TAIWANESE JOURNAL OF MATHEMATICS
Vol. 24, No. 5, pp. 1249–1258, October 2020
DOI: 10.11650/tjm/191102

Multiplicity of Solutions for a Sublinear Quasilinear Schrödinger Equation

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Abstract. In this paper, we are concerned with the multiplicity of solutions for a class of quasilinear elliptic equation arising from plasma physics. By using a dual approach, the existence of infinitely many small solutions are obtained. As a main novelty with respect to some previous results, we assume the potential V may changes sign and do not require any condition at infinity on the nonlinear term.

1. Introduction

This paper is concerned with the multiplicity of solutions for a quasilinear Schrödinger equation of the form

$$-\Delta u + V(x)u - \Delta(u^2)u = g(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N = 3, 4$ and the potential $V(x)$ may change sign.

The quasilinear Schrödinger equation arises in several models of different physical phenomena, such as in the study of superfluid films in plasma physics, in condensed matter theory, etc (see [1, 2] and the references therein). Recently, (1.1) has been extensively studied by many authors, we can for instance cite [3–8, 10–15, 17] and the references therein. For example, in [15], by using a constrained minimization argument, the existence of positive ground state solution was proved by Poppenberg, Schmitt and Wang. In [12], by a change of variables, Liu, Wang and Wang used an Orlicz space to prove the existence of soliton solution via mountain pass theorem. In [3], Colin and Jeanjean also made use of change variables to prove the existence of positive solution in the Sobolev space $H^1(\mathbb{R}^N)$. The same method of changing variables was also used recently to obtain positive solutions for the case of critical growth (see [4]). There are also some papers investigating the sublinear case. In [17], Zhou and Wu studied (1.1) with $V(x) = 0$ in a bounded domain. Liang, Gao and Li [10] treated a quasilinear Schrödinger equation with $0 < \alpha \leq V(x) \leq \gamma < \infty$.

In the present paper, without imposing any growth condition of $g(x,u)$ at infinity with respect to u, the existence of infinitely many small solutions for the problem (1.1)

Received December 1, 2018; Accepted October 30, 2019.
Communicated by Eiji Yanagida.
2010 *Mathematics Subject Classification*. 35J20, 35J62.
Key words and phrases. quasilinear, Schrödinger equation, dual approach, sublinear.
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is obtained. Furthermore, unlike [10, 17], the potential V in this paper is allowed to be sign-changing.

We introduce the following hypotheses for the problem (1.1).

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$, $\inf_{x\in\mathbb{R}^N} V(x) > -\infty$ and for each $M > 0$, $\operatorname{meas}\{x \in \mathbb{R}^N, V(x) \le M\} < +\infty$, where meas denotes the Lebesgue measure in $\mathbb{R}^N$.

(G_1) There exists a constant $\delta_1 > 0$ such that $g(x, -t) = -g(x, t)$ for all $|t| \le \delta_1$ and all $x \in \mathbb{R}^N$.

(G_2) $\lim_{t\to 0} \frac{g(x,t)}{t} = +\infty$ uniformly for $x \in \mathbb{R}^N$.

(G_3) There exist constants $\delta_2 > 0$ and $0 < s < 1$ such that $g \in C(\mathbb{R}^N \times [-\delta_2, \delta_2], \mathbb{R})$ and

$$|g(x, t)| \le b(x)|t|^s, \quad |t| \le \delta_2, \quad \forall\, x \in \mathbb{R}^N,$$

where $b(x) \in L^{2/(1-s)}(\mathbb{R}^N)$.

Our main result for (1.1) is the following theorem.

Theorem 1.1. *Suppose that $N = 3, 4$ and* (V), (G_1)–(G_3) *hold. Then the problem* (1.1) *has infinitely many solutions converging to zero.*

Throughout this paper, $C > 0$ denotes various positive constants which are not essential to our problem and may change from line to line.

2. Preliminaries and useful lemmas

Let $L^p(\Omega)$, $1 \le p \le +\infty$, $\Omega \subseteq \mathbb{R}^N$ denotes a Lebesgue space, the norm in $L^p(\Omega)$ is denoted by $|\cdot|_{p,\Omega}$. Let $H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^N$, and $H^1(\mathbb{R}^N)$ denote the usual Sobolev spaces. From (V), we know that there exist constants $V_0 > 0$ and $\sigma > 0$ such that $\widetilde{V}(x) := V(x) + V_0 > \sigma$ for all $x \in \mathbb{R}^N$.

Set

$$E(\Omega) = \left\{u \in H_0^1(\Omega) : \int_\Omega \left(|\nabla u|^2 + \widetilde{V}(x)u^2\right) dx < +\infty\right\},$$
$$E = E(\mathbb{R}^N) = \left\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \left(|\nabla u|^2 + \widetilde{V}(x)u^2\right) dx < +\infty\right\}$$

with the norms

$$\|u\|_\Omega = \|u\|_{E,\Omega} = \left(\int_\Omega \left(|\nabla u|^2 + \widetilde{V}(x)u^2\right) dx\right)^{1/2},$$
$$\|u\| = \|u\|_{E,\mathbb{R}^N} = \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + \widetilde{V}(x)u^2\right) dx\right)^{1/2},$$

respectively. Denote $2^* = 2N/(N-2)$ for $N \geq 3$. By the continuity of the embedding $E(\Omega) \hookrightarrow L^r(\Omega)$, $r \in [2, 2^*]$, $\Omega \subseteq \mathbb{R}^N$, there exist constants $\tau_r > 0$, $2 \leq r \leq 2^*$ such that $|u|_{r,\Omega} \leq \tau_r \|u\|_{E,\Omega}$, $\forall\, u \in E(\Omega)$. Moreover, similar to Lemma 3.4 in [18], we may prove that under the assumption (V), the embedding $E(\Omega) \hookrightarrow L^r(\Omega)$ is compact for $2 \leq r < 2^*$, $\Omega \subseteq \mathbb{R}^N$.

Let $\widetilde{g}(x,u) = g(x,u) + V_0 u$. We may seek to obtain solutions of (1.1) by looking for critical points of the associated functional $J\colon H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2\, dx + \frac{1}{2} \int_{\mathbb{R}^N} \widetilde{V}(x) u^2\, dx + \int_{\mathbb{R}^N} u^2 |\nabla u|^2\, dx - \int_{\mathbb{R}^N} G(x,u)\, dx,$$

where $G(x,u) = \int_0^u \widetilde{g}(x,s)\, ds$ is the primitive of function $\widetilde{g}(x,\cdot\,)$. However, this functional is not always well defined on all $H^1(\mathbb{R}^N)$. We make a change of variables $v := f^{-1}(u)$, where f is defined by $f'(t) = \frac{1}{\sqrt{1+2f^2(t)}}$ on $[0,+\infty)$ and $f(-t) = -f(t)$ on $(-\infty, 0]$.

The following properties of f can be found in [3, 7].

Lemma 2.1. *The function f satisfies the following properties:*

(1) *f is uniquely defined, C^∞ and invertible;*

(2) *$|f'(t)| \leq 1$ for all $t \in \mathbb{R}$;*

(3) *$|f(t)| \leq |t|$ for all $t \in \mathbb{R}$;*

(4) *$f(t)/2 \leq t f'(t) \leq f(t)$ for all $t > 0$;*

(5) *there exists a positive constant C_0 such that*

$$|f(t)| \geq \begin{cases} C_0 |t| & \text{if } |t| \leq 1, \\ C_0 |t|^{1/2} & \text{if } |t| \geq 1. \end{cases}$$

So, after the change of variables from J, we obtain the following functional:

$$I(v) := J(f(v)) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2\, dx + \frac{1}{2} \int_{\mathbb{R}^N} \widetilde{V}(x) f^2(v)\, dx - \int_{\mathbb{R}^N} G(x, f(v))\, dx.$$

At the end of this section, we state the following theorems which are crucial to our arguments in Section 3.

Let Γ_k denote the family of closed symmetric subsets A of E such that $0 \notin A$ and the genus $\gamma(A) \geq k$. The following critical point theorem was established in [9].

Lemma 2.2. *Let E be an infinite dimensional Banach space and $I \in C^1(E, \mathbb{R})$ satisfy* (A_1) *and* (A_2) *below.*

(A$_1$) *$I(u)$ is even, bounded from below, $I(0) = 0$ and $I(u)$ satisfies the Palais-Smale condition* (PS)*.*

(A$_2$) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u\in A_k} I(u) < 0$.*

Then $I(u)$ admits a sequence of critical points u_k such that $I(u_k) \leq 0$, $u_k \neq 0$ and $\lim_{k\to\infty} u_k = 0$.

3. Proof of main result

Let $l = \frac{1}{2}\min\{1, \delta_1, \delta_2\}$. We define an even function $h \in C^1(\mathbb{R}, \mathbb{R}^+)$ such that $0 \leq h(t) \leq 1$, $h(t) = 1$ for $|t| \leq l$; $h(t) = 0$ for $|t| \geq 2l$ and h is decreasing in $[l, 2l]$. Let $g_h(x, f(v)) = \widetilde{g}(x, f(v))h(f(v))$. Consider the cutoff functional I_h:

$$I_h(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} \widetilde{V}(x)f^2(v)\,dx - \int_{\mathbb{R}^N} G_h(x, f(v))\,dx,$$

where $G_h(x,u) = \int_0^u g_h(x,s)\,ds$. Then I_h is well defined on E and $I_h \in C^1(E,\mathbb{R})$. The critical points of I_h are weak solutions of the problem

$$-\Delta v + \widetilde{V}(x)f(v)f'(v) = g_h(x, f(v))f'(v), \quad \forall\, v \in E.$$

Then $v \in E$, satisfies $|f(v)| \leq l$, is a critical point of the functional I_h, $u = f(v) \in E$ is a weak solution of (1.1).

In order to show that critical points of I_h are solutions of (1.1), we will use the following L^∞ estimate.

Lemma 3.1. *If $\varphi \in E$ is a weak solution of problem* (1.1)*, then $\varphi \in L^\infty(\mathbb{R}^N)$. Moreover, there exists $C > 0$ such that*

$$|\varphi|_{\infty,\mathbb{R}^N} \leq C\|\varphi\|. \tag{3.1}$$

Proof. For every $\varphi \in E$ and $T > 0$, define $\varphi^T = \varphi$ for $|\varphi| \leq T$, $\varphi^T = T$ for $\varphi \geq T$ and $\varphi^T = -T$ for $\varphi \leq -T$.

Since $I_h'(\varphi) = 0$, we have

$$\int_{\mathbb{R}^N} \nabla\varphi\nabla w\,dx + \int_{\mathbb{R}^N} \widetilde{V}(x)f(\varphi)f'(\varphi)w\,dx - \int_{\mathbb{R}^N} g_h(x, f(\varphi))f'(\varphi)w\,dx = 0 \tag{3.2}$$

for all $w \in E$.

By (G$_3$), the definition of h and Lemma 2.1(2), we obtain

$$|g_h(x, f(\varphi))| \leq b(x)|f(\varphi)|^s + V_0|f(\varphi)| \leq b(x)|\varphi|^s + V_0|\varphi|. \tag{3.3}$$

For every $\varphi \in E$, there are two possibilities: (i) $b(x)|\varphi|^s \geq V_0|\varphi|$; (ii) $b(x)|\varphi|^s \leq V_0|\varphi|$.

We first assume that (i) holds. From Lemma 2.1(4), we deduce that $tf(t) > 0$, $\forall\, t \neq 0$. Taking $w = \varphi|\varphi^T|^{2r}$, $r \geq 0$ as a test function in (3.2), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} g_h(x, f(\varphi)) f'(\varphi)\varphi|\varphi^T|^{2r}\,dx \\
&= \int_{\mathbb{R}^N} \nabla\varphi\nabla(\varphi|\varphi^T|^{2r})\,dx + \int_{\mathbb{R}^N} \widetilde{V}(x) f(\varphi) f'(\varphi)\varphi|\varphi^T|^{2r}\,dx \\
&\geq \int_{\mathbb{R}^N} \nabla\varphi(\nabla\varphi|\varphi^T|^{2r} + 2r\varphi|\varphi^T|^{2r-2}\varphi^T\nabla\varphi^T)\,dx \\
&= \int_{\mathbb{R}^N} |\nabla\varphi|^2|\varphi^T|^{2r}\,dx + 2r\int_{|\varphi|\leq T} |\varphi^T|^{2r}|\nabla\varphi^T|^2\,dx \\
&= \int_{\mathbb{R}^N} |\nabla\varphi|^2|\varphi^T|^{2r}\,dx + 2r\int_{\mathbb{R}^N} |\varphi^T|^{2r}|\nabla\varphi^T|^2\,dx \\
&\geq \frac{2}{r+2}\int_{\mathbb{R}^N} \left(|\nabla\varphi|^2|\varphi^T|^{2r} + 2r|\varphi^T|^{2r}|\nabla\varphi^T|^2 + r^2|\varphi^T|^{2r}|\nabla\varphi^T|^2\right)dx \\
&\geq \frac{1}{(1+r)^2}\int_{\mathbb{R}^N} |\nabla(\varphi|\varphi^T|^r)|^2\,dx \\
&\geq \frac{C}{(1+r)^2}\left(\int_{\mathbb{R}^N} |\varphi|\varphi^T|^r|^{2^*}\,dx\right)^{2/2^*},
\end{aligned}
\tag{3.4}
$$

where we have used the Gagliardo-Nirenberg-Sobolev inequality. On the other hand, by (3.3) and the Hölder's inequality, one has

$$
\begin{aligned}
&\int_{\mathbb{R}^N} g_h(x,\varphi) f'(\varphi)\varphi|\varphi^T|^{2r}\,dx \leq 2\int_{\mathbb{R}^N} b(x)|\varphi|^s\varphi|\varphi^T|^{2r}\,dx \leq 2\int_{\mathbb{R}^N} b(x)|\varphi|^{2r+s+1}\,dx \\
&\leq 2\left(\int_{\mathbb{R}^N} |b(x)|^{2/(1-s)}\,dx\right)^{(1-s)/2}\left(\int_{\mathbb{R}^N} |\varphi|^{2(2r+s+1)/(1+s)}\,dx\right)^{(1+s)/2} \\
&= 2|b(x)|_{2/(1-s),\mathbb{R}^N}|\varphi|^{2r+s+1}_{2(2r+s+1)/(1+s),\mathbb{R}^N}.
\end{aligned}
\tag{3.5}
$$

Taking $T \to \infty$, by (3.4) and (3.5) we have

$$|\varphi|_{(r+1)2^*,\mathbb{R}^N} \leq \left(C(r+1)\right)^{1/(r+1)}\left(|\varphi|^{(2r+s+1)/2(r+1)}_{2(2r+s+1)/(1+s),\mathbb{R}^N}\right).$$

Let $r_0 = 0$ and $2^*(r_k+1) = 2(2r_{k+1}+s+1)/(1+s)$, that is $r_k = \frac{(2^*-2)(1+s)}{2^*(1+s)-4}\left(\frac{2^*(1+s)}{4}\right)^k - \frac{(2^*-2)(1+s)}{2^*(1+s)-4}$, where $2^*(1+s)/4 > 1$ since $N = 3, 4$. Note that $(C(r_i+1))^{1/(r_j+1)} \leq C(r_i+1)$, $\forall\, i < j$. By iteration we obtain

$$|\varphi|_{(r_k+1)2^*,\mathbb{R}^N} \leq e^{\zeta_k}|\varphi|^{\sigma_k}_{2,\mathbb{R}^N},$$

where $\zeta_k = \sum_{i=0}^k \frac{\ln[C(r_i+1)]}{r_i+1}$ and $\sigma_k = \prod_{i=0}^k \frac{2r_i+s+1}{2(r_i+1)}$, which are convergent as $k \to \infty$. Let $\zeta = \lim_{k\to\infty}\zeta_k$ and $\sigma = \lim_{k\to\infty}\sigma_k$. Taking the limit as $k \to \infty$ we obtain that

$$|\varphi|_\infty \leq e^{\zeta}|\varphi|^{\sigma}_{2,\mathbb{R}^N} \leq C\|\varphi\|^{\sigma}.$$

If (ii) holds, without loss of generality, we can assume that $|\varphi| \geq 1$. By (3.3), there holds $|g_h(x, f(\varphi))| \leq 2V_0|\varphi| \leq 2V_0|\varphi|^q$, where $q \in (2, 2^*)$. Then we can complete the proof by using the well-known Moser iteration. $\square$

Lemma 3.2. *I_h is bounded from below and satisfies the* (PS) *condition.*

Proof. By (G_3), the definition of h and Lemma 2.1(3), we have

$$|G_h(x, f(v))| \leq b(x)|v|^{s+1} + V_0|v|^2, \quad \forall\,(x, v) \in (\mathbb{R}^N, \mathbb{R}). \tag{3.6}$$

For any given $v \in E$. Let $\Omega = \{x \in \mathbb{R}^N : |v| \leq 1\}$. By (3.6), Lemma 2.1(3)(5) and Hölder's inequality, one has

$$\begin{aligned} I_h(v) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} \widetilde{V}(x) f^2(v)\,dx - \int_{\mathbb{R}^N} G_h(x, f(v))\,dx \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} \widetilde{V}(x) f^2(v)\,dx - \int_{\Omega} G_h(x, f(v))\,dx \\ &\geq \frac{1}{2}\int_{\Omega} |\nabla v|^2\,dx + \frac{C_0}{2}\int_{\Omega} \widetilde{V}(x) v^2\,dx - \int_{\Omega} \big(b(x)|v|^{s+1} + V_0|v|^2\big)\,dx \\ &\geq \frac{1}{2}\int_{\Omega} |\nabla v|^2\,dx + \frac{C_0}{2}\int_{\Omega} \widetilde{V}(x) v^2\,dx - \int_{\Omega} \big(b(x)|v|^{s+1} + V_0|v|^{s+1}\big)\,dx \\ &\geq C\|v\|_{\Omega}^2 - |b|_{2/(1-s),\Omega}\tau_{1+s}^{1+s}\|v\|_{\Omega}^{1+s} - V_0\tau_{1+s}^{1+s}\|v\|_{\Omega}^{1+s} \\ &\geq C\|v\|_{\Omega}^2 - |b|_{2/(1-s),\mathbb{R}^N}\tau_{1+s}^{1+s}\|v\|_{\Omega}^{1+s} - V_0\tau_{1+s}^{1+s}\|v\|_{\Omega}^{1+s}. \end{aligned} \tag{3.7}$$

This implies I_h is bounded from below.

Next, we prove that I_h satisfies the (PS) condition. Let $\{v_n\} \subset E$ be any (PS) sequence of I_h, i.e., $\{I_h(v_n)\}$ is bounded and $I_h'(v_n) \to 0$ in E^*.

For each $n \in \mathbb{N}$, set $\Omega_n = \{x \in \mathbb{R}^N : |v_n| \leq 1\}$. Then by (3.7), we have

$$C \geq I_h(v_n) \geq C\|v_n\|_{\Omega_n}^2 - |b|_{2/(1-s),\mathbb{R}^N}\tau_{1+s}^{1+s}\|v_n\|_{\Omega_n}^{1+s} - V_0\tau_{1+s}^{1+s}\|v\|_{\Omega_n}^{1+s}.$$

This implies that $\|v_n\|_{\Omega_n} \leq C$ and C is independent of n. Thus,

$$\begin{aligned} &\frac{1}{2}\int_{\Omega_n} |\nabla v_n|^2\,dx + \frac{1}{2}\int_{\Omega_n} \widetilde{V}(x) f^2(v_n)\,dx \\ &\leq I_h(v_n) + \int_{\Omega_n} G_h(x, f(v_n))\,dx \\ &\leq C + |b|_{2/(1-s),\Omega_n}\tau_{1+s}^{1+s}\|v_n\|_{\Omega_n}^{1+s} + V_0\tau_{1+s}^{1+s}\|v_n\|_{\Omega_n}^{1+s} \\ &\leq C + |b|_{2/(1-s),\mathbb{R}^N}\tau_{1+s}^{1+s}\|v_n\|_{\Omega_n}^{1+s} + V_0\tau_{1+s}^{1+s}\|v_n\|_{\Omega_n}^{1+s} \\ &\leq C, \end{aligned} \tag{3.8}$$

where C is independent of n. Similarly,

$$\begin{aligned} I_h(v_n) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v_n|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N} \widetilde{V}(x) f^2(v_n)\,dx - \int_{\Omega_n} G_h(x, f(v_n))\,dx \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega_n} |\nabla v_n|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega_n} \widetilde{V}(x) f^2(v_n)\,dx - \int_{\Omega_n} G_h(x, f(v_n))\,dx. \end{aligned}$$

Therefore,

$$\begin{aligned}(3.9)\qquad &\frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega_n}|\nabla v_n|^2\,dx+\frac{1}{2}\int_{\mathbb{R}^N\setminus\Omega_n}\widetilde{V}(x)f^2(v_n)\,dx\le I_h(v_n)+\int_{\Omega_n}G_h(x,f(v_n))\,dx\\ &\le C+|b|_{2/(1-s),\mathbb{R}^N}\tau_{1+s}^{1+s}\|v_n\|_{\Omega_n}^{1+s}+V_0\tau_{1+s}^{1+s}\|v_n\|_{\Omega_n}^{1+s}\le C,\end{aligned}$$

where C is independent of n. Then from (3.8) and (3.9), we get that $D_n^2 := \int_{\mathbb{R}^N}|\nabla v_n|^2\,dx + \int_{\mathbb{R}^N}\widetilde{V}(x)f^2(v_n)\,dx$ is bounded independent of n.

From [16], we know that there exists a constant $C>0$ such that $D_n^2\ge C\|v_n\|^2$. Then $\{v_n\}$ is bounded in E by the above arguments. Thus, up to a subsequence, we have $v_n \rightharpoonup v$ in E, $v_n\to v$ in $L^p(\mathbb{R}^N)$ for $2\le p<2^*$ and $v_n\to v$ a.e. on $\mathbb{R}^N$. By (G_3) and Lemma 2.1(2)(3), one has

$$\begin{aligned}(3.10)\qquad &\left|\int_{\mathbb{R}^N}\big(g_h(x,f(v_n))f'(v_n)-g_h(x,f(v))f'(v)\big)(v_n-v)\,dx\right|\\ &\le\int_{\mathbb{R}^N}\big(|b(x)||v_n|^s+V_0|v_n|+|b(x)||v|^s+V_0|v|\big)|(v_n-v)|\,dx\\ &\le\big(|b|_{2/(1-s),\mathbb{R}^N}|v_n|_{2,\mathbb{R}^N}^s+V_0|v_n|_{2,\mathbb{R}^N}+|b|_{2/(1-s),\mathbb{R}^N}|v|_{2,\mathbb{R}^N}^s+V_0|v|_{2,\mathbb{R}^N}\big)|v_n-v|_{2,\mathbb{R}^N}\\ &=o_n(1).\end{aligned}$$

As in the proof of Lemma 3.1 in [7] (see also Theorem 2.1 in [16]), we may prove that there exists a constant $C>0$ such that

$$(3.11)\qquad \int_{\mathbb{R}^N}|\nabla(v_n-v)|^2\,dx+\int_{\mathbb{R}^N}\widetilde{V}(x)\big(f(v_n)f'(v_n)-f(v)f'(v)\big)(v_n-v)\,dx\ge C\|v_n-v\|^2.$$

By (3.10) and (3.11), we have

$$\begin{aligned}o_n(1)&=\langle I_h'(v_n)-I_h'(v),v_n-v\rangle\\ &=\int_{\mathbb{R}^N}|\nabla(v_n-v)|^2\,dx+\int_{\mathbb{R}^N}\widetilde{V}(x)\big(f(v_n)f'(v_n)-f(v)f'(v)\big)(v_n-v)\,dx\\ &\quad-\int_{\mathbb{R}^N}\big(g_h(x,f(v_n))f'(v_n)-g_h(x,f(v))f'(v)\big)(v_n-v)\,dx\\ &\ge C\|v_n-v\|^2+o_n(1).\end{aligned}$$

Hence $v_n\to v$ in E. The proof is completed. □

In order to prove Theorem 1.1, we shall use a similar argument of [17].

Lemma 3.3. *For any $n\in\mathbb{N}$, there exist closed symmetric subsets $A_n\subset E$ such that the genus $\gamma(A_n)\ge n$ and $\sup_{v\in A_n}I_h(v)<0$.*

Proof. Let E_n be any n-dimensional subspace of E. Since all norms are equivalent in a finite dimensional space, there is a constant $\tau = \tau(E_n)$ such that

$$\|v\| \le \tau |v|_2$$

for all $v \in E_n$.

Claim: there exists a constant $\kappa > 0$ such that

$$\frac{1}{2}\int_{\mathbb{R}^N} |v|^2\,dx \ge \int_{|f(v)|>l} |v|^2\,dx \tag{3.12}$$

for all $v \in E_n$ with $\|v\| \le \kappa$. Indeed, if (3.12) is false, there exists a sequence $\{v_k\} \subset E_n \setminus \{0\}$ such that $v_k \to 0$ in E and

$$\frac{1}{2}\int_{\mathbb{R}^N} |v_k|^2\,dx < \int_{|f(v_k)|>l} |v_k|^2\,dx, \quad k \in \mathbb{N}.$$

Let $u_k = v_k/|v_k|_2$. Then

$$\frac{1}{2} < \int_{|f(v_k)|>l} |u_k|^2\,dx, \quad k \in \mathbb{N}. \tag{3.13}$$

On the other hand, we can assume that $u_k \to u$ in E since E_n is finite dimensional. Hence $u_k \to u$ in $L^2(\mathbb{R}^N)$. Moreover, it can be deduced from $v_k \to 0$ in E that

$$\operatorname{meas}\{x \in \mathbb{R}^N : |f(v_k)| > l\} \to 0, \quad k \to \infty.$$

Therefore,

$$\int_{|f(v_k)|>l} |u_k|^2\,dx \le 2\int_{\mathbb{R}^N} |u_k - u|^2\,dx + 2\int_{|f(v_k)|>l} |u|^2\,dx \to 0, \quad k \to \infty,$$

which contradicts (3.13) and hence (3.12) holds.

By (G_2), we can choose l small enough such that

$$g(x,v) \ge 4\tau^2 v$$

for all $x \in \mathbb{R}^N$ and $0 \le v \le 2l$. This inequality implies that

$$G_h(x,v) = G(x,v) \ge 2\tau^2 v^2, \quad \forall\,(x,v) \in \mathbb{R}^N \times [0,l]. \tag{3.14}$$

The assumption (G_1) implies $G_h(x,v)$ is even in v. Thus, by (3.14), we have

$$\begin{aligned} I_h(v) &\le \frac{1}{2}\|v\|^2 - \int_{\mathbb{R}^N} G_h(x,f(v))\,dx \le \frac{1}{2}\|v\|^2 - \int_{|f(v)|\le l} G_h(x,f(v))\,dx \\ &\le \frac{1}{2}\|v\|^2 - 2\tau^2 \int_{|f(v)|\le l} |v|^2\,dx \le \frac{1}{2}\|v\|^2 - 2\tau^2\left(\int_{\mathbb{R}^N} |v|^2\,dx - \int_{|f(v)|>l} |v|^2\,dx\right) \\ &\le -\frac{1}{2}\|v\|^2 \end{aligned}$$

for all $v \in E_n$ with $\|v\| \leq \kappa$.

Let $0 < \rho \leq \kappa$ and $A_n = \{v \in E_n : \|v\| = \rho\}$. We conclude that $\gamma(A_n) \geq n$ and

$$\sup_{v \in A_n} I_h(v) \leq -\frac{1}{2}\rho^2 < 0.$$

The proof is completed. □

Proof of Theorem 1.1. By (G_1) and (G_3), we get that I_h is even and $I_h(0) = 0$. Then from Lemmas 2.2, 3.2 and 3.3 we obtain that I_h has a critical sequence $\{v_n\}$ converging to 0. By Lemma 3.1, we obtain that $v_n \in L^\infty(\mathbb{R}^N)$. Moreover, from (3.1), we may get that there exists n_1 such that $|v_n|_{\infty,\mathbb{R}^N} \leq l$ for $n \geq n_1$. Then we get infinitely many small solutions of (1.1). The proof is completed. □

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