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A Class of Fourth-order Parabolic Equations with Logarithmic Nonlinearity

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Abstract. In this paper, we apply the modified potential well method and the logarithmic Sobolev inequality to study the fourth-order parabolic equation with p-Laplacian and logarithmic nonlinearity. Some results are obtained under the different initial data conditions. More precisely, we give the global existence of weak solution by combining the classical Galerkin's method with the modified potential well method, decay estimates, and blow-up in finite time when the initial energy is subcritical and critical, respectively. In addition, sufficient conditions for the global existence and blow-up of the weak solution are also provided for supercritical initial energy. These results extend and improve many results in the literature.

1. Introduction

In this paper, we study the following fourth-order parabolic equation with logarithmic nonlinearity:

$$\begin{cases} u_t + \Delta^2 u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{p-2}u\log|u| & \text{if } (x,t) \in \Omega \times (0,T), \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{if } (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{if } x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $T \in (0,+\infty]$, ν is the outward normal on $\partial\Omega$ and $u_0 \in H_0^2(\Omega)$, the parameter p satisfies the following condition:

$$\begin{cases} 2 < p < \infty & \text{if } N = 1, 2, \\ 2 < p < 2N/(N-2) & \text{if } N \geq 3. \end{cases} \tag{1.2}$$

It is well known that the fourth-order parabolic partial differential equations have many applications in the fields such as materials science, engineering, biological mathematics, image analysis, etc. Zangwill [30] gave a basic model

$$u_t = g - \nabla j + \eta \tag{1.3}$$

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with periodic boundary conditions and some initial condition $u_0 = u_0(x)$, which described a spatial variable x in the domain $\Omega = [0, L]^2$, the height $u(x,t)$ of a film in epitaxial growth. The phenomenological approach is to expand j in ∇u and powers thereof, and to keep only "sensible" terms, which yielded

$$j = A_1 \nabla u + A_2 \nabla(\Delta u) + A_3 |\nabla u|^2 \nabla u + A_4 \nabla |\nabla u|^2$$

with constants $A_1, \ldots, A_4$ in the growth law (1.3). After that, Ortiz, Repetto and Si [21] showed $A_4 = 0$ if Onsager's reciprocity relations hold, and dropped the noise term η, furthermore, they introduced a transition function which models the energetics of the boundary layer at the film/substrate interface. With the success of the model in simulating the experimental observations, the study of existence, uniqueness, and regularity of solutions is more important. King, Stein and Winkler [10] considered nonlinear parabolic problem

$$u_t + \Delta^2 u - \nabla(f(\nabla u)) = g$$

with Neumann boundary condition, and obtained existence, uniqueness and regularity of solutions under suitable conditions. Qu and Zhou [23] studied the following thin-film equation:

$$u_t + u_{xxxx} = |u|^{p-1}u - \frac{1}{|\Omega|} \int_\Omega |u|^{p-1}u \, dx. \tag{1.4}$$

By using the method of potential wells, they obtained a threshold result of global existence and blow-up for the sign-changing weak solutions. They also obtained the conditions under which the global solutions extinct in finite time. Further, Li, Gao and Han [13] added the term $-(|u_x|^{p-2}u_x)_x$ in (1.4), discussed the global existence, uniqueness, blow-up in finite time and asymptotic behavior of solutions under different initial conditions. For $N = 1$, some other results for fourth-order parabolic equations were obtained, one can refer to [5, 27]. For general dimension $N \geq 1$, Xu et al. [28] studied parabolic equation

$$u_t - q\Delta u + \Delta^2 u + f(u) = 0$$

with the same initial boundary value in (1.1), and showed that the solutions exist globally or blow up in finite time under suitable conditions by using the modified potential well method. However, they did not show whether there exist non-global solutions when the initial energy is supercritical. Until recently, Han [6] considered the following fourth-order parabolic equation with arbitrary initial energy:

$$u_t + \Delta^2 u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{q-1}u. \tag{1.5}$$

Specifically, the author gave a threshold result for the solutions to exist globally or to blow-up in finite time when the initial energy is subcritical and critical, respectively. Moreover, the decay rate of the L^2 norm was also obtained for global solutions. Sufficient conditions for the existence of global and blow-up solutions were also provided for supercritical initial energy.

In recent years, partial differential equations with logarithmic nonlinearity have been studied by many authors. Some remarkable achievements were also obtained. Chen, Luo and Liu [2] studied the following semilinear heat equation under zero Dirichlet boundary value condition:

$$u_t - \Delta u = u \log |u|, \tag{1.6}$$

and obtained the existence of global solution and blow-up at $+\infty$ under some suitable conditions by using the logarithmic Sobolev inequality and a family of potential wells, and gave the results for decay estimates of the global solutions. In the same year, Chen and Tian [3] added pseudo-parabolic term Δu_t in (1.6) to consider semilinear pseudo-parabolic equations with logarithmic nonlinearity, and obtained the existence of global solution, blow-up at $+\infty$, behavior of vacuum isolation of solutions and the asymptotic behavior of solutions. Ji, Yin and Cao [8] revealed the effect of logarithmic nonlinearity on periodic problems for semilinear heat equation and pseudo-parabolic equation with logarithmic source. Some authors considered p-Laplace equation with logarithmic nonlinearity instead of heat equation, one can refer to [1, 7, 11, 19], and the references therein. Here, we must mention potential well method, which first was proposed by Sattinger [24] to study nonlinear hyperbolic boundary-initial value problem. Since then, many authors have studied the existence of solutions for evolution equations by potential well theory [15–17, 20, 22, 26, 29]. Especially, authors [16, 17, 29] improved the results of Sattinger by introducing a family of potential wells. They not only obtained some new results on global existence and invariant sets of solutions, but also discovered the vacuum isolating of solutions.

However, when the right-hand side of (1.5) is logarithmic nonlinearity $|u|^{p-2}u \log |u|$, i.e., problem (1.1), what will happen? To our best knowledge, there is no relative work to answer this problem. There are many difficulties to deal with this problem. For example, (1) generally speaking, the higher-order problems do not admit the usual maximum principle and comparison principle, which makes some most effective methods, such as the method of upper and lower solutions, invalid any more; (2) the main difficulty is in this case that the potential well in [6] will be not suitable, since $|u|^{p-2}u \log |u|$ does not have similar properties corresponding to that one of $|u|^{q-1}u$. In this paper, we will overcome these difficulties to consider the global existence, uniqueness, decay estimates, and blow-up property of problem (1.1).

This paper is organized as follows: In Section 2, we will give the crucial logarithmic Sobolev inequality, some notations and lemmas about potential well theory. Sections 3 and 4 will be devoted to the cases $J(u_0) < d$ and $J(u_0) = d$, respectively, here $J(u_0)$ is initial energy. In Section 5, we consider the case $J(u_0) > d$.

2. Preliminaries

Throughout this paper, we denote by $\|\cdot\|_p$ the $L^p(\Omega)$ norm and $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$. We will equip $H_0^2(\Omega)$ with the norm $\|u\|_{H_0^2(\Omega)} = \|\Delta u\|_2$, which is equivalent to the standard one due to Poincaré's inequality. S is the optimal embedding constant from $H_0^2(\Omega)$ to $L^p(\Omega)$.

First, we need the following logarithmic Sobolev inequality:

Lemma 2.1. [4] *For any $u \in W^{1,p}(\mathbb{R}^N)$ with $p \in (1,+\infty)$, $u \neq 0$, and any $\mu > 0$,*

$$p\int_{\mathbb{R}^N} |u|^p \log\left(\frac{|u|}{\|u\|_{L^p(\mathbb{R}^N)}}\right) dx + \frac{N}{p}\log\left(\frac{p\mu e}{N\mathcal{L}_p}\right)\int_{\mathbb{R}^N} |u|^p\,dx \le \mu\int_{\mathbb{R}^N} |\nabla u|^p\,dx,$$

where

$$\mathcal{L}_p = \frac{p}{N}\left(\frac{p-1}{e}\right)^{p-1}\pi^{-p/2}\left[\frac{\Gamma(\frac{N}{2}+1)}{\Gamma\big(N\frac{p-1}{p}+1\big)}\right]^{p/N}.$$

For $u \in W_0^{1,p}(\Omega)$, we define $u = 0$ for $x \in \mathbb{R}^N \setminus \Omega$ such that $u \in W^{1,p}(\mathbb{R}^N)$, then the following L^p logarithmic Sobolev inequality holds for bounded domain Ω:

$$p\int_{\Omega} |u|^p \log\left(\frac{|u|}{\|u\|_p}\right) dx + \frac{N}{p}\log\left(\frac{p\mu e}{N\mathcal{L}_p}\right)\int_{\Omega} |u|^p\,dx \le \mu\int_{\Omega} |\nabla u|^p\,dx. \tag{2.1}$$

Lemma 2.2. [11] *Let ρ be a positive number. Then we have the following inequalities:*

$$\Psi^p \log\Psi \le \frac{e^{-1}}{\rho}\Psi^{p+\rho} \quad \textit{for all } \Psi \ge 1$$

and

$$|\Psi^p \log\Psi| \le (ep)^{-1} \quad \textit{for all } 0 < \Psi < 1.$$

Secondly, we need to introduce some notations and definitions of some functionals and sets.

For $u \in H_0^2(\Omega)$, define

$$J(u) = \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{p}\int_\Omega |u|^p \log|u|\,dx + \frac{1}{p^2}\|u\|_p^p,$$
$$I(u) = \|\Delta u\|_2^2 + \|\nabla u\|_p^p - \int_\Omega |u|^p \log|u|\,dx.$$

It is obvious that the functionals $J(u)$ and $I(u)$ are well-defined and continuous on $H_0^2(\Omega)$ due to the condition (1.2), and satisfy the following relation:

$$J(u) = \frac{1}{p}I(u) + \frac{1}{p^2}\|u\|_p^p + \frac{p-2}{2p}\|\Delta u\|_2^2. \tag{2.2}$$

The relation (2.2) implies that if $I(u) > 0$, then $J(u) > 0$, which plays an important role in dividing $J(u_0)$ and $I(u_0)$ into different situations. For instance, we do not need to discuss the situation $J(u_0) < 0$, $I(u_0) > 0$.

Define the Nehari manifold

$$\mathcal{N} = \{u \in H_0^2(\Omega) \mid I(u) = 0, \|\Delta u\|_2 \neq 0\}.$$

The potential well and its corresponding set are defined respectively by

$$\begin{aligned} W &= \{u \in H_0^2(\Omega) \mid I(u) > 0, J(u) < d\} \cup \{0\}, \\ V &= \{u \in H_0^2(\Omega) \mid I(u) < 0, J(u) < d\}, \end{aligned}$$

where

$$d = \inf_{0 \neq u \in H_0^2(\Omega)} \sup_{\lambda > 0} J(\lambda u) = \inf_{u \in \mathcal{N}} J(u)$$

is the depth of the potential well W.

Lemma 2.3. *The depth of the potential well W is positive.*

Proof. Fix $u \in \mathcal{N}$, according to Lemma 2.2 and (1.2), we get

$$\begin{aligned} \|\Delta u\|_2^2 + \|\nabla u\|_p^p &= \int_\Omega |u|^p \log|u|\, dx \\ &= \int_{\{x\in\Omega:|u|\geq 1\}} |u|^p \log|u|\, dx + \int_{\{x\in\Omega:|u|\leq 1\}} |u|^p \log|u|\, dx \\ &\leq \int_{\{x\in\Omega:|u|\geq 1\}} |u|^p \log|u|\, dx \leq \frac{e^{-1}}{\rho_1} \int_{\{x\in\Omega:|u|\geq 1\}} |u|^{p+\rho_1}\, dx \\ &\leq \frac{e^{-1}}{\rho_1} \|u\|_{p+\rho_1}^{p+\rho_1} \leq \frac{e^{-1}}{\rho_1} S^{p+\rho_1} \|\Delta u\|_2^{p+\rho_1} \end{aligned} \tag{2.3}$$

which implies $\|\Delta u\|_2 \geq \big(\frac{1}{(e^{-1}/\rho_1)S^{p+\rho_1}}\big)^{1/(p+\rho_1-2)}$, here we use the embedding $H_0^2(\Omega)$ to $L^{p+\rho_1}(\Omega)$, and $\rho_1 > 0$ is chosen such that $p+\rho_1 < 2N/(N-4)$ as $N \geq 5$ and ρ_1 is positive as $N \leq 4$. Since

$$\begin{aligned} J(u) &= \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{p}\int_\Omega |u|^p \log|u|\, dx + \frac{1}{p^2}\|u\|_p^p \\ &= \frac{p-2}{2p}\|\Delta u\|_2^2 + \frac{1}{p^2}\|u\|_p^p \geq \frac{p-2}{2p}\left(\frac{1}{\frac{e^{-1}}{\rho_1}S^{p+\rho_1}}\right)^{1/(p+\rho_1-2)}. \end{aligned}$$

Therefore, $d \geq \frac{p-2}{2p}\big(\frac{1}{(e^{-1}/\rho_1)S^{p+\rho_1}}\big)^{1/(p+\rho_1-2)} > 0$. □

For any $\delta > 0$, define the modified functional and Nehari manifold as follows:

$$\begin{aligned} I_\delta(u) &= \delta\|\Delta u\|_2^2 + \delta\|\nabla u\|_p^p - \int_\Omega |u|^p \log|u|\, dx, \\ \mathcal{N}_\delta &= \{u \in H_0^2(\Omega) \mid I_\delta(u) = 0, \|\Delta u\|_2 \neq 0\}. \end{aligned}$$

The corresponding modified potential well and its corresponding set are defined respectively by

$$W_\delta = \{u \in H_0^2(\Omega) \mid I_\delta(u) > 0, J(u) < d(\delta)\} \cup \{0\},$$
$$V_\delta = \{u \in H_0^2(\Omega) \mid I_\delta(u) < 0, J(u) < d(\delta)\},$$

where $d(\delta) = \inf_{u \in \mathcal{N}_\delta} J(u)$ is the depth of W_δ.

Definition 2.4 (Weak solution)**.** A function $u = u(x,t) \in L^\infty(0,T; H_0^2(\Omega))$ with $u_t \in L^2(0,T; L^2(\Omega))$ is called a weak solution to problem (1.1) if $u(x,0) = u_0(x)$ and the following equality holds:

$$(u_t, \varphi) + (\Delta u, \Delta \varphi) + (|\nabla u|^{p-2} \nabla u, \nabla \varphi) = (|u|^{p-2} u \log |u|, \varphi) \quad \text{a.e. } t > 0 \tag{2.4}$$

for any $\varphi \in H_0^2(\Omega)$. Moreover,

$$\int_0^t \|u_\tau\|_2^2 \, d\tau + J(u) = J(u_0) \quad \text{a.e. } t > 0. \tag{2.5}$$

The following lemmas, which give a series of properties of the functionals and sets defined above, will play a pivotal role in the proof of our results. The proof of these lemmas is different from that one of [2] due to the existence of logarithmic nonlinearity $|u|^{p-2} u \log |u|$. Here, we will give the specific process of proof.

Lemma 2.5. *For any $u \in H_0^2(\Omega)$ with $\|\Delta u\|_2 \neq 0$, we have*

(1) $\lim_{\lambda \to 0^+} J(\lambda u) = 0$, $\lim_{\lambda \to +\infty} J(\lambda u) = -\infty$;

(2) *there exists a unique $\lambda^* = \lambda^*(u) > 0$ such that $\frac{dJ(\lambda u)}{d\lambda}\big|_{\lambda = \lambda^*} = 0$. $J(\lambda u)$ is increasing on $0 < \lambda \leq \lambda^*$, decreasing on $\lambda^* \leq \lambda < \infty$ and takes its maximum at $\lambda = \lambda^*$;*

(3) *$I(\lambda u) > 0$ on $0 < \lambda \leq \lambda^*$, $I(\lambda u) < 0$ on $\lambda^* \leq \lambda < \infty$ and $I(\lambda^* u) = 0$.*

Proof. (1) It follows from the definition of $J(u)$ that

$$J(\lambda u) = \frac{\lambda^2}{2} \|\Delta u\|_2^2 + \frac{\lambda^p}{p} \|\nabla u\|_p^p - \frac{\lambda^p}{p} \int_\Omega |u|^p \log |u| \, dx - \frac{\lambda^p}{p} \log \lambda \|u\|_p^p + \frac{\lambda^p}{p^2} \|u\|_p^p.$$

Obviously, assertion $\lim_{\lambda \to +\infty} J(\lambda u) = -\infty$ follows from $p > 2$, and $\lim_{\lambda \to 0^+} J(\lambda u) = 0$.

(2) By a direct computation, we get

$$\frac{dJ(\lambda u)}{d\lambda} = \lambda^{p-1} \left(\lambda^{2-p} \|\Delta u\|_2^2 + \|\nabla u\|_p^p - \int_\Omega |u|^p \log |u| \, dx - \log \lambda \|u\|_p^p \right). \tag{2.6}$$

Let

$$h(\lambda) = \lambda^{2-p} \|\Delta u\|_2^2 + \|\nabla u\|_p^p - \int_\Omega |u|^p \log |u| \, dx - \log \lambda \|u\|_p^p,$$

then

$$\lim_{\lambda\to 0^+} h(\lambda) = +\infty, \qquad \lim_{\lambda\to+\infty} h(\lambda) = -\infty,$$

and

$$h'(\lambda) = (2-p)\lambda^{1-p}\|\Delta u\|_2^2 - \lambda^{-1}\|u\|_p^p < 0.$$

Therefore, there exists a unique $\lambda^* > 0$ such that $h(\lambda^*) = 0$. Moreover, (2.6) implies

$$\left.\frac{dJ(\lambda u)}{d\lambda}\right|_{\lambda=\lambda^*} = (\lambda^*)^{p-1}h(\lambda^*) = 0.$$

Since $h(\lambda) > 0$ on $(0,\lambda^*)$ and $h(\lambda) < 0$ on $(\lambda^*, +\infty)$, (2) holds.

(3) By the definition of $I(u)$, we get

$$\frac{dJ(\lambda u)}{d\lambda} = \frac{I(\lambda u)}{\lambda}.$$

It is obvious that (3) holds from (2). □

Lemma 2.6. *For any $u \in H_0^2(\Omega)$ and $\gamma(\delta) = \frac{1}{S}\left(\frac{p^2\delta e}{N\mathcal{L}_p}\right)^{N/p^2}$, we have*

(1) *if $0 \le \|\Delta u\|_2 \le \gamma(\delta)$, then $I_\delta(u) \ge 0$;*

(2) *if $I_\delta(u) < 0$, then $\|\Delta u\|_2 > \gamma(\delta)$;*

(3) *if $I_\delta(u) = 0$, then $\|\Delta u\|_2 \ge \gamma(\delta)$.*

Proof. (1) Using the logarithmic Sobolev inequality (2.1), we easily get

$$\begin{aligned} I_\delta(u) &= \delta\|\Delta u\|_2^2 + \delta\|\nabla u\|_p^p - \int_\Omega |u|^p\left(\log\frac{|u|}{\|u\|_p} + \log\|u\|_p\right)dx \\ &\ge \delta\|\Delta u\|_2^2 + \left(\delta - \frac{\mu}{p}\right)\|\nabla u\|_p^p + \left(\frac{N}{p^2}\log\left(\frac{p\mu e}{N\mathcal{L}_p}\right) - \log\|u\|_p\right)\|u\|_p^p. \end{aligned} \tag{2.7}$$

Taking $\mu = \delta p$ in (2.7), we obtain

$$I_\delta(u) \ge \left(\frac{N}{p^2}\log\left(\frac{p^2\delta e}{N\mathcal{L}_p}\right) - \log\|u\|_p\right)\|u\|_p^p. \tag{2.8}$$

If $0 \le \|\Delta u\|_2 \le \gamma(\delta)$, then $\|u\|_p \le \left(\frac{p^2\delta e}{N\mathcal{L}_p}\right)^{N/p^2}$ by embedding $H_0^2(\Omega)$ to $L^p(\Omega)$. Therefore, it follows from (2.8) that $I_\delta(u) \ge 0$.

(2) It easily follows from (1).

(3) If $I_\delta(u) = 0$, then by (2.8) we get

$$\frac{N}{p^2}\log\left(\frac{p^2\delta e}{N\mathcal{L}_p}\right) \le \log\|u\|_p. \tag{2.9}$$

It follows from the embedding $H_0^2(\Omega)$ to $L^p(\Omega)$ that

$$\|\Delta u\|_2 \ge \frac{1}{S}\|u\|_p \ge \gamma(\delta).$$

□

Lemma 2.7. *The function $d(\delta)$ satisfies the following properties:*

(1) $d(\delta) \geq \big(\frac{1-\delta}{p} + \frac{p-2}{2p}\big)(\gamma(\delta))^2 + \big(\frac{(1-\delta)\kappa_1}{p} + \frac{1}{p^2}\big)\big(\frac{p^2\delta e}{N\mathcal{L}_p}\big)^{N/p}$ *for* $0 < \delta \leq 1$. *In particular,* $d = d(1) \geq \frac{p-2}{2p}(\gamma(1))^2 + \frac{1}{p^2}\big(\frac{p^2 e}{N\mathcal{L}_p}\big)^{N/p} := m$, *where* κ_1 *is the first eigenvalue of the problem*

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \kappa |u|^{p-2}u & \text{if } x \in \Omega, \\ u = 0 & \text{if } x \in \partial\Omega; \end{cases}$$

(2) *there exists a unique* $b \in (1, \overline{\delta}]$ *such that* $d(b) = 0$ *and* $d(\delta) > 0$ *for* $1 \leq \delta < b$, *here* $\overline{\delta} = \max\big\{1 + 1/(p\kappa_1), p/2\big\}$;

(3) $d(\delta)$ *is strictly increasing on* $0 < \delta \leq 1$, *strictly decreasing on* $1 \leq \delta \leq b$ *and takes its maximum* $d = d(1)$ *at* $\delta = 1$.

Proof. (1) For any $u \in H_0^2(\Omega)$ with $\|\Delta u\|_2 \neq 0$ and $I_\delta(u) = 0$, we get $\|\Delta u\|_2 \geq \gamma(\delta)$ by Lemma 2.6(3). It follows from $\kappa_1\|u\|_p^p \leq \|\nabla u\|_p^p$ and (2.9) that

$$\begin{aligned} J(u) &= \frac{1}{p}(1-\delta)(\|\Delta u\|_2^2 + \|\nabla u\|_p^p) + \frac{1}{p}I_\delta(u) + \frac{1}{p^2}\|u\|_p^p + \frac{p-2}{2p}\|\Delta u\|_2^2 \\ &\geq \frac{1}{p}(1-\delta)\|\Delta u\|_2^2 + \frac{\kappa_1}{p}(1-\delta)\|u\|_p^p + \frac{1}{p^2}\|u\|_p^p + \frac{p-2}{2p}\|\Delta u\|_2^2 \\ &\geq \left(\frac{1-\delta}{p} + \frac{p-2}{2p}\right)(\gamma(\delta))^2 + \left(\frac{(1-\delta)\kappa_1}{p} + \frac{1}{p^2}\right)\left(\frac{p^2\delta e}{N\mathcal{L}_p}\right)^{N/p} \end{aligned}$$

for $0 < \delta \leq 1$. Therefore, we have

$$d(\delta) \geq \left(\frac{1-\delta}{p} + \frac{p-2}{2p}\right)(\gamma(\delta))^2 + \left(\frac{(1-\delta)\kappa_1}{p} + \frac{1}{p^2}\right)\left(\frac{p^2\delta e}{N\mathcal{L}_p}\right)^{N/p}$$

for $0 < \delta \leq 1$. We get $d = d(1) \geq m$ by taking $\delta = 1$.

(2) For any $u \in H_0^2(\Omega)$ with $\|\Delta u\|_2 \neq 0$ and $\delta > 0$, there exists a unique $\lambda = \lambda(\delta)$ such that $I_\delta(\lambda u) = 0$ by Lemma 2.5(3), thus $\lambda u \in \mathcal{N}_\delta$. Further, we have

$$\begin{aligned} d(\delta) \leq J(\lambda u) &= \frac{1}{p}(1-\delta)\big(\lambda^2\|\Delta u\|_2^2 + \lambda^p\|\nabla u\|_p^p\big) + \frac{1}{p}I_\delta(\lambda u) + \frac{\lambda^p}{p^2}\|u\|_p^p + \frac{(p-2)\lambda^2}{2p}\|\Delta u\|_2^2 \\ &\leq \left(\frac{1-\delta}{p} + \frac{p-2}{2p}\right)\lambda^2\|\Delta u\|_2^2 + \left(\frac{1-\delta}{p} + \frac{1}{p^2\kappa_1}\right)\lambda^p\|\nabla u\|_p^p. \end{aligned}$$

Therefore, $d(\overline{\delta}) \leq 0$. On the other hand, $d = d(1) \geq m$, $d(\delta)$ is continuous with δ, so there exists a unique b such that $d(b) = 0$ and $d(\delta) > 0$ for $1 \leq \delta < b$.

(3) Clearly, we only need to prove that for any $0 < \delta' < \delta'' < 1$ or $b > \delta' > \delta'' > 1$ and any $u \in \mathcal{N}_{\delta''}$, there exists a $v \in \mathcal{N}_{\delta'}$ and a constant $\epsilon(\delta', \delta'')$ such that $J(u) - J(v) \geq \epsilon(\delta', \delta'')$.

For $u \in \mathcal{N}_{\delta''}$, we get $I_{\delta''}(u) = 0$ (This implies $\lambda(\delta'') = 1$) and $\|\Delta u\|_2 \geq \gamma(\delta'')$ by Lemma 2.6(3). For any $u \in H_0^2(\Omega)$ with $\|\Delta u\|_2 \neq 0$ and $\delta > 0$, there exists a unique $\lambda = \lambda(\delta)$ such that $I_\delta(\lambda u) = 0$ by Lemma 2.5(3). Then, we have

$$\delta = \frac{\lambda^p \int_\Omega |u|^p \log |u| \, dx + \lambda^p \log \lambda \|u\|_p^p}{\lambda^2 \|\Delta u\|_2^2 + \lambda^p \|\nabla u\|_p^p} := f(\lambda).$$

By a direct computation, then

$$f'(\lambda) = \frac{\lambda^{p+1}}{(\lambda^2 \|\Delta u\|_2^2 + \lambda^p \|\nabla u\|_p^p)^2} F(\lambda), \tag{2.10}$$

where

$$F(\lambda) = \left[(p-2) \int_\Omega |u|^p \log |u| \, dx + (p-2) \log \lambda \|u\|_p^p + \|u\|_p^p \right] \|\Delta u\|_2^2 + \lambda^{p-2} \|u\|_p^p \|\nabla u\|_p^p.$$

Obviously,

$$F(\lambda) \to -\infty \ \text{ as } \lambda \to 0^+, \quad F(\lambda) \to +\infty \ \text{ as } \lambda \to +\infty,$$

and

$$F'(\lambda) = \frac{(p-2)\|u\|_p^p \|\Delta u\|_2^2}{\lambda} + (p-2)\lambda^{p-3} \|u\|_p^p \|\nabla u\|_p^p > 0$$

for $\lambda > 0$. Therefore, there exists a unique λ^* such that $F(\lambda^*) = 0$. It is obvious from (2.10) that $f'(\lambda^*) = 0$, and $f(\lambda)$ is decreasing on $0 < \lambda \leq \lambda^*$, increasing on $\lambda^* \leq \lambda < \infty$ and takes its minimum at $\lambda = \lambda^*$. A possible example of function $f(\lambda)$ is given as shown in Figure 2.1, where $Q_1\big(\exp\big(-\frac{\int_\Omega |u|^p \log |u| \, dx}{\|u\|_p^p}\big), 0\big)$, $Q_2\big(0, \exp\big(-\frac{\int_\Omega |u|^p \log |u| \, dx}{\|u\|_p^p}\big)\big)$.

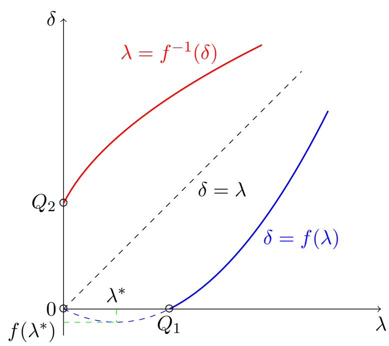

Figure 2.1: A possible example.

Next, we will prove that $\lambda = \lambda(\delta)$ is strictly increasing in $(0, +\infty)$. Since $\delta > 0$, we only consider function $f(\lambda)$ with $\lambda \in \big(\exp\big(-\frac{\int_\Omega |u|^p \log |u| \, dx}{\|u\|_p^p}\big), +\infty\big)$, i.e., the blue solid line. Obviously, red solid line is the graph of inverse function corresponding to function $f(\lambda)$. By the property of inverse function, it easily follows that $\lambda = f^{-1}(\delta)$ is strictly increasing in $(0, +\infty)$.

Choosing $v = \lambda(\delta')u$, then we have $v \in \mathcal{N}_{\delta'}$. Let $g(\lambda) = J(\lambda(\delta)u)$, then

$$\begin{aligned}\frac{dg(\lambda)}{d\lambda} &= \frac{1}{\lambda}\big[(1-\delta)\lambda^2\|\Delta u\|_2^2 + (1-\delta)\lambda^p\|\nabla u\|_p^p + I_\delta(\lambda u)\big]\\ &= (1-\delta)\lambda\|\Delta u\|_2^2 + (1-\delta)\lambda^{p-1}\|\nabla u\|_p^p.\end{aligned}$$

If $0 < \delta' < \delta'' < 1$, since λ is strictly increasing and $\lambda(\delta'') = 1$, then

$$\begin{aligned}J(u) - J(v) = g(1) - g(\lambda(\delta')) &= \int_{\lambda(\delta')}^1 \frac{dg(\lambda)}{d\lambda}\,d\lambda \ge \int_{\lambda(\delta')}^1 \lambda(1-\delta)\|\Delta u\|_2^2\,d\lambda\\ &\ge (1-\delta'')(\gamma(\delta''))^2\lambda(\delta')(1-\lambda(\delta')) = \epsilon(\delta',\delta'') > 0.\end{aligned}$$

If $b > \delta' > \delta'' > 1$, then

$$\begin{aligned}J(u) - J(v) &= \int_{\lambda(\delta')}^1 \frac{dg(\lambda)}{d\lambda}\,d\lambda \ge -\int_1^{\lambda(\delta')} \lambda(1-\delta)\|\Delta u\|_2^2\,d\lambda\\ &\ge (\delta''-1)(\gamma(\delta''))^2\lambda(\delta'')(\lambda(\delta')-1) = \epsilon(\delta',\delta'') > 0.\end{aligned}$$

Since $d(\delta)$ is continuous, $d(\delta)$ takes its maximum $d = d(1)$ at $\delta = 1$. □

Define

$$d_0 = \lim_{\delta\to 0^+} d(\delta).$$

Lemma 2.7(1) implies $d_0 \ge 0$.

Lemma 2.8. *Assume $u \in H_0^2(\Omega)$, $0 < J(u) < d$, $1 \le \delta < \widehat{\delta}$, and $\widehat{\delta} \in (1,b)$ satisfying $d(\widehat{\delta}) = J(u)$, then the sign of $I_\delta(u)$ does not change for $1 \le \delta < \widehat{\delta}$.*

Proof. Clearly, $J(u) > 0$ implies $\|\Delta u\|_2 \neq 0$. If the sign of $I_\delta(u)$ changes for $1 \le \delta < \widehat{\delta}$, then there exists a $\widetilde{\delta} \in [1,\widehat{\delta})$ such that $I_{\widetilde{\delta}}(u) = 0$. By the definition of $d(\delta)$, we get $J(u) \ge d(\widetilde{\delta})$, which contradicts $J(u) = d(\widehat{\delta}) < d(\widetilde{\delta})$ by Lemma 2.7(3). □

Obviously, the following corollary holds:

Corollary 2.9. *Assume $u \in H_0^2(\Omega)$, $d_0 < J(u) < d$, $\delta_1 < 1 < \delta_2$, and δ_1, δ_2 satisfy the equation $d(\delta) = J(u)$, then the sign of $I_\delta(u)$ does not change for $\delta_1 < \delta < \delta_2$.*

Lemma 2.10. *Assume that u is a weak solution of problem* (1.1), *then for $0 < J(u_0) < d$, there exists a $\widehat{\delta} \in (1,b)$ such that $d(\widehat{\delta}) = J(u_0)$. Furthermore,*

(1) *if $I(u_0) > 0$, then $u(x,t) \in W_\delta$ for $1 \le \delta < \widehat{\delta}$ and $0 < t < T$;*

(2) *if $I(u_0) < 0$, then $u(x,t) \in V_\delta$ for $1 \le \delta < \widehat{\delta}$ and $0 < t < T$.*

Proof. Since $0 < J(u_0) < d$, there exists $\widehat{\delta} \in (1, b)$ such that $d(\widehat{\delta}) = J(u_0)$ by Lemma 2.7. Possible situations are obtained as shown in Figure 2.2, and corresponding situations of the following Corollary 2.11 are also given.

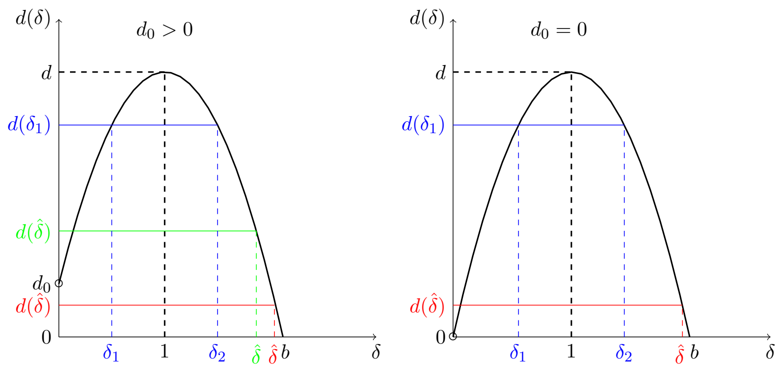

Figure 2.2: Possible situations corresponding to different d_0.

(1) For $1 \leq \delta < \widehat{\delta}$, we get

$$I_\delta(u_0) = (\delta - 1)\|\Delta u_0\|_2^2 + (\delta - 1)\|\nabla u_0\|_p^p + I(u_0) > 0.$$

It follows from Lemma 2.7(3) that

$$J(u_0) = d(\widehat{\delta}) < d(\delta).$$

Therefore, $u_0 \in W_\delta$ for $\delta \in [1, \widehat{\delta})$. Next, we prove $u(x,t) \in W_\delta$ for $1 \leq \delta < \widehat{\delta}$ and $0 < t < T$. Otherwise, there exists a $t_0 \in (0, T)$ and a $\delta_0 \in [1, \widehat{\delta})$ such that $u(t_0) \in \partial W_{\delta_0}$, i.e.,

$$I_{\delta_0}(u(t_0)) = 0, \quad \|\Delta u(t_0)\|_2 \neq 0 \quad \text{or} \quad J(u(t_0)) = d(\delta_0).$$

Clearly, $J(u(t_0)) < d(\delta_0)$ by (2.5). Therefore, $I_{\delta_0}(u(t_0)) = 0$, $\|\Delta u(t_0)\|_2 \neq 0$. The definition of $d(\delta_0)$ implies $J(u(t_0)) \geq d(\delta_0)$, which contradicts (2.5).

(2) First, we show $u_0 \in V_\delta$ for $\delta \in [1, \widehat{\delta})$. If not, there exists a $\delta^0 \in [1, \widehat{\delta})$ such that $u_0 \in V_\delta$ for $\delta \in [1, \delta_0)$ and $u_0 \in \partial V_{\delta^0}$, i.e.,

$$I_{\delta^0}(u_0) = 0 \quad \text{or} \quad J(u_0) = d(\delta^0).$$

By Lemma 2.7(3), we have $J(u_0) = d(\widehat{\delta}) < d(\delta^0)$. Therefore, we only have $I_{\delta^0}(u_0) = 0$, then

$$I_\delta(u_0) = (\delta - \delta^0)\|\Delta u_0\|_2^2 + (\delta - \delta^0)\|\nabla u_0\|_p^p + I_{\delta^0}(u_0) < 0 \quad \text{for } \delta \in [1, \delta^0).$$

By Lemma 2.6(2), we get $\|\Delta u_0\|_2 > \gamma(\delta)$ for $\delta \in [1, \delta^0)$. Further, Lemma 2.6(3) implies $\|\Delta u_0\|_2 \geq \gamma(\delta^0)$. Therefore, $u_0 \in \mathcal{N}_{\delta^0}$ is obvious. By the definition of $d(\delta^0)$, we get $d(\widehat{\delta}) = J(u_0) \geq d(\delta^0)$, which is a contradiction.

Secondly, we show $u(x,t) \in V_\delta$ for $1 \le \delta < \widehat{\delta}$ and $0 < t < T$. If not, there exist a $t_0 \in (0,T)$ and a $\delta_0 \in [1, \widehat{\delta})$ such that $u(t_0) \in \partial V_{\delta_0}$, i.e.,

$$I_{\delta_0}(u(t_0)) = 0 \quad \text{or} \quad J(u(t_0)) = d(\delta_0).$$

Clearly, $I_{\delta_0}(u(t_0)) = 0$. Assume that t_0 is the first time such that $I_{\delta_0}(u(t_0)) = 0$, then $I_{\delta_0}(u(t)) < 0$ for $0 \le t < t_0$. By Lemma 2.6(2), we get $\|\Delta u\|_2 > \gamma(\delta)$ for $0 \le t < t_0$. Further, Lemma 2.6(3) implies $\|\Delta u(t_0)\|_2 \ge \gamma(\delta_0)$. Combining with $I_{\delta_0}(u(t_0)) = 0$, we get $u(t_0) \in \mathcal{N}_{\delta_0}$. By the definition of $d(\delta_0)$, we obtain $J(u(t_0)) \ge d(\delta_0)$, which contradicts (2.5). □

Based on Lemma 2.7 and Corollary 2.9, the following corollary is obvious.

Corollary 2.11. *Assume that u is a weak solution of problem* (1.1)*, then for $d_0 < J(u_0) < d$, there exists δ_1, δ_2 such that $\delta_1 < 1 < \delta_2$ and $d(\delta_1) = d(\delta_2) = J(u_0)$. Furthermore,*

(1) *if $I(u_0) > 0$, then $u(x,t) \in W_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$;*

(2) *if $I(u_0) < 0$, then $u(x,t) \in V_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$.*

Lemma 2.12. *Assume that u is a weak solution of problem* (1.1)*, then for $J(u_0) = d$, the following statements hold:*

(1) *if $I(u_0) > 0$, then $I(u) \ge 0$ for $0 \le t < T$;*

(2) *if $I(u_0) < 0$, then $I(u) < 0$ for $0 \le t < T$.*

Proof. (1) Otherwise, there exists $t_1 \in (0,T)$ such that $I(u(t_1)) < 0$. We can find $t_0 \in (0, t_1)$ being the first point satisfying $I(u) = 0$, i.e.,

$$I(u(t_0)) = 0 \quad \text{and} \quad I(u) > 0 \quad \text{for all } 0 < t < t_0.$$

Taking $\varphi = u$ in (2.4), then we get

$$I(u) = -(u_t, u) > 0 \tag{2.11}$$

for all $0 < t < t_0$. Therefore, we get $\int_0^t \|u_\tau\|_2^2 \, d\tau > 0$. It follows from (2.5) that

$$0 < J(u) = J(u_0) - \int_0^t \|u_\tau\|_2^2 \, d\tau < d \tag{2.12}$$

for all $0 < t \le t_0$. $I(u(t_0)) = 0$ implies $\|\Delta u(t_0)\|_2 \ge \gamma(1)$ by Lemma 2.6(3), then we have $J(u(t_0)) \ge d$ by the definition of d, which contradicts (2.12).

(2) Otherwise, there exists $t_0 \in (0,T)$ such that

$$I(u(t_0)) = 0 \quad \text{and} \quad I(u) < 0 \quad \text{for all } 0 < t < t_0.$$

Obviously, (2.12) still holds. $I(u(t_0)) = 0$ implies $\|\Delta u(t_0)\|_2 \ge \gamma(1)$ by Lemma 2.6(3). Therefore, by the definition of d, we have $J(u(t_0)) \ge d$, which contradicts (2.12). □

3. The subcritical initial energy $J(u_0) < d$

In this section, we consider the global existence, decay estimate and blow-up of weak solution for problem (1.1) under $J(u_0) < d$. First, we give the following global existence theorem:

Theorem 3.1 (Global existence for $J(u_0) < d$)**.** *Let condition* (1.2) *hold,* $u_0 \in H_0^2(\Omega)$. *If* $J(u_0) < d$, $I(u_0) > 0$, *then problem* (1.1) *admits a unique global weak solution* $u \in L^\infty(0,T;H_0^2(\Omega))$ *with* $u_t \in L^2(0,T;L^2(\Omega))$ *and* $u(t) \in W$ *for* $0 \le t < T$.

Proof. The proof of this theorem is divided into two steps.

Step 1*: Global existence.* Choose a sequence $\{\varphi_i\}_{i\in\mathbb{N}}$, which is an orthonormal basis of $H_0^2(\Omega)$. Consider the following Galerkin approximation:

$$u^n(x,t) = \sum_{i=1}^n c_i^n(t)\varphi_i(x), \quad n = 1,2,\ldots,$$

where functions $c_i^n(t)\colon [0,T] \to \mathbb{R}$ satisfy the following system of ordinary differential equations:

$$\begin{aligned} (u_t^n,\varphi_j) + (\Delta u^n,\Delta\varphi_j) + (|\nabla u^n|^{p-2}\nabla u^n,\nabla\varphi_j) &= (|u^n|^{p-2}u^n\log|u^n|,\varphi_j), \\ u^n(x,0) &= \sum_{j=1}^n (u(x,0),\varphi_j)\varphi_j \end{aligned} \tag{3.1}$$

for $j = 1,2,\ldots,n$, here

$$u^n(x,0) \to u_0 \quad \text{in } H_0^2(\Omega) \text{ as } n \to \infty.$$

It follows from (3.1) that

$$\frac{dc_j^n(t)}{dt} = -(\Delta u^n,\Delta\varphi_j) - (|\nabla u^n|^{p-2}\nabla u^n,\nabla\varphi_j) + (|u^n|^{p-2}u^n\log|u^n|,\varphi_j).$$

Define

$$\begin{aligned} \mathcal{F}_j^n(\mathbf{c}^n) = &-\left(\sum_{i=1}^n c_i^n(t)\Delta\varphi_i(x),\Delta\varphi_j\right) - \left(\left|\sum_{i=1}^n c_i^n(t)\nabla\varphi_i(x)\right|^{p-2}\sum_{i=1}^n c_i^n(t)\nabla\varphi_i(x),\nabla\varphi_j\right) \\ &+ \left(\left|\sum_{i=1}^n c_i^n(t)\varphi_i(x)\right|^{p-2}\sum_{i=1}^n c_i^n(t)\varphi_i(x)\log\left|\sum_{i=1}^n c_i^n(t)\varphi_i(x)\right|,\varphi_j\right), \end{aligned}$$

and

$$\mathbf{c}^n(\cdot) = (c_j^n(\cdot))_{j=1}^n, \quad \mathcal{F}^n(\cdot) = (\mathcal{F}_j^n(\cdot))_{j=1}^n, \quad \mathbf{c}_0^n = (u_0,\varphi_j)_{j=1}^n.$$

Therefore, (3.1) can be rewritten as

$$
\begin{aligned}
&\frac{d\mathbf{c}^n(t)}{dt}=\mathcal{F}^n(\mathbf{c}^n(t)),\\
&\mathbf{c}^n(0)=\mathbf{c}_0^n.
\end{aligned}
\tag{3.2}
$$

Next, we aim to use the Peano's theorem to prove the existence of (3.2). Multiplying the first equality of (3.2) by $\mathbf{c}^n(t)$, then we get

$$
\begin{aligned}
\frac{1}{2}\frac{d|\mathbf{c}^n(t)|^2}{dt}=&-\left(\sum_{i=1}^n c_i^n(t)\Delta\varphi_i(x),\sum_{j=1}^n c_j^n(t)\Delta\varphi_j\right)\\
&-\left(\left|\sum_{i=1}^n c_i^n(t)\nabla\varphi_i(x)\right|^{p-2}\sum_{i=1}^n c_i^n(t)\nabla\varphi_i(x),\sum_{j=1}^n c_j^n(t)\nabla\varphi_j\right)\\
&+\left(\left|\sum_{i=1}^n c_i^n(t)\varphi_i(x)\right|^{p-2}\sum_{i=1}^n c_i^n(t)\varphi_i(x)\log\left|\sum_{i=1}^n c_i^n(t)\varphi_i(x)\right|,\sum_{j=1}^n c_j^n(t)\varphi_j\right).
\end{aligned}
$$

Clearly, the first and second terms on the right-hand side of the above are non-positive, then similar to (2.3)

$$
\begin{aligned}
\frac{1}{2}\frac{d|\mathbf{c}^n(t)|^2}{dt}&\le\left(\left|\sum_{i=1}^n c_i^n(t)\varphi_i(x)\right|^{p-2}\sum_{i=1}^n c_i^n(t)\varphi_i(x)\log\left|\sum_{i=1}^n c_i^n(t)\varphi_i(x)\right|,\sum_{j=1}^n c_j^n(t)\varphi_j\right)\\
&\le\frac{e^{-1}}{\rho_1}\left\|\sum_{i=1}^n c_i^n(t)\varphi_i(x)\right\|_{p+\rho_1}^{p+\rho_1}\le\frac{e^{-1}}{\rho_1}\sum_{i=1}^n|c_i^n(t)|^{p+\rho_1}\sum_{i=1}^n\|\varphi_i(x)\|_{p+\rho_1}^{p+\rho_1}\\
&\le(C(n))^{p+\rho_1}\sum_{i=1}^n|c_i^n(t)|^{p+\rho_1}\sum_{i=1}^n\|\Delta\varphi_i(x)\|_2^{p+\rho_1}\le(C(n))^{p+\rho_1}|\mathbf{c}^n(t)|^{2(p+\rho_1)},
\end{aligned}
$$

here $C(n)>0$ is a constant. Solving the above ordinary differential inequality, we get

$$
|\mathbf{c}^n(t)|\le\frac{1}{\left(|\mathbf{c}_0^n|^{-2p}-2p(C(n))^{p+\rho_1}t\right)^{1/(2p)}},\quad\forall\, t\in[0,\widetilde{T}),
$$

here $\widetilde{T}=|\mathbf{c}_0^n|^{-2p}/[2p(C(n))^{p+\rho_1}]$. There exists a sufficiently small $\varepsilon>0$ such that $\forall\, t\in[0,\widetilde{T}-\varepsilon]$, $|\mathbf{c}^n(t)|\le C$, where $C>0$ is a constant. Further, $\mathcal{F}(\mathbf{c}^n(t))$ is bounded, we denote by G_0 the boundedness. Denote

$$
T_0=0,\quad E:=\left\{(t,\mathbf{c}^n(t))\in\mathbb{R}\times\mathbb{R}^n\mid|t-T_0|\le\widetilde{T},|\mathbf{c}^n(t)-\mathbf{c}_0^n|\le C\right\},\quad\forall\, t\in[0,\widetilde{T}).
$$

Peano's theorem implies that there exists a solutions $\mathbf{c}^n(t)$ of (3.2) on $[0,T_1]$, where

$$
T_1=\min\left\{\widetilde{T}-\varepsilon,\frac{\mathbf{c}_0^n}{G_0}\right\}.
$$

For $t \in [0, T_1]$, the following inequality (3.4) still holds, then we have

$$|c_j^n(t)|^2 \le \sum_{j=1}^{n} |c_j^n(t)|^2 = \sum_{j=1}^{n} |c_j^n(t)|^2 \int_\Omega |e_j|^2 \, dx = \|u^n\|_2^2 \le S^2 \|\Delta u^n\|_2^2 < S^2 \frac{2pd}{p-2}$$

by embedding $H_0^2(\Omega)$ to $L^2(\Omega)$. Further, $\mathcal{F}(\mathbf{c}^n(t))$ is bounded. Denote by T_1 the new initial point, Peano's theorem implies that there exists a global solution $\mathbf{c}^n(t)$ of the ordinary differential equation (3.2) by repeating the similar argument.

Multiplying (3.1) by $\frac{dc_i^n(t)}{dt}$, and summing for i from 0 to n, and then integrating with respect to t from 0 to t, we get

$$\int_0^t \|u_\tau^n\|_2^2 \, d\tau + J(u^n) = J(u^n(0)).$$

It follows from $u^n(x,0) \to u_0(x)$ strongly in $H_0^2(\Omega)$ that

$$J(u^n(x,0)) \to J(u_0(x)) < d \quad \text{and} \quad I(u^n(x,0)) \to I(u_0(x)) > 0 \quad \text{as } n \to \infty.$$

Therefore, for sufficiently large n, we get

$$\int_0^t \|u_\tau^n\|_2^2 \, d\tau + J(u^n) = J(u^n(0)) < d \quad \text{and} \quad I(u^n(x,0)) > 0, \tag{3.3}$$

which implies that $u^n(x,0) \in W$.

Next, we prove $u^n(x,t) \in W$ for sufficiently large n. Otherwise, there exists a $t_0 \in (0,T)$ such that $u^n(x,t_0) \in \partial W$, i.e.,

$$I(u^n(x,t_0)) = 0, \quad \|\Delta u(x,t_0)\|_2 \ne 0 \quad \text{or} \quad J(u^n(x,t_0)) = d.$$

Clearly, $J(u^n(x,t_0)) \ne d$ by (3.3). If $I(u^n(x,t_0)) = 0$, $\|\Delta u(x,t_0)\|_2 \ne 0$, then $J(u^n(x,t_0)) \ge d$ by the definition of d, which contradicts (3.3). Therefore, $u^n(x,t) \in W$ and $I(u^n) > 0$. Recalling (3.3) and (2.2), we get

$$\int_0^t \|u_\tau^n\|_2^2 \, d\tau + \frac{1}{p} I(u^n) + \frac{1}{p^2} \|u^n\|_p^p + \frac{p-2}{2p} \|\Delta u^n\|_2^2 < d,$$

which implies

$$\begin{aligned} \|u^n\|_{H_0^2(\Omega)}^2 &< \frac{2pd}{p-2}, \\ \int_0^t \|u_\tau^n\|_2^2 \, d\tau &< d. \end{aligned} \tag{3.4}$$

Therefore, there exist a u and a subsequence of $\{u^n\}_{n\in\mathbb{N}}$ (still denoted by $\{u^n\}_{n\in\mathbb{N}}$) such that as $n \to \infty$,

$$\begin{aligned} u^n &\overset{*}{\rightharpoonup} u \quad \text{weakly * in } L^\infty(0,T; H_0^2(\Omega)); \\ u_t^n &\rightharpoonup u_t \quad \text{weakly in } L^2(0,T; L^2(\Omega)). \end{aligned}$$

Since compact embedding $H_0^2(\Omega)$ to $W_0^{1,p}(\Omega)$, further we have

$$u^n \to u \quad \text{strongly in } L^\infty(0,T;W_0^{1,p}(\Omega)).$$

According to the continuous embedding $W_0^{1,p}(\Omega)$ to $L^2(\Omega)$, then we get

$$u^n \to u \quad \text{strongly in } C([0,T];W_0^{1,p}(\Omega)) \text{ as } n\to\infty \tag{3.5}$$

by the Simon's theorem [25]. Clearly, this implies

$$\Phi(u^n) \to \Phi(u) \quad \text{a.e. } (x,t)\in\Omega\times(0,T)$$

where $\Phi(u^n) = |u^n|^{p-2}u^n \log|u^n|$.

By a direct computation, we get

$$\begin{aligned}
\int_\Omega |\Phi(u^n)|^{p'}\,dx &\le \int_{\{x\in\Omega:|u^n|\ge 1\}} |\Phi(u^n)|^{p'}\,dx + \int_{\{x\in\Omega:|u^n|<1\}} |\Phi(u^n)|^{p'}\,dx \\
&\le \left(\frac{e^{-1}}{\rho}\right)^{p'} \int_{\{x\in\Omega:|u^n|\ge 1\}} |u^n|^{p^*}\,dx + [e(p-1)]^{-p'}|\Omega| \\
&\le \left(\frac{e^{-1}}{\rho}\right)^{p'} S^{p^*}\|\Delta u^n\|_2^{p^*} + [e(p-1)]^{-p'}|\Omega| \le C(d),
\end{aligned}$$

where $p' = p/(p-1)$, $\rho_2 > 0$ is chosen such that $2N/(N-4) > p^* := (p-1+\rho_2)p'$ as $p<N$ and ρ_2 is positive as $p\ge N$.

Therefore, as $n\to\infty$, we have

$$\Phi(u^n) \overset{*}{\rightharpoonup} \Phi(u) \quad \text{weakly * in } L^\infty(0,T;L^{p'}(\Omega)).$$

Finally, we show that the limit u is a weak solution. Fix $k\in\mathbb{N}$ and choose a function $v\in C^1([0,T];C_0^\infty(\Omega))$ with the following form:

$$v = \sum_j^k l_j(t)\varphi_j,$$

here $l_j(t)\in C^1([0,T])$ with $j=1,2,\dots,k$. Taking $n\ge k$ in (3.1), multiplying the first equality of (3.1) by $l_j(t)$, summing for j from 1 to k, and then integrating with respect to t from 0 to T, we have

$$\int_0^T (u_t^n, v) + \int_0^T (\Delta u^n, \Delta v)\,dt + \int_0^T (|\nabla u^n|^{p-2}\nabla u^n, \nabla v)\,dt = \int_0^T (|u^n|^{p-2}u^n\log|u^n|, v)\,dt.$$

Letting $n\to\infty$, the following equality holds:

$$\int_0^T (u_t, v) + \int_0^T (\Delta u, \Delta v)\,dt + \int_0^T (|\nabla u|^{p-2}\nabla u, \nabla v)\,dt = \int_0^T (|u|^{p-2}u\log|u|, v)\,dt. \tag{3.6}$$

Since $C^1([0,T];C_0^\infty(\Omega))$ is dense in $L^2(0,T;H_0^2(\Omega))$, it follows that (3.6) holds for $v\in L^2(0,T;H_0^2(\Omega))$. Moreover, by the arbitrariness of $T>0$, it follows that

$$(u_t,\varphi)+(\Delta u,\Delta\varphi)+(|\nabla u|^{p-2}\nabla u,\nabla\varphi)=(|u|^{p-2}u\log|u|,\varphi)\quad \text{a.e. } t>0$$

for any $\varphi\in H_0^2(\Omega)$. In view of (3.5) and $u^n(x,0)\to u_0(x)$ strongly in $H_0^2(\Omega)$, then $u(x,0)=u_0(x)$. Assume that u is sufficiently smooth such that $u_t\in L^2(0,T;H_0^2(\Omega))$, taking $v=u_t$ in (3.6), then (2.5) holds. Since $L^2(0,T;H_0^2(\Omega))$ is dense in $L^2(0,T;L^2(\Omega))$, (2.5) holds for weak solutions of problem (1.1).

Step 2: Uniqueness of weak solution. Assume that both u, v are two weak solutions for problem (1.1), then by the definition of weak solution, for $\varphi\in H_0^2(\Omega)$, we obtain

$$\begin{aligned}(u_t,\varphi)+(\Delta u,\Delta\varphi)+(|\nabla u|^{p-2}\nabla u,\nabla\varphi)&=(|u|^{p-2}u\log|u|,\varphi),\\ (v_t,\varphi)+(\Delta v,\Delta\varphi)+(|\nabla v|^{p-2}\nabla v,\nabla\varphi)&=(|v|^{p-2}v\log|v|,\varphi).\end{aligned}$$

Subtracting the above two equalities, taking $\varphi=u-v\in H_0^2(\Omega)$, and then integrating for t from 0 to t, we have

$$\begin{aligned}&\int_0^t(\Delta u-\Delta v,\Delta u-\Delta v)\,dt+\int_0^t(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v,\nabla u-\nabla v)\,dt+\int_0^t(\varphi_t,\varphi)\,dt\\ &=\int_0^t\int_\Omega(|u|^{p-2}u\log|u|-|v|^{p-2}v\log|v|)\varphi\,dxdt.\end{aligned}$$

Clearly, the first and second terms on the left-hand side of the above equality are non-negative. By the Lipschitz continuity of $|u|^{p-2}u\log|u|$, we get

$$\int_0^t(\varphi_t,\varphi)\,dt\le C\int_0^t\int_\Omega\varphi^2\,dxdt,$$

here $C>0$ is the Lipschitz constant. Further,

$$\|\varphi\|_2^2\le C\int_0^t\int_\Omega\varphi^2\,dxdt$$

by $\varphi(x,0)=0$. Gronwall's inequality implies

$$\|\varphi\|_2^2=0.$$

Therefore, $\varphi=0$ a.e. in $\Omega\times(0,T)$. □

Here, we consider the decay estimate. Before stating the result, we need the following lemma:

Lemma 3.2. [18] *Let $E\colon \mathbb{R} \to \mathbb{R}_+$ be a non-increasing function and $\phi\colon \mathbb{R} \to \mathbb{R}_+$ be a strictly increasing function of class C^1 such that*

$$\phi(0) = 0 \ \text{ and } \ \phi(t) \to +\infty \quad \text{as } t \to +\infty.$$

Assume that there exist $\sigma \geq 0$ and $\omega > 0$ such that

$$\int_t^{+\infty} (E(s))^{1+\sigma} \phi'(s)\, ds \leq \frac{1}{\omega} (E(0))^{\sigma} E(t),$$

then E has the following decay property:

(1) *if $\sigma = 0$, then $E(t) \leq E(0) e^{1-\omega\phi(t)}$ for all $t \geq 0$;*

(2) *if $\sigma > 0$, then $E(t) \leq E(0) \left(\frac{1+\sigma}{1+\omega\sigma\phi(t)}\right)^{1/\sigma}$ for all $t \geq 0$.*

Theorem 3.3 (Decay estimate for $J(u_0) < d$)**.** *Assume that u is a global weak solution for problem* (1.1)*, $0 < J(u_0) < d$, $I(u_0) > 0$, then we have*

$$\|u\|_{W_0^{1,p}}^2 \leq \|u_0\|_{W_0^{1,p}}^2 \left(\frac{p}{2 + \omega(p-2)t}\right)^{2/(p-2)} \quad \text{for all } t \geq 0,$$

here $\omega > 0$ is obtained later.

Proof. Recalling Lemma 2.10, we know that $u(x,t) \in W_\delta$ for $1 \leq \delta < \widehat{\delta}$ and $0 < t < T$. Particularly, $I(u) > 0$ for $0 < t < T$. Therefore, it follows from (2.2), (2.11) and (2.5) that

$$\|u\|_p^p \leq p^2 J(u) < p^2 J(u_0) < p^2 d.$$

Taking $\delta = 1$ in (2.7), for $\frac{N\mathcal{L}_p}{pe}(p^2 d)^{p^2/N} < \mu < p$, we get

$$\begin{aligned} I(u) &\geq \|\Delta u\|_2^2 + \left(1 - \frac{\mu}{p}\right) \|\nabla u\|_p^p + \left(\frac{N}{p^2} \log\left(\frac{p\mu e}{N\mathcal{L}_p}\right) - \frac{1}{p} \log \|u\|_p^p\right) \|u\|_p^p \\ &\geq \left(1 - \frac{\mu}{p}\right) \|\nabla u\|_p^p + \left(\frac{N}{p^2} \log\left(\frac{p\mu e}{N\mathcal{L}_p}\right) - \frac{1}{p} \log p^2 J(u_0)\right) \|u\|_p^p \\ &\geq \frac{\alpha}{2} \|u\|_{W_0^{1,p}}^p, \end{aligned} \tag{3.7}$$

where

$$\alpha = \min\left\{1 - \frac{\mu}{p}, \frac{N}{p^2} \log\left(\frac{p\mu e}{N\mathcal{L}_p}\right) - \frac{1}{p} \log p^2 J(u_0)\right\} > 0.$$

Integrating (2.11) from t to T, then we have

$$\int_t^T I(u)\, d\tau = \frac{1}{2}\|u(t)\|_2^2 - \frac{1}{2}\|u(T)\|_2^2 \leq \frac{1}{2} \widehat{S}^2 \|u\|_{W_0^{1,p}}^2, \tag{3.8}$$

where $\widehat{S}$ is the optimal constant of the embedding $W_0^{1,p}(\Omega)$ to $L^2(\Omega)$. Combining (3.7) and (3.8), we obtain

$$\int_t^T \|u\|_{W_0^{1,p}}^p \, d\tau \le \frac{\widehat{S}^2}{\alpha}\|u\|_{W_0^{1,p}}^2.$$

Letting $T \to +\infty$, it follows that

$$\int_t^{+\infty} (E(t))^{p/2} \, d\tau \le \frac{\widehat{S}^2}{\alpha} E(t) = \frac{1}{\omega}(E(0))^{\sigma} E(t),$$

where $E(t) = \|u\|_{W_0^{1,p}}^2$, $\omega = \alpha/\big(\widehat{S}^2\|u_0\|_{W_0^{1,p}}^{2\sigma}\big)$, $\sigma = p/2 - 1$. By Lemma 3.2, we have the following estimates:

$$\|u\|_{W_0^{1,p}}^2 \le \|u_0(x)\|_{W_0^{1,p}}^2 \left(\frac{p}{2+\omega(p-2)t}\right)^{1/\sigma} \quad \text{for all } t \ge 0. \qquad \square$$

Theorem 3.4 (Blow-up for $J(u_0) < d$)**.** *Let condition* (1.2) *hold,* $u_0 \in H_0^2(\Omega)$. *If* u *is a weak solution of problem* (1.1), $J(u_0) < d$, $I(u_0) < 0$, *then there exists a finite time* T^* *such that* u *blows up in the sense of* $\lim_{t\to T^*} \int_0^t \|u\|_2^2 \, d\tau = +\infty$.

Proof. Assume that u is a global weak solution of problem (1.1) with $J(u_0) < d$, $I(u_0) < 0$, and define

$$M(t) = \int_0^t \|u\|_2^2 \, d\tau, \quad \forall\, t \ge 0,$$

then $M'(t) = \|u\|_2^2$, and

$$M''(t) = 2(u_t, u) = -2I(u). \tag{3.9}$$

Using (2.5), (3.9), (2.2) and the embedding $H_0^2(\Omega)$ to $L^2(\Omega)$, we get

$$\begin{aligned} M''(t) &= \frac{2}{p}\|u\|_p^p + (p-2)\|\Delta u\|_2^2 - 2pJ(u) \\ &\ge \frac{p-2}{S^2}\|u\|_2^2 + 2p\int_0^t \|u_\tau\|_2^2 \, d\tau - 2pJ(u_0) \\ &= \frac{p-2}{S^2} M'(t) + 2p\int_0^t \|u_\tau\|_2^2 \, d\tau - 2pJ(u_0). \end{aligned}$$

Noticing that

$$(M'(t))^2 = 4\left(\int_0^t\int_\Omega u_\tau u \, dx d\tau\right)^2 + 2\|u_0\|_2^2 M'(t) - \|u_0\|_2^4,$$

then we have

$$\begin{aligned} M''(t)M(t) - \frac{p}{2}(M'(t))^2 \ge{}& \frac{p-2}{S^2} M'(t)M(t) - 2pM(t)J(u_0) \\ &+ 2p\int_0^t \|u\|_2^2 \, d\tau \int_0^t \|u_\tau\|_2^2 \, d\tau + \frac{p}{2}\|u_0\|_2^4 \\ &- 2p\left(\int_0^t\int_\Omega u_\tau u \, dx d\tau\right)^2 - p\|u_0\|_2^2 M'(t). \end{aligned}$$

By Cauchy-Schwarz inequality

$$\left(\int_0^t\int_\Omega u_\tau u\,dxd\tau\right)^2 \le \int_0^t \|u\|_2^2\,d\tau \int_0^t \|u_\tau\|_2^2\,d\tau,$$

then

$$M''(t)M(t) - \frac{p}{2}(M'(t))^2 \ge \frac{p-2}{S^2}M'(t)M(t) - 2pM(t)J(u_0) - p\|u_0\|_2^2 M'(t). \tag{3.10}$$

Now, we discuss the following two cases:

Case 1*:* $J(u_0) \le 0$. For $J(u_0) \le 0$, (3.10) implies

$$M''(t)M(t) - \frac{p}{2}(M'(t))^2 \ge \frac{p-2}{S^2}M'(t)M(t) - p\|u_0\|_2^2 M'(t).$$

Now we show that $I(u) < 0$ for $t > 0$. Otherwise, there exists a $t_0 > 0$ such that $I(u(t_0)) = 0$ and $I(u) < 0$ for $0 \le t < t_0$. By Lemma 2.6(2)(3), we get $\|\Delta u\|_2 > \gamma(1)$ for $0 \le t < t_0$, and $\|\Delta u(t_0)\|_2 \ge \gamma(1)$. Therefore, $J(u(t_0)) \ge d$, which contradicts (2.5). Moreover, (3.9) implies $M''(t) > 0$ for $t \ge 0$. Since $M'(0) = \|u_0\|_2^2 \ge 0$, there exists a $t_0 > 0$ such that $M'(t_0) > 0$. Thus, we get

$$M(t) = M(t_0) + \int_{t_0}^t M'(\tau)\,d\tau \ge M'(t_0)(t - t_0),$$

then for any

$$t \ge t^* := \max\left\{t_0, \frac{p\|u_0\|_2^2 + \frac{p-2}{S^2}M'(t_0)t_0}{\frac{p-2}{S^2}M'(t_0)}\right\},$$

we have

$$\begin{aligned} M''(t)M(t) - \frac{p}{2}(M'(t))^2 &\ge M'(t)\left(\frac{p-2}{S^2}M(t) - p\|u_0\|_2^2\right) \\ &\ge M'(t)\left(\frac{p-2}{S^2}M'(t_0)(t-t_0) - p\|u_0\|_2^2\right) \\ &\ge 0. \end{aligned}$$

Case 2*:* $0 < J(u_0) < d$. By Lemma 2.10(2), we have $u(x,t) \in V_\delta$ for $1 \le \delta < \widehat{\delta}$ and $0 < t < T$. Lemma 2.8 implies $I_{\widehat{\delta}}(u) \le 0$. Furthermore, Lemma 2.6 implies $\|\Delta u\|_2 \ge \gamma(\widehat{\delta})$ for $t \ge 0$. It follows from (3.9) that for $t \ge 0$, we get

$$M''(t) = -2I(u) = 2(\widehat{\delta} - 1)\|\Delta u\|_2^2 + 2(\widehat{\delta} - 1)\|\nabla u\|_p^p - 2I_{\widehat{\delta}}(u) \ge C_* := 2(\widehat{\delta} - 1)(\gamma(\delta))^2,$$

which implies

$$M'(t) \ge C_* t, \quad M(t) \ge \frac{C_*}{2}t^2,$$

and $M'(t) > 0$, $M(t) > 0$ for any $t > 0$. Considering (3.10), for any

$$t \geq t^* := \max\left\{ \frac{2pJ(u_0)}{\frac{p-2}{2S^2}C_*}, \left(\frac{p\|u_0\|_2^2}{\frac{p-2}{4S^2}\frac{C_*}{2}} \right)^{1/2} \right\},$$

we have

$$\begin{aligned} M''(t)M(t) - \frac{p}{2}(M'(t))^2 &\geq M(t)\left(\frac{p-2}{2S^2}M'(t) - 2pJ(u_0) \right) \\ &\quad + M'(t)\left(\frac{p-2}{2S^2}M(t) - p\|u_0\|_2^2 \right) \\ &\geq 0. \end{aligned}$$

Based on the above discussion, for t^*, we get $M(t^*) > 0$, $M'(t^*) > 0$, thus $M(t) \to \infty$ as $t \to T^* \leq t^* + \frac{M(t^*)}{(p/2-1)M'(t^*)}$ by the Levine's concavity method [9,12,14], which contradicts our assumption. This proof is complete. $\square$

4. The critical initial energy $J(u_0) = d$

For critical initial energy $J(u_0) = d$, we still have the following global existence of weak solution by using the method of approximation.

Theorem 4.1 (Global existence for $J(u_0) = d$)**.** *Let condition* (1.2) *hold,* $u_0 \in H_0^2(\Omega)$*. If* $J(u_0) = d$*,* $I(u_0) \geq 0$*, then problem* (1.1) *admits a unique global weak solution* $u \in L^\infty(0,T;H_0^2(\Omega))$ *with* $u_t \in L^2(0,T;L^2(\Omega))$ *and* $u(t) \in W \cup \partial W$ *for* $0 \leq t < T$*.*

Proof. Let $\lambda_k = 1 - 1/k$, $k = 1, 2, \ldots$. Consider the following initial value problem:

$$\begin{cases} u_t + \Delta^2 u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{p-2}u\log|u| & \text{if } (x,t) \in \Omega \times (0,T), \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{if } (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = \lambda_k u_0(x) := u_0^k & \text{if } x \in \Omega. \end{cases} \tag{4.1}$$

Noticing that $I(u_0) \geq 0$, by Lemma 2.5(3), there exists a unique $\lambda^* = \lambda^*(u_0) \geq 1$ such that $I(\lambda^* u_0) = 0$. Using $\lambda_k < 1 \leq \lambda^*$ and Lemma 2.5(2)(3), we get $I(u_0^k) = I(\lambda_k u_0) > 0$ and $J(u_0^k) = J(\lambda_k u_0) < J(u_0) = d$. In view of Theorem 3.1, for each k, problem (4.1) admits a global weak solution $u^k \in L^\infty(0,T;H_0^2(\Omega))$ with $u_t^k \in L^2(0,T;L^2(\Omega))$ and $u^k \in W$ satisfying

$$\int_0^t \|u_\tau^k\|_2^2\,d\tau + J(u^k) = J(u_0^k) < d.$$

Applying the similar argument in Theorem 3.1, there exist a subsequence of $\{u^k\}_{k\in\mathbb{N}}$ and a function u such that u is a weak solution of problem (1.1) with $I(u) \geq 0$ and $J(u) \leq d$ for $0 \leq t < T$. The proof of uniqueness for weak solution is the same as that one in Theorem 3.1. $\square$

Theorem 4.2 (Decay estimate for $J(u_0)=d$)**.** *Assume that u is a global weak solution for problem* (1.1), *$J(u_0)=d$, $I(u_0)>0$, then the following statements hold:*

(1) *if $I(u)>0$ for $0<t<T$, then*

$$\|u\|^2_{W_0^{1,p}} \le \|u_0\|^2_{W_0^{1,p}} \left(\frac{p}{2+\omega_1(p-2)t}\right)^{2/(p-2)} \quad \text{for all } t\ge t_0>0,$$

here ω_1 is obtained later;

(2) *if $I(u)>0$ for $0<t<t^*$ and $I(u(t^*))=0$, then weak solution $u(x,t)$ vanishes in finite time t^*.*

Proof. Recalling Lemma 2.12, we have $I(u)\ge 0$ for $0<t<T$.

(1) For $I(u)>0$ for $0<t<\infty$, by the same argument of (2.12), we have

$$0<J(u(t_0))=J(u_0)-\int_0^{t_0}\|u_\tau\|_2^2\,d\tau<d$$

for any $t_0>0$. Taking $t=t_0$ as the initial time, by the same argument of Theorem 3.3, we get

$$\|u\|^2_{W_0^{1,p}} \le \|u_0\|^2_{W_0^{1,p}} \left(\frac{p}{2+\omega_1(p-2)t}\right)^{2/(p-2)} \quad \text{for all } t\ge t_0>0,$$

where the value of α_1 in $\omega_1=\alpha_1/\big[\widehat{S}^2(E(0))^\sigma\big]$ is as follows:

$$\alpha_1=\min\left\{1-\frac{\mu}{p},\frac{N}{p^2}\log\left(\frac{p\mu e}{N\mathcal{L}_p}\right)-\frac{1}{p}\log p^2J(u(t_0))\right\}>0.$$

(2) Assume that $I(u)>0$ for $0<t<t^*$ and $I(u(t^*))=0$, then (2.11) implies $u_t\ne 0$ for $0<t<t^*$. Therefore, (2.5) implies

$$J(u(t^*))=d-\int_0^{t^*}\|u_\tau\|_2^2\,d\tau<d.$$

By the definition of d, we easily know $\|\Delta u(t^*)\|_2=0$, which implies $u(t^*)=0$. Therefore, $u(t)\equiv 0$ for $t\ge t^*$, i.e., weak solution $u(x,t)$ vanishes in finite time t^*. □

Theorem 4.3 (Blow-up for $J(u_0)=d$)**.** *Let condition* (1.2) *hold, $u_0\in H_0^2(\Omega)$. If u is a weak solution of problem* (1.1), *$J(u_0)=d$, $I(u_0)<0$, then there exists a finite time T^* such that u blows up in the sense of $\lim_{t\to T^*}\int_0^t\|u\|_2^2\,d\tau=+\infty$.*

Proof. By the same argument in (3.10), we get

$$M''(t)M(t)-\frac{p}{2}(M'(t))^2\ge\frac{p-2}{S^2}M'(t)M(t)-2pdM(t)-p\|u_0\|_2^2M'(t).$$

Since $J(u_0) = d$, $I(u_0) < 0$, by the continuity of $J(u)$ and $I(u)$ with respect to t, there exists a t_0 such that $J(u(x,t)) > 0$ and $I(u(x,t)) < 0$ for $0 < t \le t_0$. By the same argument of (2.12), we have

$$0 < J(u(t_0)) = d - \int_0^{t_0} \|u_\tau\|_2^2 \, d\tau < d.$$

Taking $t = t_0$ as the initial time, then we get $u(x,t) \in V_\delta$ for $1 \le \delta < \widehat{\delta}$ and $t_0 < t$ by Lemma 2.10(2). Therefore, $I_\delta(u) < 0$, $\|\Delta u\|_2 > \gamma(\delta)$ for $1 \le \delta < \widehat{\delta}$ and $t_0 < t$ by Lemma 2.6(2). Furthermore, recalling Lemma 2.8, we get $I_{\widehat{\delta}}(u) \le 0$ and $\|\Delta u\|_2 \ge \gamma(\widehat{\delta})$ for $t_0 < t$. The rest of this proof is the same as that one of Case 2 in Theorem 3.4, here we omit the specific process. □

5. The supercritical initial energy $J(u_0) > d$

In this section, we will give some sufficient conditions for global existence of weak solutions and blow-up in finite time. Before stating our theorem, we need some sets and lemmas.

Define

$$\mathcal{N}_+ = \{u \in H_0^2(\Omega) \mid I(u) > 0\}, \quad \mathcal{N}_- = \{u \in H_0^2(\Omega) \mid I(u) < 0\},$$

and the (open) sublevels of J

$$J^\varsigma = \{u \in H_0^2(\Omega) \mid J(u) < \varsigma\}.$$

Obviously, by the definition of $J(u)$, $\mathcal{N}$, J^ς and d, we get

$$\mathcal{N}^\varsigma := \mathcal{N} \cap J^\varsigma = \left\{ u \in \mathcal{N} \;\middle|\; \frac{1}{p^2}\|u\|_p^p + \frac{p-2}{2p}\|\Delta u\|_2^2 < \varsigma \right\} \ne \emptyset, \quad \forall \varsigma > d.$$

For $\varsigma > d$, define

$$\lambda_\varsigma = \inf\{\|u\|_2 \mid u \in \mathcal{N}^\varsigma\}, \quad \Lambda_\varsigma = \sup\{\|u\|_2 \mid u \in \mathcal{N}^\varsigma\}.$$

Clearly, λ_ς is non-increasing and Λ_ς is non-decreasing.

For convenience, we introduce the following sets:

$$\mathcal{B} = \{u_0 \in H_0^2(\Omega) \mid \text{the solution } u = u(t) \text{ of problem (1.1) blows up in finite time}\};$$
$$\mathcal{G}_0 = \{u_0 \in H_0^2(\Omega) \mid u(t) \to 0 \text{ in } H_0^2(\Omega) \text{ as } t \to \infty\}.$$

Lemma 5.1. *If condition* (1.2) *holds, then*

(1) 0 *is away from both* $\mathcal{N}$ *and* $\mathcal{N}_-$, *i.e.,* $\operatorname{dist}(0,\mathcal{N}) > 0$ *and* $\operatorname{dist}(0,\mathcal{N}_-) > 0$*;*

(2) *for any $\varsigma > 0$, the set $J^{\varsigma} \cap \mathcal{N}_+$ is bounded in $H_0^2(\Omega)$.*

Proof. (1) For $u \in \mathcal{N}$, by the definition of d, (2.2), and the embedding $H_0^2(\Omega)$ to $L^p(\Omega)$, we have

$$\begin{aligned} d &\le \frac{1}{2}\|\Delta u\|_2^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{p}\int_\Omega |u|^p \log|u|\,dx + \frac{1}{p^2}\|u\|_p^p \\ &= \frac{1}{p^2}\|u\|_p^p + \frac{p-2}{2p}\|\Delta u\|_2^2 \le \frac{S^p}{p^2}\|\Delta u\|_2^p + \frac{p-2}{2p}\|\Delta u\|_2^2, \end{aligned}$$

which implies that there exists a constant $\rho > 0$ such that $\operatorname{dist}(0,\mathcal{N}) = \inf_{u\in\mathcal{N}} \|\Delta u\|_2 \ge \rho$.

For $u \in \mathcal{N}_-$, we get $\|\Delta u\|_2 \ne 0$, and similar to (2.3), we have

$$\|\Delta u\|_2^2 + \|\nabla u\|_p^p < \int_\Omega |u|^p \log|u|\,dx \le \frac{e^{-1}}{\rho_1} S^{p+\rho_1}\|\Delta u\|_2^{p+\rho_1}$$

which implies $\|\Delta u\|_2 > \big(\frac{1}{(e^{-1}/\rho_1)S^{p+\rho_1}}\big)^{1/(p+\rho_1-2)}$. Therefore, $\operatorname{dist}(0,\mathcal{N}_-) = \inf_{u\in\mathcal{N}_-} \|\Delta u\|_2 > 0$.

(2) For any $u \in J^{\varsigma} \cap \mathcal{N}_+$, then $J(u) < \varsigma$ and $I(u) > 0$. Therefore, it follows from (2.2) that

$$\varsigma > J(u) = \frac{1}{p}I(u) + \frac{1}{p^2}\|u\|_p^p + \frac{p-2}{2p}\|\Delta u\|_2^2 > \frac{p-2}{2p}\|\Delta u\|_2^2,$$

which yields $\|\Delta u\|_2^2 < \frac{2p}{p-2}\varsigma$. Therefore, the set $J^{\varsigma} \cap \mathcal{N}_+$ is bounded in $H_0^2(\Omega)$. □

Lemma 5.2. *For any $\varsigma > d$, λ_ς and Λ_ς satisfy*

$$0 < \lambda_\varsigma \le \Lambda_\varsigma < +\infty.$$

Proof. If $u \in \mathcal{N}^{\varsigma}$, then by (2.3) and Gagliardo-Nirenberg inequality, we obtain

$$\begin{aligned} \|\Delta u\|_2^2 + \|\nabla u\|_p^p &= \int_\Omega |u|^p \log|u|\,dx \le \frac{e^{-1}}{\rho_1}\|u\|_{p+\rho_1}^{p+\rho_1} \\ &\le \frac{e^{-1}}{\rho_1} C(N,p)\|\Delta u\|_2^{\alpha(p+\rho_1)}\|u\|_2^{(1-\alpha)(p+\rho_1)}, \end{aligned} \tag{5.1}$$

where $\alpha \in (0,1)$ due to $p < 2N/(N-4) - \rho_1$ when $N \ge 5$, $C(N,p) > 0$ is a constant. (5.1) can be written as

$$\|u\|_2^{(1-\alpha)(p+\rho_1)} \ge \frac{\rho_1}{e^{-1}}\frac{1}{C(N,p)}\|\Delta u\|_2^{2-\alpha(p+\rho_1)}. \tag{5.2}$$

Clearly, the left-hand side of (5.2) is bounded and away from 0 by Lemma 5.1(1) and the definition of $\mathcal{N}^{\varsigma}$. Therefore, we get $\lambda_\varsigma > 0$ by the definition of λ_ς. Using the embedding $H_0^2(\Omega)$ to $L^2(\Omega)$, then $\|u\|_2^2 \le S^2\|\Delta u\|_2^2$. Recalling the definition of $\mathcal{N}^{\varsigma}$, it is obvious that $\Lambda_\varsigma < +\infty$. □

Theorem 5.3. *Let condition* (1.2) *hold,* $u_0 \in H_0^2(\Omega)$*. If* $J(u_0) > d$*, then the following statements hold:*

(1) *if* $u_0 \in \mathcal{N}_+$ *and* $\|u_0\|_2 \leq \lambda_{J(u_0)}$*, then* $u_0 \in \mathcal{G}_0$*;*

(2) *if* $u_0 \in \mathcal{N}_-$ *and* $\|u_0\|_2 \geq \Lambda_{J(u_0)}$*, then* $u_0 \in \mathcal{B}$*.*

Proof. Denote by $T(u_0)$ the maximal existence time of the solutions for problem (1.1). If there exists a global solution, i.e., $T(u_0) = \infty$, we denote by

$$\omega(u_0) = \bigcap_{t \geq 0} \overline{\{u(\iota) : \iota \geq t\}}^{H_0^2(\Omega)}$$

the ω-limit of $u_0 \in H_0^2(\Omega)$.

(1) If $u_0 \in \mathcal{N}_+$ and $\|u_0\|_2 \leq \lambda_{J(u_0)}$, then we claim that $u \in \mathcal{N}_+$ for all $t \in [0, T(u_0))$. By contradiction, there exists a $t_0 \in (0, T(u_0))$ such that $u \in \mathcal{N}_+$ for $t \in [0, t_0)$ and $u(t_0) \in \mathcal{N}$. Therefore, $u_t \neq 0$ for $\Omega \times (0, t_0)$ from (2.11). It follows from (2.5) that $J(u(t_0)) < J(u_0)$, which implies $u(t_0) \in J^{J(u_0)}$. Further, $u(t_0) \in \mathcal{N}^{J(u_0)}$. By the definition of $\lambda_{J(u_0)}$, we obtain

$$\|u(t_0)\|_2 \geq \lambda_{J(u_0)}. \tag{5.3}$$

Noticing that $I(u(t)) > 0$ for $t \in [0, t_0)$, it follows from (2.11) that

$$\|u(t_0)\|_2 < \|u_0\|_2 \leq \lambda_{J(u_0)},$$

which contradicts (5.3). Therefore, $u \in \mathcal{N}_+$ for all $t \in [0, T(u_0))$, which implies $u(t) \in J^{J(u_0)}$ for all $t \in [0, T(u_0))$ by (2.11) and (2.5). Lemma 5.1(2) shows that $u(t)$ remains bounded in $H_0^2(\Omega)$ for $t \in [0, T(u_0))$, and the boundedness of $\|u\|_{H_0^2(\Omega)}$ is independent of t, moreover, $T(u_0) = +\infty$, $u \in \mathcal{N}_+ \cap J^{J(u_0)}$ for $0 \leq t < \infty$. For any $\omega \in \omega(u_0)$, then

$$\|\omega\|_2 < \lambda_{J(u_0)}, \quad J(\omega) < J(u_0),$$

by (2.5) and (2.11). Noticing that the definition of $\lambda_{J(u_0)}$, we obtain $\omega(u_0) \cap \mathcal{N} = \emptyset$. Therefore, $\omega(u_0) = \{0\}$, i.e., $u_0 \in \mathcal{G}_0$.

(2) If $u_0 \in \mathcal{N}_-$ and $\|u_0\|_2 \geq \Lambda_{J(u_0)}$, then we claim that $u \in \mathcal{N}_-$ for all $t \in [0, T(u_0))$. By contradiction, there exists a $t^0 \in (0, T(u_0))$ such that $u \in \mathcal{N}_-$ for $t \in [0, t^0)$ and $u(t^0) \in \mathcal{N}$. Similar to Case (1), we get $J(u(t^0)) < J(u_0)$, which implies $u(t^0) \in J^{J(u_0)}$. Further, $u(t^0) \in \mathcal{N}^{J(u_0)}$. By the definition of $\Lambda_{J(u_0)}$, we obtain

$$\|u(t^0)\|_2 \leq \Lambda_{J(u_0)}. \tag{5.4}$$

Noticing that $I(u(t)) < 0$ for $t \in [0, t^0)$, it follows from (2.11) that

$$\|u(t^0)\|_2 > \|u_0\|_2 \geq \Lambda_{J(u_0)},$$

which contradicts (5.4). Suppose $T(u_0)=\infty$, then for any $\omega\in\omega(u_0)$,

$$\|\omega\|_2>\Lambda_{J(u_0)},\quad J(\omega)<J(u_0),$$

by (2.5) and (2.11). Recalling the definition of $\Lambda_{J(u_0)}$, we obtain $\omega(u_0)\cap\mathcal{N}=\emptyset$. Therefore, $\omega(u_0)=\{0\}$, which contradicts $\operatorname{dist}(0,\mathcal{N}_-)>0$ in Lemma 5.1(1). Therefore, $\omega(u_0)=\emptyset$, $T(u_0)<\infty$. □

Corollary 5.4. *Let condition* (1.2) *hold, $u_0\in H_0^2(\Omega)$, $J(u_0)>d$. If $p^2|\Omega|^{(p-2)/2}J(u_0)<\|u_0\|_2^p$, then $u_0\in\mathcal{N}_-\cap\mathcal{B}$.*

Proof. By Hölder's inequality, we get

$$p^2|\Omega|^{(p-2)/2}J(u_0)<\|u_0\|_2^p\le\|u_0\|_p^p|\Omega|^{(p-2)/2}. \tag{5.5}$$

Combining (2.2) and (5.5), we obtain

$$\begin{aligned}J(u_0)&=\frac{1}{p}I(u_0)+\frac{1}{p^2}\|u_0\|_p^p+\frac{p-2}{2p}\|\Delta u_0\|_2^2\\&\ge\frac{1}{p}I(u_0)+\frac{1}{p^2}\|u_0\|_p^p>\frac{1}{p}I(u_0)+J(u_0),\end{aligned}$$

which implies $I(u_0)<0$, i.e., $u_0\in\mathcal{N}_-$.

To prove $u_0\in\mathcal{B}$, we only need to prove $\|u_0\|_2\ge\Lambda_{J(u_0)}$ by Theorem 5.3(2). For any $u\in\mathcal{N}^{J(u_0)}$, it follows from (2.2) that

$$\begin{aligned}\|u\|_2^p\le\|u\|_p^p|\Omega|^{(p-2)/2}&<|\Omega|^{(p-2)/2}p^2\left(\frac{1}{p}I(u)+\frac{1}{p^2}\|u\|_p^p+\frac{p-2}{2p}\|\Delta u\|_2^2\right)\\&=|\Omega|^{(p-2)/2}p^2J(u)<|\Omega|^{(p-2)/2}p^2J(u_0).\end{aligned}$$

Therefore, taking the supremum of above inequality over $\mathcal{N}^{J(u_0)}$, we easily obtain

$$\Lambda_{J(u_0)}^p<|\Omega|^{(p-2)/2}p^2J(u_0)<\|u_0\|_2^p,$$

i.e., $\|u_0\|_2>\Lambda_{J(u_0)}$. Therefore, $u_0\in\mathcal{N}_-\cap\mathcal{B}$. □

The following theorem indicates that there exist blow-up solutions to problem (1.1) for any supercritical initial energy.

Theorem 5.5. *Let condition* (1.2) *hold, $u_0\in H_0^2(\Omega)$. For any $M>d$, then there exists $u_M\in\mathcal{N}_-$ such that $J(u_M)=M$ and $u_M\in\mathcal{B}$.*

Proof. Assume that $M>d$ and Ω_1, Ω_2 are two arbitrary disjoint open subdomains of Ω. Furthermore, we assume that $\nu\in H_0^2(\Omega_1)$ is an arbitrary nonzero function, then we take ζ large enough such that

$$J(\zeta\nu)=\frac{\zeta^2}{2}\|\Delta u\|_2^2+\frac{\zeta^p}{p}\|\nabla u\|_p^p-\frac{\zeta^p}{p}\int_\Omega|u|^p\log|u|\,dx-\frac{\zeta^p}{p}\log|\zeta|\int_\Omega|u|^p\,dx+\frac{\zeta^p}{p^2}\|u\|_p^p\le0,$$

and

$$\|\zeta\nu\|_2^p > |\Omega|^{(p-2)/2} p^2 M.$$

We fix such a number $\zeta > 0$ and choose a function $\mu \in H_0^2(\Omega_2)$ satisfying $M = J(\mu) + J(\zeta\nu)$. Extend ν and μ to be 0 in $\Omega \setminus \Omega_1$ and $\Omega \setminus \Omega_2$. Set $u_M = \zeta\nu + \mu$, then $M = J(\mu + \zeta\nu) = J(u_M)$ and it follows that

$$\|u_M\|_2^p \geq \|\zeta\nu\|_2^p > |\Omega|^{(p-2)/2} p^2 J(u_M).$$

By Corollary 5.4, then $u_M \in \mathcal{N}_- \cap \mathcal{B}$. This completes the proof of this theorem. $\square$

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