{1,\perp} = \mathcal{O}(r_\varepsilon^2).$$

If we define the parameter $t_1 \in \mathbb{R}$ by

$$t_1 = \frac{1}{\ln r_\varepsilon}\big[4\ln\tau_1 + \ln\gamma + \mathcal{E}_1(\widetilde{x}^1, \widetilde{\boldsymbol{x}})\big],$$

then the systems found by projecting (3.36) gather in this equality

$$T^1_{1,\varepsilon} = \big(t_1, \lambda_1, \varphi^1_{1,0}, \widetilde{\varphi}^1_{1,0}, \varphi^1_{1,1}, \widetilde{\varphi}^1_{1,1}, \widetilde{\psi}^1_{1,1}, \overline{\nabla}\mathcal{E}_1(\widetilde{x}^1, \widetilde{\boldsymbol{x}}), \varphi^{1,\perp}_1, \widetilde{\varphi}^{1,\perp}_1, \psi^{1,\perp}_1, \widetilde{\psi}^{1,\perp}_1\big) = \mathcal{O}(r^2_\varepsilon). \tag{3.38}$$

As usual, the terms $\mathcal{O}(r_\varepsilon^2)$ depend nonlinearly on all the variables on the left side, but is bounded (in the appropriate norm) by a constant (independent of ε and κ) times r_ε^2, provided $\varepsilon \in (0, \varepsilon_\kappa)$.

- On $\partial B_{r_\varepsilon}(\widetilde{x}^1)$, we have

$$\begin{aligned}(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(x) = &-\frac{\lambda_3}{\xi} G(x, \widetilde{x}^3) + G(x, \widetilde{x}^2) + h^1_2\left(R^1_\varepsilon \frac{x - \widetilde{x}^1}{r_\varepsilon}\right) + H^{\mathrm{int}}\left(\varphi^1_2, \psi^1_2; \frac{x - \widetilde{x}^1}{r_\varepsilon}\right)\\ &- (1 + \lambda_2) G(x, \widetilde{x}^2) - H^{\mathrm{ext}}\left(\widetilde{\varphi}^1_2, \widetilde{\psi}^1_2; \frac{x - \widetilde{x}^1}{r_\varepsilon}\right) + \mathcal{O}(r^2_\varepsilon).\end{aligned}$$

In the same manner as above, we will solve on S^3 the following system

$$\begin{aligned}(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) &= 0, &\quad \partial_r(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) &= 0,\\ \Delta(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) &= 0, &\quad \partial_r\Delta(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) &= 0.\end{aligned} \tag{3.39}$$

We decompose

$$\begin{aligned}\varphi^1_2 &= \varphi^1_{2,0} + \varphi^1_{2,1} + \varphi^{1,\perp}_2, &\quad \psi^1_2 &= 8\varphi^1_{2,0} + 12\varphi^1_{2,1} + \psi^{1,\perp}_2,\\ \widetilde{\varphi}^1_2 &= \widetilde{\varphi}^1_{2,0} + \widetilde{\varphi}^1_{2,1} + \widetilde{\varphi}^{1,\perp}_2, &\quad \widetilde{\psi}^1_2 &= \widetilde{\psi}^1_{2,1} + \widetilde{\psi}^{1,\perp}_2\end{aligned}$$

with $\varphi^1_{2,0}, \widetilde{\varphi}^1_{2,0} \in \mathbb{E}_0$, $\varphi^1_{2,1}, \widetilde{\varphi}^1_{2,1}, \widetilde{\psi}^1_{2,1} \in \mathbb{E}_1$ and $\varphi^{1,\perp}_2$, $\widetilde{\varphi}^{1,\perp}_2$, $\psi^{1,\perp}_2$, $\widetilde{\psi}^{1,\perp}_2$ belong to $(L^2(S^3))^\perp$.

Projecting the set of equations (3.39) over $\mathbb{E}_0$, we get

$$\varphi^1_{2,0} - \widetilde{\varphi}^1_{2,0} + \mathcal{O}(r^2_\varepsilon) = 0, \quad 2\varphi^1_{2,0} + 2\widetilde{\varphi}^1_{2,0} + \mathcal{O}(r^2_\varepsilon) = 0, \quad 8\varphi^1_{2,0} + \mathcal{O}(r^2_\varepsilon) = 0.$$

From the L^2-projection of (3.39) over $\mathbb{E}_1$, we obtain the system of equations

$$\begin{aligned}\varphi^1_{2,1} - \widetilde{\varphi}^1_{2,1} + \mathcal{O}(r^2_\varepsilon) &= 0, &\quad 3\varphi^1_{2,1} + 3\widetilde{\varphi}^1_{2,1} + \frac{1}{2}\widetilde{\psi}^1_{2,1} + \mathcal{O}(r^2_\varepsilon) &= 0,\\ 15\varphi^1_{2,1} - 3\widetilde{\varphi}^1_{2,1} - \widetilde{\psi}^1_{2,1} + \mathcal{O}(r^2_\varepsilon) &= 0, &\quad 15\varphi^1_{2,1} + 15\widetilde{\varphi}^1_{2,1} + \frac{18}{4}\widetilde{\psi}^1_{2,1} + \mathcal{O}(r^2_\varepsilon) &= 0.\end{aligned}$$

Finally, we consider the L^2-projection onto $(L^2(S^3))^\perp$. This yields the system

$$\begin{aligned}\varphi^{1,\perp}_2 - \widetilde{\varphi}^{1,\perp}_2 + \mathcal{O}(r^2_\varepsilon) &= 0, &\quad \partial_r\big(H^{\mathrm{int}}_{\varphi^{1,\perp}_2, \psi^{1,\perp}_2} - H^{\mathrm{ext}}_{\widetilde{\varphi}^{1,\perp}_2, \widetilde{\psi}^{1,\perp}_2}\big) + \mathcal{O}(r^2_\varepsilon) &= 0,\\ \psi^{1,\perp}_2 - \widetilde{\psi}^{1,\perp}_2 + \mathcal{O}(r^2_\varepsilon) &= 0, &\quad \partial_r\Delta\big(H^{\mathrm{int}}_{\varphi^{1,\perp}_2, \psi^{1,\perp}_2} - H^{\mathrm{ext}}_{\widetilde{\varphi}^{1,\perp}_2, \widetilde{\psi}^{1,\perp}_2}\big) + \mathcal{O}(r^2_\varepsilon) &= 0.\end{aligned}$$

Using again Lemma 3.8, the above system can be rewritten as

$$\varphi^{1,\perp}_2 = \mathcal{O}(r^2_\varepsilon), \quad \widetilde{\varphi}^{1,\perp}_2 = \mathcal{O}(r^2_\varepsilon), \quad \psi^{1,\perp}_2 = \mathcal{O}(r^2_\varepsilon) \quad \text{and} \quad \widetilde{\psi}^{1,\perp}_2 = \mathcal{O}(r^2_\varepsilon).$$

Then the systems found by projecting (3.39) gather in this equality

$$T^1_{2,\varepsilon} = \big(\varphi^1_{2,0}, \widetilde{\varphi}^1_{2,0}, \varphi^1_{2,1}, \widetilde{\varphi}^1_{2,1}, \widetilde{\psi}^1_{2,1}, \varphi_2^{1,\perp}, \widetilde{\varphi}_2^{1,\perp}, \psi_2^{1,\perp}, \widetilde{\psi}_2^{1,\perp}\big) = \mathcal{O}(r_\varepsilon^2). \tag{3.40}$$

• On $\partial B_{r_\varepsilon}(\widetilde{x}^2)$, we have

$$\begin{aligned}
&(1-\xi)(u_{\text{int},1}-u_{\text{ext},1})(x) + (1-\gamma)(u_{\text{int},2}-u_{\text{ext},2})(x)\\
&= -4(2-\gamma-\xi)\ln\tau_2 + 8(2-\gamma-\xi)\lambda_2 \ln|x-\widetilde{x}^2|\\
&\quad + (1-\xi)h_1^2\left(R_\varepsilon^2\frac{x-\widetilde{x}^2}{r_\varepsilon}\right) + (1-\gamma)h_2^2\left(R_\varepsilon^2\frac{x-\widetilde{x}^2}{r_\varepsilon}\right)\\
&\quad + (1-\xi)H^{\text{int}}\left(\varphi_1^2,\psi_1^2;\frac{x-\widetilde{x}^2}{r_\varepsilon}\right) + (1-\gamma)H^{\text{int}}\left(\varphi_2^2,\psi_2^2;\frac{x-\widetilde{x}^2}{r_\varepsilon}\right)\\
&\quad - (1-\xi)H^{\text{ext}}\left(\widetilde{\varphi}_1^2,\widetilde{\psi}_1^2;\frac{x-\widetilde{x}^2}{r_\varepsilon}\right) - (1-\gamma)H^{\text{ext}}\left(\widetilde{\varphi}_2^2,\widetilde{\psi}_2^2;\frac{x-\widetilde{x}^2}{r_\varepsilon}\right)\\
&\quad - \left[(2-\gamma-\xi)H(x,\widetilde{x}^2) + \frac{1-\xi}{\gamma}G(x,\widetilde{x}^1) + \frac{1-\gamma}{\xi}G(x,\widetilde{x}^3)\right] + \mathcal{O}\left(\frac{\varepsilon^2\tau_2^{-2}}{|x-\widetilde{x}^2|^2}\right) + \mathcal{O}(r_\varepsilon^2).
\end{aligned}$$

We denote by

$$\varphi^2 = (1-\xi)\varphi_1^2 + (1-\gamma)\varphi_2^2, \quad \psi^2 = (1-\xi)\psi_1^2 + (1-\gamma)\psi_2^2,$$
$$\widetilde{\varphi}^2 = (1-\xi)\widetilde{\varphi}_1^2 + (1-\gamma)\widetilde{\varphi}_2^2, \quad \widetilde{\psi}^2 = (1-\xi)\widetilde{\psi}_1^2 + (1-\gamma)\widetilde{\psi}_2^2, \quad h^2 = (1-\xi)h_1^2 + (1-\gamma)h_2^2.$$

Then we have

$$\begin{aligned}
&(1-\xi)(u_{\text{int},1}-u_{\text{ext},1})(x) + (1-\gamma)(u_{\text{int},2}-u_{\text{ext},2})(x)\\
&= -4(2-\gamma-\xi)\ln\tau_2 + 8(2-\gamma-\xi)\lambda_2 \ln|x-\widetilde{x}^2|\\
&\quad + h^2\left(R_\varepsilon^2\frac{x-\widetilde{x}^2}{r_\varepsilon}\right) + H^{\text{int}}\left(\varphi^2,\psi^2;\frac{x-\widetilde{x}^2}{r_\varepsilon}\right) - H^{\text{ext}}\left(\widetilde{\varphi}^2,\widetilde{\psi}^2;\frac{x-\widetilde{x}^2}{r_\varepsilon}\right)\\
&\quad - \left[(2-\gamma-\xi)H(x,\widetilde{x}^2) + \frac{1-\xi}{\gamma}G(x,\widetilde{x}^1) + \frac{1-\gamma}{\xi}G(x,\widetilde{x}^3)\right] + \mathcal{O}\left(\frac{\varepsilon^2\tau_2^{-2}}{|x-\widetilde{x}^2|^2}\right) + \mathcal{O}(r_\varepsilon^2).
\end{aligned} \tag{3.41}$$

Next, even though all functions are defined on $\partial B_{r_\varepsilon}(\widetilde{x}^2)$ in (3.34), it will be more convenient to solve on S^3, the following set of equations

$$\begin{aligned}
\big((1-\xi)(u_{\text{int},1}-u_{\text{ext},1}) + (1-\gamma)(u_{\text{int},2}-u_{\text{ext},2})\big)(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) &= 0,\\
\partial_r\big((1-\xi)(u_{\text{int},1}-u_{\text{ext},1}) + (1-\gamma)(u_{\text{int},2}-u_{\text{ext},2})\big)(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) &= 0,\\
\Delta\big((1-\xi)(u_{\text{int},1}-u_{\text{ext},1}) + (1-\gamma)(u_{\text{int},2}-u_{\text{ext},2})\big)(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) &= 0,\\
\partial_r\Delta\big((1-\xi)(u_{\text{int},1}-u_{\text{ext},1}) + (1-\gamma)(u_{\text{int},2}-u_{\text{ext},2})\big)(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) &= 0.
\end{aligned} \tag{3.42}$$

Since the boundary data are chosen to satisfy (3.5) or (3.6). We decompose

$$\varphi^2 = \varphi_0^2 + \varphi_1^2 + \varphi^{2,\perp}, \qquad \psi^2 = 8\varphi_0^2 + 12\varphi_1^2 + \psi^{2,\perp},$$
$$\widetilde{\varphi}^2 = \widetilde{\varphi}_0^2 + \widetilde{\varphi}_1^2 + \widetilde{\varphi}^{2,\perp}, \qquad \widetilde{\psi}^2 = \widetilde{\psi}_1^2 + \widetilde{\psi}^{2,\perp},$$

where $\varphi_0^2, \widetilde{\varphi}_0^2 \in \mathbb{E}_0 = \mathbb{R}$ are constant on S^3, φ_1^2, $\widetilde{\varphi}_1^2$, $\widetilde{\psi}_1^2$ belong to $\mathbb{E}_1 = \operatorname{Span}\{e_1, e_2, e_3, e_4\}$ and $\varphi^{2,\perp}$, $\widetilde{\varphi}^{2,\perp}$, $\psi^{2,\perp}$, $\widetilde{\psi}^{2,\perp}$ are $L^2(S^3)$ orthogonal to $\mathbb{E}_0$ and $\mathbb{E}_1$.

We insist that, for $x \in S^3$, both equations (3.41) involve the same relation of the parameter τ_2 and the appropriate energy $\mathcal{E}_2$. Then we have

$$\begin{aligned}&(1-\xi)(u_{\text{int},1} - u_{\text{ext},1})(x) + (1-\gamma)(u_{\text{int},2} - u_{\text{ext},2})(\widetilde{x}^2 + r_\varepsilon x)\\ &= -4(2-\gamma-\xi)\ln\tau_2 + 8(2-\gamma-\xi)\lambda_2 \ln r_\varepsilon|x| + H^{\text{int}}(\varphi^2,\psi^2;x) - H^{\text{ext}}(\widetilde{\varphi}^2,\widetilde{\psi}^2;x)\\ &\quad - \left[(2-\gamma-\xi)H(\widetilde{x}^2,\widetilde{x}^2) + \frac{1-\xi}{\gamma}G(\widetilde{x}^2,\widetilde{x}^1) + \frac{1-\gamma}{\xi}G(\widetilde{x}^2,\widetilde{x}^3)\right] + \mathcal{O}(r_\varepsilon^2).\end{aligned}$$

Projecting the set of equations (3.42) over $\mathbb{E}_0$, we get

$$\begin{aligned}-4(2-\gamma-\xi)\ln\tau_2 + 8(2-\gamma-\xi)\lambda_2\ln r_\varepsilon + \varphi_0^2 - \widetilde{\varphi}_0^2 - \mathcal{E}_2(\widetilde{x}^2,\widetilde{x}) + \mathcal{O}(r_\varepsilon^2) &= 0,\\ 8(2-\gamma-\xi)\lambda_2 + 2\varphi_0^2 + 2\widetilde{\varphi}_0^2 + \mathcal{O}(r_\varepsilon^2) &= 0,\\ 16(2-\gamma-\xi)\lambda_2 + 8\varphi_0^2 + \mathcal{O}(r_\varepsilon^2) &= 0,\\ -32(2-\gamma-\xi)\lambda_2 + \mathcal{O}(r_\varepsilon^2) &= 0,\end{aligned} \tag{3.43}$$

where

$$\mathcal{E}_2(\,\cdot\,,\widetilde{\boldsymbol{x}}) := (2-\gamma-\xi)H(\,\cdot\,,\widetilde{x}^2) + \frac{1-\xi}{\gamma}G(\,\cdot\,,\widetilde{x}^1) + \frac{1-\gamma}{\xi}G(\,\cdot\,,\widetilde{x}^3).$$

The system (3.43) can be simply written as

$$\lambda_2 = \mathcal{O}(r_\varepsilon^2), \quad \varphi_0^2 = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_0^2 = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \frac{1}{\ln r_\varepsilon}\left[4\ln\tau_2 + \frac{\mathcal{E}_2(\widetilde{x}^2,\widetilde{\boldsymbol{x}})}{2-\gamma-\xi}\right] = \mathcal{O}(r_\varepsilon^2).$$

We are now in a position to define τ_2^- and τ_2^+. In fact, according to the above analysis, as ε tends to 0, we expect that $\widetilde{x}^i$ will converge to x^i for $i \in \{1,2,3\}$ and τ_2 will converge to τ_2^* satisfying

$$4\ln\tau_2^* = -\frac{\mathcal{E}_2(x^2,\boldsymbol{x})}{2-\gamma-\xi}.$$

Hence it is enough to choose τ_2^- and τ_2^+ in such a way that

$$4\ln(\tau_2^-) < -\frac{\mathcal{E}_2(x^2,\boldsymbol{x})}{2-\gamma-\xi} < 4\ln(\tau_2^+).$$

Consider now the projection of (3.42) over $\mathbb{E}_1$. Given a smooth function f defined in Ω, we identify its gradient $\nabla f = (\partial_{x_1} f, \ldots, \partial_{x_4} f)$ with the element of $\mathbb{E}_1$,

$$\overline{\nabla} f = \sum_{i=1}^{4} \partial_{x_i} f e_i.$$

Keeping these notations in mind, we obtain the system of equations

$$\begin{aligned}\varphi_1^2-\widetilde{\varphi}_1^2-\overline{\nabla}\mathcal{E}_2(\widetilde{x}^2,\widetilde{\boldsymbol{x}})+\mathcal{O}(r_\varepsilon^2)&=0,\\ 3\varphi_1^2+3\widetilde{\varphi}_1^2+\frac{1}{2}\widetilde{\psi}_1^2-\overline{\nabla}\mathcal{E}_2(\widetilde{x}^2,\widetilde{\boldsymbol{x}})+\mathcal{O}(r_\varepsilon^2)&=0,\\ 15\varphi_1^2-3\widetilde{\varphi}_1^2-\widetilde{\psi}_1^2+\mathcal{O}(r_\varepsilon^2)&=0,\\ 15\varphi_1^2+15\widetilde{\varphi}_1^2+\frac{18}{4}\widetilde{\psi}_1^2+\mathcal{O}(r_\varepsilon^2)&=0,\end{aligned}$$

which can be simplified as follows

$$\varphi_1^2=\mathcal{O}(r_\varepsilon^2),\quad \widetilde{\varphi}_1^2=\mathcal{O}(r_\varepsilon^2),\quad \widetilde{\psi}_1^2=\mathcal{O}(r_\varepsilon^2)\quad\text{and}\quad \overline{\nabla}\mathcal{E}_2(\widetilde{x}^2,\widetilde{\boldsymbol{x}})=\mathcal{O}(r_\varepsilon^2).$$

Finally, we consider the projection onto $L^2(S^3)^\perp$. This yields the system

$$\begin{aligned}\varphi^{2,\perp}-\widetilde{\varphi}^{2,\perp}+\mathcal{O}(r_\varepsilon^2)&=0, &\quad \partial_r\big(H^{\mathrm{int}}_{\varphi^{2,\perp},\psi^{2,\perp}}-H^{\mathrm{ext}}_{\widetilde{\varphi}^{2,\perp},\widetilde{\psi}^{2,\perp}}\big)+\mathcal{O}(r_\varepsilon^2)&=0,\\ \psi^{2,\perp}-\widetilde{\psi}^{2,\perp}+\mathcal{O}(r_\varepsilon^2)&=0, &\quad \partial_r\Delta\big(H^{\mathrm{int}}_{\varphi^{2,\perp},\psi^{2,\perp}}-H^{\mathrm{ext}}_{\widetilde{\varphi}^{2,\perp},\widetilde{\psi}^{2,\perp}}\big)+\mathcal{O}(r_\varepsilon^2)&=0.\end{aligned}$$

Thanks to the result of Lemma 3.8, this last system can be rewritten as

$$\varphi^{2,\perp}=\mathcal{O}(r_\varepsilon^2),\quad \widetilde{\varphi}^{2,\perp}=\mathcal{O}(r_\varepsilon^2),\quad \psi^{2,\perp}=\mathcal{O}(r_\varepsilon^2)\quad\text{and}\quad \widetilde{\psi}^{2,\perp}=\mathcal{O}(r_\varepsilon^2).$$

If we define the parameter $t_2\in\mathbb{R}$ by

$$t_2=\frac{1}{\ln r_\varepsilon}\left[4\ln\tau_2+\frac{\mathcal{E}_2(\widetilde{x}^2,\widetilde{\boldsymbol{x}})}{2-\gamma-\xi}\right],$$

then the systems found by projecting (3.42) gather in this equality

$$T^2_{c,\varepsilon}=\big(t_2,\lambda_2,\varphi_0^2,\widetilde{\varphi}_0^2,\varphi_1^2,\widetilde{\varphi}_1^2,\widetilde{\psi}_1^2,\overline{\nabla}\mathcal{E}_2(\widetilde{x}^2,\widetilde{\boldsymbol{x}}),\varphi^{2,\perp},\widetilde{\varphi}^{2,\perp},\psi^{2,\perp},\widetilde{\psi}^{2,\perp}\big)=\mathcal{O}(r_\varepsilon^2). \tag{3.44}$$

As usual, the terms $\mathcal{O}(r_\varepsilon^2)$ depend nonlinearly on all the variables on the left side, but is bounded (in the appropriate norm) by a constant (independent of ε and κ) times r_ε^2, provided $\varepsilon\in(0,\varepsilon_\kappa)$.

- On $\partial B_{r_\varepsilon}(\widetilde{x}^3)$, we have

$$\begin{aligned}(u_{\mathrm{int},1}-u_{\mathrm{ext},1})(x)=&-\frac{\lambda_1}{\gamma}G(x,\widetilde{x}^1)-\lambda_2G(x,\widetilde{x}^2)+h_1^3\left(R_\varepsilon^3\frac{x-\widetilde{x}^3}{r_\varepsilon}\right)\\&+H^{\mathrm{int}}\left(\varphi_1^3,\psi_1^3;\frac{x-\widetilde{x}^3}{r_\varepsilon}\right)-H^{\mathrm{ext}}\left(\widetilde{\varphi}_1^3,\widetilde{\psi}_1^3;\frac{x-\widetilde{x}^3}{r_\varepsilon}\right)+\mathcal{O}(r_\varepsilon^2).\end{aligned}$$

In the same manner as above, we will solve on S^3 the following system

$$\begin{aligned}(u_{\mathrm{int},1}-u_{\mathrm{ext},1})(\widetilde{x}^3+r_\varepsilon\,\cdot\,)&=0, &\quad \partial_r(u_{\mathrm{int},1}-u_{\mathrm{ext},1})(\widetilde{x}^3+r_\varepsilon\,\cdot\,)&=0,\\ \Delta(u_{\mathrm{int},1}-u_{\mathrm{ext},1})(\widetilde{x}^3+r_\varepsilon\,\cdot\,)&=0, &\quad \partial_r\Delta(u_{\mathrm{int},1}-u_{\mathrm{ext},1})(\widetilde{x}^3+r_\varepsilon\,\cdot\,)&=0.\end{aligned} \tag{3.45}$$

We decompose

$$\varphi_1^3 = \varphi_{1,0}^3 + \varphi_{1,1}^3 + \varphi_1^{3,\perp}, \qquad \psi_1^3 = 8\varphi_{1,0}^3 + 12\varphi_{1,1}^3 + \psi_1^{3,\perp},$$
$$\widetilde{\varphi}_1^3 = \widetilde{\varphi}_{1,0}^3 + \widetilde{\varphi}_{1,1}^3 + \widetilde{\varphi}_1^{3,\perp}, \qquad \widetilde{\psi}_1^3 = \widetilde{\psi}_{1,1}^3 + \widetilde{\psi}_1^{3,\perp}$$

with $\varphi_{1,0}^3, \widetilde{\varphi}_{1,0}^3 \in \mathbb{E}_0$, $\varphi_{1,1}^3, \widetilde{\varphi}_{1,1}^3, \widetilde{\psi}_{1,1}^3 \in \mathbb{E}_1$ and $\varphi_1^{3,\perp}$, $\widetilde{\varphi}_1^{3,\perp}$, $\psi_1^{3,\perp}$, $\widetilde{\psi}_1^{3,\perp}$ belong to $(L^2(S^3))^\perp$.

Projecting the set of equations (3.45) over $\mathbb{E}_0$, we get

$$\varphi_{1,0}^3 - \widetilde{\varphi}_{1,0}^3 + \mathcal{O}(r_\varepsilon^2) = 0, \quad 2\varphi_{1,0}^3 + 2\widetilde{\varphi}_{1,0}^3 + \mathcal{O}(r_\varepsilon^2) = 0, \quad 8\varphi_{1,0}^3 + \mathcal{O}(r_\varepsilon^2) = 0.$$

From the L^2-projection of (3.45) over $\mathbb{E}_1$, we obtain the system of equations

$$\varphi_{1,1}^3 - \widetilde{\varphi}_{1,1}^3 + \mathcal{O}(r_\varepsilon^2) = 0, \qquad 3\varphi_{1,1}^3 + 3\widetilde{\varphi}_{1,1}^3 + \frac{1}{2}\widetilde{\psi}_{1,1}^3 + \mathcal{O}(r_\varepsilon^2) = 0,$$
$$15\varphi_{1,1}^3 - 3\widetilde{\varphi}_{1,1}^3 - \widetilde{\psi}_{1,1}^3 + \mathcal{O}(r_\varepsilon^2) = 0, \qquad 15\varphi_{1,1}^3 + 15\widetilde{\varphi}_{1,1}^3 + \frac{18}{4}\widetilde{\psi}_{1,1}^3 + \mathcal{O}(r_\varepsilon^2) = 0.$$

Finally, we consider the L^2-projection onto $(L^2(S^3))^\perp$. This yields the system

$$\varphi_1^{3,\perp} - \widetilde{\varphi}_1^{3,\perp} + \mathcal{O}(r_\varepsilon^2) = 0, \qquad \partial_r\big(H^{\mathrm{int}}_{\varphi_1^{3,\perp},\psi_1^{3,\perp}} - H^{\mathrm{ext}}_{\widetilde{\varphi}_1^{3,\perp},\widetilde{\psi}_1^{3,\perp}}\big) + \mathcal{O}(r_\varepsilon^2) = 0,$$
$$\psi_1^{3,\perp} - \widetilde{\psi}_1^{3,\perp} + \mathcal{O}(r_\varepsilon^2) = 0, \qquad \partial_r\Delta\big(H^{\mathrm{int}}_{\varphi_1^{3,\perp},\psi_1^{3,\perp}} - H^{\mathrm{ext}}_{\widetilde{\varphi}_1^{3,\perp},\widetilde{\psi}_1^{3,\perp}}\big) + \mathcal{O}(r_\varepsilon^2) = 0.$$

Using again Lemma 3.8, the above system can be rewritten as

$$\varphi_1^{3,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_1^{3,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \psi_1^{3,\perp} = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \widetilde{\psi}_1^{3,\perp} = \mathcal{O}(r_\varepsilon^2).$$

Then the systems found by projecting (3.45) gather in this equality

$$T_{1,\varepsilon}^3 = \big(\varphi_{1,0}^3, \widetilde{\varphi}_{1,0}^3, \varphi_{1,1}^3, \widetilde{\varphi}_{1,1}^3, \widetilde{\psi}_{1,1}^3, \varphi_1^{3,\perp}, \widetilde{\varphi}_1^{3,\perp}, \psi_1^{3,\perp}, \widetilde{\psi}_1^{3,\perp}\big) = \mathcal{O}(r_\varepsilon^2). \tag{3.46}$$

- On $\partial B_{r_\varepsilon}(\widetilde{x}^3)$, we have

$$\begin{aligned}
&(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(x)\\
&= -\frac{4}{\xi}\ln\tau_3 + \frac{8\lambda_3}{\xi}\ln|x-\widetilde{x}^3| - \frac{1-\xi}{\gamma\xi}G(x,\widetilde{x}^1) - \frac{\ln\xi}{\xi}\\
&\quad + h_2^3\left(R_\varepsilon^3\frac{x-\widetilde{x}^3}{r_\varepsilon}\right) + H^{\mathrm{int}}\left(\varphi_2^3,\psi_2^3;\frac{x-\widetilde{x}^3}{r_\varepsilon}\right) - H^{\mathrm{ext}}\left(\widetilde{\varphi}_2^3,\widetilde{\psi}_2^3;\frac{x-\widetilde{x}^3}{r_\varepsilon}\right)\\
&\quad - \frac{1+\lambda_3}{\xi}H(x,\widetilde{x}^3) - \left(1+\lambda_2-\frac{1-\xi}{\xi}\right)G(x,\widetilde{x}^2) + \mathcal{O}\left(\frac{\varepsilon^2\tau_3^{-2}}{|x-\widetilde{x}^3|^2}\right) + \mathcal{O}(r_\varepsilon^2).
\end{aligned} \tag{3.47}$$

Next, even though all functions are defined on $\partial B_{r_\varepsilon}(\widetilde{x}^3)$ in (3.34), it will be more convenient to solve on S^3 the following set of equations

$$\begin{aligned}
(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^3 + r_\varepsilon\,\cdot\,) = 0, \qquad \partial_r(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^3 + r_\varepsilon\,\cdot\,) = 0,\\
\Delta(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^3 + r_\varepsilon\,\cdot\,) = 0, \qquad \partial_r\Delta(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^3 + r_\varepsilon\,\cdot\,) = 0.
\end{aligned} \tag{3.48}$$

Since the boundary data are chosen to satisfy (3.5) or (3.6). We decompose

$$\varphi_2^3 = \varphi_{2,0}^3 + \varphi_{2,1}^3 + \varphi_2^{3,\perp}, \qquad \psi_2^3 = 8\varphi_{2,0}^3 + 12\varphi_{2,1}^3 + \psi_2^{3,\perp},$$
$$\widetilde{\varphi}_2^3 = \widetilde{\varphi}_{2,0}^3 + \widetilde{\varphi}_{2,1}^3 + \widetilde{\varphi}_2^{3,\perp}, \qquad \widetilde{\psi}_2^3 = \widetilde{\psi}_{2,1}^3 + \widetilde{\psi}_2^{3,\perp},$$

where $\varphi_{2,0}^3, \widetilde{\varphi}_{2,0}^3 \in \mathbb{E}_0 = \mathbb{R}$ are constant on S^3, $\varphi_{2,1}^3$, $\widetilde{\varphi}_{2,1}^3$, $\widetilde{\psi}_{2,1}^3$ belong to $\mathbb{E}_1 = \mathrm{Span}\{e_1, e_2, e_3, e_4\}$ and $\varphi_2^{3,\perp}$, $\widetilde{\varphi}_2^{3,\perp}$, $\psi_2^{3,\perp}$, $\widetilde{\psi}_2^{3,\perp}$ are $L^2(S^3)$ orthogonal to $\mathbb{E}_0$ and $\mathbb{E}_1$.

Using (3.47), we have for $x \in S^3$

$$\begin{aligned}
&(u_{\mathrm{int},2} - u_{\mathrm{ext},2})(\widetilde{x}^3 + r_\varepsilon x) \\
&= -\frac{4}{\xi}\ln\tau_3 + \frac{8\lambda_3}{\xi}\ln(r_\varepsilon|x|) - \frac{1}{\xi}\left(H(\widetilde{x}^3,\widetilde{x}^3) + G(\widetilde{x}^3,\widetilde{x}^2) + \frac{1-\xi}{\gamma}G(\widetilde{x}^3,\widetilde{x}^1)\right) \\
&\quad + H^{\mathrm{int}}(\varphi_2^3, \psi_2^3; x) - H^{\mathrm{ext}}(\widetilde{\varphi}_2^3, \widetilde{\psi}_2^3; x) - \frac{\ln\xi}{\xi} \\
&\quad - \frac{\lambda_3}{\xi}H(\widetilde{x}^3,\widetilde{x}^3) - \lambda_2 G(\widetilde{x}^3,\widetilde{x}^2) + \mathcal{O}(r_\varepsilon^2).
\end{aligned}$$

Then, the projection of the set equations (3.48) over $\mathbb{E}_0$ will yield

$$\begin{aligned}
-4\ln\tau_3 + 8\lambda_3\ln r_\varepsilon - \ln\xi + \xi\varphi_{2,0}^3 - \xi\widetilde{\varphi}_{2,0}^3 - \mathcal{E}_3(\widetilde{x}^3, \widetilde{\boldsymbol{x}}) + \mathcal{O}(r_\varepsilon^2) &= 0, \\
8\lambda_3 + 2\xi\varphi_{2,0}^3 + 2\xi\widetilde{\varphi}_{2,0}^3 + \mathcal{O}(r_\varepsilon^2) &= 0, \\
16\lambda_3 + 8\xi\varphi_{2,0}^3 + \mathcal{O}(r_\varepsilon^2) &= 0, \\
-32\lambda_3 + \mathcal{O}(r_\varepsilon^2) &= 0,
\end{aligned} \tag{3.49}$$

where

$$\mathcal{E}_3(\,\cdot\,, \widetilde{\boldsymbol{x}}) := H(\,\cdot\,, \widetilde{x}^3) + G(\,\cdot\,, \widetilde{x}^2) + \frac{1-\xi}{\gamma}G(\,\cdot\,, \widetilde{x}^1).$$

The system (3.49) can be simply written as

$$\lambda_3 = \mathcal{O}(r_\varepsilon^2), \quad \varphi_{2,0}^3 = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_{2,0}^3 = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \frac{1}{\ln r_\varepsilon}\left[4\ln\tau_3 + \ln\xi + \mathcal{E}_3(\widetilde{x}^3, \widetilde{\boldsymbol{x}})\right] = \mathcal{O}(r_\varepsilon^2).$$

We are now in a position to define τ_3^- and τ_3^+. In fact, according to the above analysis, as ε tends to 0, we expect that $\widetilde{x}^i$ will converge to x^i for $i \in \{1,2,3\}$ and τ_3 will converge to τ_3^* satisfying

$$4\ln\tau_3^* = -\ln\xi - \mathcal{E}_3(x^3, \boldsymbol{x}).$$

Hence it is enough to choose τ_3^- and τ_3^+ in such a way that

$$4\ln(\tau_3^-) < -\ln\xi - \mathcal{E}_3(x^3, \boldsymbol{x}) < 4\ln(\tau_3^+).$$

Consider now the projection of (3.48) over $\mathbb{E}_1$. Given a smooth function f defined in Ω, we identify its gradient $\nabla f = (\partial_{x_1} f, \dots, \partial_{x_4} f)$ with the element of $\mathbb{E}_1$,

$$\overline{\nabla} f = \sum_{i=1}^4 \partial_{x_i} f e_i.$$

Keeping these notations in mind, we obtain the system of equations

$$\begin{aligned}\varphi^3_{2,1} - \widetilde{\varphi}^3_{2,1} - \overline{\nabla}\mathcal{E}_3(\widetilde{x}^3, \widetilde{\boldsymbol{x}}) + \mathcal{O}(r^2_\varepsilon) &= 0,\\ 3\varphi^3_{2,1} + 3\widetilde{\varphi}^3_{2,1} + \frac{1}{2}\widetilde{\psi}^3_{2,1} - \overline{\nabla}\mathcal{E}_3(\widetilde{x}^3, \widetilde{\boldsymbol{x}}) + \mathcal{O}(r^2_\varepsilon) &= 0,\\ 15\varphi^3_{2,1} - 3\widetilde{\varphi}^3_{2,1} - \widetilde{\psi}^3_{2,1} + \mathcal{O}(r^2_\varepsilon) &= 0,\\ 15\varphi^3_{2,1} + 15\widetilde{\varphi}^3_{2,1} + \frac{18}{4}\widetilde{\psi}^3_{2,1} + \mathcal{O}(r^2_\varepsilon) &= 0,\end{aligned}$$

which can be simplified as follows

$$\varphi^3_{2,1} = \mathcal{O}(r^2_\varepsilon), \quad \widetilde{\varphi}^3_{2,1} = \mathcal{O}(r^2_\varepsilon), \quad \widetilde{\psi}^3_{2,1} = \mathcal{O}(r^2_\varepsilon) \quad \text{and} \quad \overline{\nabla}\mathcal{E}_3(\widetilde{x}^3, \widetilde{\boldsymbol{x}}) = \mathcal{O}(r^2_\varepsilon).$$

Finally, we consider the projection onto $L^2(S^3)^\perp$. This yields the system

$$\begin{aligned}&\varphi^{3,\perp}_2 - \widetilde{\varphi}^{3,\perp}_2 + \mathcal{O}(r^2_\varepsilon) = 0, \qquad \partial_r\big(H^{\text{int}}_{\varphi^{3,\perp}_2, \psi^{3,\perp}_2} - H^{\text{ext}}_{\widetilde{\varphi}^{3,\perp}_2, \widetilde{\psi}^{3,\perp}_2}\big) + \mathcal{O}(r^2_\varepsilon) = 0,\\ &\psi^{3,\perp}_2 - \widetilde{\psi}^{3,\perp}_2 + \mathcal{O}(r^2_\varepsilon) = 0, \qquad \partial_r\Delta\big(H^{\text{int}}_{\varphi^{3,\perp}_2, \psi^{3,\perp}_2} - H^{\text{ext}}_{\widetilde{\varphi}^{3,\perp}_2, \widetilde{\psi}^{3,\perp}_2}\big) + \mathcal{O}(r^2_\varepsilon) = 0.\end{aligned}$$

Thanks to the result of Lemma 3.8, this last system can be rewritten as

$$\varphi^{3,\perp}_2 = \mathcal{O}(r^2_\varepsilon), \quad \widetilde{\varphi}^{3,\perp}_2 = \mathcal{O}(r^2_\varepsilon), \quad \psi^{3,\perp}_2 = \mathcal{O}(r^2_\varepsilon) \quad \text{and} \quad \widetilde{\psi}^{3,\perp}_2 = \mathcal{O}(r^2_\varepsilon).$$

If we define the parameter $t_3 \in \mathbb{R}$ by

$$t_3 = \frac{1}{\ln r_\varepsilon}\big[4\ln\tau_3 + \ln\xi + \mathcal{E}_3(\widetilde{x}^3, \widetilde{\boldsymbol{x}})\big],$$

then the systems found by projecting (3.48) gather in this equality

$$T^3_{2,\varepsilon} = \big(t_3, \lambda_3, \varphi^3_{2,0}, \widetilde{\varphi}^3_{2,0}, \varphi^3_{2,1}, \widetilde{\varphi}^3_{2,1}, \widetilde{\psi}^3_{2,1}\overline{\nabla}\mathcal{E}_3(\widetilde{x}^3, \widetilde{\boldsymbol{x}}), \varphi^{3,\perp}_2, \widetilde{\varphi}^{3,\perp}_2, \psi^{3,\perp}_2, \widetilde{\psi}^{3,\perp}_2\big) = \mathcal{O}(r^2_\varepsilon). \tag{3.50}$$

As usual, the terms $\mathcal{O}(r^2_\varepsilon)$ depend nonlinearly on all the variables on the left side, but is bounded (in the appropriate norm) by a constant (independent of ε and κ) times r^2_ε, provided $\varepsilon \in (0, \varepsilon_\kappa)$.

We recall that $\boldsymbol{d} = r_\varepsilon(\widetilde{\boldsymbol{x}} - \boldsymbol{x})$, in addition the previous systems can be written as for $i = 1, 2, 3$:

$$\big(\boldsymbol{d}, t_i, \lambda_i, \varphi^i, \widetilde{\varphi}^i, \psi^i, \widetilde{\psi}^i, \overline{\nabla}\mathcal{E}_i\big) = \mathcal{O}(r^2_\varepsilon).$$

Combining (3.38), (3.40), (3.44), (3.46) and (3.50), we have

$$T_{i,\varepsilon} = (T^1_{i,\varepsilon}, T^2_{i,\varepsilon}, T^3_{i,\varepsilon}) = \big(\mathcal{O}(r^2_\varepsilon), \mathcal{O}(r^2_\varepsilon), \mathcal{O}(r^2_\varepsilon)\big) \quad \text{for } i = 1, 2. \tag{3.51}$$

Then the nonlinear mapping which appears on the right-hand side of (3.51) is continuous, compact. In addition, reducing ε_κ if necessary, this nonlinear mapping sends the ball of radius κr^2_ε (for the natural product norm) into itself, provided κ is fixed large enough. Applying Schauder's fixed point Theorem in the ball of radius κr^2_ε in the product space where the entries live, we obtain the existence of a solution of equation (3.51). This completes the proof of Theorem 1.5.

4. Proof of Theorem 1.6

4.1. Construction of the approximate solution

4.1.1. Ansatz and first estimates

We define another ansatz for solution of (1.1):

$$\widetilde{u}_1(x)=\begin{cases}\frac{1}{\gamma}u_{\varepsilon,\tau_1}(x-x^1)-\frac{1-\gamma}{\gamma}G(x,x^2)-\frac{1-\gamma}{\gamma\xi}G(x,x^3)-\frac{\ln\gamma}{\gamma}, & x\in B_{r_\varepsilon}(x^1),\\ u_{\varepsilon,\tau_2}(x-x^2)+\frac{1-\gamma}{\gamma(2-\gamma-\xi)}G(x,x^1)-\frac{1-\gamma}{\xi(2-\gamma-\xi)}G(x,x^3), & x\in B_{r_\varepsilon}(x^2),\\ \frac{1}{\gamma}G(x,x^1)+G(x,x^2), & x\in\Omega\setminus\bigcup_{i=1}^2 B_{r_\varepsilon}(x^i)\end{cases}$$

and

$$\widetilde{u}_2(x)=\begin{cases}\frac{1}{\xi}u_{\varepsilon,\tau_3}(x-x^3)-\frac{1-\xi}{\xi}G(x,x^2)-\frac{1-\xi}{\gamma\xi}G(x,x^1)-\frac{\ln\xi}{\xi}, & x\in B_{r_\varepsilon}(x^3),\\ u_{\varepsilon,\tau_2}(x-x^2)-\frac{1-\xi}{\gamma(2-\gamma-\xi)}G(x,x^1)+\frac{1-\xi}{\xi(2-\gamma-\xi)}G(x,x^3), & x\in B_{r_\varepsilon}(x^2),\\ \frac{1}{\xi}G(x,x^3)+G(x,x^2), & x\in\Omega\setminus\bigcup_{i=2}^3 B_{r_\varepsilon}(x^i).\end{cases}$$

Therefore, in $B_{r_\varepsilon}(x^1)$, there holds

$$\Delta^2\widetilde{u}_1-\rho^4e^{\gamma\widetilde{u}_1+(1-\gamma)\widetilde{u}_2}=0,$$

$$\Delta^2\widetilde{u}_2-\rho^4e^{\xi\widetilde{u}_2+(1-\xi)\widetilde{u}_1}=\frac{-384\varepsilon^4\tau_1^{4\frac{1-\xi}{\gamma}}(1+\varepsilon^2)^{4\frac{1-\gamma-\xi}{\gamma}}}{\gamma^{\frac{1-\xi}{\gamma}}(\varepsilon^2+\tau_1^2|x-x^1|^2)^{4\frac{1-\xi}{\gamma}}}e^{\frac{\gamma+\xi-1}{\gamma}G(x,x^2)+\frac{\gamma+\xi-1}{\gamma\xi}G(x,x^3)}.$$

Then, for $r=|x-x^1|$ and $0<\delta<(\gamma+\xi-1)/\gamma$, we have

$$\begin{aligned}\left\|\Delta^2\widetilde{u}_2-\rho^4e^{\xi\widetilde{u}_2+(1-\xi)\widetilde{u}_1}\right\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(B_{r_\varepsilon}(x^1))}&\le C\sup_{r<r_\varepsilon}\frac{\tau_1^{4\frac{1-\xi}{\gamma}}\varepsilon^4}{\gamma^{\frac{1-\xi}{\gamma}}(1+\varepsilon^2)^{4-4\frac{1-\xi}{\gamma}}}\frac{r^{4-\delta}}{\varepsilon^{8\frac{1-\xi}{\gamma}}\left(1+\left(\frac{\tau_1}{\varepsilon}r\right)^2\right)^{4\frac{1-\xi}{\gamma}}}\\ &\le C\sup_{r<R_\varepsilon^1}\frac{\varepsilon^{8-8\frac{1-\xi}{\gamma}-\delta}}{(1+\varepsilon^2)^{4-4\frac{1-\xi}{\gamma}}}\frac{r^{4-\delta}}{(1+r^2)^{4\frac{1-\xi}{\gamma}}}\\ &\le C\sup_{r<R_\varepsilon^1}\varepsilon^{8-8\frac{1-\xi}{\gamma}-\delta}S(r),\end{aligned}$$

where $S(r)=\frac{r^{4-\delta}}{(1+r^2)^{4\frac{1-\xi}{\gamma}}}$.

If $4-\delta-8(1-\xi)/\gamma\le 0$, then S is bounded on $\mathbb{R}_+$, hence

$$\left\|\Delta^2\widetilde{u}_2-\rho^4e^{\xi\widetilde{u}_2+(1-\xi)\widetilde{u}_1}\right\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(B_{r_\varepsilon}(x^1))}\le C\varepsilon^{8-8\frac{1-\xi}{\gamma}-\delta}\le Cr_\varepsilon^2.$$

If $4-\delta-8(1-\xi)/\gamma>0$, $\sup_{[0,r_\varepsilon/\varepsilon)}S(r)=S(r_\varepsilon/\varepsilon)$, then

$$\left\|\Delta^2\widetilde{u}_2-\rho^4e^{\xi\widetilde{u}_2+(1-\xi)\widetilde{u}_1}\right\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(B_{r_\varepsilon}(x^1))}\le Cr_\varepsilon^2\quad\text{as }\varepsilon^{8-8\frac{1-\xi}{\gamma}-\delta}S\left(\frac{r_\varepsilon}{\varepsilon}\right)\le Cr_\varepsilon^2.$$

On the other hand, in $B_{r_\varepsilon}(x^2)$, there holds

$$\Delta^2\widetilde{u}_1 - \rho^4 e^{\gamma\widetilde{u}_1+(1-\gamma)\widetilde{u}_2} = \frac{-384\varepsilon^4\tau_2^4}{(\varepsilon^2+\tau_2^2|x-x^2|^2)^4}\left(e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)}\left(\frac{1}{\gamma}G(x,x^1)-\frac{1}{\xi}G(x,x^3)\right)}-1\right),$$
$$\Delta^2\widetilde{u}_2 - \rho^4 e^{\xi\widetilde{u}_2+(1-\xi)\widetilde{u}_1} = \frac{-384\varepsilon^4\tau_2^4}{(\varepsilon^2+\tau_2^2|x-x^2|^2)^4}\left(e^{\frac{(1-\xi)(\gamma+\xi-1)}{(2-\gamma-\xi)}\left(\frac{-1}{\gamma}G(x,x^1)+\frac{1}{\xi}G(x,x^3)\right)}-1\right).$$

Then, for $r=|x-x^2|$, $\mu\in(1,2)$ and using the condition (1.6), we have the following estimates

$$\left\|\Delta^2\widetilde{u}_1 - \rho^4 e^{\gamma\widetilde{u}_1+(1-\gamma)\widetilde{u}_2}\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(B_{r_\varepsilon}(x^2))} \le Cr_\varepsilon^2$$

and

$$\left\|\Delta^2\widetilde{u}_1 - \rho^4 e^{\xi\widetilde{u}_1+(1-\xi)\widetilde{u}_2}\right\|_{\mathcal{C}^{0,\alpha}_{\mu-4}(B_{r_\varepsilon}(x^2))} \le Cr_\varepsilon^2.$$

Finally, in $B_{r_\varepsilon}(x^3)$, there holds

$$\Delta^2\widetilde{u}_1 - \rho^4 e^{\gamma\widetilde{u}_1+(1-\gamma)\widetilde{u}_2} = \frac{-384\varepsilon^4\tau_3^{4\frac{1-\gamma}{\xi}}(1+\varepsilon^2)^{4(1-\xi-\gamma)/\xi}}{\xi^{\frac{1-\gamma}{\xi}}(\varepsilon^2+\tau_3^2|x-x^3|^2)^{4\frac{1-\gamma}{\xi}}}e^{\frac{\gamma+\xi-1}{\xi}G(x,x^2)+\frac{\gamma+\xi-1}{\xi\gamma}G(x,x^1)},$$
$$\Delta^2\widetilde{u}_2 - \rho^4 e^{\xi\widetilde{u}_2+(1-\xi)\widetilde{u}_1} = 0.$$

Then, for $r=|x-x^3|$ and $0<\delta<(\gamma+\xi-1)/\xi$, we have the same estimates

$$\left\|\Delta^2\widetilde{u}_1 - \rho^4 e^{\gamma\widetilde{u}_1+(1-\gamma)\widetilde{u}_2}\right\|_{\mathcal{C}^{0,\alpha}_{\delta-4}(B_{r_\varepsilon}(x^3))} \le Cr_\varepsilon^2.$$

4.2. The nonlinear interior problem

Here, we are interested to study the system

$$\Delta^2 u_1 = \rho^4 e^{\gamma u_1+(1-\gamma)u_2}, \quad \Delta^2 u_2 = \rho^4 e^{\xi u_2+(1-\xi)u_1}.$$

Using the following transformations

$$\begin{cases} v_1(x) = u_1\left(\frac{\varepsilon}{\tau_1}x\right) + \frac{8}{\gamma}\ln\varepsilon - \frac{4}{\gamma}\ln\left(\frac{\tau_1(1+\varepsilon^2)}{2}\right) & \text{in } B_{r_\varepsilon}(x^1),\\ v_2(x) = u_2\left(\frac{\varepsilon}{\tau_1}x\right) & \text{in } B_{r_\varepsilon}(x^1),\end{cases}$$
$$\begin{cases} v_1(x) = u_1\left(\frac{\varepsilon}{\tau_2}x\right) + 8\ln\varepsilon - 4\ln\left(\frac{\tau_2(1+\varepsilon^2)}{2}\right) & \text{in } B_{r_\varepsilon}(x^2),\\ v_2(x) = u_2\left(\frac{\varepsilon}{\tau_2}x\right) + 8\ln\varepsilon - 4\ln\left(\frac{\tau_2(1+\varepsilon^2)}{2}\right) & \text{in } B_{r_\varepsilon}(x^2)\end{cases}$$

and

$$\begin{cases} v_1(x) = u_1\left(\frac{\varepsilon}{\tau_3}x\right) & \text{in } B_{r_\varepsilon}(x^3),\\ v_2(x) = u_2\left(\frac{\varepsilon}{\tau_3}x\right) + \frac{8}{\xi}\ln\varepsilon - \frac{4}{\xi}\ln\left(\frac{\tau_3(1+\varepsilon^2)}{2}\right) & \text{in } B_{r_\varepsilon}(x^3).\end{cases}$$

So the previous systems can be written as

$$\begin{cases} \Delta^2 v_1 = 24e^{\gamma v_1+(1-\gamma)v_2} & \text{in } B_{R^1_\varepsilon}(x^1), \\ \Delta^2 v_2 = 24C_{1,\varepsilon}^{4\frac{\gamma+\xi-1}{\gamma}} \varepsilon^{8\frac{\gamma+\xi-1}{\gamma}} e^{\xi v_2+(1-\xi)v_1} & \text{in } B_{R^1_\varepsilon}(x^1), \end{cases} \tag{4.1}$$

$$\begin{cases} \Delta^2 v_1 = 24e^{\gamma v_1+(1-\gamma)v_2} & \text{in } B_{R^2_\varepsilon}(x^2), \\ \Delta^2 v_2 = 24e^{\xi v_2+(1-\xi)v_1} & \text{in } B_{R^2_\varepsilon}(x^2) \end{cases} \tag{4.2}$$

and

$$\begin{cases} \Delta^2 v_1 = 24C_{3,\varepsilon}^{4\frac{\gamma+\xi-1}{\xi}} \varepsilon^{8\frac{\gamma+\xi-1}{\xi}} e^{\gamma v_1+(1-\gamma)v_2} & \text{in } B_{R^3_\varepsilon}(x^3), \\ \Delta^2 v_2 = 24e^{\xi v_2+(1-\xi)v_1} & \text{in } B_{R^3_\varepsilon}(x^3), \end{cases} \tag{4.3}$$

where $C_{i,\varepsilon} = \frac{2}{\tau_i(1+\varepsilon^2)}$ for $i = 1, 3$. Here $\tau_i > 0$ is a constant which will be fixed later.

Given $\varphi^i := (\varphi^i_1, \varphi^i_2) \in (\mathcal{C}^{4,\alpha}(S^3))^2$ and $\psi^i := (\psi^i_1, \psi^i_2) \in (\mathcal{C}^{2,\alpha}(S^3))^2$ such that (φ^i_1, ψ^i_1) and (φ^i_2, ψ^i_2) satisfy (3.5). We denote by $\overline{u} = u_{\varepsilon=1,\tau_i=1}$.

In $B_{R^1_\varepsilon}(x^1)$ and $B_{R^3_\varepsilon}(x^3)$, we reproduce exactly the same as in the proof of Theorem 1.5, so we have the following propositions.

Proposition 4.1. *Given $\kappa > 0$, $\mu \in (1, 2)$ and $\delta \in \big(0, \min\{(\frac{\gamma+\xi-1}{\gamma}), (\frac{\gamma+\xi-1}{\xi})\}\big)$, there exist $\varepsilon_\kappa > 0$, $c_\kappa > 0$ and $\gamma_0 \in (0, 1)$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\gamma \in (\gamma_0, 1)$, for all τ_1 in some fixed compact subset of $[\tau_1^-, \tau_1^+] \subset (0, \infty)$ and for φ^1_j and ψ^1_j satisfying* (3.5) *and* (3.13)*, there exists a unique (h^1_1, h^1_2) $(:= (h_{1,\varepsilon,\tau_1,\varphi^1_1,\psi^1_1}, h_{2,\varepsilon,\tau_1,\varphi^1_2,\psi^1_2}))$ solution of* (3.12) *such that*

$$\|(h^1_1, h^1_2)\|_{C^{4,\alpha}_\mu(\mathbb{R}^4)\times C^{4,\alpha}_\delta(\mathbb{R}^4)} \le 2c_\kappa r^2_\varepsilon.$$

Hence

$$\begin{aligned} v_1(x) :=& \frac{1}{\gamma}\overline{u}(x - x^1) - \frac{1-\gamma}{\gamma} G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) - \frac{1-\gamma}{\gamma\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) \\ &- \frac{\ln\gamma}{\gamma} + h^1_1(x) + H^{\text{int}}\left(\varphi^1_1, \psi^1_1; \frac{x - x^1}{R^1_\varepsilon}\right), \\ v_2(x) :=& \frac{1}{\xi} G\left(\frac{\varepsilon x}{\tau_1}, x^3\right) + G\left(\frac{\varepsilon x}{\tau_1}, x^2\right) + h^1_2(x) + H^{\text{int}}\left(\varphi^1_2, \psi^1_2; \frac{x - x^1}{R^1_\varepsilon}\right) \end{aligned}$$

solves (4.1) *in $B_{R^1_\varepsilon}(x^1)$.*

Proposition 4.2. *Given $\kappa > 0$, $\mu \in (1, 2)$ and $\delta \in \big(0, \min\{(\frac{\gamma+\xi-1}{\gamma}), (\frac{\gamma+\xi-1}{\xi})\}\big)$, there exist $\varepsilon_\kappa > 0$, $c_\kappa > 0$ and $\xi_0 \in (0, 1)$ such that for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\xi \in (\xi_0, 1)$, for all τ_3 in some fixed compact subset of $[\tau_3^-, \tau_3^+] \subset (0, \infty)$ and for φ^3_j and ψ^3_j satisfying* (3.5) *and* (3.17)*, there exists a unique (h^3_1, h^3_2) $(:= (h_{1,\varepsilon,\tau_3,\varphi^3_1,\psi^3_1}, h_{2,\varepsilon,\tau_3,\varphi^3_2,\psi^3_2}))$ solution of* (3.12) *such that*

$$\|(h^3_1, h^3_2)\|_{C^{4,\alpha}_\delta(\mathbb{R}^4)\times C^{4,\alpha}_\mu(\mathbb{R}^4)} \le 2c_\kappa r^2_\varepsilon.$$

Hence

$$\begin{aligned} v_1(x) &:= \frac{1}{\gamma} G\left(\frac{\varepsilon x}{\tau_3}, x^1\right) + G\left(\frac{\varepsilon x}{\tau_3}, x^2\right) + h_1^3(x) + H^{\rm int}\left(\varphi_1^3, \psi_1^3; \frac{x - x^3}{R_\varepsilon^3}\right), \\ v_2(x) &:= \frac{1}{\xi}\overline{u}(x - x^3) - \frac{1-\xi}{\xi} G\left(\frac{\varepsilon x}{\tau_3}, x^2\right) - \frac{1-\xi}{\gamma\xi} G\left(\frac{\varepsilon x}{\tau_3}, x^1\right) \\ &\qquad - \frac{\ln \xi}{\xi} + h_2^3(x) + H^{\rm int}\left(\varphi_2^3, \psi_2^3; \frac{x - x^3}{R_\varepsilon^3}\right) \end{aligned}$$

solves (4.3) *in* $B_{R_\varepsilon^3}(x^3)$.

In $B_{R_\varepsilon^2}(x^2)$ we look for a solution of (4.2) of the form

$$\begin{aligned} v_1(x) &= \overline{u}(x - x^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) \\ &\qquad + H^{\rm int}\left(\varphi_1^2, \psi_1^2; \frac{x - x^2}{R_\varepsilon^2}\right) + h_1^2(x), \\ v_2(x) &= \overline{u}(x - x^2) - \frac{1-\xi}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) + \frac{1-\xi}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, x^3\right) \\ &\qquad + H^{\rm int}\left(\varphi_2^2, \psi_2^2; \frac{x - x^2}{R_\varepsilon^2}\right) + h_2^2(x). \end{aligned}$$

This amounts to solve the equations

(4.4)

$$\mathbb{L}h_1^2 = \frac{384}{(1+r^2)^4}\left[e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)}\left(\frac{1}{\gamma}G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) - \frac{1}{\xi}G\left(\frac{\varepsilon x}{\tau_2}, x^3\right)\right) + \gamma\left(h_1^2 + H^{\rm int}_{\varphi_1^2,\psi_1^2}\right) + (1-\gamma)\left(h_2^2 + H^{\rm int}_{\varphi_2^2,\psi_2^2}\right)} - h_1^2 - 1\right],$$

$$\mathbb{L}h_2^2 = \frac{384}{(1+r^2)^4}\left[e^{\frac{(1-\xi)(\gamma+\xi-1)}{(2-\gamma-\xi)}\left(-\frac{1}{\gamma}G\left(\frac{\varepsilon x}{\tau_2}, x^1\right) + \frac{1}{\xi}G\left(\frac{\varepsilon x}{\tau_2}, x^3\right)\right) + \xi\left(h_2^2 + H^{\rm int}_{\varphi_2^2,\psi_2^2}\right) + (1-\xi)\left(h_1^2 + H^{\rm int}_{\varphi_1^2,\psi_1^2}\right)} - h_2^2 - 1\right].$$

We denote by

$$\mathbb{L}h_1^2 = \mathcal{T}_3(h_1^2, h_2^2) \quad \text{and} \quad \mathbb{L}h_2^2 = \mathcal{T}_4(h_1^2, h_2^2).$$

To find a solution of (4.4), it is enough to find a fixed point (h_1^2, h_2^2) in a small ball of $\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)$, solutions of

$$\begin{aligned} h_1^2 &= \mathcal{G}_\mu \circ \xi_\mu \circ \mathcal{T}_3(h_1^2, h_2^2) = \mathcal{N}_2(h_1^2, h_2^2), \\ h_2^2 &= \mathcal{G}_\mu \circ \xi_\mu \circ \mathcal{T}_4(h_1^2, h_2^2) = \mathcal{M}_2(h_1^2, h_2^2). \end{aligned} \tag{4.5}$$

Then, we have the following result.

Lemma 4.3. *Let* $\mu \in (1,2)$, γ_0 *and* $\xi_0 \in (0,1)$. *Given* $\kappa > 0$, *there exist* $\varepsilon_\kappa > 0$ *and* $c_\kappa > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_\kappa)$, $\gamma \in (\gamma_0, 1)$ *and* $\xi \in (\xi_0, 1)$. *We have*

$$\|\mathcal{N}_2(0,0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2, \quad \|\mathcal{M}_2(0,0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2,$$

$$\|\mathcal{N}_2(h_1^2, h_2^2) - \mathcal{N}_2(k_1^2, k_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa(1-\gamma)\|(h_1^2, h_2^2) - (k_1^2, k_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)},$$

$$\|\mathcal{M}_2(h_1^2, h_2^2) - \mathcal{M}_2(k_1^2, k_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa(1-\xi)\|(h_1^2, h_2^2) - (k_1^2, k_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}$$

provided (h_1^2, h_2^2), (k_1^2, k_2^2) *in* $\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) \times \mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)$ *satisfying*

$$\|(h_1^2, h_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2 \quad \textit{and} \quad \|(k_1^2, k_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2. \tag{4.6}$$

Proof. The proof of the first and the second estimates follows from the asymptotic behavior of H^{int} together with the assumption on the norms of φ_j^2 and ψ_j^2 given by (3.20) and it follows from the estimate of H^{int}, given by Lemma 3.7, that

$$\left\| H^{\text{int}}_{\varphi_j^2,\psi_j^2}\left(\frac{r}{R_\varepsilon^2}\,\cdot\right)\right\|_{\mathcal{C}^{4,\alpha}(\overline{B}_2(0)-B_1(0))} \le Cr^2(R_\varepsilon^2)^{-2}\big(\|\varphi_j^2\|_{\mathcal{C}^{4,\alpha}(S^3)} + \|\psi_j^2\|_{\mathcal{C}^{2,\alpha}(S^3)}\big)$$

for all $r \le R_\varepsilon^2/2$. Then by (3.20), we get

$$\left\| H^{\text{int}}_{\varphi_j^2,\psi_j^2}\left(\frac{r}{R_\varepsilon^2}\,\cdot\right)\right\|_{\mathcal{C}^{4,\alpha}(\overline{B}_2(0)-B_1(0))} \le c_\kappa \varepsilon^2 r^2.$$

On the other hand, using (1.6) we obtain

$$\begin{aligned}
&\sup_{r\le R_\varepsilon^2} r^{4-\mu}|\mathcal{T}_3(0,0)| \\
&\le \sup_{r\le R_\varepsilon^2} \frac{384 r^{4-\mu}}{(1+r^2)^4}\left| e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)}\left(\frac{1}{\gamma}G\left(\frac{\varepsilon x}{\tau_2},x^1\right)-\frac{1}{\xi}G\left(\frac{\varepsilon x}{\tau_2},x^3\right)\right)} e^{\gamma H^{\text{int}}_{\varphi_1^2,\psi_1^2}+(1-\gamma)H^{\text{int}}_{\varphi_2^2,\psi_2^2}} - 1\right| \\
&\le c \sup_{r\le R_\varepsilon^2} \frac{384 r^{4-\mu} r^2\varepsilon^2}{(1+r^2)^4}.
\end{aligned}$$

Making use of Proposition 3.2 together with (3.4), for $\mu \in (1,2)$, we get that there exists c_κ such that

$$\|\mathcal{N}_2(0,0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2.$$

For the second estimate, we use the same techniques to prove

$$\|\mathcal{M}_2(0,0)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa r_\varepsilon^2.$$

To derive the third estimate, for (h_1^2, h_2^2), (k_1^2, k_2^2) verifying (4.6), we have

$$\begin{aligned}
&\sup_{r\le R_\varepsilon^2} r^{4-\mu}|\mathcal{T}_3(h_1^2,h_2^2) - \mathcal{T}_3(k_1^2,k_2^2)| \\
&\le \sup_{r\le R_\varepsilon^2} \frac{384 r^{4-\mu}}{(1+r^2)^4}\Big| e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)}\left(\frac{1}{\gamma}G\left(\frac{\varepsilon x}{\tau_2},x^1\right)-\frac{1}{\xi}G\left(\frac{\varepsilon x}{\tau_2},x^3\right)\right)} e^{\gamma h_1^2+\gamma H^{\text{int}}_{\varphi_1^2,\psi_1^2}+(1-\gamma)h_2^2+(1-\gamma)H^{\text{int}}_{\varphi_2^2,\psi_2^2}} - h_1^2 \\
&\qquad - e^{\frac{(1-\gamma)(\gamma+\xi-1)}{(2-\gamma-\xi)}\left(\frac{1}{\gamma}G\left(\frac{\varepsilon x}{\tau_2},x^1\right)-\frac{1}{\xi}G\left(\frac{\varepsilon x}{\tau_2},x^3\right)\right)} e^{\gamma k_1^2+\gamma H^{\text{int}}_{\varphi_1^2,\psi_1^2}+(1-\gamma)k_2^2+(1-\gamma)H^{\text{int}}_{\varphi_2^2,\psi_2^2}} + k_1^2\Big| \\
&\le c \sup_{r\le R_\varepsilon^2} \frac{384 r^{4-\mu}}{(1+r^2)^4}\big|(\gamma-1)(h_1^2-k_1^2) + (1-\gamma)(h_2^2-k_2^2)\big| \\
&\le c \sup_{r\le R_\varepsilon^2} \frac{384 r^{4-\mu}}{(1+r^2)^4}(1-\gamma)\big[r^\mu\|h_1^2-k_1^2\|_{\mathcal{C}_\mu^{4,\alpha}} + r^\mu\|h_2^2-k_2^2\|_{\mathcal{C}_\mu^{4,\alpha}}\big].
\end{aligned}$$

We conclude that

$$\|\mathcal{N}_2(h_1^2,h_2^2)-\mathcal{N}_2(k_1^2,k_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa(1-\gamma)\|(h_1^2,h_2^2)-(k_1^2,k_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}. \tag{4.7}$$

Similarly, we get

$$\|\mathcal{M}_2(h_1^2,h_2^2)-\mathcal{M}_2(k_1^2,k_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le c_\kappa(1-\xi)\|(h_1^2,h_2^2)-(k_1^2,k_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)}. \tag{4.8}$$

□

Then there exist γ_0 and $\xi_0\in(0,1)$ such that $c_\kappa(1-\gamma)\le 1/2$ and $c_\kappa(1-\xi)\le 1/2$ for all $\gamma\in(\gamma_0,1)$ and $\xi\in(\xi_0,1)$. Therefore (4.7) and (4.8) are enough to show that

$$(h_1^2,h_2^2)\mapsto(\mathcal{N}_2(h_1^2,h_2^2),\mathcal{M}_2(h_1^2,h_2^2))$$

is a contraction from the ball

$$\left\{(h_1^2,h_2^2)\in\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4) : \|(h_1^2,h_2^2)\|_{\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)\times\mathcal{C}_\mu^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2\right\}$$

into itself. Then applying a contraction mapping argument, we obtain the following proposition.

Proposition 4.4. *Given $\kappa>0$, $\mu\in(1,2)$, $\gamma_0\in(0,1)$ and $\xi_0\in(0,1)$, there exist $\varepsilon_\kappa>0$ and $c_\kappa>0$ such that for all $\varepsilon\in(0,\varepsilon_\kappa)$, $\gamma\in(\gamma_0,1)$ and $\xi\in(\xi_0,1)$, for all τ_2 in some fixed compact subset of $[\tau_2^-,\tau_2^+]\subset(0,\infty)$ and for φ_j^2 and ψ_j^2 satisfying* (3.5) *and* (3.20)*, there exists a unique (h_1^2,h_2^2) $(:=(h_{1,\varepsilon,\tau_2,\varphi_1^2,\psi_1^2},h_{2,\varepsilon,\tau_2,\varphi_2^2,\psi_2^2}))$ solution of* (4.5) *such that*

$$\|(h_1^2,h_2^2)\|_{C_\mu^{4,\alpha}(\mathbb{R}^4)\times C_\mu^{4,\alpha}(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2.$$

Hence

$$\begin{aligned} v_1(x) &:= \overline{u}(x-x^2)+\frac{1-\gamma}{\gamma(2-\gamma-\xi)}G\left(\frac{\varepsilon x}{\tau_2},x^1\right)-\frac{1-\gamma}{\xi(2-\gamma-\xi)}G\left(\frac{\varepsilon x}{\tau_2},x^3\right)\\ &\quad + h_1^2(x)+H^{\mathrm{int}}\left(\varphi_1^2,\psi_1^2;\frac{x-x^2}{R_\varepsilon^2}\right),\\ v_2(x) &:= \overline{u}(x-x^2)-\frac{1-\xi}{\gamma(2-\gamma-\xi)}G\left(\frac{\varepsilon x}{\tau_2},x^1\right)+\frac{1-\xi}{\xi(2-\gamma-\xi)}G\left(\frac{\varepsilon x}{\tau_2},x^3\right)\\ &\quad + h_2^2(x)+H^{\mathrm{int}}\left(\varphi_2^2,\psi_2^2;\frac{x-x^2}{R_\varepsilon^2}\right) \end{aligned}$$

solves (4.2) *in $B_{R_\varepsilon^2}(x^2)$.*

Remark also that the functions (h_1^i,h_2^i) $(:=(h^i_{1,\varepsilon,\tau_i,\varphi_1^i,\psi_1^i},h^i_{2,\varepsilon,\tau_i,\varphi_2^i,\psi_2^i}))$, for $i\in\{1,2,3\}$, depend continuously on the parameter τ_i.

4.3. The nonlinear exterior problem

By the same arguments as in the proof of Theorem 1.5, exterior problem, we obtain the following proposition.

Proposition 4.5. *Given $\kappa > 0$, there exists $\varepsilon_\kappa > 0$ (depending on κ) such that for any $\varepsilon \in (0, \varepsilon_\kappa)$, λ_i and $\widetilde{x}^i$ satisfying* (3.29) *and functions $\widetilde{\varphi}^i_j$ and $\widetilde{\psi}^i_j$ satisfying* (3.6) *and* (3.28)*, there exists a unique $(\widetilde{v}_1, \widetilde{v}_2)$ $(:= (\widetilde{v}_{1,\varepsilon,\lambda_1,\lambda_2,\widetilde{\boldsymbol{x}},\widetilde{\varphi}^i_1,\widetilde{\psi}^i_1}, \widetilde{v}_{2,\varepsilon,\lambda_2,\lambda_3,\widetilde{\boldsymbol{x}},\widetilde{\varphi}^i_2,\widetilde{\psi}^i_2}))$ solution of* (3.27) *so that for v_k $(k = 1, 2)$ defined by*

$$v_1(x) := \frac{1+\lambda_1}{\gamma} G(x, \widetilde{x}^1) + (1+\lambda_2) G(x, \widetilde{x}^2) + \sum_{i=1}^{3} \chi_{r_0}(x - \widetilde{x}^i) H^{\text{ext}}\left(\widetilde{\varphi}^i_1, \widetilde{\psi}^i_1; \frac{x - \widetilde{x}^i}{r_\varepsilon}\right) + \widetilde{v}_1(x),$$

$$v_2(x) := \frac{1+\lambda_3}{\xi} G(x, \widetilde{x}^3) + (1+\lambda_2) G(x, \widetilde{x}^2) + \sum_{i=1}^{3} \chi_{r_0}(x - \widetilde{x}^i) H^{\text{ext}}\left(\widetilde{\varphi}^i_2, \widetilde{\psi}^i_2; \frac{x - \widetilde{x}^i}{r_\varepsilon}\right) + \widetilde{v}_2(x)$$

solves (3.24) *in $\overline{\Omega}_{r_\varepsilon}(\widetilde{\boldsymbol{x}})$. In addition, we have*

$$\|(\widetilde{v}_1, \widetilde{v}_2)\|_{C^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))} \le 2 c_\kappa r_\varepsilon^2.$$

4.4. The nonlinear Cauchy-data matching

We will gather the results of the previous sections. Using the previous notations, assume that $\widetilde{\boldsymbol{x}} := (\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3) \in \Omega^3$ are given close to $\boldsymbol{x} := (x^1, x^2, x^3)$. Assume also that

$$\boldsymbol{\tau} := (\tau_1, \tau_2, \tau_3) \in [\tau_1^-, \tau_1^+] \times [\tau_2^-, \tau_2^+] \times [\tau_3^-, \tau_3^+] \subset (0, \infty)^3$$

are given (the values of τ_l^- and τ_l^+, for $l = 1, 2, 3$ will be fixed later). First, we consider some set of boundary data $\boldsymbol{\varphi}^i := (\varphi^i_1, \varphi^i_2) \in (\mathcal{C}^{4,\alpha}(S^3))^2$ and $\boldsymbol{\psi}^i := (\psi^i_1, \psi^i_2) \in (\mathcal{C}^{2,\alpha}(S^3))^2$. According to the result of Propositions 3.10, 3.11 and 3.13 and provided $\varepsilon \in (0, \varepsilon_\kappa)$, we can find, $u_{\text{int}} := (u_{\text{int},1}, u_{\text{int},2})$ a solution of (3.7) in $B_{r_\varepsilon}(\widetilde{x}^1) \cup B_{r_\varepsilon}(\widetilde{x}^2) \cup B_{r_\varepsilon}(\widetilde{x}^3)$, which can be decomposed as

$$u_{\text{int},1}(x) := \begin{cases} \frac{1}{\gamma} u_{\varepsilon,\tau_1}(x - \widetilde{x}^1) - \frac{1-\gamma}{\gamma} G(x, \widetilde{x}^2) - \frac{1-\gamma}{\gamma\xi} G(x, \widetilde{x}^3) \\ \quad - \frac{\ln\gamma}{\gamma} + H^{\text{int}}_{\varphi^1_1,\psi^1_1}\left(\frac{x-\widetilde{x}^1}{r_\varepsilon}\right) + h^1_1\left(\frac{R^1_\varepsilon(x-\widetilde{x}^1)}{r_\varepsilon}\right) & \text{in } B_{r_\varepsilon}(\widetilde{x}^1), \\ u_{\varepsilon,\tau_2}(x - \widetilde{x}^2) + \frac{1-\gamma}{\gamma(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, \widetilde{x}^1\right) - \frac{1-\gamma}{\xi(2-\gamma-\xi)} G\left(\frac{\varepsilon x}{\tau_2}, \widetilde{x}^3\right) \\ \quad + H^{\text{int}}_{\varphi^2_1,\psi^2_1}\left(\frac{x-\widetilde{x}^2}{r_\varepsilon}\right) + h^2_1\left(\frac{R^2_\varepsilon(x-\widetilde{x}^2)}{r_\varepsilon}\right) & \text{in } B_{r_\varepsilon}(\widetilde{x}^2), \\ \frac{1}{\gamma} G(x, \widetilde{x}^1) + G(x, \widetilde{x}^2) + H^{\text{int}}_{\varphi^3_1,\psi^3_1}\left(\frac{x-\widetilde{x}^3}{r_\varepsilon}\right) + h^3_1\left(\frac{R^3_\varepsilon(x-\widetilde{x}^3)}{r_\varepsilon}\right) & \text{in } B_{r_\varepsilon}(\widetilde{x}^3) \end{cases}$$

and

$$u_{\mathrm{int},2}(x) := \begin{cases} \frac{1}{\xi}G(x,\widetilde{x}^3) + G(x,\widetilde{x}^2) + H^{\mathrm{int}}_{\varphi_2^1,\psi_2^1}\big(\frac{x-\widetilde{x}^1}{r_\varepsilon}\big) + h_2^1\big(\frac{R_\varepsilon^1(x-\widetilde{x}^1)}{r_\varepsilon}\big) & \text{in } B_{r_\varepsilon}(\widetilde{x}^1), \\ u_{\varepsilon,\tau_2}(x-\widetilde{x}^2) - \frac{1-\xi}{\gamma(2-\gamma-\xi)}G\big(\frac{\varepsilon x}{\tau_2},\widetilde{x}^1\big) + \frac{1-\xi}{\xi(2-\gamma-\xi)}G\big(\frac{\varepsilon x}{\tau_2},\widetilde{x}^3\big) & \\ \quad +H^{\mathrm{int}}_{\varphi_2^2,\psi_2^2}\big(\frac{x-\widetilde{x}^2}{r_\varepsilon}\big) + h_2^2\big(\frac{R_\varepsilon^2(x-\widetilde{x}^2)}{r_\varepsilon}\big) & \text{in } B_{r_\varepsilon}(\widetilde{x}^2), \\ \frac{1}{\xi}u_{\varepsilon,\tau_3}(x-\widetilde{x}^3) - \frac{1-\xi}{\xi}G(x,\widetilde{x}^2) - \frac{1-\xi}{\gamma\xi}G(x,\widetilde{x}^1) & \\ \quad -\frac{\ln\xi}{\xi} + H^{\mathrm{int}}_{\varphi_2^3,\psi_2^3}\big(\frac{x-\widetilde{x}^3}{r_\varepsilon}\big) + h_2^3\big(\frac{R_\varepsilon^3(x-\widetilde{x}^3)}{r_\varepsilon}\big) & \text{in } B_{r_\varepsilon}(\widetilde{x}^3), \end{cases}$$

where for $i \in \{1,2,3\}$ and $j \in \{1,2\}$, $R_\varepsilon^i = \tau_i \frac{r_\varepsilon}{\varepsilon}$ and the functions h_j^i satisfy

$$\|(h_1^1,h_2^1)\|_{\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2, \quad \|(h_1^2,h_2^2)\|_{(\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4))^2} \le 2c_\kappa r_\varepsilon^2$$

and

$$\|(h_1^3,h_2^3)\|_{\mathcal{C}^{4,\alpha}_\delta(\mathbb{R}^4)\times\mathcal{C}^{4,\alpha}_\mu(\mathbb{R}^4)} \le 2c_\kappa r_\varepsilon^2.$$

Similarly, given some boundary data $\widetilde{\varphi}_j^i \in C^{4,\alpha}(S^3)$, $\widetilde{\psi}_j^i \in C^{2,\alpha}(S^3)$ satisfying (3.6), $(\lambda_1,\lambda_2,\lambda_3) \in \mathbb{R}^3$ satisfying (3.29), provided $\varepsilon \in (0,\varepsilon_\kappa)$, by Proposition 3.15, we find a solution $u_{\mathrm{ext}} := (u_{\mathrm{ext},1}, u_{\mathrm{ext},2})$ of (3.7) in $\overline{\Omega} \setminus (B_{r_\varepsilon}(\widetilde{x}^1) \cup B_{r_\varepsilon}(\widetilde{x}^2)) \cup B_{r_\varepsilon}(\widetilde{x}^3))$ which can be decomposed as

$$\begin{aligned} u_{\mathrm{ext},1}(x) &:= \frac{1+\lambda_1}{\gamma}G(x,\widetilde{x}^1) + (1+\lambda_2)G(x,\widetilde{x}^2) \\ &\quad + \sum_{i=1}^3 \chi_{r_0}(x-\widetilde{x}^i)H^{\mathrm{ext}}\left(\widetilde{\varphi}_1^i, \widetilde{\psi}_1^i; \frac{x-\widetilde{x}^i}{r_\varepsilon}\right) + \widetilde{v}_1(x), \\ u_{\mathrm{ext},2}(x) &:= \frac{1+\lambda_3}{\xi}G(x,\widetilde{x}^3) + (1+\lambda_2)G(x,\widetilde{x}^2) \\ &\quad + \sum_{i=1}^3 \chi_{r_0}(x-\widetilde{x}^i)H^{\mathrm{ext}}\left(\widetilde{\varphi}_2^i, \widetilde{\psi}_2^i; \frac{x-\widetilde{x}^i}{r_\varepsilon}\right) + \widetilde{v}_2(x) \end{aligned}$$

with $\widetilde{v}_1, \widetilde{v}_2 \in C^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}}))$ satisfying

$$\|(\widetilde{v}_1,\widetilde{v}_2)\|_{(C^{4,\alpha}_\nu(\overline{\Omega}^*(\widetilde{\boldsymbol{x}})))^2} \le 2c_\kappa r_\varepsilon^2.$$

It remains to determine the parameters and the boundary data in such a way that the function equal to u_{int} in $B_{r_\varepsilon}(\widetilde{x}^1) \cup B_{r_\varepsilon}(\widetilde{x}^2) \cup B_{r_\varepsilon}(\widetilde{x}^3)$ and equal to u_{ext} in $\overline{\Omega}_{r_\varepsilon}(\widetilde{\boldsymbol{x}})$ is a smooth function. This amounts to find the boundary data and the parameters so that, for each $j = 1,2$,

$$(4.9) \qquad \begin{aligned} u_{\mathrm{int},j} &= u_{\mathrm{ext},j}, & \partial_r u_{\mathrm{int},j} &= \partial_r u_{\mathrm{ext},j}, \\ \Delta u_{\mathrm{int},j} &= \Delta u_{\mathrm{ext},j}, & \partial_r \Delta u_{\mathrm{int},j} &= \partial_r \Delta u_{\mathrm{ext},j} \end{aligned}$$

on $\partial B_{r_\varepsilon}(\widetilde{x}^1)$, $\partial B_{r_\varepsilon}(\widetilde{x}^2)$ and $\partial B_{r_\varepsilon}(\widetilde{x}^3)$.

Suppose that (4.9) is verified, this provides that for each ε small enough $u_\varepsilon \in \mathcal{C}^{4,\alpha}$ (which is obtained by patching together the functions $u_{\rm int}$ and the function $u_{\rm ext}$), a weak solution of our system and elliptic regularity theory implies that this solution is in fact smooth. That will complete the proof since, as ε tends to 0, the sequence of solutions we have obtain satisfies the required singular limit behaviors.

Before we proceed, the following remarks are due. First it will be convenient to observe that the function u_{ε,τ_i} can be expanded as

$$u_{\varepsilon,\tau_i}(x) = -4\ln\tau_i - 8\ln|x| + \mathcal{O}\left(\frac{\varepsilon^2\tau_i^{-2}}{|x|^2}\right) \quad \text{on } \partial B_{r_\varepsilon}(0).$$

• On $\partial B_{r_\varepsilon}(\widetilde{x}^1)$, according to the proof of Theorem 1.5 and since when ε tend to 0, it is enough to choose τ_1^- and τ_1^+ in such a way that

$$4\ln(\tau_1^-) < -\ln\gamma - \mathcal{E}_1(x^1,\boldsymbol{x}) < 4\ln(\tau_1^+),$$

where

$$\mathcal{E}_1(\,\cdot\,,\widetilde{\boldsymbol{x}}) := H(\,\cdot\,,\widetilde{x}^1) + G(\,\cdot\,,\widetilde{x}^2) + \frac{1-\gamma}{\xi}G(\,\cdot\,,\widetilde{x}^3).$$

Also using the fact that

$$\begin{aligned}\varphi_1^1 &= \varphi_{1,0}^1 + \varphi_{1,1}^1 + \varphi_1^{1,\perp}, &\qquad \psi_1^1 &= 8\varphi_{1,0}^1 + 12\varphi_{1,1}^1 + \psi_1^{1,\perp},\\ \widetilde{\varphi}_1^1 &= \widetilde{\varphi}_{1,0}^1 + \widetilde{\varphi}_{1,1}^1 + \widetilde{\varphi}_1^{1,\perp}, &\qquad \widetilde{\psi}_1^1 &= \widetilde{\psi}_{1,1}^1 + \widetilde{\psi}_1^{1,\perp},\end{aligned}$$

where $\varphi_{1,0}^1, \widetilde{\varphi}_{1,0}^1 \in \mathbb{E}_0 = \mathbb{R}$ are constant on S^3, $\varphi_{1,1}^1$, $\widetilde{\varphi}_{1,1}^1$, $\widetilde{\psi}_{1,1}^1$ belong to $\mathbb{E}_1 = \mathrm{Span}\{e_1, e_2, e_3, e_4\}$ and $\varphi_1^{1,\perp}$, $\widetilde{\varphi}_1^{1,\perp}$, $\psi_1^{1,\perp}$, $\widetilde{\psi}_1^{1,\perp}$ are $L^2(S^3)$ orthogonal to $\mathbb{E}_0$ and $\mathbb{E}_1$. We can prove that

$$\begin{aligned}(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) &= 0, &\qquad \partial_r(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) &= 0,\\ \Delta(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) &= 0, &\qquad \partial_r\Delta(u_{\mathrm{int},1} - u_{\mathrm{ext},1})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) &= 0\end{aligned}$$

on S^3 yield to

(4.10)
$$T_{1,\varepsilon}^1 = \big(t_1, \lambda_1, \varphi_{1,0}^1, \widetilde{\varphi}_{1,0}^1, \varphi_{1,1}^1, \widetilde{\varphi}_{1,1}^1, \widetilde{\psi}_{1,1}^1, \overline{\nabla}\mathcal{E}_1(\widetilde{x}^1,\widetilde{\boldsymbol{x}}), \varphi_1^{1,\perp}, \widetilde{\varphi}_1^{1,\perp}, \psi_1^{1,\perp}, \widetilde{\psi}_1^{1,\perp}\big) = \mathcal{O}(r_\varepsilon^2),$$

where

$$t_1 = \frac{1}{\ln r_\varepsilon}\big[4\ln\tau_1 + \ln\gamma + \mathcal{E}_1(\widetilde{x}^1,\widetilde{\boldsymbol{x}})\big].$$

Finally, using the fact that

$$\begin{aligned}\varphi_2^1 &= \varphi_{2,0}^1 + \varphi_{2,1}^1 + \varphi_2^{1,\perp}, &\qquad \psi_2^1 &= 8\varphi_{2,0}^1 + 12\varphi_{2,1}^1 + \psi_2^{1,\perp},\\ \widetilde{\varphi}_2^1 &= \widetilde{\varphi}_{2,0}^1 + \widetilde{\varphi}_{2,1}^1 + \widetilde{\varphi}_2^{1,\perp}, &\qquad \widetilde{\psi}_2^1 &= \widetilde{\psi}_{2,1}^1 + \widetilde{\psi}_2^{1,\perp}\end{aligned}$$

with $\varphi^1_{2,0}, \widetilde{\varphi}^1_{2,0} \in \mathbb{E}_0$, $\varphi^1_{2,1}, \widetilde{\varphi}^1_{2,1}, \widetilde{\psi}^1_{2,1} \in \mathbb{E}_1$ and $\varphi_2^{1,\perp}$, $\widetilde{\varphi}_2^{1,\perp}$, $\psi_2^{1,\perp}$, $\widetilde{\psi}_2^{1,\perp}$ belong to $(L^2(S^3))^\perp$. We can prove that

$$\begin{aligned}(u_{\text{int},2} - u_{\text{ext},2})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) = 0, &\qquad \partial_r(u_{\text{int},2} - u_{\text{ext},2})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) = 0,\\ \Delta(u_{\text{int},2} - u_{\text{ext},2})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) = 0, &\qquad \partial_r\Delta(u_{\text{int},2} - u_{\text{ext},2})(\widetilde{x}^1 + r_\varepsilon\,\cdot\,) = 0\end{aligned}$$

on S^3 yield to

$$T^1_{2,\varepsilon} = \left(\varphi^1_{2,0}, \widetilde{\varphi}^1_{2,0}, \varphi^1_{2,1}, \widetilde{\varphi}^1_{2,1}, \widetilde{\psi}^1_{2,1}, \varphi_2^{1,\perp}, \widetilde{\varphi}_2^{1,\perp}, \psi_2^{1,\perp}, \widetilde{\psi}_2^{1,\perp}\right) = \mathcal{O}(r_\varepsilon^2). \tag{4.11}$$

As usual, the terms $\mathcal{O}(r_\varepsilon^2)$ depend nonlinearly on all the variables on the left side, but is bounded (in the appropriate norm) by a constant (independent of ε and κ) times r_ε^2, provided $\varepsilon \in (0, \varepsilon_\kappa)$.

- On $\partial B_{r_\varepsilon}(\widetilde{x}^2)$, we have

$$\begin{aligned}(u_{\text{int},1} - u_{\text{ext},1})(x) = {}& -4\ln\tau_2 + 8\lambda_2 \ln|x - \widetilde{x}^2| + h_1^2\left(R_\varepsilon^2 \frac{x - \widetilde{x}^2}{r_\varepsilon}\right)\\ &+ H^{\text{int}}\left(\varphi_1^2, \psi_1^2; \frac{x - \widetilde{x}^2}{r_\varepsilon}\right) - H^{\text{ext}}\left(\widetilde{\varphi}_1^2, \widetilde{\psi}_1^2; \frac{x - \widetilde{x}^2}{r_\varepsilon}\right)\\ &- \left[(2 - \gamma - \xi)H(x, \widetilde{x}^2) + \frac{1-\xi}{\gamma} G(x, \widetilde{x}^1) + \frac{1-\gamma}{\xi} G(x, \widetilde{x}^3)\right]\\ &+ \mathcal{O}\left(\frac{\varepsilon^2 \tau_2^{-2}}{|x - \widetilde{x}^2|^2}\right) + \mathcal{O}(r_\varepsilon^2)\end{aligned} \tag{4.12}$$

and

$$\begin{aligned}(u_{\text{int},2} - u_{\text{ext},2})(x) = {}& -4\ln\tau_2 + 8\lambda_2 \ln|x - \widetilde{x}^2| + h_2^2\left(R_\varepsilon^2 \frac{x - \widetilde{x}^2}{r_\varepsilon}\right)\\ &+ H^{\text{int}}\left(\varphi_2^2, \psi_2^2; \frac{x - \widetilde{x}^2}{r_\varepsilon}\right) - H^{\text{ext}}\left(\widetilde{\varphi}_2^2, \widetilde{\psi}_2^2; \frac{x - \widetilde{x}^2}{r_\varepsilon}\right)\\ &- \left[(2 - \gamma - \xi)H(x, \widetilde{x}^2) + \frac{1-\xi}{\gamma} G(x, \widetilde{x}^1) + \frac{1-\gamma}{\xi} G(x, \widetilde{x}^3)\right]\\ &+ \mathcal{O}\left(\frac{\varepsilon^2 \tau_2^{-2}}{|x - \widetilde{x}^2|^2}\right) + \mathcal{O}(r_\varepsilon^2).\end{aligned} \tag{4.13}$$

Next, even though all functions are defined on $\partial B_{r_\varepsilon}(\widetilde{x}^2)$ in (4.9), it will be more convenient to solve on S^3, for $i = 1, 2$, the following set of equations

$$\begin{aligned}(u_{\text{int},i} - u_{\text{ext},i})(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) = 0, &\qquad \partial_r(u_{\text{int},i} - u_{\text{ext},i})(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) = 0,\\ \Delta(u_{\text{int},i} - u_{\text{ext},i})(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) = 0, &\qquad \partial_r\Delta(u_{\text{int},i} - u_{\text{ext},i})(\widetilde{x}^2 + r_\varepsilon\,\cdot\,) = 0.\end{aligned} \tag{4.14}$$

Since the boundary data are chosen to satisfy (3.5) or (3.6). We decompose

$$\begin{aligned}\varphi_i^2 = \varphi^2_{i,0} + \varphi^2_{i,1} + \varphi_i^{2,\perp}, &\qquad \psi_i^2 = 8\varphi^2_{i,0} + 12\varphi^2_{i,1} + \psi_i^{2,\perp},\\ \widetilde{\varphi}_i^2 = \widetilde{\varphi}^2_{i,0} + \widetilde{\varphi}^2_{i,1} + \widetilde{\varphi}_i^{2,\perp}, &\qquad \widetilde{\psi}_i^2 = \widetilde{\psi}^2_{i,1} + \widetilde{\psi}_i^{2,\perp},\end{aligned}$$

where $\varphi^1_{i,0}, \widetilde{\varphi}^1_{i,0} \in \mathbb{E}_0 = \mathbb{R}$ are constant on S^3, $\varphi^1_{i,1}$, $\widetilde{\varphi}^1_{i,1}$, $\widetilde{\psi}^1_{i,1}$ belong to $\mathbb{E}_1 = \mathrm{Span}\{e_1, e_2, e_3, e_4\}$ and $\varphi^{1,\perp}_i$, $\widetilde{\varphi}^{1,\perp}_i$, $\psi^{1,\perp}_i$, $\widetilde{\psi}^{1,\perp}_i$ are $L^2(S^3)$ orthogonal to $\mathbb{E}_0$ and $\mathbb{E}_1$.

We insist that, for $x \in S^3$, both equations (4.12) and (4.13) involve the same relation of the parameter τ_2 and the appropriate energy $\mathcal{E}_2$. Then we have

$$\begin{aligned}(u_{\mathrm{int},i} - u_{\mathrm{ext},i})(\widetilde{x}^2 + r_\varepsilon x) = {} & -4\ln\tau_2 + 8\lambda_2 \ln r_\varepsilon |x| + H^{\mathrm{int}}(\varphi^2_i, \psi^2_i, x) - H^{\mathrm{ext}}(\widetilde{\varphi}^2_i, \widetilde{\psi}^2_i, x) \\ & - \left[(2-\gamma-\xi)H(\widetilde{x}^2, \widetilde{x}^2) + \frac{1-\xi}{\gamma} G(\widetilde{x}^2, \widetilde{x}^1) + \frac{1-\gamma}{\xi} G(\widetilde{x}^2, \widetilde{x}^3)\right] \\ & + \mathcal{O}(r^2_\varepsilon).\end{aligned}$$

Projecting the set of equations (4.14) over $\mathbb{E}_0$, we get

$$\begin{aligned} -4\ln\tau_2 + 8\lambda_2 \ln r_\varepsilon + \varphi^2_{i,0} - \widetilde{\varphi}^2_{i,0} - \mathcal{E}_2(\widetilde{x}^2, \widetilde{x}) + \mathcal{O}(r^2_\varepsilon) &= 0, \\ 8\lambda_2 + 2\varphi^2_{i,0} + 2\widetilde{\varphi}^2_{i,0} + \mathcal{O}(r^2_\varepsilon) &= 0, \\ 16\lambda_2 + 8\varphi^2_{i,0} + \mathcal{O}(r^2_\varepsilon) &= 0, \\ -32\lambda_2 + \mathcal{O}(r^2_\varepsilon) &= 0, \end{aligned} \tag{4.15}$$

where

$$\mathcal{E}_2(\,\cdot\,, \widetilde{\boldsymbol{x}}) := (2-\gamma-\xi)H(\,\cdot\,, \widetilde{x}^2) + \frac{1-\xi}{\gamma} G(\,\cdot\,, \widetilde{x}^1) + \frac{1-\gamma}{\xi} G(\,\cdot\,, \widetilde{x}^3).$$

The system (4.15) can be simply written as

$$\lambda_2 = \mathcal{O}(r^2_\varepsilon), \quad \varphi^2_{i,0} = \mathcal{O}(r^2_\varepsilon), \quad \widetilde{\varphi}^2_{i,0} = \mathcal{O}(r^2_\varepsilon) \quad \text{and} \quad \frac{1}{\ln r_\varepsilon}\big[4\ln\tau_2 + \mathcal{E}_2(\widetilde{x}^2, \widetilde{\boldsymbol{x}})\big] = \mathcal{O}(r^2_\varepsilon).$$

We are now in a position to define τ_2^- and τ_2^+. In fact, according to the above analysis, as ε tends to 0, we expect that $\widetilde{x}^i$ will converge to x^i for $i \in \{1,2,3\}$ and τ_2 will converge to τ_2^* satisfying

$$4\ln\tau_2^* = -\mathcal{E}_2(x^2, \boldsymbol{x}).$$

Hence it is enough to choose τ_2^- and τ_2^+ in such a way that

$$4\ln(\tau_2^-) < -\mathcal{E}_2(x^2, \boldsymbol{x}) < 4\ln(\tau_2^+).$$

Consider now the projection of (4.14) over $\mathbb{E}_1$. Given a smooth function f defined in Ω, we identify its gradient $\nabla f = (\partial_{x_1} f, \ldots, \partial_{x_4} f)$ with the element of $\mathbb{E}_1$,

$$\overline{\nabla} f = \sum_{i=1}^4 \partial_{x_i} f e_i.$$

Keeping these notations in mind, we obtain the system of equations

$$\varphi^2_{i,1} - \widetilde{\varphi}^2_{i,1} - \overline{\nabla}\mathcal{E}_2(\widetilde{x}^2, \widetilde{\boldsymbol{x}}) + \mathcal{O}(r^2_\varepsilon) = 0,$$

$$
\begin{aligned}
3\varphi_{i,1}^2 + 3\widetilde{\varphi}_{i,1}^2 + \frac{1}{2}\widetilde{\psi}_{i,1}^2 - \overline{\nabla}\mathcal{E}_2(\widetilde{x}^2, \widetilde{\boldsymbol{x}}) + \mathcal{O}(r_\varepsilon^2) = 0,\\
15\varphi_{i,1}^2 - 3\widetilde{\varphi}_{i,1}^2 - \widetilde{\psi}_{i,1}^2 + \mathcal{O}(r_\varepsilon^2) = 0,\\
15\varphi_{i,1}^2 + 15\widetilde{\varphi}_{i,1}^2 + \frac{18}{4}\widetilde{\psi}_{i,1}^2 + \mathcal{O}(r_\varepsilon^2) = 0,
\end{aligned}
$$

which can be simplified as follows

$$
\varphi_{i,1}^2 = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_{i,1}^2 = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\psi}_{i,1}^2 = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \overline{\nabla}\mathcal{E}_2(\widetilde{x}^2, \widetilde{\boldsymbol{x}}) = \mathcal{O}(r_\varepsilon^2).
$$

Finally, we consider the projection onto $L^2(S^3)^\perp$. This yields the system

$$
\begin{aligned}
\varphi_i^{2,\perp} - \widetilde{\varphi}_i^{2,\perp} + \mathcal{O}(r_\varepsilon^2) = 0, \qquad & \partial_r\big(H^{\text{int}}_{\varphi_i^{2,\perp},\psi_i^{2,\perp}} - H^{\text{ext}}_{\widetilde{\varphi}_i^{2,\perp},\widetilde{\psi}_i^{2,\perp}}\big) + \mathcal{O}(r_\varepsilon^2) = 0,\\
\psi_i^{2,\perp} - \widetilde{\psi}_i^{2,\perp} + \mathcal{O}(r_\varepsilon^2) = 0, \qquad & \partial_r\Delta\big(H^{\text{int}}_{\varphi_i^{2,\perp},\psi_i^{2,\perp}} - H^{\text{ext}}_{\widetilde{\varphi}_i^{2,\perp},\widetilde{\psi}_i^{2,\perp}}\big) + \mathcal{O}(r_\varepsilon^2) = 0.
\end{aligned}
$$

Thanks to the result of Lemma 3.8, this last system can be rewritten as

$$
\varphi_i^{2,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \widetilde{\varphi}_i^{2,\perp} = \mathcal{O}(r_\varepsilon^2), \quad \psi_i^{2,\perp} = \mathcal{O}(r_\varepsilon^2) \quad \text{and} \quad \widetilde{\psi}_i^{2,\perp} = \mathcal{O}(r_\varepsilon^2).
$$

If we define the parameter $t_2 \in \mathbb{R}$ by

$$
t_2 = \frac{1}{\ln r_\varepsilon}\big[4\ln\tau_2 + \mathcal{E}_2(\widetilde{x}^2, \widetilde{\boldsymbol{x}})\big],
$$

then the systems found by projecting (4.14) gather in this equality

$$
T_{c,\varepsilon}^2 = \big(t_2, \lambda_2, \varphi_{i,0}^2, \widetilde{\varphi}_{i,0}^2, \varphi_{i,1}^2, \widetilde{\varphi}_{i,1}^2, \widetilde{\psi}_{i,1}^2, \overline{\nabla}\mathcal{E}_2(\widetilde{x}^2, \widetilde{\boldsymbol{x}}), \varphi_i^{2,\perp}, \widetilde{\varphi}_i^{2,\perp}, \psi_i^{2,\perp}, \widetilde{\psi}_i^{2,\perp}\big) = \mathcal{O}(r_\varepsilon^2) \tag{4.16}
$$

for $i = 1, 2$. As usual, the terms $\mathcal{O}(r_\varepsilon^2)$ depend nonlinearly on all the variables on the left side, but is bounded (in the appropriate norm) by a constant (independent of ε and κ) times r_ε^2, provided $\varepsilon \in (0, \varepsilon_\kappa)$.

- On $\partial B_{r_\varepsilon}(\widetilde{x}^3)$, according to the proof of Theorem 1.5, using the fact that

$$
\begin{aligned}
\varphi_1^3 = \varphi_{1,0}^3 + \varphi_{1,1}^3 + \varphi_1^{3,\perp}, \qquad & \psi_1^3 = 8\varphi_{1,0}^3 + 12\varphi_{1,1}^3 + \psi_1^{3,\perp},\\
\widetilde{\varphi}_1^3 = \widetilde{\varphi}_{1,0}^3 + \widetilde{\varphi}_{1,1}^3 + \widetilde{\varphi}_1^{3,\perp}, \qquad & \widetilde{\psi}_1^3 = \widetilde{\psi}_{1,1}^3 + \widetilde{\psi}_1^{3,\perp}
\end{aligned}
$$

with $\varphi_{1,0}^3, \widetilde{\varphi}_{1,0}^3 \in \mathbb{E}_0$, $\varphi_{1,1}^3, \widetilde{\varphi}_{1,1}^3, \widetilde{\psi}_{1,1}^3 \in \mathbb{E}_1$ and $\varphi_1^{3,\perp}$, $\widetilde{\varphi}_1^{3,\perp}$, $\psi_1^{3,\perp}$, $\widetilde{\psi}_1^{3,\perp}$ belong to $(L^2(S^3))^\perp$. We can prove that

$$
T_{1,\varepsilon}^3 = \big(\varphi_{1,0}^3, \widetilde{\varphi}_{1,0}^3, \varphi_{1,1}^3, \widetilde{\varphi}_{1,1}^3, \widetilde{\psi}_{1,1}^3, \varphi_1^{3,\perp}, \widetilde{\varphi}_1^{3,\perp}, \psi_1^{3,\perp}, \widetilde{\psi}_1^{3,\perp}\big) = \mathcal{O}(r_\varepsilon^2). \tag{4.17}
$$

On other hand, according to the proof of Theorem 1.5 and since when ε tend to 0, its enough to choose τ_1^- and τ_1^+ is such a way that

$$
4\ln(\tau_3^-) < -\ln\xi - \mathcal{E}_3(x^3, \boldsymbol{x}) < 4\ln(\tau_3^+),
$$

where

$$\mathcal{E}_3(\,\cdot\,,\widetilde{\boldsymbol{x}}) := H(\,\cdot\,,\widetilde{x}^3) + G(\,\cdot\,,\widetilde{x}^2) + \frac{1-\xi}{\gamma} G(\,\cdot\,,\widetilde{x}^1).$$

Also using the fact that

$$\begin{aligned} \varphi_2^3 &= \varphi_{2,0}^3 + \varphi_{2,1}^3 + \varphi_2^{3,\perp}, & \psi_2^3 &= 8\varphi_{2,0}^3 + 12\varphi_{2,1}^3 + \psi_2^{3,\perp}, \\ \widetilde{\varphi}_2^3 &= \widetilde{\varphi}_{2,0}^3 + \widetilde{\varphi}_{2,1}^3 + \widetilde{\varphi}_2^{3,\perp}, & \widetilde{\psi}_2^3 &= \widetilde{\psi}_{2,1}^3 + \widetilde{\psi}_2^{3,\perp}, \end{aligned}$$

where $\varphi_{2,0}^3, \widetilde{\varphi}_{2,0}^3 \in \mathbb{E}_0 = \mathbb{R}$ are constant on S^3, $\varphi_{2,1}^3$, $\widetilde{\varphi}_{2,1}^3$, $\widetilde{\psi}_{2,1}^3$ belong to $\mathbb{E}_1 = \operatorname{Span}\{e_1, e_2, e_3, e_4\}$ and $\varphi_2^{3,\perp}$, $\widetilde{\varphi}_2^{3,\perp}$, $\psi_2^{3,\perp}$, $\widetilde{\psi}_2^{3,\perp}$ are $L^2(S^3)$ orthogonal to $\mathbb{E}_0$ and $\mathbb{E}_1$. We can prove that

$$\begin{aligned} (u_{\mathrm{int},i} - u_{\mathrm{ext},i})(\widetilde{x}^3 + r_\varepsilon\,\cdot\,) &= 0, & \partial_r(u_{\mathrm{int},i} - u_{\mathrm{ext},i})(\widetilde{x}^3 + r_\varepsilon\,\cdot\,) &= 0, \\ \Delta(u_{\mathrm{int},i} - u_{\mathrm{ext},i})(\widetilde{x}^3 + r_\varepsilon\,\cdot\,) &= 0, & \partial_r\Delta(u_{\mathrm{int},i} - u_{\mathrm{ext},i})(\widetilde{x}^3 + r_\varepsilon\,\cdot\,) &= 0 \end{aligned}$$

on S^3 yield to

$$T_{2,\varepsilon}^3 = \big(t_3, \lambda_3, \varphi_{2,0}^3, \widetilde{\varphi}_{2,0}^3, \varphi_{2,1}^3, \widetilde{\varphi}_{2,1}^3, \widetilde{\psi}_{2,1}^3 \overline{\nabla}\mathcal{E}_3(\widetilde{x}^3, \widetilde{\boldsymbol{x}}), \varphi_2^{3,\perp}, \widetilde{\varphi}_2^{3,\perp}, \psi_2^{3,\perp}, \widetilde{\psi}_2^{3,\perp}\big) = \mathcal{O}(r_\varepsilon^2), \tag{4.18}$$

where

$$t_3 = \frac{1}{\ln r_\varepsilon}\big[4\ln\tau_3 + \ln\xi + \mathcal{E}_3(\widetilde{x}^3, \widetilde{\boldsymbol{x}})\big].$$

As usual, the terms $\mathcal{O}(r_\varepsilon^2)$ depend nonlinearly on all the variables on the left side, but is bounded (in the appropriate norm) by a constant (independent of ε and κ) times r_ε^2, provided $\varepsilon \in (0, \varepsilon_\kappa)$.

We recall that $\boldsymbol{d} = r_\varepsilon(\widetilde{\boldsymbol{x}} - \boldsymbol{x})$, in addition the previous systems can be written as for $i = 1, 2, 3$:

$$\big(\boldsymbol{d}, t_i, \lambda_i, \varphi^i, \widetilde{\varphi}^i, \psi^i, \widetilde{\psi}^i, \overline{\nabla}\mathcal{E}_i\big) = \mathcal{O}(r_\varepsilon^2).$$

Combining (4.10), (4.11), (4.16), (4.17) and (4.18), we have

$$T_{i,\varepsilon} = (T_{i,\varepsilon}^1, T_{i,\varepsilon}^2, T_{i,\varepsilon}^3) = (\mathcal{O}(r_\varepsilon^2), \mathcal{O}(r_\varepsilon^2), \mathcal{O}(r_\varepsilon^2)) \quad \text{for } i = 1, 2. \tag{4.19}$$

Then the nonlinear mapping which appears on the right-hand side of (4.19) is continuous, compact. In addition, reducing ε_κ if necessary, this nonlinear mapping sends the ball of radius κr_ε^2 (for the natural product norm) into itself, provided κ is fixed large enough. Applying Schauder's fixed point theorem in the ball of radius κr_ε^2 in the product space where the entries live, we obtain the existence of a solution of equation (4.19). This completes the proof of Theorem 1.6.

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