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Backward Stability and Divided Invariance of an Attractor for the Delayed Navier-Stokes Equation

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Abstract. We study backward stability of a pullback attractor especially for a delay equation. We introduce a new concept of a backward attractor, which is defined by a compact, pullback attracting and dividedly invariant family. We then show the equivalence between existence of a backward attractor and backward stability of the pullback attractor, and present some criteria by using the backward limit-set compactness of the system. In the application part, we consider the Navier-Stokes equation with a nonuniform Lipschitz delay term and a backward tempered force. Based on the fact that the delay does not change the backward bounds of the velocity field and external forces, we establish the backward-uniform estimates and obtain a backward attractor, which leads to backward stability of the pullback attractor. Some special cases of variable delay and distributed delay are discussed.

1. Introduction

The initial motivation of this work is to consider the longtime stability of a pullback attractor, especially for a delayed PDE [1–3, 13, 15, 33, 34, 36, 37, 39].

A pullback attractor $\{\mathcal{P}(t) : t \in \mathbb{R}\}$ is called **backward stable** (resp. forward stable) if there is a nonempty compact set K such that

$$\operatorname{dist}(\mathcal{P}(t), K) \to 0 \quad \text{as } t \to -\infty \quad (\text{resp. as } t \to +\infty). \tag{1.1}$$

For a delayed equation, the attractor itself closely relates to the stability of the solution [17, 30, 40]. So, the subject like (1.1) means twice-stability or twice-attraction. The minimum (if exists) among all compact sets (like K) satisfying (1.1) will be called a twice attractor.

Due to the pullback direction of the attractor, we focus on the backward stability. Some abstract criteria for backward stability (1.1) has been recently established by using

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uniform compactness [20, 21] or (weaker) backward compactness [25, 26] of the pullback attractor, and applied to non-delay equations [9, 14, 35].

In this paper, we will establish new criteria for the backward stability from a completely different viewpoint. Our idea is to generalize the invariance of the attractor to the so-called **divided invariance**, which means that a family $\mathcal{A}(\cdot)$ of sets is the backward union of another invariant family (see Definition 2.1).

A family $\mathcal{A}(\cdot)$ of compact sets is called a **backward attractor** if it is pullback attracting and dividedly invariant (instead of the invariance in the definition of a pullback attractor).

Such a backward attractor (if exists) is always unique (Proposition 2.2), although the pullback attractor may not be unique [7, 19].

We provide an equivalent definition of a backward attractor in Theorem 2.4. In particular, a backward attractor is **backward attracting**, which means that it is pullback attracting in the whole past, and gives the real meaning of a backward attractor.

Now, we come back the subject of backward stability and prove that a backward attractor $\mathcal{A}(\cdot)$ is always backward stable in the sense of (1.1).

More importantly, we prove in Theorem 2.7 that the existence of a (unique) backward attractor is a necessary and sufficient condition to ensure both existence and backward stability of the (minimal) pullback attractor.

Some criteria in terms of the dynamical system (evolution process) are established in Theorem 2.10 for the existence of a backward attractor, which leads to backward stability of the pullback attractor.

The abstract criteria are applied to the following 2D Navier-Stokes equation with variable delays:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f(x,t) + g(t,u_t) \quad \text{in } \Omega\times(t_0,\infty), \\ \operatorname{div} u = 0 \quad \text{in } \Omega\times(t_0,\infty), \quad u(x,t) = 0 \quad \text{on } \partial\Omega\times(t_0,\infty), \\ u(x,t_0+\theta) = \phi(x,\theta), \quad x\in\Omega,\ \theta\in[-h,0]. \end{cases}$$

Given $\phi \in C([-h,0],H) := C_H$, where H is a subspace of $\mathbb{L}^2(\Omega)$, the solution $u \in C([t_0,+\infty),H)$ has a delay expansion such that $u \in C([t_0-h,+\infty),H)$. Therefore, the delay shift $u_t\colon C([t_0,+\infty),H) \to C_H$ is well-defined by $u_t(\theta) = u(t+\theta)$ for $\theta \in [-h,0]$ and $t \geq t_0$.

Replacing the uniform Lipschitz condition as given in [4,30], we give a weaker assumption that $g\colon \mathbb{R}\times C_H \to H$ is pointwise Lipschitz continuous with a Lipschitz variable $L_g(\cdot)$ instead of the Lipschitz constant L_g, see Assumption (G3).

This weak assumption is suitable for more models with variable delays such as $G(t,u(t-\rho(t)))$, which is more general than $G(u(t-\rho(t)))$ used in [1,12,23,30,32] and the references therein. Also, it contains the case of distributed delays [4, 16], see the last example of the present paper.

Now, the tempered conditions of $L_g(\cdot)$ and $f(\,\cdot\,,\cdot\,)$ are enough to ensure the existence of a pullback attractor.

In order to obtain backward stability of the pullback attractor, we needs some stronger assumptions, e.g., both $L_g(\cdot)$ and $f(\,\cdot\,,\cdot\,)$ are backward tempered.

Under these assumptions, we find an important fact that the delay does not change the backward bounds of external forces and the velocity field of the fluid. So, we can provide a priori estimate in C_H such that the estimate is uniform in the past.

The backward-uniform estimate provides a backward absorbing set. Also, the Ascoli-Arzelà theorem shows that the evolution process is backward limit-set compact in C_H.

So, the abstract results can be applied to show both existence and backward stability of the minimal pullback attractor by proving the existence of a backward attractor (see Theorem 4.1).

2. Backward stability of a non-autonomous attractor

Let (X, d) be a complete metric space. A family of operators $S(t, s)\colon X \to X$ for $t \geq s$ is called an *evolution process* if

- $S(s, s) = I_X$ and $S(t, s) = S(t, \tau)S(\tau, s)$ for all $t \geq \tau \geq s$,
- $(t, x) \to S(t, s)x$ is continuous from $[s, +\infty) \times X$ to X for each $s \in \mathbb{R}$.

2.1. Divided invariance and backward attractors

We use $\mathcal{D}(\cdot)$ to denote a time-dependent family $\{\mathcal{D}(t) : t \in \mathbb{R}\}$ of nonempty sets in X. A family $\mathcal{D}(\cdot)$ is called *invariant* under the process if $S(t, s)\mathcal{D}(s) = \mathcal{D}(t)$ for all $t \geq s$. A family $\mathcal{P}(\cdot)$ is called *pullback attracting* if

$$\lim_{r \to +\infty} \operatorname{dist}(S(t, t - r)B, \mathcal{P}(t)) = 0, \quad \forall\, t \in \mathbb{R},\ B \in \mathfrak{B},$$

where $\mathfrak{B}$ denotes all of bounded sets in X and $\operatorname{dist}(\,\cdot\,,\cdot\,)$ denotes the Hausdorff semi-metric.

Recall that a *pullback attractor* $\mathcal{P}(\cdot)$ is defined by a compact, pullback attracting and invariant family.

We generalize invariance to divided invariance and introduce the concept of a backward attractor.

Definition 2.1. A family $\mathcal{A}(\cdot)$ is called **dividedly invariant** if it is the backward union of an invariant family $\mathcal{D}(\cdot)$, that is, $\mathcal{A}(t) = \overline{\bigcup_{s \leq t} \mathcal{D}(s)}$ for all $t \in \mathbb{R}$. A family $\mathcal{A}(\cdot)$ is called a **backward attractor** if it is compact, pullback attracting and dividedly invariant.

Notice that pullback attractors may not be unique (see [6, 19]) and so one often use the *minimal* pullback attractor (see [7, 8, 11, 31, 38]). However, we have the following uniqueness result for a backward attractor.

Proposition 2.2. *A process $S(\cdot,\cdot)$ has at most one backward attractor.*

Proof. Let $\mathcal{A}_1(\cdot)$ and $\mathcal{A}_2(\cdot)$ be two backward attractors. By Definition 2.1, there are two invariant families $\mathcal{D}_1(\cdot)$ and $\mathcal{D}_2(\cdot)$ such that

$$\mathcal{A}_1(t) = \overline{\bigcup_{s\le t} \mathcal{D}_1(s)}, \quad \mathcal{A}_2(t) = \overline{\bigcup_{s\le t} \mathcal{D}_2(s)}, \quad \forall\, t \in \mathbb{R}. \tag{2.1}$$

From (2.1), we see that $\mathcal{A}_2(\cdot)$ is increasing, that is $\mathcal{A}_2(t_1) \subset \mathcal{A}_2(t_2)$ if $t_1 \le t_2$.

We now fix a pair (s,t) with $s \le t$ and so $\mathcal{A}_2(t) \supset \mathcal{A}_2(s)$. Hence, by (2.1) and the invariance of $\mathcal{D}_1(\cdot)$,

$$\begin{aligned} \operatorname{dist}(\mathcal{D}_1(s), \mathcal{A}_2(t)) &= \operatorname{dist}(S(s, s-r)\mathcal{D}_1(s-r), \mathcal{A}_2(t)) \\ &\le \operatorname{dist}(S(s, s-r)\mathcal{A}_1(s), \mathcal{A}_2(s)), \quad \forall\, r \ge 0. \end{aligned} \tag{2.2}$$

Notice that $\mathcal{A}_1(s)$ is compact and so $\mathcal{A}_1(s) \in \mathfrak{B}$, which can be pullback attracted by $\mathcal{A}_2(\cdot)$. Letting $r \to +\infty$ in (2.2), we obtain $\mathcal{D}_1(s) \subset \mathcal{A}_2(t)$ for all $s \le t$ and thus, by (2.1), $\mathcal{A}_1(t) \subset \mathcal{A}_2(t)$. The opposite inclusion is similar to prove and so $\mathcal{A}_1(\cdot) = \mathcal{A}_2(\cdot)$ as desired. □

We then prove a backward attractor is backward attracting, which is the real meaning of a backward attractor.

Definition 2.3. $\mathcal{A}(\cdot)$ is called a **backward attractor** if it is the minimum among all of compact and backward attracting families, where $\mathcal{A}(\cdot)$ is called **backward attracting** if

$$\lim_{r\to+\infty} \operatorname{dist}(S(s, s-r)B, \mathcal{A}(t)) = 0, \quad \forall\, s \le t,\ B \in \mathfrak{B}. \tag{2.3}$$

The backward attraction (in the past) is stronger than the pullback attraction (at current). However, the backward attraction is weaker than the backward equi-attraction, the latter means that the convergence in (2.3) is uniform in $s \in (-\infty, t]$, see [27].

We prove both definitions are indeed equivalent.

Theorem 2.4. $\mathcal{A}(\cdot)$ *is the minimal, compact and backward attracting family if and only if it is compact, dividedly invariant and pullback attracting.*

Proof. Sufficiency. Suppose $\mathcal{A}(\cdot)$ is compact, dividedly invariant and pullback attracting. The divided invariance implies that there is an invariant family $\mathcal{D}(\cdot)$ such that $\mathcal{A}(t) = \overline{\bigcup_{s\le t} \mathcal{D}(s)}$ for all $t \in \mathbb{R}$.

We show that $\mathcal{A}(\cdot)$ is backward attracting. Notice that $\mathcal{A}(s) \subset \mathcal{A}(t)$ for all $s \le t$. Hence, by the pullback attraction at s with $s \le t$, we have

$$\operatorname{dist}(S(s, s-\tau)B, \mathcal{A}(t)) \le \operatorname{dist}(S(s, s-\tau)B, \mathcal{A}(s)) \to 0$$

as $\tau \to +\infty$ for each $B \in \mathfrak{B}$. This means that $\mathcal{A}(t)$ pullback attracts B at any $s \le t$. Thereby, $\mathcal{A}(\cdot)$ is backward attracting, and compact by the assumption.

We need to prove the minimality. Indeed, we assume $\mathcal{K}(\cdot)$ is another compact and backward attracting family. Let $B_0 = \bigcup_{s \le t} \mathcal{D}(s)$ with a fixed $t \in \mathbb{R}$. Then $B_0 \subset \mathcal{A}(t)$ and thus B_0 is a bounded set. So, B_0 can be backward attracted by $\mathcal{K}(t)$, which together with the invariance of $\mathcal{D}(\cdot)$ implies that, for all $\sigma \le t$,

$$\begin{aligned}\operatorname{dist}(\mathcal{D}(\sigma), \mathcal{K}(t)) &= \operatorname{dist}(S(\sigma, \sigma-\tau)\mathcal{D}(\sigma-\tau), \mathcal{K}(t)) \\ &\le \operatorname{dist}(S(\sigma, \sigma-\tau)B_0, \mathcal{K}(t)) \to 0 \quad \text{as } \tau \to +\infty.\end{aligned}$$

By the compactness of $\mathcal{K}(\cdot)$, we have $\mathcal{D}(\sigma) \subset \mathcal{K}(t)$ for all $\sigma \le t$, and thus

$$\mathcal{A}(t) = \overline{\bigcup_{\sigma \le t} \mathcal{D}(\sigma)} \subset \overline{\mathcal{K}(t)} = \mathcal{K}(t).$$

Thereby, $\mathcal{A}(\cdot)$ is indeed a backward attractor in the sense of Definition 2.3.

Necessity. Suppose $\mathcal{A}(\cdot)$ is minimal, compact and backward attracting, then it is obviously pullback attracting. It suffices to prove that $\mathcal{A}(\cdot)$ is dividedly invariant.

Let $t \in \mathbb{R}$ be fixed and define a new family $\mathcal{A}^t(\cdot)$ (from $\mathcal{A}(\cdot)$) by

$$\mathcal{A}^t(s) \equiv \mathcal{A}(t) \text{ if } s \le t, \quad \text{and} \quad \mathcal{A}^t(s) = \mathcal{A}(s) \text{ if } s > t.$$

By the backward attraction (2.3) of $\mathcal{A}(\cdot)$, if $s \le t$ and $B \in \mathfrak{B}$,

$$\operatorname{dist}(S(s, s-\tau)B, \mathcal{A}^t(s)) = \operatorname{dist}(S(s, s-\tau)B, \mathcal{A}(t)) \to 0$$

as $\tau \to +\infty$. If $s > t$, then

$$\operatorname{dist}(S(s, s-\tau)B, \mathcal{A}^t(s)) = \operatorname{dist}(S(s, s-\tau)B, \mathcal{A}(s)) \to 0$$

as $\tau \to +\infty$. Hence, $\mathcal{A}^t(\cdot)$ is pullback attracting. It is obvious that $\mathcal{A}^t(\cdot)$ is compact.

By [6, Theorem 2.12], there is a (minimal) pullback attractor $\mathcal{P}(\cdot)$, which is the minimum among all of compact and pullback attracting families. Hence, $\mathcal{P}(s) \subset \mathcal{A}^t(s) = \mathcal{A}(t)$ for all $s \le t$, where t is fixed as above. Thereby,

$$\bigcup_{s \le t} \mathcal{P}(s) \subset \mathcal{A}(t) \quad \text{and thus} \quad \overline{\bigcup_{s \le t} \mathcal{P}(s)} \subset \mathcal{A}(t), \quad \forall\, t \in \mathbb{R}. \tag{2.4}$$

In order to prove the opposite inclusion in (2.4), we let $\mathcal{K}(t) := \overline{\bigcup_{s\le t}\mathcal{P}(s)}$ for all $t\in\mathbb{R}$. By (2.4), $\mathcal{K}(\cdot)$ is compact. On the other hand, for all $\sigma \le t$ and $B\in\mathfrak{B}$,

$$\mathrm{dist}(S(\sigma,\sigma-\tau)B,\mathcal{K}(t)) \le \mathrm{dist}(S(\sigma,\sigma-\tau)B,\mathcal{P}(\sigma)) \to 0$$

as $\tau\to+\infty$. Hence, $\mathcal{K}(\cdot)$ is compact and backward attracting. By the minimality of $\mathcal{A}(\cdot)$, we have

$$\mathcal{A}(t)\subset\mathcal{K}(t)=\overline{\bigcup_{s\le t}\mathcal{P}(s)} \quad \text{and so} \quad \mathcal{A}(t)=\overline{\bigcup_{s\le t}\mathcal{P}(s)}, \quad \forall\, t\in\mathbb{R}. \tag{2.5}$$

Therefore, by the invariance of the pullback attractor, $\mathcal{A}(\cdot)$ is dividedly invariant as desired. □

2.2. Backward stability of a non-autonomous attractor

In this subsection, we discuss backward stability of a backward attractor and particularly a pullback attractor.

Definition 2.5. A non-autonomous attractor (or a time-dependent family generally) $\mathcal{A}(\cdot)$ is called **backward stable** if there is a nonempty compact set K such that

$$\lim_{t\to-\infty}\mathrm{dist}(\mathcal{A}(t),K)=0. \tag{2.6}$$

The minimum (if exists) among all compact sets (like K) satisfying (2.6) is called a **twice-attractor**.

Lemma 2.6. *A backward attractor $\mathcal{A}(\cdot)$ is always backward stable with a twice-attractor A, given by the α-limit set*

$$A=\bigcap_{t\le 0}\overline{\bigcup_{s\le t}\mathcal{A}(s)} =: \alpha(\mathcal{A}(\cdot)). \tag{2.7}$$

Proof. By the divided invariance, $\mathcal{A}(\cdot)$ is an increasing family. Hence, $\mathrm{dist}(\mathcal{A}(t),\mathcal{A}(0))\to 0$ as $t\to-\infty$, which means $\mathcal{A}(\cdot)$ is backward stable (because $\mathcal{A}(0)$ is a compact set).

We then prove $\alpha(\mathcal{A}(\cdot))$ is the twice-attractor for $\mathcal{A}(\cdot)$. Since $\mathcal{A}(\cdot)$ is an increasing family of compact sets, it follows from (2.7) that $\alpha(\mathcal{A}(\cdot))=\bigcap_{t\le 0}\mathcal{A}(t)$. By the theorem of nested compact sets, $\alpha(\mathcal{A}(\cdot))$ is nonempty compact. We then prove that $\alpha(\mathcal{A}(\cdot))$ attracts $\mathcal{A}(\cdot)$ at negative infinity:

$$\lim_{t\to-\infty}\mathrm{dist}(\mathcal{A}(t),\alpha(\mathcal{A}(\cdot)))=0. \tag{2.8}$$

Suppose (2.8) is not true, then there are $\delta>0$, $0\ge t_n\to-\infty$ and $x_n\in\mathcal{A}(t_n)$ such that

$$\mathrm{dist}(x_n,\alpha(\mathcal{A}(\cdot)))\ge\delta, \quad \forall\, n\in\mathbb{N}. \tag{2.9}$$

Note that $\{x_n\}_n \subset \bigcup_{t\le 0}\mathcal{A}(t) = \mathcal{A}(0)$, we know $\{x_n\}_n$ is pre-compact. Passing to a subsequence, we have $x_{n^*} \to x$ as $n^* \to \infty$. Since $x_{n^*} \in \mathcal{A}(t_{n^*})$, we have $x \in \alpha(\mathcal{A}(\cdot))$, which contradicts with (2.9).

We prove the minimality. Suppose K is another compact set satisfying (2.6) and $x \in \alpha(\mathcal{A}(\cdot))$. Then, there are $t_n \to -\infty$ and $x_n \in \mathcal{A}(t_n)$ such that $x_n \to x$, and thus

$$\operatorname{dist}(x, K) \le d(x, x_n) + \operatorname{dist}(\mathcal{A}(t_n), K) \to 0 \quad \text{as } n \to \infty.$$

Therefore $x \in K$ and thus $\alpha(\mathcal{A}(\cdot)) \subset K$ as desired. □

The following result establish some new criterion for backward stability of a pullback attractor via the existence of a backward attractor.

Theorem 2.7. *The following two assertions for an evolution process $S(\,\cdot\,,\cdot\,)$ are equivalent.*

(a) *The process has a backward attractor $\mathcal{A}(\cdot)$.*

(b) *The process has a minimal pullback attractor $\mathcal{P}(\cdot)$ such that it is backward stable.*

Moreover, both backward attractor and pullback attractor have the same twice attractor given by $\alpha(\mathcal{P}(\cdot)) = \alpha(\mathcal{A}(\cdot))$.

Proof. (a) ⇒ (b). Suppose there is a backward attractor $\mathcal{A}(\cdot)$, then $\mathcal{A}(\cdot)$ is compact and pullback attracting. By [6, Theorem 2.12], there is a minimal pullback attractor $\mathcal{P}(\cdot)$.

By the same proof as in Theorem 2.4, we know (2.5) holds true, that is, $\mathcal{A}(t) = \overline{\bigcup_{s\le t}\mathcal{P}(s)}$ for all $t \in \mathbb{R}$. In particular, $\operatorname{dist}(\mathcal{P}(t), \mathcal{A}(0)) = 0$ for all $t \le 0$ and thus $\mathcal{P}(\cdot)$ is backward stable.

(b) ⇒ (a). Suppose there is a minimal pullback attractor $\mathcal{P}(\cdot)$ with the backward stability. We will prove that there is a backward attractor $\mathcal{A}(\cdot)$ given by

$$\mathcal{A}(t) := \overline{\bigcup_{s\le t}\mathcal{P}(s)}, \quad \forall\, t \in \mathbb{R}. \tag{2.10}$$

From (2.10), the invariance of $\mathcal{P}(\cdot)$ implies that $\mathcal{A}(\cdot)$ is dividedly invariant, and the pullback attraction of $\mathcal{P}(\cdot)$ implies that the larger set $\mathcal{A}(\cdot)$ is still pullback attracting. It suffices to prove $\mathcal{A}(\cdot)$ is compact.

Suppose $\{x_n\}_n \subset \bigcup_{s\le t}\mathcal{P}(s)$ with a fixed t, then there are $t_n \le t$ such that $x_n \in \mathcal{P}(t_n)$ for all $n \in \mathbb{N}$.

If $t_0 := \inf\{t_n\} > -\infty$, then, by [26, Lemma 2.3], a pullback attractor is locally compact and so $\bigcup_{s\in[t_0,t]}\mathcal{P}(s)$ is compact, which implies $\{x_n\}_n$ is pre-compact.

If $\inf\{t_n\} = -\infty$, then we assume without loss of generality that $t_n \downarrow -\infty$. Notice that $\mathcal{P}(\cdot)$ is assumed to be backward robust, and thus there is a nonempty compact set K such that

$$\mathrm{dist}(x_n, K) \le \mathrm{dist}(\mathcal{P}(t_n), K) \to 0 \quad \text{as } n \to \infty.$$

From this, one can prove that $\{x_n\}_n$ has a convergent subsequence still.

In a word, $\bigcup_{s\le t} \mathcal{P}(s)$ is pre-compact and thus $\mathcal{A}(t) := \overline{\bigcup_{s\le t} \mathcal{P}(s)}$ is compact as desired.

By Lemma 2.6, $\mathcal{A}(\cdot)$ has always a twice attractor given by $\alpha(\mathcal{A}(\cdot))$. By the same method as in Lemma 2.6, one can prove that if the pullback attractor $\mathcal{P}(\cdot)$ is backward robust then $\mathcal{P}(\cdot)$ has a twice attractor given by $\alpha(\mathcal{P}(\cdot))$.

Finally, we prove $\alpha(\mathcal{A}(\cdot)) = \alpha(\mathcal{P}(\cdot))$. By $\mathcal{A}(\cdot) \supset \mathcal{P}(\cdot)$, we obtain $\alpha(\mathcal{A}(\cdot)) \supset \alpha(\mathcal{P}(\cdot))$ immediately.

On the contrary, let $x \in \alpha(\mathcal{A}(\cdot))$. By the definition of the α-limit set, there are $t_n \downarrow -\infty$ and $x_n \in \mathcal{A}(t_n)$ such that $x_n \to x$. By the above proof, we know $\mathcal{A}(t_n) = \overline{\bigcup_{s\le t_n} \mathcal{P}(s)}$ and so $x_n \in \overline{\bigcup_{s\le t_n} \mathcal{P}(s)}$. Hence, we can choose $s_n \le t_n$ and $y_n \in \mathcal{P}(s_n)$ such that $d(y_n, x_n) \le 1/n$. Since $s_n \to -\infty$, $y_n \in \mathcal{P}(s_n)$ and $y_n \to x$, we have $x \in \alpha(\mathcal{P}(\cdot))$. Therefore, $\alpha(\mathcal{A}(\cdot)) \subset \alpha(\mathcal{P}(\cdot))$ and thus $\alpha(\mathcal{A}(\cdot)) = \alpha(\mathcal{P}(\cdot))$. □

2.3. Criteria in terms of the process

For the purpose of application, we need to establish the criteria for the existence of a backward attractor, which leads to the backward robustness of the pullback attractor in view of Theorem 2.7.

Definition 2.8. A process $S(\,\cdot\,,\cdot\,)$ in X is said to be **backward limit-set compact** if

$$\lim_{\sigma\to+\infty} \kappa_X \left(\bigcup_{\tau\ge\sigma} \bigcup_{s\le t} S(s, s-\tau)B \right) = 0, \quad \forall\, t \in \mathbb{R},\ \forall\, B \in \mathfrak{B},$$

where the Kuratowski measure $\kappa_X(D)$ is the largest lower bound of r such that D has an r-net.

Recall that a family $\mathcal{K}(\cdot)$ is *pullback absorbing* if for each $t \in \mathbb{R}$ and $B \in \mathfrak{B}$ there is a $\tau_0 = \tau_0(t, B) > 0$ such that $S(t, t-\tau)B \subset \mathcal{K}(t)$ for all $\tau \ge \tau_0$. We extend it to the backward absorption.

Definition 2.9. A family $\mathcal{K}(\cdot)$ is called **backward absorbing** if, for each pair (s, t) with $s \le t$ and $B \in \mathfrak{B}$, there is a $\tau_0 = \tau_0(s, t, B) > 0$ such that

$$S(s, s-\tau)B \subset \mathcal{K}(t), \quad \forall\, \tau \ge \tau_0.$$

A mapping $\xi\colon \mathbb{R}\to X$ is called a complete orbit for the process if $S(t,s)\xi(s)=\xi(t)$ for all $t\geq s$, and ξ is called backward compact if $\{\xi(s): s\leq t\}$ is pre-compact for each $t\in\mathbb{R}$. We denote by

$$\digamma_{\mathrm{bc}} := \{\xi : \xi \text{ is a backward compact complete orbit}\}.$$

We recall the notations of usual and backward ω-limit sets by

$$\omega(D,t)=\bigcap_{\sigma>0}\overline{\bigcup_{\tau\geq\sigma}S(t,t-\tau)D},\quad \Omega(D,t)=\bigcap_{\sigma>0}\overline{\bigcup_{\tau\geq\sigma}\bigcup_{s\leq t}S(s,s-\tau)D}. \tag{2.11}$$

Theorem 2.10. *Suppose a process $S(\cdot,\cdot)$ is backward limit-set compact. Then the followings are equivalent.*

(i) *The process has a unique backward attractor $\mathcal{A}(\cdot)$.*

(ii) *The process has a bounded and backward absorbing family $\mathcal{K}(\cdot)$.*

(iii) *The process has an increasing, bounded and pullback absorbing family $\mathcal{K}_0(\cdot)$.*

(iv) *The process has a (minimal) pullback attractor $\mathcal{P}(\cdot)$ with backward stability.*

In either case, both $\mathcal{A}(\cdot)$ and $\mathcal{P}(\cdot)$ have the same twice attractor $A=\alpha(\mathcal{P}(\cdot))=\alpha(\mathcal{A}(\cdot))$, and the backward attractor is given by

$$\mathcal{A}(t)=\overline{\{\xi(s):\xi\in\digamma_{\mathrm{bc}}, s\leq t\}}=\overline{\bigcup_{s\leq t}\omega(\mathcal{K}(s),s)}\subset\Omega(\mathcal{K}(t),t). \tag{2.12}$$

Proof. (ii) $\Rightarrow$ (i). Suppose $\mathcal{K}(\cdot)$ is bounded and backward absorbing. We first prove $\Omega(\mathcal{K}(t),t)$ (as in (2.11)) is compact for each $t\in\mathbb{R}$. Indeed, let

$$D_\sigma=\bigcup_{\tau\geq\sigma}\bigcup_{s\leq t}S(s,s-\tau)\mathcal{K}(t),\quad \forall\,\sigma\geq 0.$$

Then, $\{\overline{D_\sigma}\}_{\sigma>0}$ is a decreasing family of sets. By the backward limit-set compactness, $\kappa_X(\overline{D_\sigma})=\kappa_X(D_\sigma)\to 0$ as $\sigma\to+\infty$. By [24, Lemma 2.7], the intersection $\bigcap_{\sigma>0}\overline{D_\sigma}$ (it is just $\Omega(\mathcal{K}(t),t)$) is nonempty compact as desired.

We then prove the family $\Omega(\mathcal{K}(\cdot),\cdot)$ is backward attracting, that is,

$$\lim_{\tau\to+\infty}\operatorname{dist}(S(s,s-\tau)B,\Omega(\mathcal{K}(t),t))=0,\quad \forall\,s\leq t,\ B\in\mathfrak{B}.$$

If the above limit is not true, then, there are $\delta>0$, $s\leq t$, $\tau_n\uparrow+\infty$ and a bounded sequence $\{x_n\}\subset X$ such that

$$\operatorname{dist}(S(s,s-\tau_n)x_n,\Omega(\mathcal{K}(t),t))\geq\delta,\quad \forall\,n\in\mathbb{N}. \tag{2.13}$$

Note that $\mathcal{K}(\cdot)$ is backward absorbing, for each $k \in \mathbb{N}$, we can sequentially choose $\tau_{n_k} \geq k + \tau_{n_{k-1}}$ such that

$$y_{n_k} := S(s-k, s-k-(\tau_{n_k}-k))x_{n_k} \in \mathcal{K}(t).$$

Since $\mathcal{K}(t)$ is bounded, by the backward limit-set compactness of the process, we know that

$$\kappa_X\{S(s,s-k)y_{n_k}\}_{k=k_0}^{\infty} \to 0 \quad \text{as } k_0 \to \infty.$$

Passing to a subsequence, we have $S(s,s-k^*)y_{n_{k^*}} \to y$ as $k^* \to \infty$. By the definition of the backward limit-set, we have $y \in \Omega(\mathcal{K}(t),t)$ in view of the fact $s \leq t$. By the process property,

$$S(s,s-\tau_{n_{k^*}})x_{n_{k^*}} = S(s,s-k^*)y_{n_{k^*}} \to y \in \Omega(\mathcal{K}(t),t),$$

which contradicts with (2.13).

Finally, we show the existence of a backward attractor. By the previous proof, $\Omega(\mathcal{K}(\cdot),\cdot)$ is nonempty compact and backward attracting (and so pullback attracting). By [29], there is a minimal pullback attractor $\mathcal{P}(\cdot)$, given by $\mathcal{P}(t) = \omega(\mathcal{K}(t),t)$ for all $t \in \mathbb{R}$. We then define a family $\mathcal{A}(\cdot)$ by

$$\mathcal{A}(t) := \overline{\bigcup_{s\leq t} \mathcal{P}(s)} = \overline{\bigcup_{s\leq t} \omega(\mathcal{K}(s),s)}, \quad \forall\, t \in \mathbb{R}. \tag{2.14}$$

By the definition, $\mathcal{A}(\cdot)$ is dividedly invariant in view of invariance of $\mathcal{P}(\cdot)$, and $\mathcal{A}(\cdot)$ is pullback attracting in view of $\mathcal{A}(\cdot) \supset \mathcal{P}(\cdot)$.

It suffices to prove $\mathcal{A}(t)$ is compact. For this end, we claim $\omega(\mathcal{K}(s),s) \subset \omega(\mathcal{K}(t),s)$ for all $s \leq t$. Indeed, if $x \in \omega(\mathcal{K}(s),s)$, then there are $\tau_n \to +\infty$ and $y_n \in \mathcal{K}(s)$ such that $S(s,s-\tau_n)y_n \to x$. By the backward absorption of $\mathcal{K}(\cdot)$, for each $m \in \mathbb{N}$, there is $n_m > \max(m, n_{m-1})$ such that

$$z_m := S(s-m, s-\tau_{n_m})y_{n_m} \in \mathcal{K}(t), \quad \text{and} \quad S(s,s-m)z_m = S(s,s-\tau_{n_m})y_{n_m} \to x$$

as $m \to \infty$, which implies $x \in \omega(\mathcal{K}(t),s)$ as desired. Hence,

$$\begin{aligned}\mathcal{A}(t) = \overline{\bigcup_{s\leq t}\omega(\mathcal{K}(s),s)} \subset \overline{\bigcup_{s\leq t}\omega(\mathcal{K}(t),s)} &= \overline{\bigcup_{s\leq t}\bigcap_{\sigma>0}\overline{\bigcup_{\tau\geq\sigma} S(s,s-\tau)\mathcal{K}(t)}} \\ &\subset \bigcap_{\sigma>0}\overline{\bigcup_{\tau\geq\sigma}\bigcup_{s\leq t} S(s,s-\tau)\mathcal{K}(t)} = \Omega(\mathcal{K}(t),t).\end{aligned}$$

By the previous proof, $\Omega(\mathcal{K}(t),t)$ is compact, so $\mathcal{A}(t)$ is compact and obviously nonempty. Therefore, $\mathcal{A}(\cdot)$ is the unique backward attractor in the sense of Definition 2.1. Moreover, by (2.14), the last equality in (2.12) holds true.

(i) $\Rightarrow$ (iii). For a $\delta > 0$, we let $\mathcal{K}_0(t)$ be the δ-neighborhood $N_\delta(\mathcal{A}(t))$ for all $t \in \mathbb{R}$. By compactness and attraction of $\mathcal{A}(\cdot)$, we know $\mathcal{K}_0(\cdot)$ is bounded and pullback absorbing. As a dividedly invariant family, $\mathcal{A}(\cdot)$ is increasing and thus $\mathcal{K}_0(\cdot)$ is increasing.

(iii) $\Rightarrow$ (ii). It suffices to prove that the increasing and pullback absorbing family $\mathcal{K}_0(\cdot)$ is backward absorbing. Indeed, consider a pair (s,t) with $s \le t$ and $B \in \mathfrak{B}$. By the increasingly pullback absorption at s, there is a $\tau_0 = \tau_0(s,B) > 0$ such that

$$S(s, s-\tau)B \subset \mathcal{K}_0(s) \subset \mathcal{K}_0(t), \quad \forall \, \tau \ge \tau_0.$$

The assertion (i) $\Leftrightarrow$ (iv) follows from Theorem 2.7 immediately.

It suffices to prove the first equality in (2.12). We first consider the minimal pullback attractor $\mathcal{P}(\cdot)$. By (2.14), $\mathcal{P}(\cdot)$ is backward bounded, i.e., $\bigcup_{s\le t} \mathcal{P}(s)$ is bounded. Then, by the abstract result in [6, 22],

$$\mathcal{P}(s) = \{\xi(s) : \xi \text{ is a backward bounded complete orbit}\}, \quad \forall \, s \in \mathbb{R}.$$

Let ξ be a backward bounded complete orbit, we prove that ξ is backward compact. Let $B_t := \{\xi(s) : s \le t\}$, which is a bounded set. By the backward attraction of $\mathcal{A}(\cdot)$, for $s \le t$,

$$\begin{aligned} \operatorname{dist}(\xi(s), \mathcal{A}(t)) &= \operatorname{dist}(S(s, s-\tau)\xi(s-\tau), \mathcal{A}(t)) \\ &\le \operatorname{dist}(S(s, s-\tau)B_t, \mathcal{A}(t)) \to 0 \quad \text{as } \tau \to \infty, \end{aligned}$$

which implies $\xi(s) \in \mathcal{A}(t)$ for all $s \le t$ and thus $B_t \subset \mathcal{A}(t)$. By the compactness of $\mathcal{A}(t)$, we know B_t is pre-compact. Therefore,

$$\mathcal{P}(t) = \{\xi(t) : \xi \in \mathcal{F}_{\mathrm{bc}}\}, \quad \mathcal{A}(t) = \overline{\bigcup_{s \le t} \mathcal{P}(s)} = \overline{\{\xi(s) : \xi \in \mathcal{F}_{\mathrm{bc}}, s \le t\}}.$$

The proof is complete. □

Remark 2.11. If X is a uniform convex Banach space, then the backward limit-set compactness is equivalent to the backward flattening, which means the pullback flattening property [18, 19] is uniform in the past.

3. Navier-Stokes equations with variable delays

3.1. The continuous process from the delayed equation

Let Ω be a bounded 2D-domain with a smooth boundary Γ. The delay Navier-Stokes equation can be read as

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f(x,t) + g(t, u_t) \quad \text{in } \Omega \times (t_0, \infty), \\ \operatorname{div} u = 0 \quad \text{on } \Omega \times (t_0, \infty), \quad u(x,t) = 0 \quad \text{on } \Gamma \times (t_0, \infty), \\ u_{t_0}(x, \theta) := u(x, t_0 + \theta) = \phi(x, \theta), \quad x \in \Omega, \ \theta \in [-h, 0], \end{cases} \tag{3.1}$$

where $\nu > 0$ is the kinematic viscosity, $h > 0$ is the time of memory effect, u is the velocity field of the fluid, p denotes the pressure.

Let H be the closure of $\mathcal{V}$ in $\mathbb{L}^2(\Omega) = (L^2(\Omega))^2$, with $\mathbb{L}^2$-norm $\|\cdot\|$ and inner product $(\cdot\,,\cdot)$, where

$$\mathcal{V} = \{u \in \mathbb{C}_0^\infty(\Omega) : \operatorname{div} u = 0\}, \quad (u,v) = \sum_{i=1}^{2} \int_\Omega u_i(x)v_i(x)\,dx.$$

Let V be the closure of $\mathcal{V}$ in $\mathbb{H}_0^1(\Omega)$ with norm $\|\cdot\|_V$, and inner product $((\cdot\,,\cdot))$, i.e.,

$$((u,v)) = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}\,dx, \quad \|u\|_V^2 = ((u,u)), \quad \forall\, u,v \in V.$$

We denote $C_H = C([-h,0],H)$, $L_H^2 = L^2(-h,0;H)$ and similarly C_V, L_V^2. For each $t \geq t_0$, we define the delay shift u_t by

$$u_t\colon C([t_0-h,+\infty),H) \to C_H, \quad u_t(\theta) = u(t+\theta), \quad \forall\, \theta \in [-h,0].$$

We make the assumptions as follows.

Hypothesis G. The nonlinear delay mapping $g\colon \mathbb{R}\times C_H \to \mathbb{L}^2(\Omega)$ satisfies

(G1) For each $\xi \in C_H$, the mapping $t \to g(t,\xi)$ is measurable from $\mathbb{R}$ to $\mathbb{L}^2(\Omega)$.

(G2) $g(t,0) = 0$ for all $t \in \mathbb{R}$.

(G3) There is a positive continuous function $L_g(\cdot)$ with the backward translation boundedness:

$$\widetilde{L_g}(t) := \sup_{s\leq t} \int_{s-1}^{s} L_g^2(r)\,dr < +\infty, \quad \forall\, t \in \mathbb{R} \tag{3.2}$$

such that for all $\xi,\eta \in C_H$ and $t \in \mathbb{R}$,

$$\|g(t,\xi) - g(t,\eta)\| \leq L_g(t)\|\xi-\eta\|_{C_H}.$$

(G4) There are $m_0 > 0$, $C_g > 0$ such that for all $m \in [0,m_0]$ and $u,v \in C([t_0-h,t];H)$,

$$\int_{t_0}^{t} e^{ms}\|g(s,u_s) - g(s,v_s)\|^2\,ds \leq C_g^2 \int_{t_0-h}^{t} e^{ms}\|u(s)-v(s)\|^2\,ds.$$

Hypothesis F. The force $f \in L_{\text{loc}}^2(\mathbb{R},\mathbb{L}^2(\Omega))$ satisfies the backward translation boundedness:

$$\sup_{s\leq t} \int_{s-1}^{s} \|f(r)\|^2\,dr < +\infty, \quad \forall\, t \in \mathbb{R}.$$

We remark here that the condition (G3) is more general than the uniform Lipschitz condition (i.e., $L_g(\cdot) \equiv L_g$) in [4].

On the other hand, by the same method as given in [10, 28, 41], one can prove the following equivalences.

Lemma 3.1. *f satisfies Hypothesis* F *if and only if it is backward tempered:*

$$F_\gamma(t) := \sup_{s\le t}\int_{-\infty}^{s} e^{\gamma(r-s)}\|f(r)\|^2\,dr < +\infty, \quad \forall\,\gamma>0,\ t\in\mathbb{R}.$$

In particular,

$$e^{-\gamma}\sup_{s\le t}\int_{s-1}^{s}\|f(r)\|^2\,dr \le F_\gamma(t) \le \frac{1}{1-e^{-\gamma}}\sup_{s\le t}\int_{s-1}^{s}\|f(r)\|^2\,dr. \tag{3.3}$$

Similarly, $L_g(\cdot)$ satisfies (3.2) *if and only if it is backward tempered.*

Let $(V^*, \|\cdot\|_{V^*})$ be the dual space of V. We consider $A\colon V\to V^*$ by $\langle Au, v\rangle = ((u,v))$. Let $P\colon \mathbb{L}^2(\Omega)\to H$ be the projector and $D(A) = \mathbb{H}^2(\Omega)\cap V$, then $Au = -P\Delta u$ for $u\in D(A)$. We also consider the bilinear form $B\colon V\times V\to V^*$ by

$$\langle B(u,v), w\rangle = b(u,v,w) = \sum_{i,j=1}^{2}\int_\Omega u_i\frac{\partial v_j}{\partial x_i} w_j\,dx$$

and $B(u) = B(u,u)$. Note that if $u\in D(A)$ then $B(u)\in H^* = H$ by using the first inequality below:

$$\begin{cases} b(u,v,v) = 0,\\ |b(u,v,w)| \le c_0\|u\|^{1/2}\|u\|_V^{1/2}\|v\|_V^{1/2}\|Av\|^{1/2}\|w\|,\\ |b(u,v,w)| \le c_0\|u\|^{1/2}\|u\|_V^{1/2}\|v\|_V\|w\|^{1/2}\|w\|_V^{1/2}. \end{cases} \tag{3.4}$$

As usual, replacing (3.1), we consider the following distribution problem: given $t_0\in\mathbb{R}$,

$$\begin{cases} \text{Find } u\in L^2(t_0-h,T;H)\cap L^2(t_0,T;V)\cap L^\infty(t_0,T;H), \quad \forall\,T>t_0,\\ \text{s.t. } \frac{d}{dt}u(t) + \nu Au(t) + B(u(t)) = f(t) + g(t,u_t) \quad \text{in } \mathcal{D}'(t_0,\infty;V^*),\\ u(t_0+\theta) = \phi(\theta) \quad \text{for all } \theta\in[-h,0]. \end{cases} \tag{3.5}$$

The well-posedness of (3.5) can be established by using the same method as given in [12, 23, 30, 32] (see also [42, 43] for the non-delayed NS equation).

Proposition 3.2. *Let $\phi\in C_H$ and $g\colon \mathbb{R}\times C_H\to\mathbb{L}^2(\Omega)$ satisfy* (G1)–(G4)*. If $f\in L^2_{\text{loc}}(\mathbb{R};V^*)$, then the problem* (3.5) *has a unique solution such that*

$$u\in C([t_0-h,T];H)\cap L^2(t_0,T;V).$$

If $f\in L^2_{\text{loc}}(\mathbb{R};\mathbb{L}^2(\Omega))$, then

$$u\in C([t_0+\epsilon,T];V)\cap L^2(t_0+\epsilon,T;D(A)), \quad \forall\,T>t_0+\epsilon>t_0.$$

Furthermore, if $u_0=\phi(0)\in V$ and $f\in L^2_{\text{loc}}(\mathbb{R};\mathbb{L}^2(\Omega))$, then for all $T>t_0$,

$$u\in C([t_0,T];V)\cap L^2(t_0,T;D(A)) \quad \textit{and} \quad u'\in L^2(t_0,T;H).$$

The above proposition indicates the existence of an evolution process $U(\,\cdot\,,\cdot\,)$ on C_H defined by

$$U(t,t_0)\colon C_H \to C_H, \quad U(t,t_0)\phi = u_t(\,\cdot\,;t_0,\phi), \quad \forall\, t \geq t_0, \tag{3.6}$$

where $u_t(\,\cdot\,;t_0,\phi)$ is the delay t-shift of the solution $u(\,\cdot\,;t_0,\phi)$ with initial data $t_0 \in \mathbb{R}$ and $\phi \in C_H$.

By the local integrability of $L_g(\cdot)$, the similar method as in [4, 30] can prove the continuity of $U(t,t_0)\colon C_H \to C_H$.

3.2. Backward absorbing sets in C_H

Let $\lambda_1 > 0$ be the first eigenvalue of the Stokes operator A.

Lemma 3.3. *Let all conditions* (G1)–(G4) *and Hypothesis* F *be satisfied. If $\nu\lambda_1 > C_g$, then, for each $t \in \mathbb{R}$ and bounded set $D \subset C_H$, there exists a $\tau_0 := \tau_0(t,D) \geq 2h+1$ such that for all $\tau \geq \tau_0$ and $\phi \in D$,*

$$\sup_{s \leq t} \sup_{r \in [s-h-1,s]} \|u_r(\,\cdot\,;s-\tau,\phi)\|_{C_H}^2 \leq 1 + c_1 F(t), \tag{3.7}$$

where $c_1 = e^{\gamma_2(2h+1)}/\gamma_1$ with $\gamma_1 := \nu\lambda_1 - C_g > 0$ and $\gamma_2 := \min\{m_0, \gamma_1/2\}$ (m_0 is given in (G4)*), and $F(\cdot)$ is an increasing function defined by*

$$F(t) := \sup_{s \leq t} \int_{-\infty}^{s} e^{\gamma_2(r-s)} \|f(r)\|^2 \, dr < +\infty, \quad \forall\, t \in \mathbb{R}.$$

In addition, let $\widetilde{L_g}(\cdot)$ be given as in (G3)*, then, for all $\tau \geq \tau_0$ and $\phi \in D$,*

$$\sup_{s \leq t} \sup_{\theta \in [-h,0]} \int_{s+\theta-1}^{s+\theta} \|u(r;s-\tau,\phi)\|_V^2 \, dr \leq c_2(1+F(t))(1+\widetilde{L_g}(t)). \tag{3.8}$$

Proof. Multiplying (3.5) by u, by $b(u,u,u) = 0$, we have

$$\frac{d}{dr}\|u(r)\|^2 + 2\nu\|u(r)\|_V^2 = 2(f(r),u(r)) + 2(g(r,u_r),u(r)).$$

By the Cauchy-Schwartz inequality and the Young inequality, we obtain

$$\frac{d}{dr}\|u(r)\|^2 + 2\nu\|u(r)\|_V^2 \leq \frac{\|f(r)\|^2}{\gamma_1} + (\gamma_1 + C_g)\|u(r)\|^2 + \frac{1}{C_g}\|g(r,u_r)\|^2. \tag{3.9}$$

By the Poincaré inequality, we further obtain

$$\frac{d}{dr}\|u(r)\|^2 \leq \frac{\|f(r)\|^2}{\gamma_1} + (\gamma_1 + C_g - 2\nu\lambda_1)\|u(r)\|^2 + \frac{1}{C_g}\|g(r,u_r)\|^2.$$

Multiplying by $e^{\gamma_2 r}$ in the above inequality, we obtain

$$\frac{d}{dr}(e^{\gamma_2 r}\|u(r)\|^2) \le \frac{e^{\gamma_2 r}}{\gamma_1}\|f(r)\|^2 + (\gamma_1+\gamma_2+C_g-2\nu\lambda_1)e^{\gamma_2 r}\|u(r)\|^2 + \frac{e^{\gamma_2 r}}{C_g}\|g(r,u_r)\|^2.$$

Let $t\in\mathbb{R}$ be fixed. By the integrals of the above inequality with respect to $r\in[s-\tau,s+\sigma]$, where $s\le t$, $\tau\ge 2h+1$ and $\sigma\in[-(2h+1),0]$, we obtain

$$\begin{aligned}\|u(s+\sigma;s-\tau,\phi)\|^2 \le{}& e^{-\gamma_2(\sigma+\tau)}\|\phi(0)\|^2 + \frac{1}{\gamma_1}\int_{s-\tau}^{s+\sigma} e^{\gamma_2(r-s-\sigma)}\|f(r)\|^2\,dr\\ &+ (\gamma_1+\gamma_2+C_g-2\nu\lambda_1)\int_{s-\tau}^{s+\sigma} e^{\gamma_2(r-s-\sigma)}\|u(r)\|^2\,dr\\ &+ \frac{1}{C_g}\int_{s-\tau}^{s+\sigma} e^{\gamma_2(r-s-\sigma)}\|g(r,u_r)\|^2\,dr.\end{aligned}\tag{3.10}$$

We estimate the last term in (3.10). Since $\gamma_2\le m_0$, by (G4) and (G2), we obtain for all $\sigma\in[-2h-1,0]$,

$$\begin{aligned}&\frac{1}{C_g}\int_{s-\tau}^{s+\sigma} e^{\gamma_2(r-s-\sigma)}\|g(r,u_r)\|^2\,dr \le C_g\int_{s-\tau-h}^{s+\sigma} e^{\gamma_2(r-s-\sigma)}\|u(r)\|^2\,dr\\ &= C_g\int_{s-\tau-h}^{s-\tau} e^{\gamma_2(r-s-\sigma)}\|u(r;s-\tau,\phi)\|^2\,dr + C_g\int_{s-\tau}^{s+\sigma} e^{\gamma_2(r-s-\sigma)}\|u(r)\|^2\,dr\\ &= C_g e^{-\gamma_2(\sigma+\tau)}\int_{-h}^{0} e^{\gamma_2\theta}\|\phi(\theta)\|^2\,d\theta + C_g\int_{s-\tau}^{s+\sigma} e^{\gamma_2(r-s-\sigma)}\|u(r)\|^2\,dr\\ &\le e^{-\gamma_2(\sigma+\tau)}C_g h\|\phi\|_{C_H}^2 + C_g\int_{s-\tau}^{s+\sigma} e^{\gamma_2(r-s-\sigma)}\|u(r)\|^2\,dr.\end{aligned}\tag{3.11}$$

Substituting (3.11) into (3.10) and noticing $(\gamma_1+\gamma_2+C_g-2\nu\lambda_1)+C_g=\gamma_2-\gamma_1\le-\gamma_1/2<0$, we have for all $\sigma\in[-2h-1,0]$,

$$\begin{aligned}\|u(s+\sigma;s-\tau,\phi)\|^2 &\le e^{-\gamma_2(\sigma+\tau)}(1+C_gh)\|\phi\|_{C_H}^2 + \frac{1}{\gamma_1}\int_{s-\tau}^{s+\sigma} e^{\gamma_2(r-s-\sigma)}\|f(r)\|^2\,dr\\ &\le e^{-\gamma_2\tau}e^{\gamma_2(2h+1)}(1+C_gh)\|D\|_{C_H}^2 + \frac{e^{\gamma_2(2h+1)}}{\gamma_1}\int_{-\infty}^{s} e^{\gamma_2(r-s)}\|f(r)\|^2\,dr.\end{aligned}$$

Let $\tau_0=2h+1+\log(1+C_gh)+\log(\|D\|_{C_H}^2+1)$, then, for all $\tau\ge\tau_0$ and $\phi\in D$,

$$\begin{aligned}&\sup_{s\le t}\sup_{\sigma\in[-2h-1,0]}\|u(s+\sigma;s-\tau,\phi)\|^2\\ &\le 1+c_1\sup_{s\le t}\int_{-\infty}^{s} e^{\gamma_2(r-s)}\|f(r)\|^2\,dr := 1+c_1F(t),\end{aligned}\tag{3.12}$$

where $c_1=e^{\gamma_2(2h+1)}/\gamma_1$. By (3.12) again,

$$\begin{aligned}&\sup_{s\le t}\sup_{r\in[s-h-1,s]}\max_{\theta\in[-h,0]}\|u(r+\theta;s-\tau,\phi)\|^2\\ &=\sup_{s\le t}\sup_{\sigma\in[-2h-1,0]}\|u(s+\sigma;s-\tau,\phi)\|^2\le 1+c_1F(t),\end{aligned}$$

which proves (3.7) as desired.

In order to prove (3.8), we integrate (3.9) with respect to $r \in [s+\theta-1, s+\theta]$ with $s \le t$ and $\theta \in [-h, 0]$. The result is

$$\begin{aligned}&\|u(s+\theta)\|^2 - \|u(s+\theta-1)\|^2 + 2\nu \int_{s+\theta-1}^{s+\theta} \|u(r; s-\tau, \phi)\|_V^2 \, dr \\ &\le \frac{1}{\gamma_1} \int_{s+\theta-1}^{s+\theta} \|f(r)\|^2 \, dr + (\gamma_1 + C_g) \int_{s+\theta-1}^{s+\theta} \|u(r)\|^2 \, dr + \frac{1}{C_g} \int_{s+\theta-1}^{s+\theta} \|g(r, u_r)\|^2 \, dr\end{aligned}$$

and thus for all $s \le t$ and $\theta \in [-h, 0]$,

$$\begin{aligned}\int_{s+\theta-1}^{s+\theta} \|u(r; s-\tau, \phi)\|_V^2 \, dr \le c_3 \|u(s+\theta-1)\|^2 + c_4 \int_{s+\theta-1}^{s+\theta} \|u(r)\|^2 \, dr \\ + c_5 \int_{s+\theta-1}^{s+\theta} \|f(r)\|^2 \, dr + c_6 \int_{s+\theta-1}^{s+\theta} \|g(r, u_r)\|^2 \, dr.\end{aligned}$$

Note that (3.12) is true for all $\sigma \in [-2h-1, 0]$, we have for all $\tau \ge \tau_0$ and $\phi \in D$,

$$\sup_{s \le t} \sup_{\theta \in [-h,0]} \|u(s+\theta-1; s-\tau, \phi)\|^2 \le 1 + c_1 F(t),$$
$$\sup_{s \le t} \sup_{\theta \in [-h,0]} \int_{s+\theta-1}^{s+\theta} \|u(r; s-\tau, \phi)\|^2 \, dr \le 1 + c_1 F(t).$$

By Lemma 3.1 and Hypothesis F,

$$\sup_{s \le t} \sup_{\theta \in [-h,0]} \int_{s+\theta-1}^{s+\theta} \|f(r)\|^2 \, dr = \sup_{s \le t} \int_{s-1}^{s} \|f(r)\|^2 \, dr \le e^{\gamma_2} F(t) < \infty. \tag{3.13}$$

By the assumptions (G2) and (G3), it follows from (3.7) that

$$\begin{aligned}&\sup_{s \le t} \sup_{\theta \in [-h,0]} \int_{s+\theta-1}^{s+\theta} \|g(r, u_r)(\,\cdot\,; s-\tau, \phi)\|^2 \, dr \\ &\le \sup_{s \le t} \sup_{\theta \in [-h,0]} \int_{s+\theta-1}^{s+\theta} L_g^2(r) \|u_r(\,\cdot\,; s-\tau, \phi)\|_{C_H}^2 \, dr \\ &\le \sup_{s \le t} \sup_{r \in [s-h-1, s]} \|u_r(\,\cdot\,; s-\tau, \phi)\|_{C_H}^2 \cdot \sup_{s \le t} \sup_{\theta \in [-h,0]} \int_{s+\theta-1}^{s+\theta} L_g^2(r) \, dr \\ &\le (1 + c_1 F(t)) \sup_{s \le t} \int_{s-1}^{s} L_g^2(r) \, dr := (1 + c_1 F(t)) \widetilde{L_g}(t) < \infty.\end{aligned}$$

Therefore, (3.8) holds true. By Lemma 3.1, $F(\cdot)$ is finite and increasing. □

Remark 3.4. The time-delay in (3.13) does not change the backward bound of the integral of the force $f(\cdot)$, and this fact is still true for $L_g(\cdot)$.

3.3. Further backward absorption

We consider the absorption in $C_V = C([-h,0],V)$.

Lemma 3.5. *Under the same assumptions as given in Lemma* 3.3 *with the bounded set* $D \subset C_H$ *and* $\tau_0 \geq 2h+1$*, we have for all* $\tau \geq \tau_0$ *and* $\phi \in D$*,*

$$\sup_{s\leq t} \|u_s(\,\cdot\,; s-\tau,\phi)\|^2_{C_V} \leq e^{(1+F(t))\alpha(t)}\alpha(t), \quad \forall\, t \in \mathbb{R}, \tag{3.14}$$

where $\alpha(t) = c_7(1+F(t))(1+\widetilde{L_g}(t))$.

Proof. Taking the inner product of (3.5) with Au in $\mathbb{L}^2(\Omega)$, we have

$$\frac{d}{dr}\|u\|^2_V + 2\nu\|Au\|^2 + b(u,u,Au) = 2(f(r),Au) + 2(g(r,u_r),Au). \tag{3.15}$$

By the assumptions (G2)–(G3) and the Young inequality,

$$\begin{aligned} 2|(g(r,u_r),Au)| &\leq \frac{\nu}{4}\|Au\|^2 + \frac{4}{\nu}\|g(r,u_r)\|^2 \\ &\leq \frac{\nu}{4}\|Au\|^2 + \frac{4}{\nu}L_g^2(r)\|u_r\|^2_{C_H}. \end{aligned} \tag{3.16}$$

The Young inequality also gives $2|(f(r),Au)| \leq \frac{\nu}{4}\|Au\|^2 + \frac{4}{\nu}\|f(r)\|^2$. By (3.4),

$$2|b(u,u,Au)| \leq 2c_0\|u\|^{1/2}\|u\|_V\|Au\|^{3/2} \leq \frac{\nu}{2}\|Au\|^2 + c_8\|u\|^2\|u\|^4_V. \tag{3.17}$$

Substituting the estimates (3.16) and (3.17) into (3.15), we obtain

$$\frac{d}{dr}\|u\|^2_V + \nu\|Au\|^2 \leq c_8\|u\|^2\|u\|^4_V + \frac{4}{\nu}(\|f(r)\|^2 + L_g^2(r)\|u_r\|^2_{C_H}). \tag{3.18}$$

In particular,

$$\frac{d}{dr}\|u\|^2_V \leq c_8(\|u\|^2\|u\|^2_V)\|u\|^2_V + \frac{4}{\nu}(\|f(r)\|^2 + L_g^2(r)\|u_r\|^2_{C_H}). \tag{3.19}$$

Next, we need to use the *Uniform Gronwall Inequality*: If the nonnegative functions y, z_1, z_2 satisfy $y'(r) \leq z_1(r)y(r) + z_2(r)$ for all $r \geq s-\tau$, where $\tau \geq 2h+1$, then for all $\theta \in [-h,0]$,

$$y(s+\theta) \leq e^{\int_{s+\theta-1}^{s+\theta} z_1(r)\,dr}\left(\int_{s+\theta-1}^{s+\theta} y(r)\,dr + \int_{s+\theta-1}^{s+\theta} z_2(r)\,dr\right).$$

We apply the uniform Gronwall inequality on (3.19) with

$$y(r) = \|u(r;s-\tau,\phi)\|^2_V, \quad z_1(r) = c_8(\|u(r)\|^2\|u(r)\|^2_V),$$
$$z_2(r) = \frac{4}{\nu}(\|f(r)\|^2 + L_g^2(r)\|u_r\|^2_{C_H}).$$

The result is that for all $\tau \geq \tau_0$ ($\geq 2h+1$) and $\phi \in D$,

$$\|u(s+\theta; s-\tau, \phi)\|_V^2 \leq e^{c_8 I_1(s,\theta)} \left(I_2(s,\theta) + \frac{4}{\nu} I_3(s,\theta) \right), \quad \forall\, s \leq t,\ \theta \in [-h, 0]. \tag{3.20}$$

We first consider I_1, which is defined by

$$I_1(s,\theta) := \int_{s+\theta-1}^{s+\theta} \|u(r; s-\tau, \phi)\|^2 \|u(r; s-\tau, \phi)\|_V^2 \, dr.$$

By (3.7) and (3.8) in Lemma 3.3, we have

$$\begin{aligned}
&\sup_{s\leq t} \sup_{\theta\in[-h,0]} I_1(s,\theta) \\
&\leq \left(\sup_{s\leq t} \sup_{\sigma\in[-h-1,0]} \|u(s+\sigma; s-\tau, \phi)\|^2 \right) \left(\sup_{s\leq t} \sup_{\theta\in[-h,0]} \int_{s+\theta-1}^{s+\theta} \|u(r; s-\tau, \phi)\|_V^2 \, dr \right) \\
&\leq c(1+F(t))^2 (1+\widetilde{L_g}(t)).
\end{aligned} \tag{3.21}$$

By (3.8) again,

$$\begin{aligned}
\sup_{s\leq t} \sup_{\theta\in[-h,0]} I_2(s,\theta) &:= \sup_{s\leq t} \sup_{\theta\in[-h,0]} \int_{s+\theta-1}^{s+\theta} \|u(r; s-\tau, \phi)\|_V^2 \, dr \\
&\leq c_2 (1+F(t))(1+\widetilde{L_g}(t)).
\end{aligned} \tag{3.22}$$

By (3.7) in Lemma 3.3 and (3.3) in Lemma 3.1,

$$\begin{aligned}
&\sup_{s\leq t} \sup_{\theta\in[-h,0]} I_3(s,\theta) \\
&:= \sup_{s\leq t} \sup_{\theta\in[-h,0]} \int_{s+\theta-1}^{s+\theta} (\|f(r)\|^2 + L_g^2(r) \|u_r\|_{C_H}^2) \, dr \\
&\leq \sup_{s\leq t} \int_{s-1}^{s} \|f(r)\|^2 \, dr + \sup_{s\leq t} \left(\sup_{r\in[s-h-1,s]} \|u_r\|_{C_H}^2 \right) \cdot \int_{s-1}^{s} L_g^2(r) \, dr \\
&\leq e^{\gamma_2} F(t) + (1+c_1 F(t)) \widetilde{L_g}(t).
\end{aligned} \tag{3.23}$$

We substitute (3.21)–(3.23) into (3.20) to obtain

$$\sup_{s\leq t} \|u_s(\,\cdot\,; s-\tau, \phi)\|_{C_V}^2 = \sup_{s\leq t} \sup_{\theta\in[-h,0]} \|u(s+\theta; s-\tau, \phi)\|_V^2 \leq e^{(1+F(t))\alpha(t)} \alpha(t)$$

for all $t \in \mathbb{R}$, $\tau \geq \tau_0$ and $\phi \in D$. The proof is complete. □

Lemma 3.6. *Under the same assumptions as given in Lemma 3.3 with same number $\tau_0 = \tau_0(t, D) \geq 2h+1$, we have for $\tau \geq \tau_0$, $\phi \in D$ and $-h \leq \theta_1 < \theta_2 \leq 0$,*

$$\sup_{s\leq t} \int_{s+\theta_1}^{s+\theta_2} \|Au(r; s-\tau, \phi)\|^2 \, dr \leq \beta_1(t) |\theta_2 - \theta_1| + \beta_2(t), \tag{3.24}$$

where both $\beta_1(\cdot)$ *and* $\beta_2(\cdot)$ *are finite, increasing and defined by*

$$\beta_1(t) = \frac{c_8}{\nu}(1 + c_1F(t))e^{2(1+F(t))\alpha(t)}\alpha^2(t) \quad \textit{and} \quad \beta_2(t) = c_9 e^{(1+F(t))\alpha(t)}\alpha(t),$$

where c_9 *is a positive constant and* c_8 *is given* (3.20).

Proof. It follows from (3.18) that

$$\|Au(r)\|^2 \le -\frac{1}{\nu}\frac{d}{dr}\|u(r)\|_V^2 + \frac{c_8}{\nu}\|u(r)\|^2\|u(r)\|_V^4 + \frac{4}{\nu^2}(\|f(r)\|^2 + L_g^2(r)\|u_r\|_{C_H}^2).$$

Integrating the above inequality on $[s+\theta_1, s+\theta_2]$, where $s \le t$, $\theta_1, \theta_2 \in [-h, 0]$, we obtain for all $\tau \ge \tau_0$ and $\phi \in D$,

$$\begin{aligned} &\sup_{s\le t}\int_{s+\theta_1}^{s+\theta_2} \|Au(r; s-\tau, \phi)\|^2\, dr \\ &\le \frac{1}{\nu}\sup_{s\le t}\|u(s+\theta_1; s-\tau, \phi)\|_V^2 + \frac{c_8}{\nu}\sup_{s\le t}\int_{s+\theta_1}^{s+\theta_2}\|u(r)\|^2\|u(r)\|_V^4\, dr \\ &\quad + \frac{4}{\nu^2}\sup_{s\le t}\int_{s+\theta_1}^{s+\theta_2}(\|f(r)\|^2 + L_g^2(r)\|u_r\|_{C_H}^2)\, dr. \end{aligned} \tag{3.25}$$

We estimate each term. By (3.14) in Lemma 3.5,

$$\sup_{s\le t}\|u(s+\theta_1; s-\tau, \phi)\|_V^2 \le \sup_{s\le t}\|u_s(\,\cdot\,; s-\tau, \phi)\|_{C_V}^2 \le e^{(1+F(t))\alpha(t)}\alpha(t).$$

By (3.14) and (3.7),

$$\begin{aligned} &\sup_{s\le t}\int_{s+\theta_1}^{s+\theta_2}\|u(r)\|^2\|u(r)\|_V^4\, dr \\ &\le \left(\sup_{s\le t}\|u_s(\,\cdot\,; s-\tau, \phi)\|_{C_H}^2\right)\left(\sup_{s\le t}\|u_s(\,\cdot\,; s-\tau, \phi)\|_{C_V}^2\right)^2|\theta_2-\theta_1| \\ &\le (1+c_1F(t))e^{2(1+F(t))\alpha(t)}\alpha^2(t)|\theta_2-\theta_1|. \end{aligned}$$

By (3.7) in Lemma 3.3,

$$\begin{aligned} &\sup_{s\le t}\int_{s+\theta_1}^{s+\theta_2}(\|f(r)\|^2 + L_g^2(r)\|u_r(\,\cdot\,; s-\tau, \phi)\|_{C_H}^2)\, dr \\ &\le \sup_{s\le t}\int_{s-h}^{s}\|f(r)\|^2\, dr + \sup_{s\le t}\sup_{r\in[s-h,s]}\|u_r\|_{C_H}^2\sup_{s\le t}\int_{s-h}^{s}L_g^2(r)\, dr \\ &\le e^{\gamma_2 h}F(t) + (1+c_1F(t))(h+1)\sup_{s\le t}\int_{s-1}^{s}L_g^2(r)\, dr \le c\alpha(t). \end{aligned}$$

We substitute all above inequalities into (3.25) to obtain (3.24) as desired. □

4. Backward stability of pullback attractor and backward attractor

For the following main theorem, the difficulty is to verify the backward limit-set compactness via the Ascoli-Arzelà theorem.

Theorem 4.1. *Assume* (G1)–(G4), *Hypothesis* F *and* $\nu\lambda_1 > C_g$. *Then, the delayed Navier-Stokes equation possesses a pullback attractor* $\mathcal{P}(\cdot)$ *and a backward attractor* $\mathcal{A}(\cdot)$ *such that they are backward stable in* C_H*:*

$$\lim_{t\to-\infty} \mathrm{dist}_{C_H}(\mathcal{P}(t), A) = 0, \quad \lim_{t\to-\infty} \mathrm{dist}_{C_H}(\mathcal{A}(t), A) = 0, \tag{4.1}$$

where $A = \alpha(\mathcal{P}(\cdot)) = \alpha(\mathcal{A}(\cdot))$ *is the minimal compact set satisfying* (4.1). *The backward attractor is given by*

$$\mathcal{A}(t) = \overline{\{\xi(s) : \xi \in \mathcal{F}_{\mathrm{bc}}, s \le t\}} = \overline{\bigcup_{s\le t} \mathcal{P}(s)},$$

where $\mathcal{F}_{\mathrm{bc}}$ *denotes the set of all backward compact complete orbits.*

Proof. By Lemma 3.3, the process $U(\,\cdot\,,\cdot\,)$ defined by (3.6) has an increasing, bounded and pullback absorbing brochette $\mathcal{K}(\cdot)$ defined by

$$\mathcal{K}(t) := \left\{ w \in C_H : \|w\|^2_{C_H} \le 1 + \frac{e^{\gamma_2 h}}{\gamma_1} F(t) \right\}, \quad \forall\, t \in \mathbb{R}.$$

It suffices to verify that the process $U(\,\cdot\,,\cdot\,)$ is backward limit-set compact in C_H. In fact, we will prove a stronger result that the closure $\overline{\mathcal{Z}}$ is compact in C_H, where

$$\begin{aligned}\mathcal{Z} := \mathcal{Z}(t, \tau_0, D) &= \bigcup_{s\le t} \bigcup_{\tau\ge\tau_0} U(s, s-\tau)D \\ &= \{u_s(\,\cdot\,; s-\tau, \phi) \in C_H \mid s \le t, \tau \ge \tau_0, \phi \in D\},\end{aligned}$$

and $\tau_0 := \tau_0(t, D)$ is given in Lemma 3.3. By the Ascoli-Arzelà theorem, we need to verify two points.

Pointwise compactness: For each $\theta \in [-h, 0]$, the set

$$\mathcal{Z}(\theta) = \{u_s(\,\cdot\,; s-\tau, \phi)(\theta) \in H \mid s \le t, \tau \ge \tau_0, \phi \in D\}$$

is pre-compact in H. Indeed, by Lemma 3.5, $\|\mathcal{Z}(\theta)\|^2_V \le e^{(1+F(t))\alpha(t)}\alpha(t) < +\infty$, which means the set $\mathcal{Z}(\theta)$ is bounded in V. By the compactness of the Sobolev embedding $V \hookrightarrow H$, we know $\mathcal{Z}(\theta)$ is pre-compact in H as desired.

Equi-continuity: For each $\varepsilon > 0$, there is a $\delta > 0$ such that if $|\theta_1 - \theta_2| < \delta$ with $-h \le \theta_1 < \theta_2 \le 0$, then,

$$\sup_{s\le t} \sup_{\tau\ge\tau_0} \sup_{\phi\in D} \|u_s(\,\cdot\,; s-\tau, \phi)(\theta_1) - u_s(\,\cdot\,; s-\tau, \phi)(\theta_2)\| < \varepsilon. \tag{4.2}$$

Indeed, by (3.5), we have for all $s \le t$, $\tau \ge \tau_0$ and $\phi \in D$,

$$\begin{aligned} &\|u_s(\,\cdot\,; s-\tau,\phi)(\theta_1) - u_s(\,\cdot\,; s-\tau,\phi)(\theta_2)\| \\ &= \|u(s+\theta_1; s-\tau,\phi) - u(s+\theta_2; s-\tau,\phi)\| \\ &\le \int_{s+\theta_1}^{s+\theta_2} \|u'(r; s-\tau,\phi)\|\,dr \\ &\le \int_{s+\theta_1}^{s+\theta_2} (\nu\|Au(r)\| + \|B(u(r))\| + \|f(r)\| + \|g(r,u_r)\|)\,dr. \end{aligned} \tag{4.3}$$

By (3.24) in Lemma 3.6,

$$\begin{aligned} &\nu \int_{s+\theta_1}^{s+\theta_2} \|Au(r; s-\tau,\phi)\|\,dr \\ &\le \nu \left(\int_{s+\theta_1}^{s+\theta_2} \|Au(r; s-\tau,\phi)\|^2\,dr \right)^{1/2} |\theta_1-\theta_2|^{1/2} \\ &\le c\beta_1^{1/2}(t)|\theta_1-\theta_2| + c\beta_2^{1/2}(t)|\theta_1-\theta_2|^{1/2}. \end{aligned} \tag{4.4}$$

By the second formula in (3.4), it follows from (3.14) and (4.4) that

$$\begin{aligned} &\int_{s+\theta_1}^{s+\theta_2} \|B(u(r; s-\tau,\phi))\|\,dr \\ &\le \int_{s+\theta_1}^{s+\theta_2} c_0\lambda_1^{-1/2}\|u(r)\|_V\|Au(r)\|\,dr \\ &\le c \sup_{\theta\in[-h,0]} \|u(s+\theta; s-\tau,\phi)\|_V \int_{s+\theta_1}^{s+\theta_2} \|Au(r)\|\,dr \\ &\le ce^{\frac{1}{2}(1+F(t))\alpha(t)}\alpha^{1/2}(t)(\beta_1^{1/2}(t)|\theta_1-\theta_2| + \beta_2^{1/2}(t)|\theta_1-\theta_2|^{1/2}). \end{aligned} \tag{4.5}$$

By Hypothesis F,

$$\int_{s+\theta_1}^{s+\theta_2} \|f(r)\|\,dr \le \left(\int_{s+\theta_1}^{s+\theta_2} \|f(r)\|^2\,dr \right)^{1/2} |\theta_1-\theta_2|^{1/2} \le cF^{1/2}(t)|\theta_1-\theta_2|^{1/2}. \tag{4.6}$$

Finally, by Hypotheses (G2) and (G3),

$$\begin{aligned} &\int_{s+\theta_1}^{s+\theta_2} \|g(r, u_r(\,\cdot\,; s-\tau,\phi)\|\,dr \le \int_{s+\theta_1}^{s+\theta_2} L_g(r)\|u_r(\,\cdot\,; s-\tau,\phi)\|_{C_H}\,dr \\ &\le (1+c_1F(t))^{1/2} \int_{s+\theta_1}^{s+\theta_2} L_g(r)\,dr \le c(1+F(t))^{1/2}(\widetilde{L_g}(t))^{1/2}|\theta_1-\theta_2|^{1/2}. \end{aligned} \tag{4.7}$$

Since $|\theta_1-\theta_2|^{1/2} \le h^{1/2}$ (it is bounded), we substitute (4.4)–(4.7) into (4.3) to obtain that there is an increasing positive function $\beta(\cdot)$ such that

$$\sup_{s\le t}\sup_{\tau\ge\tau_0}\sup_{\phi\in D} \|u_s(\,\cdot\,; s-\tau,\phi)(\theta_1) - u_s(\,\cdot\,; s-\tau,\phi)(\theta_2)\| \le \beta(t)|\theta_1-\theta_2|^{1/2},$$

which proves (4.2) as desired. Therefore, the abstract Theorem 2.10 can be applied. □

To close the paper, we provide two special examples for variable delay and distribution delay.

Generalizing the form $G_0(u(t-\rho(t)))$ as given in the literature (e.g., [4, 5, 30]), we consider the more general variable delay $G(t, u(t-\rho(t)))$, more precisely, for each $u \in C([t_0-h,\infty), H)$,

$$g(t,u_t)(x) := G(t, u(t-\rho(t))(x)), \quad \forall\, t \geq t_0,\ x \in \Omega.$$

The function $G\colon \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$ satisfies $G(t,0)=0$ and

$$|G(t,y)-G(t,z)|_{\mathbb{R}^2} \leq L_G(t)|y-z|_{\mathbb{R}^2}, \quad \forall\, t \in \mathbb{R},\ y,z \in \mathbb{R}^2, \tag{4.8}$$

where $L_G(\cdot)$ is nonnegative, continuous and increasing. We assume $\rho(\cdot) \in C^1(\mathbb{R})$ is nonnegative such that

$$h := \sup_{t\in\mathbb{R}} \rho(t) \in (0,+\infty), \quad \rho_* := \sup_{t\in\mathbb{R}} \rho'(t) \in (-\infty, 1).$$

Then, the conditions (G1)–(G2) are satisfied. Since $L_G(\cdot)$ is increasing, it is easy to show that $L_G(\cdot)$ is backward translation bounded and thus the condition (G3) holds true as follows:

$$\begin{aligned}\|g(t,u_t)-g(t,v_t)\|^2 &= \int_\Omega |G(t,u(t-\rho(t))(x)) - G(t,v(t-\rho(t))(x))|_{\mathbb{R}^2}^2\,dx \\ &\leq L_G^2(t)\|u(t-\rho(t))-v(t-\rho(t))\|^2 \leq L_G^2(t)\|u_t-v_t\|_{C_H}^2.\end{aligned}$$

In order to verify the condition (G4), we further assume that $L := \lim_{t\to+\infty} L_G(t) < +\infty$ and $\nu\lambda_1 > L(1-\rho_*)^{-1/2}$. In this case, we can take an $m_0 > 0$ such that

$$\nu\lambda_1 > Le^{m_0 h/2}(1-\rho_*)^{-1/2} =: C_g.$$

Now, for $m \in [0, m_0]$ and $t \geq t_0$,

$$\begin{aligned}&\int_{t_0}^t e^{mr}\|g(r,u_r)-g(r,v_r)\|^2\,dr \\ &= \int_{t_0}^t e^{mr}\|G(r,u(r-\rho(r))) - G(r,v(r-\rho(r)))\|^2\,dr \\ &\leq \int_{t_0}^t e^{mr}L_G^2(r)\|u(r-\rho(r))-v(r-\rho(r))\|^2\,dr \\ &\leq L_G^2(t)\int_{t_0-\rho(t_0)}^{t-\rho(t)} \frac{e^{m(\sigma+h)}}{1-\rho_*}\|u(\sigma)-v(\sigma)\|^2\,d\sigma \\ &\leq C_g^2\int_{t_0-h}^t e^{m\sigma}\|u(\sigma)-v(\sigma)\|^2\,d\sigma.\end{aligned}$$

On the other hand, we consider a distributed delay: for any $\widehat{h} > 0$,

$$g(t, u_t)(x) = \int_{-\widehat{h}}^{0} G(s, u(t+s)(x))\, ds, \quad \forall\, t \geq t_0,\ x \in \Omega,$$

where $G(\cdot)$ is given by (4.8). Note that $L_G(\cdot)$ in (4.8) is continuous, we have $L_G \in L^2(-\widehat{h}, 0)$. We then calculate as follows:

$$\begin{aligned}
&\int_{t_0}^{t} e^{mr} \|g(r, u_r) - g(r, v_r)\|^2\, dr \\
&= \int_{t_0}^{t} e^{mr} \int_{\Omega} \left| \int_{-\widehat{h}}^{0} (G(s, u(s+r)(x)) - G(s, v(s+r)(x)))\, ds \right|_{\mathbb{R}^2}^2 dxdr \\
&\leq \int_{t_0}^{t} e^{mr} \int_{\Omega} \left(\int_{-\widehat{h}}^{0} |L_G(s)| |u(s+r)(x) - v(s+r)(x)|_{\mathbb{R}^2}\, ds \right)^2 dxdr \\
&\leq \int_{t_0}^{t} e^{mr} \int_{\Omega} \int_{-\widehat{h}}^{0} L_G^2(s)\, ds \int_{-\widehat{h}}^{0} |u(s+r)(x) - v(s+r)(x)|_{\mathbb{R}^2}^2\, dsdxdr \\
&= \|L_G(\cdot)\|_{L^2(-\widehat{h},0)}^2 \int_{-\widehat{h}}^{0} \int_{t_0+s}^{t+s} e^{m(\sigma - s)} \|u(\sigma) - v(\sigma)\|^2\, d\sigma ds \\
&\leq \|L_G(\cdot)\|_{L^2(-\widehat{h},0)}^2 \widehat{h} e^{m\widehat{h}} \int_{t_0-\widehat{h}}^{t} e^{m\sigma} \|u(\sigma) - v(\sigma)\|^2\, d\sigma.
\end{aligned}$$

If we assume $\nu\lambda_1 > \|L_G(\cdot)\|_{L^2(-\widehat{h},0)} \widehat{h}^{1/2}$, then we can take some $m_0 > 0$ such that

$$\nu\lambda_1 > \|L_G(\cdot)\|_{L^2(-\widehat{h},0)} \widehat{h}^{1/2} e^{m_0 \widehat{h}/2} := C_g,$$

and the condition (G4) holds true for all $m \in [0, m_0]$.

If we further assume f satisfies Hypothesis F in above two examples, then, Theorem 4.1 ensures the existence of a backward attractor and backward stability of the pullback attractor.

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