

The Explicit Formulae and Evaluations of Ramanujan’s Remarkable Product of Theta-functions

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Abstract. On pages 338 and 339 in his first notebook, Ramanujan defined remarkable product of theta-functions $a_{m,n}$ and also recorded eighteen explicit values depending on two parameters m and n . All these values have been established by Berndt et al. In this paper, we establish a new general formulae for the explicit evaluations of $a_{3m,3}$ and $a_{m,9}$ by using P - Q mixed modular equation and values for certain class invariant of Ramanujan. Using these formulae, we calculate some new explicit values of $a_{3m,3}$ for $m = 2, 7, 13, 17, 25, 37$ and $a_{m,9}$ for $m = 17, 37$.

1. Introduction

The following definitions of theta functions φ , ψ and f with $|q| < 1$ are classical [1]:

$$\begin{aligned}\varphi(q) &= f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_{\infty}^2 (q^2; q^2)_{\infty}, \\ \psi(q) &= f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}^2}, \\ f(-q) &= f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_{\infty},\end{aligned}$$

where $(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n)$.

For $q = e^{-\pi\sqrt{n}}$, Weber-Ramanujan class invariant [2, (1.3), p. 183] is defined by

$$(1.1) \quad G_n = 2^{-1/4} q^{-1/24} \chi(q) = \frac{f(q)}{2^{1/4} q^{1/24} f(-q^2)},$$

where n is a positive rational number and $\chi(q) = (-q; q^2)_{\infty}$.

Now we shall recall the definition of modular equation from [1]. The complete elliptic integral of the first kind $K(k)$ of modulus k is defined by

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(1/2)_n^2}{(n!)^2} k^{2n} = \frac{\pi}{2} \varphi^2 \left(e^{-\pi \frac{K'}{K}} \right), \quad (0 < k < 1)$$

Received May 31, 2018; Accepted July 28, 2019.

Communicated by Yifan Yang.

2010 *Mathematics Subject Classification*. Primary: 33D10; Secondary: 11F20, 14K25.

Key words and phrases. modular equation, class invariants, remarkable product of theta-function.

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and let $K' = K(k')$, where $k' = \sqrt{1 - k^2}$ is called the complementary modulus of k . Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l and l' respectively. Suppose that the equality

$$(1.2) \quad n \frac{K'}{K} = \frac{L'}{L}$$

holds for some positive integer n , then a modular equation of degree n is a relation between the moduli k and l which is implied by equation (1.2). Ramanujan defined his modular equation involving α and β , where $\alpha = k^2$ and $\beta = l^2$. Then we say β is of degree n over α .

On page 338 in his first notebook [6], Ramanujan defines

$$(1.3) \quad a_{m,n} = \frac{nq^{(n-1)/4}\psi^2(q^n)\varphi^2(-q^{2n})}{\psi^2(q)\varphi^2(-q^2)},$$

where $q = e^{-\pi\sqrt{m/n}}$ and m, n are positive rationals then, on pages 338 and 339, he offers a list of 18 particular values. All these 18 values have been established by Berndt, Chan and Zhang [3].

M. S. Naika and B. N. Dharmendra [5] have given an alternative form of (1.3) and their properties as below:

$$(1.4) \quad a_{m,n} = \frac{nq^{(n-1)/4}\psi^2(-q^n)\varphi^2(q^n)}{\psi^2(-q)\varphi^2(q)}, \quad a_{1,n} = 1 \quad \text{and} \quad a_{m,n} = a_{n,m}.$$

They have proved some general theorems to calculate explicit values of $a_{m,n}$.

In this article, we establish a certain P - Q mixed modular equation, which will be used along with class invariant to find general formulae for the explicit evaluations of the Ramanujan’s remarkable product of theta functions. Further, explicit values of $a_{3m,3}$ for $m = 2, 7, 13, 17, 25, 37$ and $a_{m,9}$ for $m = 17, 37$ have been evaluated.

2. Preliminary results

In this section, we have listed some lemmas which play a vital role in establishing our main results.

Lemma 2.1. [1, Entry 24(iii), p. 39]

$$(2.1) \quad f(q)f(-q^2) = \psi(-q)\varphi(q).$$

Lemma 2.2. [8, Theorem 3.2.4] *If $P = \frac{f(-q)}{q^{1/24}f(-q^2)}$ and $Q = \frac{f(-q^3)}{q^{1/8}f(-q^6)}$, then*

$$(2.2) \quad (PQ)^3 + \left(\frac{2}{PQ}\right)^3 = \left(\frac{Q}{P}\right)^6 - \left(\frac{P}{Q}\right)^6.$$

Lemma 2.3. [7, Theorem 2.3] *If $P = \frac{q^{1/12}f(-q)f(-q^6)}{f(-q^2)f(-q^3)}$ and $Q = \frac{q^{1/4}f(-q^3)f(-q^{18})}{f(-q^6)f(-q^9)}$, then*

$$(PQ)^6 + \frac{1}{(PQ)^6} - 10 \left[(PQ)^3 + \frac{1}{(PQ)^3} \right] + 20 = \left[\left(\frac{P}{Q} \right)^6 + \left(\frac{Q}{P} \right)^6 \right] \left[(PQ)^3 + \frac{1}{(PQ)^3} - 1 \right].$$

Lemma 2.4. [7, Theorem 2.4] *If $P = \frac{f(-q)f(-q^2)}{q^{1/4}f(-q^3)f(-q^6)}$ and $Q = \frac{f(-q^3)f(-q^6)}{q^{3/4}f(-q^9)f(-q^{18})}$, then*

$$\frac{9}{PQ} + PQ = \left(\frac{P}{Q} \right)^{-2} - 3.$$

Lemma 2.5. [7, Theorem 2.6] *If $P = \frac{q^{5/12}f(-q)f(-q^{18})}{f(-q^3)f(-q^6)}$ and $Q = \frac{q^{1/12}f(-q^2)f(-q^9)}{f(-q^3)f(-q^6)}$, then*

$$\left(\frac{P}{Q} \right)^3 + \left(\frac{Q}{P} \right)^3 = \frac{1}{(PQ)^2} + 1.$$

Lemma 2.6. [4, Main result (vi)] *If $P = \frac{q^{1/2}f(-q)f(-q^9)}{f(-q^3)f(-q^3)}$ and $Q = \frac{q^{1/3}f(-q^2)f(-q^{18})}{f(-q^6)f(-q^6)}$, then*

$$\left(\frac{P}{Q} \right)^3 + \left(\frac{Q}{P} \right)^3 = \frac{1}{PQ} - 3PQ.$$

3. General formulae for the explicit evaluations of $a_{3m,3}$ and $a_{9,n}$

In this section, we establish some general formulae for the explicit evaluations of $a_{3m,3}$ and $a_{9,n}$ by using the P - Q modular equation and class invariant.

Theorem 3.1. *If $P = \frac{f(-q)f(-q^9)}{q^{5/12}f(-q^2)f(-q^{18})}$ and $Q = \frac{f(-q^3)f(-q^3)}{q^{1/4}f(-q^6)f(-q^6)}$, then*

$$\left(\frac{P}{Q} \right)^3 - \left(\frac{Q}{P} \right)^3 = Q^2 - \frac{4}{Q^2}.$$

Proof. Replacing q by q^3 in (2.2), we obtain

$$\begin{aligned} (3.1) \quad & \left(\frac{f(-q^3)f(-q^9)}{q^{1/2}f(-q^6)f(-q^{18})} \right)^3 + 8 \left(\frac{f(-q^3)f(-q^9)}{q^{1/2}f(-q^6)f(-q^{18})} \right)^{-3} \\ & = \left(\frac{q^{1/4}f(-q^3)f(-q^{18})}{f(-q^6)f(-q^9)} \right)^{-6} - \left(\frac{q^{1/4}f(-q^3)f(-q^{18})}{f(-q^6)f(-q^9)} \right)^6. \end{aligned}$$

Now multiplying (2.2) with (3.1), we deduce that

$$(3.2) \quad x^2 - 8x - 2 = (PQ)^3 + \left(\frac{4}{PQ} \right)^3 + \left(\frac{P}{Q} \right)^6 + \left(\frac{Q}{P} \right)^6,$$

where $x = \left(\frac{q^{1/3}f(-q)f(-q^{18})}{f(-q^2)f(-q^9)} \right)^3 + \left(\frac{q^{1/3}f(-q)f(-q^{18})}{f(-q^2)f(-q^9)} \right)^{-3}$.

Lemma 2.3 may be written in the form

$$x^2 - 10x + 18 - y(x - 1) = 0,$$

where $y = \left(\frac{P}{Q}\right)^6 + \left(\frac{Q}{P}\right)^6$.

Solving this for x , we obtain

$$(3.3) \quad x = \frac{y + 10 + \sqrt{y^2 + 16y + 28}}{2}.$$

Using (3.3) in (3.2), we deduce, after a straightforward, lengthy calculation, that

$$\begin{aligned} & (Q^6 - P^5Q^3 + 4PQ^3 - P^6)(Q^6 + P^3Q^5 - 4P^3Q - P^6) \\ & \times (Q^{12} + P^5Q^9 - 4PQ^9 + P^{10}Q^6 + 2P^6Q^6 + 16P^2Q^6 - P^{11}Q^3 + 4P^7Q^3 + P^{12}) \\ & \times (Q^{12} - P^3Q^{11} + P^6Q^{10} + 4P^3Q^7 + 2P^6Q^6 + P^9Q^5 + 16P^6Q^2 - 4P^9Q + P^{12}) = 0. \end{aligned}$$

We observe that the second factor of the above equation vanishes for $q \rightarrow 0$ and other factors do not vanish for that specific value. Hence, we have

$$(3.4) \quad Q^6 + P^3Q^5 - 4P^3Q - P^6 = 0.$$

Dividing (3.4) by P^3Q^3 , this completes the proof of the theorem. □

Theorem 3.2. *If m is a positive rational and*

$$(3.5) \quad \Delta = G_m^4 + G_m^{-4},$$

then

$$(3.6) \quad \begin{aligned} a_{3m,3} &= \frac{1}{\sqrt{3}} \left(\sqrt{\frac{\Delta^2 - 1}{3}} - \sqrt{\frac{\Delta^2 - 4}{3}} \right) \\ & \times \left(\sqrt{\frac{\Delta^2 + 2 + \sqrt{(\Delta^2 - 1)(\Delta^2 - 4)}}{6}} - \sqrt{\frac{\Delta^2 - 4 + \sqrt{(\Delta^2 - 1)(\Delta^2 - 4)}}{6}} \right)^2. \end{aligned}$$

Proof. Applying Theorem 3.1 in Lemma 2.6 and changing q to $-q$, we have

$$(3.7) \quad \frac{q^{5/6} f(q) f(-q^2) f(q^9) f(-q^{18})}{f^2(q^3) f^2(-q^6)} = \frac{\sqrt{z^2 - 4} - \sqrt{z^2 - 16}}{6},$$

where $z = \left(\frac{f(q^3)}{q^{1/8} f(-q^6)}\right)^4 + 4\left(\frac{f(q^3)}{q^{1/8} f(-q^6)}\right)^{-4}$.

Using (1.1) and (2.1) in (3.7) with $q = e^{-\pi\sqrt{m/9}}$, we obtain that

$$(3.8) \quad \frac{q^{5/6} \psi(-q) \varphi(q) \psi(-q^9) \varphi(q^9)}{\psi^2(-q^3) \varphi^2(q^3)} = \frac{\sqrt{\Delta^2 - 1} - \sqrt{\Delta^2 - 4}}{3}.$$

Lemma 2.4 replace q by $-q$ and using the equation (3.7), we arrive at the following equation

$$\frac{f(q)f(-q^2)}{qf(q^9)f(-q^{18})} + 9 \left(\frac{f(q)f(-q^2)}{qf(q^9)f(-q^{18})} \right)^{-1} = \left(\frac{\sqrt{z^2 - 4} + \sqrt{z^2 - 16}}{2} \right)^2 + 3.$$

Once again, we can use (1.1) and (2.1) in above and solving the quadratic equation, we deduce that

$$(3.9) \quad \frac{\psi(-q)\varphi(q)}{q\psi(-q^9)\varphi(q^9)} = \left(\sqrt{\frac{\Delta^2 + 2 + \sqrt{(\Delta^2 - 1)(\Delta^2 - 4)}}{2}} + \sqrt{\frac{\Delta^2 - 4 + \sqrt{(\Delta^2 - 1)(\Delta^2 - 4)}}{2}} \right)^2.$$

By combining (3.8) and (3.9) with definition of $a_{m,n}$, we complete the proof. □

Corollary 3.3. *If m is a positive rational and Δ is define as in Theorem 3.2, then*

$$a_{m,9} = \left(\sqrt{\frac{\Delta^2 + 2 + \sqrt{(\Delta^2 - 1)(\Delta^2 - 4)}}{6}} - \sqrt{\frac{\Delta^2 - 4 + \sqrt{(\Delta^2 - 1)(\Delta^2 - 4)}}{6}} \right)^4.$$

Proof. Employing (1.4) in (3.9), it is not difficult to deduce our corollary. □

4. Explicit evaluations of $a_{3m,3}$ and $a_{m,9}$

Using Theorem 3.2 and Corollary 3.3, several values of $a_{3m,3}$ and $a_{m,9}$ have evaluated when the class invariant G_m is known. In this section, we compute values of $a_{3m,3}$ for $m = 2, 7, 13, 17, 25, 37$ and $a_{m,9}$ for $m = 17, 37$.

Theorem 4.1. *We have*

$$(4.1) \quad a_{6,3} = \frac{1}{3}(\sqrt{2} + 1)^{1/2}(\sqrt{3} - \sqrt{2}) \left(\frac{\sqrt{3} + 1}{\sqrt{2}} \right).$$

Proof. Letting $m = 2$, $G_2 = \left(\frac{\sqrt{2}+1}{2}\right)^{1/8}$ (see [8, p. 114]), and using this value in (3.5), we find that

$$(4.2) \quad \Delta^2 = \frac{5\sqrt{2} + 1}{2} \quad \text{and also} \quad \sqrt{(\Delta^2 - 1)(\Delta^2 - 4)} = \frac{4\sqrt{2} - 5}{2}.$$

Employing (4.2) in (3.6) with $m = 2$, we find, after a straightforward, lengthy calculation, that

$$a_{6,3} = \frac{1}{3} \left(5\sqrt{2} - 4 - \sqrt{57 - 40\sqrt{2}} \right)^{1/2} \left(\frac{3\sqrt{2} - 2}{2} - \sqrt{\frac{9 - 6\sqrt{2}}{2}} \right).$$

By [2, (9.5), p. 284], we have

$$\sqrt{57 - 40\sqrt{2}} = 4\sqrt{2} - 5 \quad \text{and} \quad \sqrt{\frac{9 - 6\sqrt{2}}{2}} = \frac{2\sqrt{3} - \sqrt{6}}{2}.$$

Thus, after further simplification, we arrive at (4.1). □

Theorem 4.2. *We have*

$$(4.3) \quad a_{21,3} = \frac{1}{\sqrt{3}} \left(\frac{\sqrt{7} - \sqrt{3}}{2} \right) \left(\sqrt{\frac{5 + \sqrt{21}}{8}} - \sqrt{\frac{\sqrt{21} - 3}{8}} \right)^4.$$

Proof. Let $m = 7$, $G_7 = 2^{1/4}$ of [2, p. 189] and using this value in (3.5), we obtain that

$$\Delta = \frac{5}{2} \quad \text{and also} \quad \sqrt{(\Delta^2 - 1)(\Delta^2 - 4)} = \frac{3\sqrt{21}}{4}.$$

Employing these values in (3.6), we readily find that

$$(4.4) \quad \begin{aligned} a_{21,3} &= \frac{1}{\sqrt{3}} \left(\frac{\sqrt{7} - \sqrt{3}}{2} \right) \left(\sqrt{\frac{11 + \sqrt{21}}{8}} - \sqrt{\frac{3 + \sqrt{21}}{8}} \right)^2 \\ &= \frac{1}{\sqrt{3}} \left(\frac{\sqrt{7} - \sqrt{3}}{2} \right) \left(\frac{7 + \sqrt{21}}{4} - \sqrt{\left(\frac{7 + \sqrt{21}}{4} \right)^2 - 1} \right). \end{aligned}$$

Now we apply Lemma 9.10 in [2, p. 292] with $r = (7 + \sqrt{21})/4$. Then $t = (\sqrt{21} + 1)/8$ and so

$$(4.5) \quad \frac{7 + \sqrt{21}}{4} - \sqrt{\left(\frac{7 + \sqrt{21}}{4} \right)^2 - 1} = \left(\sqrt{\frac{5 + \sqrt{21}}{8}} - \sqrt{\frac{\sqrt{21} - 3}{8}} \right)^4.$$

From (4.4) and (4.5), we deduce (4.3). □

Theorem 4.3. *We have*

$$\begin{aligned} a_{39,3} &= \frac{1}{\sqrt{3}} \left(\frac{\sqrt{3} - 1}{\sqrt{2}} \right)^2 \left(\sqrt{\frac{5 + 2\sqrt{3}}{2}} - \sqrt{\frac{3 + 2\sqrt{3}}{2}} \right)^2, \\ a_{51,3} &= \frac{1}{3} (\sqrt{17} - 4)^{1/2} \left(\sqrt{\frac{15 + 3\sqrt{17}}{4}} - \sqrt{\frac{11 + 3\sqrt{17}}{4}} \right)^2, \\ a_{75,3} &= \frac{1}{\sqrt{3}} \left(\frac{\sqrt{5} - \sqrt{3}}{\sqrt{2}} \right)^2 \left(\sqrt{\frac{17 + 4\sqrt{15}}{2}} - \sqrt{\frac{15 + 4\sqrt{15}}{2}} \right)^2, \end{aligned}$$

$$\begin{aligned}
 a_{111,3} &= \frac{1}{\sqrt{3}} \left(\frac{\sqrt{3}-1}{\sqrt{2}} \right)^4 \left(\sqrt{25+14\sqrt{3}} - \sqrt{24+14\sqrt{3}} \right)^2, \\
 a_{17,9} &= \left(\sqrt{\frac{15+3\sqrt{17}}{4}} - \sqrt{\frac{11+3\sqrt{17}}{4}} \right)^4, \\
 a_{37,9} &= \left(\sqrt{25+14\sqrt{3}} - \sqrt{24+14\sqrt{3}} \right)^4.
 \end{aligned}$$

Proof. Employing class invariant for $m = 7, 13, 17, 25, 37$ (see [2, pp. 189–191]) in Theorem 3.2 and Corollary 3.3, we obtain all the above values. Since the proof is analogous to the previous theorems and so we omit the details. \square

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