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Exceptional Set of Waring-Goldbach Problem with Unequal Powers of Primes

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Abstract. In this paper, it is proved that with at most $O(N^{17/42+\varepsilon})$ exceptions, all even positive integer n, $n \in [N/2, N]$, can be represented in the form $p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4$, where p_1, p_2, p_3, p_4, p_5, p_6 are prime numbers. This improves a recent result $O(N^{13/16+\varepsilon})$ due to Zhang and Li [13].

1. Introduction

Let $n, k_1, k_2, \ldots, k_s$ be natural numbers such that $2 \leq k_1 \leq k_2 \leq \cdots \leq k_s$, $n > s$. Waring's problem of mixed type concerns the representation of a natural number n as the form

$$n = x_1^{k_1} + x_2^{k_2} + \cdots + x_s^{k_s}.$$

Not very much is known the results of this kind. For references in this aspect, we refer the reader to Section P12 of LeVeques *Reviews in number theory*, the bibliography in Vaughan [10] and the recent papers by J. Brüdern [2,3] and by T. D. Wooley [12].

In 1970, Vaughan [9] obtained the asymptotic formula for the number of representations of an integer as the sum of two squares, two cube, and two fourth powers. Let $\widetilde{R}(n)$ denote the number of representations of the integer n in the shape

$$n = x_1^2 + x_2^2 + x_3^3 + x_4^3 + x_5^4 + x_6^4$$

with $x_i \in \mathbb{N}$ $(1 \leq i \leq 6)$, and let

$$\Theta_{2,3,4}(n) = \sum_{q=1}^{\infty} \frac{1}{q^6} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \prod_{i=1}^{3} \left(\sum_{x_i=1}^{q} e\left(\frac{ax_i^{i+1}}{q} \right) \right)^2 e\left(-\frac{an}{q} \right).$$

Hence, Vaughan [9] proved that

$$\widetilde{R}(n) = \frac{\Gamma^2(\frac{3}{2})\Gamma^2(\frac{4}{3})\Gamma^2(\frac{5}{4})}{\Gamma(\frac{13}{6})} \Theta_{2,3,4}(n) n^{7/6} + O(n^{7/6-1/96+\varepsilon}).$$

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In view of Vaughan's result, it is reasonable to propose the conjecture that every sufficiently large even integer n can be expressed as the sum of two squares, two cube and two fourth powers of primes. That is, for sufficiently large even integer n, the equation

$$n = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4 \tag{1.1}$$

is solvable in primes p_j $(1 \leq j \leq 6)$. Here and in the sequel, the letter p, with or without subscripts, always stands for a prime number. This conjecture is perhaps out of reach at present times. It is possible, however, to obtain a weaker result with p_1 replaced by an almost-prime. Let $\mathcal{P}_r$ denote an almost-prime with at most r prime factors, counted according to multiplicity. In 2015, Lü [7] proved that for every sufficiently large even integer n, the equation

$$n = x^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4$$

is solvable with x being an almost-prime $\mathcal{P}_6$. On the other hand, in 2017, Liu [6] proved that every sufficiently large even integer n can be represented as two squares of primes, two cubes of primes, two fourth powers of primes and 41 powers of 2, i.e.,

$$n = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^4 + p_6^4 + 2^{v_1} + \cdots + 2^{v_{41}}.$$

Let $E(N)$ denote the number of positive even integer n, $n \in [N/2, N]$, which can not be represented as (1.1). In 2018, Zhang and Li [13] considered the exceptional set of the problem (1.1) and got

$$E(N) \ll N^{13/16+\varepsilon}.$$

In this paper, we sharpen the above result and establish the following result.

Theorem 1.1. *Let $E(N)$ be defined as above. Then, for any $\varepsilon > 0$, we have*

$$E(N) \ll N^{17/42+\varepsilon}.$$

Remark 1.2. Note that $13/16 \approx 0.8125$ and $17/42 \approx 0.4048$, Theorem 1.1 improves the result of Zhang and Li [13] to approximately half of the original. We establish Theorem 1.1 by means of the circle method in combination with some new ideas of Liu [5] and Zhang and Li [13]. Especially, the method from Liu [5] plays an important role in dealing with the minor arcs.

Actually, the method of Liu [5] derives from Wooley [11]. Utilizing this method and combining with the key Lemma 2.2, we obtain the better estimates about $\mathfrak{m}_1$ and $\mathfrak{m}_2$ than the results of [13]. Consequently, we can make this improvement.

Notations. Throughout this paper, ε and A always denote positive constants which are arbitrary small and sufficiently large, respectively, which may not be the same at different occurrences. $e(x) = e^{2\pi i x}$; $f(x) \ll g(x)$ means that $f(x) = O(g(x))$; $f(x) \asymp g(x)$ means that $f(x) \ll g(x) \ll f(x)$. N is a sufficiently large integer, and thus we use L to denote both $\log N$ and $\log n$. The letter c, with or without subscripts or superscripts, always denotes a positive constant.

2. Outline of method

Throughout, we assume that N is a sufficiently large positive integer. In order to apply the circle method, we set

$$P = N^{9/80-2\varepsilon}, \quad Q = N^{71/80+\varepsilon}. \tag{2.1}$$

For any integers a, q satisfying

$$1 \le a \le q \le Q, \quad (a,q) = 1,$$

by Dirichlet's lemma on rational approximation (see Lemma 2 on page 142 of Karatsuba [4]), we define the major arc $\mathfrak{M}$ and minor arc $\mathfrak{m}$ as usual, namely

$$\mathfrak{M} = \bigcup_{1\le q\le P} \bigcup_{\substack{1\le a\le q\\(a,q)=1}} \mathfrak{M}(a,q), \quad \mathfrak{m} = \left[\frac{1}{Q}, 1+\frac{1}{Q}\right] \setminus \mathfrak{M}, \tag{2.2}$$

where

$$\mathfrak{M}(a,q) = \left\{\alpha \in [0,1] : \left|\alpha - \frac{a}{q}\right| \le \frac{1}{qQ}\right\}.$$

For $k = 2, 3, 4$, we set

$$f_k(\alpha) = \sum_{X_k < p \le 2X_k} (\log p) e(p^k \alpha), \tag{2.3}$$

where $X_k = (\frac{N}{16})^{1/k}$. Let

$$R(n) = \sum_{\substack{n=p_1^2+p_2^2+p_3^3+p_4^3+p_5^4+p_6^4\\ X_2<p_1,p_2\le 2X_2, X_3<p_3,p_4\le 2X_3, X_4<p_5,p_6\le 2X_4}} (\log p_1)\cdots(\log p_6).$$

Then

$$\begin{aligned} R(n) &= \int_0^1 \left(\prod_{k=2}^4 f_k^2(\alpha)\right) e(-n\alpha)\, d\alpha = \int_{1/Q}^{1+1/Q} \left(\prod_{k=2}^4 f_k^2(\alpha)\right) e(-n\alpha)\, d\alpha \\ &= \left\{\int_{\mathfrak{M}} + \int_{\mathfrak{m}}\right\} \left(\prod_{k=2}^4 f_k^2(\alpha)\right) e(-n\alpha)\, d\alpha. \end{aligned}$$

In order to describe the contribution from the major arcs, we introduce some notations. Let

$$C_k(q,a) = \sum_{\substack{m=1\\(m,q)=1}}^{q} e\left(\frac{am^k}{q}\right) \quad \text{and} \quad B(n,q) = \sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(-\frac{an}{q}\right)\prod_{k=2}^{4} C_k^2(q,a).$$

The singular series is defined by (see [13, (3.1)])

$$\Theta(n) = \sum_{q=1}^{\infty} \frac{B(n,q)}{\varphi^6(q)}. \tag{2.4}$$

We define the singular integral as (see [13, (3.9)])

$$\Im(n) := \sum_{\substack{m_1+\cdots+m_6=n\\ X_2^2<m_1,m_2\le(2X_2)^2\\ X_3^3<m_3,m_4\le(2X_3)^3\\ X_4^4<m_5,m_6\le(2X_4)^4}} (m_1m_2)^{-1/2}(m_3m_4)^{-1/2}(m_5m_6)^{-1/2}. \tag{2.5}$$

Lemma 2.1. (see [13, Proposition 2.1]) *Let the major arcs $\mathfrak{M}$ be defined as in* (2.2) *with P and Q defined in* (2.1)*. Then, for $n \in [N/2, N]$ and any $A > 0$, there holds*

$$\int_{\mathfrak{M}} \left(\prod_{k=2}^{4} f_k^2(\alpha)\right) e(-n\alpha)\, d\alpha = \frac{1}{576}\Theta(n)\Im(n) + O(n^{7/6}L^{-A}),$$

where $\Theta(n)$ is the singular series defined in (2.4)*, which is absolutely convergent and satisfies*

$$0 < c^* \le \Theta(n) \ll d(n),$$

where $d(n)$ and c^* denote Dirichlet's divisor function and some fixed constant, respectively; while $\Im(n)$ is defined by (2.5) *and satisfies*

$$\Im(n) \asymp N^{7/6}.$$

Next, we need the following lemmas to handle the minor arcs. In this paper, we divide $\mathfrak{m}$ into $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Let

$$\Re(a,q) = \left[\frac{a}{q} - \frac{1}{qN^{5/6}}, \frac{a}{q} + \frac{1}{qN^{5/6}}\right], \quad \Re = \bigcup_{1\le q\le N^{1/6}} \bigcup_{\substack{1\le a\le q\\(a,q)=1}} \Re(a,q).$$

We define

$$\mathfrak{m}_1 = \mathfrak{m} \cap \Re, \quad \mathfrak{m}_2 = \mathfrak{m} \setminus \Re. \tag{2.6}$$

Lemma 2.2. *Let* $f_2(\alpha)$ *and* $f_4(\alpha)$ *be defined in* (2.3). *We have*

$$\int_0^1 |f_2(\alpha) f_4^2(\alpha)|^2 \, d\alpha \ll N^{1+\varepsilon}.$$

Proof. This lemma actually derives from Brüdern [1, Lemma 1]. From [1, Lemma 1], we have

$$\int_0^1 \left| \prod_{i=1}^{s} f'_{k_i}(\alpha) \right|^2 d\alpha \ll N^{1/k_1+\cdots+1/k_s+\varepsilon}, \tag{2.7}$$

where $2 \le k_1 \le \cdots \le k_s$ are natural numbers satisfying

$$\sum_{i=j+1}^{s} \frac{1}{k_i} \le \frac{1}{k_j}, \quad 1 \le j \le s-1, \tag{2.8}$$

and

$$f'_k(\alpha) = \sum_{x \le N^{1/k}} e(\alpha x^k).$$

Obviously, when we substitute

$$f^*_k(\alpha) = \sum_{p \le N^{1/k}} e(\alpha p^k)$$

for $f'_k(\alpha)$, (2.7) is true.

Utilizing partial summation formula, we know that $f_k(\alpha)$ differs from $f^*_k(\alpha)$ by a $\log N$ (see [8, (29), p. 326]). Hence, taking $k_0 = 1$, $k_1 = 2$, $k_2 = 4$, $k_3 = 4$ satisfying (2.8), we can obtain

$$\int_0^1 |f_2(\alpha) f_4^2(\alpha)|^2 \, d\alpha \ll N^{1/2+1/4+1/4+\varepsilon} = N^{1+\varepsilon}. \qquad \square$$

Remark 2.3. Lemma 2.2 is very important to estimate the integral over $\mathfrak{m}_j$, $j = 1, 2$.

Lemma 2.4. (see [13, Lemma 6.6]) *Suppose that* $\alpha \in \mathfrak{m}_1$. *Then we have*

$$f_3(\alpha) \ll N^{133/480+\varepsilon}.$$

Lemma 2.5. (see [13, Lemma 6.7]) *Suppose that* $\alpha \in \mathfrak{m}_2$. *Then we have*

$$f_3(\alpha) \ll N^{13/42+\varepsilon}.$$

3. Auxiliary estimates

We are now equipped to establish the auxiliary estimates in this paper, and we initiate our proof by recalling the Farey dissections (2.2) and (2.6) that

$$R(n) = \left\{ \int_{\mathfrak{M}} + \int_{\mathfrak{m}_1} + \int_{\mathfrak{m}_2} \right\} \left(\prod_{k=2}^{4} f_k^2(\alpha) \right) e(-n\alpha)\, d\alpha,$$

where $f_k(\alpha)$ is defined in (2.3).

From Lemma 2.1, we can get the evaluation of the integral over $\mathfrak{M}$. Next we will compute the estimation of the integrals over $\mathfrak{m}_j$, $j = 1, 2$.

3.1. The integrals over $\mathfrak{m}_j$, $j = 1, 2$

We denote by $Z_j(N)$ the set of even integers n, $N/2 < n \le N$, for which the inequality

$$\left| \int_{m_j} f_2^2(\alpha) f_3^2(\alpha) f_4^2(\alpha) e(-\alpha n)\, d\alpha \right| \ge n^{7/6} L^{-A} \tag{3.1}$$

holds. For simplicity, we abbreviate the cardinality of $Z_j(N)$ to Z_j. Next, define the complex number $\xi_j(n)$ by taking $\xi_j(n) = 0$ for $n \notin Z_j(N)$, and for $n \in Z_j(N)$ by means of the equation

$$\left| \int_{m_j} f_2^2(\alpha) f_3^2(\alpha) f_4^2(\alpha) e(-\alpha n)\, d\alpha \right| = \xi_j(n) \int_{m_j} f_2^2(\alpha) f_3^2(\alpha) f_4^2(\alpha) e(-\alpha n)\, d\alpha. \tag{3.2}$$

Plainly, one has $|\xi_j(n)| = 1$ whenever $\xi_j(n)$ is nonzero. Thus, we have

$$\sum_{n \in Z_j(N)} \xi_j(n) \int_{m_j} f_2^2(\alpha) f_3^2(\alpha) f_4^2(\alpha) e(-\alpha n)\, d\alpha = \int_{m_j} f_2^2(\alpha) f_3^2(\alpha) f_4^2(\alpha) K_j(\alpha)\, d\alpha, \tag{3.3}$$

where the exponential sum $K_j(\alpha)$ is defined by

$$K_j(\alpha) = \sum_{n \in Z_j(N)} \xi_j(n) e(-\alpha n).$$

Let

$$I_j = \int_{m_j} f_2^2(\alpha) f_3^2(\alpha) f_4^2(\alpha) K_j(\alpha)\, d\alpha.$$

By (3.1)–(3.3), we get

$$I_j \ge \sum_{n \in Z_j(N)} n^{7/6} L^{-A} \gg Z_j N^{7/6} L^{-A}. \tag{3.4}$$

Next, we will use the following lemma to compute Z_j.

Lemma 3.1. (see [11, Lemma 2.1] with $k = 2$ or [5, (3.6)]) *Let $f_2(\alpha)$ be defined in* (2.3) *and K_j be defined above. Then*

$$\int_0^1 |f_2(\alpha) K_j(\alpha)|^2\, d\alpha \ll N^{\varepsilon} (Z_j N^{1/2} + Z_j^2).$$

3.2. The integrals over Z_j

We now establish our estimate for Z_j. An application of Cauchy-Schwarz inequality yields the inequality

$$I_1 \ll \left(\max_{\alpha\in\mathfrak{m}_1} |f_3^2(\alpha)|\right)\left(\int_0^1 |f_2(\alpha)K_1(\alpha)|^2\,d\alpha\right)^{1/2}\left(\int_0^1 |f_2(\alpha)f_4^2(\alpha)|^2\,d\alpha\right)^{1/2}.$$

Combining Lemmas 2.2, 2.4 and 3.1, we find that

$$\begin{aligned} I_1 &\ll (N^{133/480+\varepsilon})^2(N^{\varepsilon}(Z_1N^{1/2}+Z_1^2))^{1/2}(N^{1+\varepsilon})^{1/2} \\ &\ll N^{253/240+\varepsilon}(Z_1^{1/2}N^{1/4+\varepsilon}+Z_1N^{\varepsilon}) \ll Z_1^{1/2}N^{313/240+\varepsilon}+Z_1N^{253/240+\varepsilon}. \end{aligned} \tag{3.5}$$

Hence, (3.4) and (3.5) reveal that

$$Z_1 \ll N^{11/40+\varepsilon}. \tag{3.6}$$

We use the same method to compute Z_2, so

$$I_2 \ll \left(\max_{\alpha\in\mathfrak{m}_2} |f_3^2(\alpha)|\right)\left(\int_0^1 |f_2(\alpha)K_2(\alpha)|^2\,d\alpha\right)^{1/2}\left(\int_0^1 |f_2(\alpha)f_4^2(\alpha)|^2\,d\alpha\right)^{1/2}. \tag{3.7}$$

Combining Lemmas 2.2, 2.5 and 3.1, we find that

$$I_2 \ll Z_2^{1/2}N^{115/84+\varepsilon}+Z_2N^{47/42+\varepsilon}.$$

Hence, (3.4) and (3.7) reveal that

$$Z_2 \ll N^{17/42+\varepsilon}. \tag{3.8}$$

4. Proof of Theorem 1.1

Let $Z(N)$ denote the number of even integers n in the interval $[N/2, N]$ such that the following asymptotic formula

$$R(n) = \frac{1}{576}\Theta(n)\mathfrak{S}(n) + O(n^{7/6}L^{-A})$$

fails to hold. On recalling (3.6) and (3.8), we arrive at the conclusion that

$$Z(N) \ll Z_1 + Z_2 \ll N^{17/42+\varepsilon}.$$

Hence $E(N) \ll N^{17/42+\varepsilon}$.

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