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A Family of Threefolds of General Type with Canonical Map of High Degree

Davide Frapporti* and Christian Gleissner

Abstract. In this note we provide a two-dimensional family of smooth minimal threefolds of general type with canonical map of degree 96, improving the previous known bound of 72.

1. Introduction

In this paper we consider smooth varieties of general type. In the case of curves it is classically known that the canonical map is either an embedding or a degree 2 map onto $\mathbb{P}^1$, the latter happens precisely when the curve is hyperelliptic.

In higher dimensions the situation is much less clear and it is natural to ask: "Assuming that the canonical map is generically finite, is its degree universally bounded? If so, what is the maximal possible value of the *canonical degree*?" In this paper by canonical degree we mean the degree of the canonical map.

For surfaces Beauville [2] showed that the canonical degree is at most 36, and equality holds if and only if $p_g = 3$, $q = 0$, $K^2 = 36$ and the canonical system is base point free.

In the threefold case, Hacon [13] established 576 as a bound for the canonical degree. Later on in [7] this bound was improved to 360, which can be achieved if and only if $p_g = 4$, $q_1 = 2$, $\chi(\mathcal{O}) = -5$, $K^3 = 360$ and the canonical system is base point free. In [13] Hacon also explained that if one allows terminal singularities, there is no bound for the canonical degree, presenting an infinite series of threefolds with index 2 terminal singularities and arbitrarily high canonical degree.

Beauville and Hacon's proofs rely heavily on the Miyaoka-Yau inequality, which cannot be used to control the canonical degree in higher dimensions.

For surfaces, as well as for Gorenstein minimal threefolds, the maximal value can be achieved only if the Miyaoka-Yau inequality becomes an equality, i.e., for ball-quotients. Unfortunately these are notoriously hard to handle and it is not clear if the bounds are sharp. In [21] Yeung claims the existence of a surface realizing degree 36, but the proof seems to have a gap, as pointed out by the Mathscinet Reviewer (MR Number:

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*Corresponding author.

MR3673651). More recently Rito [20] used the Borisov-Keum equations of a fake projective plane to construct a surface having canonical map of degree 36.

Leaving ball-quotients aside, one can still try to construct examples having high canonical degree. For surfaces, the highest canonical degrees obtained so far are: 16 due to Persson (see [18], double covering of a Campedelli surface), 24 due to Rito (see [19], abelian covering of a blow-up of $\mathbb{P}^2$), and 32 due to the second author together with Pignatelli and Rito (see [11], product-quotient surface). In dimension 3 the situation is less established. The highest known values of the canonical degree are 32, 48, 64 and 72, which can be achieved simply by taking the product $C \times S$, where C is a hyperelliptic curve and S is one of the above mentioned surfaces. Other examples with canonical degree 32 and 64 were constructed in [3].

In this paper we set a new record for canonical degree of a smooth threefold:

Theorem 1.1 (Theorem 3.3)**.** *There exists a two dimensional family of smooth threefolds X with canonical degree* 96 *and whose canonical image is a quadric.*

The family lives in a 6-dimensional family of threefolds isogenous to a product. Threefolds isogenous to a product are a special case of *product-quotient* varieties (see [5, 9, 10]), i.e., varieties birational to the quotient of the product of smooth projective curves by the action of a finite group.

In recent years the intensive work on product-quotient varieties has produced several interesting examples, e.g., the mentioned example of a surface of general type with canonical map of degree 32; new topological types for surface of general type, in particular a family of surfaces of general type with $K^2 = 7$, $p_g = q = 2$ (see [4] and the references therein); and recently the first examples of rigid but not infinitesimally rigid compact complex manifolds [1]. For other interesting applications see [6, 8, 12, 15, 16].

We point out that we originally found this 2-dimensional family using the technology developed in the papers mentioned above, together with the theory of abelian coverings (see [14, 17]). However, we decided not to use the language of product-quotients here, but to give a simple and self-contained description using explicit equations instead.

The paper is organized as follows: in Section 2 we construct the 6-dimensional family and in Section 3 we prove the main theorem.

2. The construction

Let $C_{a,b}$ be the curve in $\mathbb{P}^4$ defined by the equations

$$x_2^2 = x_0^2 - x_1^2, \quad x_3^2 = x_0^2 - ax_1^2, \quad x_4^2 = x_0^2 - bx_1^2, \tag{2.1}$$

where $a, b \in \mathbb{C} \setminus \{0,1\}$, $a \neq b$. The curve $C_{a,b}$ is then a smooth canonical curve of genus 5. We note that $C_{a,b}$ is invariant under the $(\mathbb{Z}_2)^4$-action on $\mathbb{P}^4$ given by

$$e_i \colon x_i \mapsto -x_i, \quad x_j \mapsto x_j, \quad i \neq j$$

where $\{e_1, \ldots, e_4\}$ is the standard basis of $(\mathbb{Z}_2)^4$ and $e_0 := e_1 + e_2 + e_3 + e_4$.

The map $\pi \colon C_{a,b} \to \mathbb{P}^1$, $\pi(x_0 : x_1 : x_2 : x_3 : x_4) = (x_0^2 : x_1^2)$ is a covering of the projective line branched in 5 points:

$$p_0 = [0:1], \quad p_1 = [1:0], \quad p_2 = [1:1], \quad p_3 = [a:1] \quad \text{and} \quad p_4 = [b:1].$$

Since this map has degree 16 and is $(\mathbb{Z}_2)^4$-equivariant, it is the quotient map, and the stabilizer of a ramification point in $\pi^{-1}(p_i)$ is the group of order 2 generated by e_i.

We consider now the threefold $T := C_{a_1,b_1} \times C_{a_2,b_2} \times C_{a_3,b_3} \subset \mathbb{P}^4_{\mathbf{x}} \times \mathbb{P}^4_{\mathbf{y}} \times \mathbb{P}^4_{\mathbf{z}}$ with the "twisted" $(\mathbb{Z}_2)^4$-action defined by

$$e_i(\mathbf{x}, \mathbf{y}, \mathbf{z}) := (e_i \cdot \mathbf{x}, (Ae_i) \cdot \mathbf{y}, (A^2 e_i) \cdot \mathbf{z}),$$

where $\mathbf{x} := (x_0 : x_1 : x_2 : x_3 : x_4)$ (similarly for $\mathbf{y}$ and $\mathbf{z}$) and $A \in \mathrm{GL}(4, \mathbb{Z}_2)$ is the matrix

$$A := \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A^2 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

Remark 2.1. (1) A is chosen so that

$$\{e_0, \ldots, e_4, Ae_0, \ldots, Ae_4, A^2 e_0, \ldots, A^2 e_4\} = (\mathbb{Z}_2)^4 \setminus \{0\}.$$

(2) A has order 3 and satisfies $I + A + A^2 = 0$.

The rest of the paper is devoted to the study of the canonical map of the quotient threefold

$$X := (C_{a_1,b_1} \times C_{a_2,b_2} \times C_{a_3,b_3})/(\mathbb{Z}_2)^4.$$

Lemma 2.2. *The threefold X is smooth of general type, with ample canonical class K_X and invariants*

$$\chi(\mathcal{O}_X) = -4, \quad K_X^3 = -48\chi(\mathcal{O}_X) = 192.$$

Proof. The e_i are the unique non-trivial elements of $(\mathbb{Z}_2)^4$ having fixed points on the curve. According to Remark 2.1, $Ae_j \neq e_k$ for all j, k, therefore the $(\mathbb{Z}_2)^4$-action on the product T is free and X is smooth.

Since T is of general type, with ample canonical class and the action is free, the quotient X is also of general type with ample canonical class. Moreover, by the freeness of the action it follows

$$16\chi(\mathcal{O}_X) = \chi(\mathcal{O}_T) = \prod_{i=1}^{3}(1 - g(C_{a_i,b_i})) = -4^3,$$

$$16K_X^3 = K_T^3 = 6\prod_{i=1}^{3}(2g(C_{a_i,b_i}) - 2) = -48\chi(\mathcal{O}_T). \qquad \square$$

Remark 2.3. Since C_{a_i,b_i} is a canonical curve, the restriction $H^0(\mathbb{P}^4, \mathcal{O}(1)) \to H^0(C_{a_i,b_i}, K_{C_{a_i,b_i}})$ is an isomorphism. Therefore

$$H^0(C_{a_1,b_1}, K_{C_{a_1,b_1}}) = \langle x_0, x_1, x_2, x_3, x_4\rangle,$$
$$H^0(C_{a_2,b_2}, K_{C_{a_2,b_3}}) = \langle y_0, y_1, y_2, y_3, y_4\rangle,$$
$$H^0(C_{a_3,b_3}, K_{C_{a_2,b_3}}) = \langle z_0, z_1, z_2, z_3, z_4\rangle.$$

Lemma 2.4. *The canonical system of X is spanned by*

$$s_i := x_i y_i z_i, \quad i = 0, \ldots, 4.$$

In particular $p_g(X) = 5$.

Proof. By Künneth's formula we have the following isomorphism

$$H^0(K_T) \cong H^0(K_{C_{a_1,b_1}}) \otimes H^0(K_{C_{a_2,b_2}}) \otimes H^0(K_{C_{a_3,b_3}}) = \langle x_i y_j z_k\rangle_{i,j,k}.$$

The $(\mathbb{Z}_2)^4$-action on the product induces an action on $H^0(K_T)$ via pull-back:

$$e_\alpha^*(x_i y_j z_k) = e_\alpha^*(x_i) \cdot (Ae_\alpha)^*(y_j) \cdot (A^2 e_\alpha)^*(z_k) = (-1)^{n_\alpha(i,j,k)} x_i y_j z_k,$$

where $n_\alpha(i,j,k) = I_{i\alpha} + A_{j\alpha} + A^2_{k\alpha}$ for $\alpha = 1, 2, 3, 4$; and $I_{0\alpha} := 0$, $A_{0\alpha} := 0$, $A^2_{0\alpha} := 0$ for all α. By the freeness of the action it holds

$$H^0(X, K_X) = H^0(T, K_T)^{(\mathbb{Z}_2)^4} = \langle x_i y_j z_k \mid n_\alpha(i,j,k) = 0, \forall\, \alpha\rangle,$$

and we conclude

$$H^0(X, K_X) = \langle x_0y_0z_0, x_1y_1z_1, x_2y_2z_2, x_3y_3z_3, x_4y_4z_4\rangle,$$

since $I_{i\alpha} + A_{j\alpha} + A^2_{k\alpha} = 0$ for all α if and only if $i = j = k$ (cf. Remark 2.1). $\square$

Remark 2.5. In a similar way one can determine all the other Hodge numbers of X:

$$q_2(X) = 0, \quad q_1(X) = 0, \quad h^{1,1}(X) = 3, \quad h^{2,1}(X) = 15.$$

3. The canonical map

Theorem 3.1. *The canonical system $|K_X|$ is base point free, and the canonical image is a hypersurface in $\mathbb{P}^4$.*

Proof. It follows from (2.1) and the conditions on the parameters a_k and b_k that x_i, x_j (resp. y_i, y_j and z_i, z_j) with $i \neq j$ cannot vanish simultaneously on C_{a_1,b_1} (resp. C_{a_2,b_2} and C_{a_3,b_3}). Hence the sections $s_i = x_i y_i z_i$ have no common zeros and the canonical system $|K_X|$ is base point free.

Since K_X is ample, the image of the canonical map is a threefold, otherwise there would exist a curve $C \subset X$ with $K_X.C = 0$. □

Note that the construction of X depends on 6 parameters a_1, b_1, a_2, b_2, a_3, b_3. We now show that there is a 2-dimensional subfamily whose elements have canonical degree 96. Since

$$(\deg \varphi_{K_X}) \cdot \deg(\varphi_{K_X}(X)) = K_X^3 = 192 = 96 \cdot 2,$$

and the $\varphi_{K_X}(X)$ is non-degenerate, we observe that 96 is the maximal possible canonical degree. This maximum is achieved if and only if the image is a quadric in $\mathbb{P}^4$, i.e., the sections s_i satisfy a quadratic relation.

Proposition 3.2. *The sections s_i satisfy a non-trivial quadratic relation of the form*

$$\lambda_0 s_0^2 + \lambda_1 s_1^2 + \lambda_2 s_2^2 + \lambda_3 s_3^2 + \lambda_4 s_4^2 = 0 \tag{3.1}$$

if and only if $a_1 = a_2 = a_3$ and $b_1 = b_2 = b_3$.

Proof. Using the equations of the curves (2.1), the existence of a non-trivial quadratic relation as in (3.1) is equivalent to vanishing of the following expression

$$\begin{gathered}\lambda_0 x_0^2 y_0^2 z_0^2 + \lambda_1 x_1^2 y_1^2 z_1^2 + \lambda_2 (x_0^2 - x_1^2)(y_0^2 - y_1^2)(z_0^2 - z_1^2) \\ + \lambda_3 (x_0^2 - a_1 x_1^2)(y_0^2 - a_2 y_1^2)(z_0^2 - a_3 z_1^2) + \lambda_4 (x_0^2 - b_1 x_1^2)(y_0^2 - b_2 y_1^2)(z_0^2 - b_3 z_1^2)\end{gathered}$$

for some $0 \neq (\lambda_0, \ldots, \lambda_4) \in \mathbb{C}^5$.

In other words, there exists a non-trivial quadratic relation as in (3.1) if and only if the following system of linear equations has a non-trivial solution:

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & -a_1a_2a_3 & -b_1b_2b_3 \\ 0 & 0 & 1 & a_1 & b_1 \\ 0 & 0 & 1 & a_2 & b_2 \\ 0 & 0 & 1 & a_3 & b_3 \\ 0 & 0 & 1 & a_1a_2 & b_1b_2 \\ 0 & 0 & 1 & a_2a_3 & b_2b_3 \\ 0 & 0 & 1 & a_1a_3 & b_1b_3 \end{pmatrix} \cdot \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = 0.$$

This means that the matrix has rank at most 4. Using elementary matrix transformations under using the conditions $a_i, b_i \notin \{0,1\}$ and $a_i \neq b_i$, the matrix becomes

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & -a_1a_2a_3 & -b_1b_2b_3 \\ 0 & 0 & 1 & a_1 & b_1 \\ 0 & 0 & 0 & 1 & \frac{b_1(b_2-1)}{a_1(a_2-1)} \\ 0 & 0 & 0 & 0 & (b_3-1)(a_2-1)-(a_3-1)(b_2-1) \\ 0 & 0 & 0 & 0 & (b_2-b_1)a_1(a_2-1)-(a_2-a_1)b_1(b_2-1) \\ 0 & 0 & 0 & 0 & (b_3-b_1)a_1(a_2-1)-(a_3-a_1)b_1(b_2-1) \\ 0 & 0 & 0 & 0 & (b_2b_3-b_1)a_1(a_2-1)-(a_2a_3-a_1)b_1(b_2-1) \end{pmatrix}$$

which has rank 4 if and only if the following equations hold:

$$(b_3-1)(a_2-1) = (a_3-1)(b_2-1), \tag{3.2a}$$

$$(b_2-b_1)a_1(a_2-1) = (a_2-a_1)b_1(b_2-1), \tag{3.2b}$$

$$(b_3-b_1)a_1(a_2-1) = (a_3-a_1)b_1(b_2-1), \tag{3.2c}$$

$$(b_2b_3-b_1)a_1(a_2-1) = (a_2a_3-a_1)b_1(b_2-1). \tag{3.2d}$$

We claim that the system of equations (3.2) is equivalent to the system

$$a_1 = a_2 = a_3 \quad \text{and} \quad b_1 = b_2 = b_3, \tag{3.3}$$

under the conditions on the a_i's and b_i's. Clearly, it suffices to prove that a solution of (3.2) is also a solution of (3.3). We distinguish two cases:

Case $b_1 = b_2$. Equation (3.2b) implies $a_1 = a_2$. We resolve (3.2a) after a_3 and substitute it in (3.2c):

$$(b_3 - b_2)a_2(a_2 - 1) = \frac{(a_2 - 1)(b_3 - b_2)}{(b_2 - 1)} b_2(b_2 - 1).$$

The factors $(a_2 - 1)$ and $(b_2 - 1)$ cancel out and the equation becomes

$$(b_3 - b_2)(a_2 - b_2) = 0.$$

Since $a_2 \neq b_2$, we conclude $b_2 = b_3$. This implies $a_1 = a_3$ using (3.2c) again.

Case $b_1 \neq b_2$. By (3.2b) we have $a_1 \neq a_2$ and we are allowed to divide (3.2d) and (3.2c) by (3.2b):

$$\frac{b_2 b_3 - b_1}{b_2 - b_1} = \frac{a_2 a_3 - a_1}{a_2 - a_1} \quad \text{and} \quad \frac{b_3 - b_1}{b_2 - b_1} = \frac{a_3 - a_1}{a_2 - a_1}.$$

Furthermore we can assume that $b_3 \neq b_1$, since $b_3 = b_1$ would imply $b_1 = b_2 = b_3$ similarly to the previous case. We rewrite the fractions above as

$$b_2 b_3(a_2 - a_1) - b_1(a_2 - a_1) = a_2 a_3(b_2 - b_1) - a_1(b_2 - b_1)$$

and

$$b_3(a_2 - a_1) - b_1(a_2 - a_1) = a_3(b_2 - b_1) - a_1(b_2 - b_1)$$

and subtract them:

$$(b_2 b_3 - b_3)(a_2 - a_1) = (a_2 a_3 - a_3)(b_2 - b_1). \tag{3.4}$$

Next we resolve (3.2b) after $(a_2 - a_1)$ and substitute in (3.4). This simplifies to $b_3 a_1 = a_3 b_1$. We use this equation to rewrite (3.2c) as

$$(b_3 - b_1)a_1(a_2 - b_2) = 0.$$

Since $a_2 \neq 0$ and $a_2 \neq b_2$, we conclude that $b_1 = b_3$, a contradiction. □

As consequence we get:

Theorem 3.3. *There exists a two dimensional family of smooth threefolds X of general type with canonical degree* 96 *and whose canonical image is a quadric.*

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Davide Frapporti and Christian Gleissner
University of Bayreuth, Lehrstuhl Mathematik VIII; Universitätsstrasse 30, D-95447 Bayreuth, Germany
E-mail address: `davide.frapporti@uni-bayreuth.de`,
`christian.gleissner@uni-bayreuth.de`