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Complete Cotorsion Pairs in Exact Categories

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Abstract. We generalize a theorem of Saorín-Šťovíček on complete cotorsion pairs in exact categories. Our proof is based on a generalized small object argument due to Chorny. As a consequence, we cover some examples which are not covered by the result of Saorín-Šťovíček.

1. Introduction

Ever since Salce introduced the notion of a cotorsion pair in the late 1970's in [13], the significance of complete cotorsion pairs has been widely understood in approximation theory [7] and the theory of closed exact model categories [10]. One fundamental result on complete cotorsion pairs is due to Eklof and Trlifaj, they proved that any cotorsion pair cogenerated by a set of modules is complete in [4]. It is a generalization of the corresponding result of Göbel and Shelah on abelian groups in [6].

In [14], Saorín and Šťovíček proved that a cotorsion pair cogenerated by a homological set is complete in their *efficient* exact categories. The proof of [14] used a variant of the small object argument due to Hovey. A key point of an efficient exact category is the axiom stating that transfinite compositions of inflations exist and are inflations. However, this axiom seems superfluous. For example, given a Grothendieck category $\mathcal{G}$ with a generator G, although the category $\mathrm{Ch}(\mathcal{G}_G)$ of chain complexes of the G-exact category $\mathcal{G}_G$ considered in [5] is not necessarily efficient, Gillespie still gave some complete cotorsion pairs cogenerated by a homological set in $\mathrm{Ch}(\mathcal{G}_G)$. This motivates us to generalize Saorín and Šťovíček's result in theory by removing the efficient axiom.

Another motivation of this paper is to understand Christensen and Hovey's relative closed model structure of the chain complexes category $\mathrm{Ch}(\mathcal{A})$ of a bicomplete abelian category $\mathcal{A}$ determined by a projective class in [3], through Hovey Correspondence between exact model structures and complete cotorsion pairs in [10].

Our main result is the following.

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Theorem 1.1. *Let $\mathcal{A}$ be an exact category. Let $\mathcal{I}$ be a homological class of inflations that permits the generalized small object argument and put $\mathcal{F} = (\mathrm{Cok}(\mathcal{I}))^{\perp}$. The following statements hold true:*

(1) *The following assertions are equivalent:*

 (a) *Each relative $\mathcal{I}$-cell complex with codomain in $\mathcal{F}$ is an inflation.*
 (b) *All relative $\mathcal{I}$-cell complexes are inflations.*
 (c) *$({}^{\perp}\mathcal{F}, \mathcal{F})$ is a right complete cotorsion pair in $\mathcal{A}$.*

 In that case the mentioned cotorsion pair is complete if and only if ${}^{\perp}\mathcal{F}$ is a class of generators in $\mathcal{A}$.

(2) *Consider the following conditions:*

 (a) *Each morphism in $\mathcal{I}$-inj with domain in $\mathrm{Cell}(\mathcal{I})$ is a deflation.*
 (b) *All morphisms in $\mathcal{I}$-inj are deflations.*
 (c) *$(\mathrm{Cof}(\mathcal{I}), \mathcal{F})$ is a left complete cotorsion pair in $\mathcal{A}$.*

 The implications (b) $\Rightarrow$ (a) $\Rightarrow$ (c) hold true and, when $\mathcal{A}$ is weakly idempotent complete, all assertions are equivalent. In that case the mentioned cotorsion pair is complete if and only if $\mathcal{F}$ is a class of cogenerators of $\mathcal{A}$.

The contents of the paper are as follows. In Section 2, we define the necessary notation and prove Chorny's generalized small object argument. In Section 3, we prove our main result and give examples of complete cotorsion pairs in the categories of chain complexes of (relative) exact categories. Throughout the paper, all colimits in concern are small colimits.

2. A generalized small object argument

In this section, we recall a generalized small object argument due to Chorny in [2]. We follow the sequence of lemmas from [9], extending them to work for the general case. The proofs are essentially the same but we include the general versions here for clarity and convenience of the reader.

2.1. Relative $\mathcal{I}$-cell complexes

Let $\mathcal{C}$ be a category. Suppose $i\colon A \to B$ and $p\colon X \to Y$ are morphisms in $\mathcal{C}$. Given a morphism $(f, g)\colon i \to p$, i.e., a commutative diagram in $\mathcal{C}$ of the following form

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle i}\downarrow & \overset{h}{\nearrow} & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & Y \end{array}$$

a *lift* or *lifting* in the diagram is a morphism $h\colon B \to X$ such that $h \circ i = f$ and $p \circ h = g$. A morphism $i\colon A \to B$ is said to have the *left lifting property* with respect to another morphism $p\colon X \to Y$ and p is said to have the *right lifting property* with respect to i if a lift exists in any diagram of the above form.

Definition 2.1. [9, Definition 2.1.7] Let $\mathcal{I}$ be a class of morphisms in a category $\mathcal{C}$.

(1) A morphism is *$\mathcal{I}$-injective* if it has the right lifting property with respect to every morphism in $\mathcal{I}$. The class of $\mathcal{I}$-injective morphisms is denoted $\mathcal{I}$-inj.

(2) A morphism is an *$\mathcal{I}$-cofibration* if it has the left lifting property with respect to every $\mathcal{I}$-injective morphism. The class of $\mathcal{I}$-cofibrations is denoted $\mathcal{I}$-cof.

If $\mathcal{C}$ has an initial object 0, an object $A \in \mathcal{C}$ is *$\mathcal{I}$-cofibrant* if the morphism $0 \to A \in \mathcal{I}$-cof. The collection of $\mathcal{I}$-cofibrants is denoted $\mathrm{Cof}(\mathcal{I})$.

Let $\mathcal{C}$ be a category and λ an ordinal. A functor $X : \lambda \to \mathcal{C}$ (i.e., a diagram

$$X_0 \to X_1 \to X_2 \to \cdots \to X_\alpha \to \cdots \quad (\alpha < \lambda)$$

in $\mathcal{C}$) is called a *λ-sequence* if for every limit ordinal $\gamma < \lambda$ the colimit $\mathrm{colim}_{\alpha<\gamma} X_\alpha$ exists and the induced morphism $\mathrm{colim}_{\alpha<\gamma} X_\alpha \to X_\gamma$ is an isomorphism.

If a colimit of a λ-sequence X exists, then the morphism $X_0 \to \mathrm{colim}_{\alpha<\lambda} X_\alpha$ is called the *transfinite composition* of X.

If $\mathcal{D}$ is a collection of morphisms of $\mathcal{C}$ and λ is an ordinal, then a *λ-sequence of morphisms in $\mathcal{D}$* is a λ-sequence $X_0 \to X_1 \to X_2 \to \cdots \to X_\alpha \to \cdots$ $(\alpha < \lambda)$ in $\mathcal{C}$ such that each morphism $X_\alpha \to X_{\alpha+1}$ is in $\mathcal{D}$ for $\alpha + 1 < \lambda$. A *transfinite composition of morphisms in $\mathcal{D}$* is the transfinite composition of a λ-sequence of morphisms in $\mathcal{D}$.

Definition 2.2. [9, Definition 2.1.9] Let $\mathcal{I}$ be a class of morphisms in a category $\mathcal{C}$. Assume that the transfinite compositions of pushouts of morphisms in $\mathcal{I}$ exist. A *relative $\mathcal{I}$-cell complex* is a transfinite composition of pushouts of morphisms in $\mathcal{I}$.

The collection of relative $\mathcal{I}$-cell complexes is denoted $\mathcal{I}$-cell. Note that $\mathcal{I}$-cell contains all isomorphisms. If $\mathcal{C}$ has an initial object 0, an object $A \in \mathcal{C}$ is an *$\mathcal{I}$-cell complex* if the morphism $0 \to A \in \mathcal{I}$-cell. The collection of $\mathcal{I}$-cell complexes is denoted $\mathrm{Cell}(\mathcal{I})$.

Lemma 2.3. *Let $\mathcal{C}$ be a category and $\mathcal{I}$ a class of morphisms in $\mathcal{C}$. If the transfinite compositions of pushouts of morphisms in $\mathcal{I}$ exist, then $\mathcal{I}$-cell $\subseteq \mathcal{I}$-cof.*

Proof. Assume that we have a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{g} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle j} \\ D & \xrightarrow{h} & B \end{array}$$

where $f \in \mathcal{I}$-cell and $j \in \mathcal{I}$-inj.

Let $f: C \to D$ be the transfinite composition of the λ-sequence

$$C = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_\alpha \xrightarrow{f_\alpha} X_{\alpha+1} \to \cdots \quad (\alpha + 1 < \lambda)$$

where all morphisms f_α are pushouts of morphisms in $\mathcal{I}$. Let $\tau_\alpha: X_\alpha \to D = \operatorname{colim}_{\alpha<\lambda} X_\alpha$ be the colimit morphism for all $\alpha < \lambda$. We will construct the morphism $u_\alpha: X_\alpha \to A$ by transfinite induction such that $j \circ u_\alpha = h \circ \tau_\alpha$ and $u_{\alpha+1} \circ f_\alpha = u_\alpha$.

For $\alpha = 1$, let $u_0 = g$. Assume that f_0 is the pushout of $i: E \to F$ in I:

$$\begin{array}{ccccc} E & \longrightarrow & X_0 & \xrightarrow{g} & A \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f_0} & \nearrow & \downarrow{\scriptstyle j} \\ F & \xrightarrow{v} & X_1 & \xrightarrow{h\circ\tau_1} & B. \end{array}$$

Since $j \in \mathcal{I}$-inj, there is a lifting $v: F \to A$. It induces a morphism $u_1: X_1 \to A$ such that $j \circ u_1 = h \circ \tau_1$ and $u_1 \circ f_0 = g$ by the universal property of pushouts. Assume now that we have defined $u_\alpha: X_\alpha \to A$ for all $\alpha < \beta$. If β is a limit ordinal, let $u_\beta: X_\beta = \operatorname{colim}_{\alpha<\beta} X_\alpha \to A$ be the induced morphism by u_α for $\alpha < \beta$, then $j \circ u_\beta = h \circ \tau_\beta$. If β has a predecessor α, then replace $f_0: X_0 \to X_1$ by f_α in the case of $\alpha = 0$, we can construct a morphism $u_{\alpha+1}: X_{\alpha+1} \to A$ satisfying $u_{\alpha+1} \circ f_\alpha = u_\alpha$ and $j \circ u_{\alpha+1} = h \circ \tau_{\alpha+1}$, which completes our transfinite induction. Therefore, let $u: D = \operatorname{colim}_{\alpha<\lambda} X_\alpha \to A$ be the induced morphism by all morphisms u_α, then $j \circ u = h$ and $u \circ f = g$ by the universal property of colimits. So $f \in \mathcal{I}$-cof. $\square$

Lemma 2.4. *Let $\mathcal{C}$ be a category and $\mathcal{I}$ a class of morphisms in $\mathcal{C}$. If the transfinite compositions of pushouts of morphisms in $\mathcal{I}$ exist, then the transfinite compositions of morphisms of* $\mathcal{I}$-cell *exist and belong to* $\mathcal{I}$-cell.

Proof. Let λ be an ordinal and X be a λ-sequence

$$X_0 \to X_1 \to X_2 \to \cdots \to X_\alpha \to \cdots \quad (\alpha < \lambda)$$

such that each morphism $X_\alpha \to X_{\alpha+1}$ for $\alpha + 1 < \lambda$ is the transfinite composition of the γ_α-sequence

$$X_\alpha = W_0^\alpha \to W_1^\alpha \to W_2^\alpha \to \cdots \to W_\beta^\alpha \to \cdots \quad (\beta < \gamma_\alpha)$$

of pushouts of morphisms in $\mathcal{I}$. By *interpolating* these sequences for all $\alpha < \lambda$ into the λ-sequence X [8, Definition 10.2.11], we get a μ-sequence $Y: \mu \to \mathcal{C}$ of pushouts of morphisms in $\mathcal{I}$ by Propositions 10.2.8 and 10.2.13 in [8]. By assumption, the transfinite composition of the μ-sequence Y exists, that is, $\operatorname{colim}_{\gamma<\mu} Y_\gamma$ exists. By the construction of Y, we have $\operatorname{colim}_{\alpha<\lambda} X_\alpha = \operatorname{colim}_{\gamma<\mu} Y_\gamma$ and the transfinite composition $X_0 \to \operatorname{colim}_{\alpha<\lambda} X_\alpha$ is the transfinite composition $Y_0 \to \operatorname{colim}_{\gamma<\mu} Y_\gamma$, and it is a relative $\mathcal{I}$-cell complex. $\square$

2.2. Chorny's generalized small object argument

Recall that, the *cofinality* of a limit ordinal λ, denoted by $\mathrm{cf}(\lambda)$, is the smallest cardinal κ such that there exists a subset T of λ with $|T| = \kappa$ and $\sup(T) = \lambda$.

Let $\mathcal{C}$ be a category. Let κ be a cardinal and $\mathcal{D}$ a class of morphisms in $\mathcal{C}$. An object A of $\mathcal{C}$ is said to be *κ-small relative to $\mathcal{D}$* if for every ordinal λ with $\mathrm{cf}(\lambda) > \kappa$ and every λ-sequence $X\colon \lambda \to \mathcal{C}$ of morphisms in $\mathcal{D}$, the natural morphism

$$\mathrm{colim}_{\alpha<\lambda} \mathrm{Hom}_{\mathcal{C}}(A, X_\alpha) \to \mathrm{Hom}_{\mathcal{C}}(A, \mathrm{colim}_{\alpha<\lambda} X_\alpha)$$

is an isomorphism. An object A in $\mathcal{C}$ is called *small relative to $\mathcal{D}$* if it is κ-small relative to $\mathcal{D}$ for some cardinal κ.

Definition 2.5. Let $\mathcal{C}$ be a category. We say a class $\mathcal{I}$ of morphisms in $\mathcal{C}$ *permits the generalized small object argument* if the following conditions hold:

(i) The transfinite compositions of pushouts of morphisms in $\mathcal{I}$ exist.

(ii) There is a cardinal κ, such that the domains of morphisms of $\mathcal{I}$ are κ-small relative to $\mathcal{I}$-cell.

(iii) For every morphism f in $\mathcal{C}$, there is a morphism $g \in \mathcal{I}$-cell equipped with a morphism $s\colon g \to f$, such that any morphism $i \to f$ with $i \in \mathcal{I}$ factors through s.

Lemma 2.6. *Let $\mathcal{I}$ be a class of morphisms of a category $\mathcal{C}$ such that the transfinite compositions of pushouts of morphisms in $\mathcal{I}$ exist. Then any pushout of morphisms of* $\mathcal{I}$-cell *exists and belongs to* $\mathcal{I}$-cell.

Proof. Assume that $f\colon A \to B$ is a relative $\mathcal{I}$-cell complex. Then there is an ordinal λ and a λ-sequence

$$A = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \to X_\beta \xrightarrow{f_\beta} X_{\beta+1} \to \cdots \quad (\beta + 1 < \lambda)$$

such that every f_β is a pushout of a morphism in $\mathcal{I}$ and f is the transfinite composition of X. Let $g_0\colon A \to E_0$ be any morphism in $\mathcal{C}$.

We claim that there is a commutative diagram:

$$\begin{array}{ccccccccc} X_0 & \xrightarrow{f_0} & X_1 & \xrightarrow{f_1} & \cdots & \longrightarrow & X_\beta & \xrightarrow{f_\beta} & \cdots \\ {\scriptstyle g_0}\downarrow & & {\scriptstyle g_1}\downarrow & & & & {\scriptstyle g_\beta}\downarrow & & \\ E_0 & \xrightarrow{h_0} & E_1 & \xrightarrow{h_1} & \cdots & \longrightarrow & E_\beta & \xrightarrow{h_\beta} & \cdots \end{array} \tag{2.1}$$

such that each square is a pushout. In fact, since f_0 is a pushout of a morphism in $\mathcal{I}$, say $i\colon C \to D$, i.e., we have a pushout diagram

$$\begin{array}{ccc} C & \xrightarrow{i} & D \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ X_0 & \xrightarrow{f_0} & X_1. \end{array}$$

By assumption, the pushout of i along $g_0 \circ s$ exists and, it induces a pushout

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & X_1 \\ {\scriptstyle g_0}\downarrow & & \downarrow{\scriptstyle g_1} \\ E_0 & \xrightarrow{h_0} & E_1. \end{array}$$

Assume that we have defined E_α and $g_\alpha \colon X_\alpha \to E_\alpha$ for all $\alpha < \beta$. If β is a limit ordinal, define $E_\beta = \operatorname{colim}_{\alpha<\beta} E_\alpha$, and define g_β to be the morphism induced by g_α. If β has a predecessor α, i.e., $\beta = \alpha + 1$, define $E_\beta = E_\alpha \coprod_{X_\alpha} X_{\alpha+1}$, and g_β to be the pushout of g_α along f_α. Therefore we have a λ-sequence

$$E_0 \xrightarrow{h_0} E_1 \xrightarrow{h_1} \cdots \to E_\beta \xrightarrow{h_\beta} E_{\beta+1} \to \cdots \quad (\beta + 1 < \lambda).$$

Its transfinite composition $E_0 \to \operatorname{colim}_{\alpha<\lambda} E_\alpha$ exists by assumption and belongs to $\mathcal{I}$-cell by construction. The commutative diagram (2.1) induces a desired pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B = \operatorname{colim}_{\alpha<\lambda} X_\alpha \\ {\scriptstyle g_0}\downarrow & & \downarrow \\ E_0 & \longrightarrow & \operatorname{colim}_{\alpha<\lambda} E_\alpha \end{array}$$

in $\mathcal{C}$. □

Now we can prove the following generalized Quillen small object argument due to Chorny [2, Theorem 1.1].

Theorem 2.7 (The generalized small object argument)**.** *Let $\mathcal{C}$ be a category and $\mathcal{I}$ a class of morphisms in $\mathcal{C}$. Suppose that $\mathcal{I}$ permits the generalized small object argument. Then every morphism $f \colon X \to Y$ in $\mathcal{C}$ admits a factorization $f = \delta(f) \circ \gamma(f)$, where $\gamma(f) \in \mathcal{I}$-cell and $\delta(f) \in \mathcal{I}$-inj.*

Proof. By Lemmas 2.4 and 2.6, the proof of Theorem 1.1 in [2] works here. □

3. Proof of the main result

In this section, we recall the definition of complete cotorsion pairs in exact categories and prove our main result.

3.1. Cotorsion pairs in exact categories

The concept of an exact category is due to D. Quillen [12], a simple axiomatic description can be found in [11, Appendix A]. Roughly speaking, an *exact category* is an additive category $\mathcal{A}$ equipped with a class $\mathcal{E}$ of kernel-cokernel sequences $A \overset{s}{\rightarrowtail} B \overset{t}{\twoheadrightarrow} C$ in $\mathcal{A}$ such that s is the kernel of t and t is the cokernel of s. The class $\mathcal{E}$ satisfies exact axioms, for details, we refer the reader to [1, Definition 2.1]. Given an exact category $\mathcal{A}$, we will call a kernel-cokernel sequence a *conflation* if it is in $\mathcal{E}$. The morphism s in a conflation $A \overset{s}{\rightarrowtail} B \overset{t}{\twoheadrightarrow} C$ is called an *inflation* and the morphism t is called a *deflation*.

Given an exact category $\mathcal{A}$, a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects of $\mathcal{A}$ is called a *cotorsion pair* in $\mathcal{A}$ if

$$\mathcal{T} = {}^{\perp}\mathcal{F} := \{T \in \mathcal{A} \mid \operatorname{Ext}^1_{\mathcal{A}}(T, F) = 0, \forall\, F \in \mathcal{F}\}$$

and

$$\mathcal{F} = \mathcal{T}^{\perp} := \{F \in \mathcal{A} \mid \operatorname{Ext}^1_{\mathcal{A}}(T, F) = 0, \forall\, T \in \mathcal{T}\}.$$

A cotorsion pair $(\mathcal{T}, \mathcal{F})$ is called *right complete* if for each $A \in \mathcal{A}$ there exists a conflation $A \rightarrowtail F \twoheadrightarrow T$ such that $T \in \mathcal{T}$ and $F \in \mathcal{F}$. *Left complete* cotorsion is defined dually. We call a cotorsion pair *complete* if it is both right complete and left complete.

3.2. Eklof's lemma

Given a class $\mathcal{S}$ of objects in an exact category $\mathcal{A}$, an object A of $\mathcal{A}$ is called a *transfinite extension of* $\mathcal{S}$ if the morphism $0 \to A$ is the transfinite composition of a λ-sequence

$$X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} \cdots \to X_\beta \xrightarrow{i_\beta} X_{\beta+1} \to \cdots \quad (\beta + 1 < \lambda)$$

such that each i_β is an inflation with a cokernel in $\mathcal{S}$.

The following lemma is proved in [10, Lemma 6.2], called *Eklof's Lemma.*

Lemma 3.1. *Let $\mathcal{A}$ be an exact category and $F \in \mathcal{A}$. Then ${}^{\perp}F$ is closed under retracts and, if A is a transfinite extension of ${}^{\perp}F$, then it is in ${}^{\perp}F$.*

3.3. The main result

Given a class $\mathcal{I}$ of inflations in the exact category $\mathcal{A}$, let $\operatorname{Cok}(\mathcal{I}) = \{A \in \mathcal{A} \mid A \cong \operatorname{Coker}(i)$ for some $i \in \mathcal{I}\}$. Following [14, Definition 2.3], $\mathcal{I}$ is called a *homological* class if the morphism $A \to 0$ belongs to $\mathcal{I}$-inj (i.e., the map $i^*\colon \operatorname{Hom}_{\mathcal{A}}(D, A) \to \operatorname{Hom}_{\mathcal{A}}(C, A)$ is surjective for all $i\colon C \to D$ in $\mathcal{I}$) implies $A \in \operatorname{Cok}(\mathcal{I})^{\perp}$.

Let $\mathcal{A}$ be an exact category. A collection $\mathcal{X}$ of objects of $\mathcal{A}$ is called a *class of generators* of $\mathcal{A}$ if for any object $A \in \mathcal{A}$, there is a deflation $\coprod_{s\in S} X_s \twoheadrightarrow A$, where all X_s are in $\mathcal{X}$ and S is a set. The notation of a *class of cogenerators* of $\mathcal{A}$ is defined dually.

Recall that an exact category is called *weakly idempotent complete* (*WIC*, for short) if every split monomorphism is an inflation.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. (1) (a) $\Rightarrow$ (c). Given any object $A \in \mathcal{A}$, since $\mathcal{I}$ permits the generalized small object argument, we can factor $A \to 0$ as the composition $A \xrightarrow{f} B \to 0$, with $f \in \mathcal{I}$-cell and $B \to 0 \in \mathcal{I}$-inj. Since $\mathcal{I}$ is homological, we have $B \in \mathcal{F}$. Therefore f is an inflation by the condition (a). By Lemma 2.6, $C := \mathrm{Coker}(f)$ belongs to $\mathrm{Cell}(\mathcal{I})$. Thus C is a transfinite extension of $\mathrm{Cok}(\mathcal{I})$ and hence in ${}^{\perp}\mathcal{F}$ by Lemma 3.1. So we have a conflation $A \overset{f}{\rightarrowtail} B \twoheadrightarrow C$, where $B \in \mathcal{F}$ and $C \in {}^{\perp}\mathcal{F}$. Therefore $({}^{\perp}\mathcal{F}, \mathcal{F})$ is a right complete cotorsion pair in $\mathcal{A}$.

(c) $\Rightarrow$ (b). Fix any relative $\mathcal{I}$-cell complex $\iota\colon X \to Y$ and, using the right completeness of the cotorsion pair, fix an inflation $j\colon X \rightarrowtail F$, where $F \in \mathcal{F}$ and $\mathrm{Coker}(j) \in {}^{\perp}\mathcal{F}$. Since $\mathcal{I}$ is homological, $F \to 0$ is in $\mathcal{I}$-inj. Then it has the right lifting property with respect to ι, so that we get a morphism $h\colon Y \to X$ such that $h \circ \iota = j$. By Lemma 2.6, the pushout of $\iota\colon X \to Y$ along the morphism $X \to 0$ exists, Thus ι has a cokernel, and then it is an inflation by the Obscure Axiom.

(b) $\Rightarrow$ (a) is clear.

The *only if* part of the last statement is trivial. For the *if* part, let A be any object in $\mathcal{A}$. Since ${}^{\perp}\mathcal{F}$ is a class of generators of $\mathcal{A}$ and closed under coproducts by Lemma 3.1, there exists a deflation $p\colon G \twoheadrightarrow A$ with $G \in {}^{\perp}\mathcal{F}$. Let $K = \ker p$. Since $({}^{\perp}\mathcal{F}, \mathcal{F})$ is right complete in $\mathcal{A}$, there is a conflation $K \rightarrowtail F \twoheadrightarrow C$, where $F \in \mathcal{F}$ and $C \in {}^{\perp}\mathcal{F}$. By Proposition 2.12 in [1], there is a commutative diagram of conflations:

$$
\begin{array}{ccccc}
K & \rightarrowtail & F & \twoheadrightarrow & C \\
\downarrow & & \downarrow & & \| \\
G & \rightarrowtail & T & \twoheadrightarrow & C \\
{\scriptstyle p}\downarrow & & \downarrow & & \\
A & = & A & &
\end{array}
$$

such that the upper-left square is a pushout diagram. Since ${}^{\perp}\mathcal{F}$ is closed under extensions, we know that $T \in {}^{\perp}\mathcal{F}$. Thus $F \rightarrowtail T \twoheadrightarrow A$ is a desired conflation. Therefore $({}^{\perp}\mathcal{F}, \mathcal{F})$ is a complete cotorsion pair in $\mathcal{A}$.

(2) (b) $\Rightarrow$ (a) is clear.

(a) $\Rightarrow$ (c). For each $A \in \mathcal{A}$, we claim that there is a conflation $K \rightarrowtail U \twoheadrightarrow A$ such that $K \in \mathcal{F}$ and $U \in \mathrm{Cell}(\mathcal{I})$. In fact, since $\mathcal{I}$ permits the generalized small object argument, we can factor $0 \to A$ as a composition $0 \to U \xrightarrow{f} A$, where $U \in \mathrm{Cell}(\mathcal{I})$ and $f \in \mathcal{I}$-inj. Since ${}^{\perp}\mathcal{F}$ is closed under transfinite extensions by Lemma 3.1, we know that $\mathrm{Cell}(\mathcal{I}) \subseteq {}^{\perp}\mathcal{F}$. By the condition (a), f is a deflation. Since $\mathcal{I}$-inj is closed under pullback, we know that $K = \ker f \to 0 \in \mathcal{I}$-inj. Then $K \in \mathcal{F}$. So if one proves ${}^{\perp}\mathcal{F} = \mathrm{Cof}(\mathcal{I})$, then we are done.

If one takes $A \in {}^{\perp}\mathcal{F}$ in the above argument, then the given conflation splits. So A is a direct summand of U. Since $\mathrm{Cell}(\mathcal{I}) \subseteq \mathrm{Cof}(\mathcal{I})$ and $\mathrm{Cof}(\mathcal{I})$ is closed under retracts, we know that $A \in \mathrm{Cof}(\mathcal{I})$. Thus ${}^{\perp}\mathcal{F} \subseteq \mathrm{Cof}(\mathcal{I})$. On the other hand, by using the general small object argument, it is easy to see that $\mathrm{Cof}(\mathcal{I})$ consists of retracts of $\mathrm{Cell}(\mathcal{I})$. Therefore $\mathrm{Cof}(\mathcal{I}) \subseteq {}^{\perp}\mathcal{F}$ since ${}^{\perp}\mathcal{F}$ is closed under retracts by Lemma 3.1. So we have ${}^{\perp}\mathcal{F} = \mathrm{Cof}(\mathcal{I})$.

Assume now that $\mathcal{A}$ is WIC in order to prove (c) $\Rightarrow$ (b). Let $p\colon X \to Y$ be a morphism in $\mathcal{I}$-inj and fix a deflation $q\colon T \twoheadrightarrow Y$, where $T \in \mathrm{Cof}(\mathcal{I})$. By taking the pullback of q and p, one gets that the opposite $q'\colon Z \to X$ of q in that pullback is a deflation. It is well known that $\mathcal{I}$-inj is closed under pullback (see, for example, Lemma 7.2.11 in [8]), so the morphism p' is also in $\mathcal{I}$-inj. But the morphism $0 \to T$ is an $\mathcal{I}$-cofibration since $T \in \mathrm{Cof}(\mathcal{I})$. It follows that p' has the right lifting property with respect to $0 \to T$ (along the identity on T). Therefore there exists a morphism $s\colon T \to Z$ such that $p' \circ s = 1_T$. Then p' is a retraction and, by the WIC condition on $\mathcal{A}$, it is a deflation. Thus p is a deflation by using Proposition 7.6 in [1] to $p \circ q' = q \circ p'$.

For the final statement of assertion (2), we only need to prove the *if* part since the *only if* part is obvious.

Assume now that $\mathcal{F}$ is a class of cogenerators of $\mathcal{A}$. For each $A \in \mathcal{A}$, we claim that there is a conflation $A \rightarrowtail F \twoheadrightarrow C$ such that $F \in \mathcal{F}$ and $C \in {}^{\perp}\mathcal{F}$. In fact, this can be proved dually with the proof of the last statement of the assertion (1). Thus $(\mathrm{Cof}(\mathcal{I}), \mathcal{F})$ is right complete, and then it is complete in $\mathcal{A}$. □

Example 3.2. Let $\mathcal{A}$ be a complete and cocomplete abelian category and $\mathcal{P}$ a class of objects in $\mathcal{A}$. A short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{A}$ is called $\mathcal{P}$-*exact* if

$$0 \to \mathrm{Hom}_{\mathcal{A}}(P, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(P, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(P, C) \to 0$$

is a short exact sequence of abelian groups for any $P \in \mathcal{P}$. We use $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$ to denote a $\mathcal{P}$-exact sequence and, let $\mathcal{E}_{\mathcal{P}}$ be the class of all $\mathcal{P}$-exact sequences. The morphism f in a $\mathcal{P}$-exact sequence $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$ is called a $\mathcal{P}$-*inflation* and the morphism g is called a $\mathcal{P}$-*deflation*.

Assume now $\mathcal{P}$ is a *projective class* of $\mathcal{A}$, i.e., for each object A there is a $\mathcal{P}$-deflation $P \twoheadrightarrow A$ with $P \in \mathcal{P}$ (see [3, Denfinition 1.1]). Then $(\mathcal{A}, \mathcal{E}_{\mathcal{P}})$ is an exact category. We will use $\mathcal{A}_{\mathcal{P}}$ to denote this exact category.

We denote $\mathrm{Ch}(\mathcal{A}_{\mathcal{P}})$ the exact category of all chain complexes of objects of $\mathcal{A}$ with the form $\cdots \to X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \to \cdots$ and conflations degreewise lie in $\mathcal{A}_{\mathcal{P}}$. Given an object $A \in \mathcal{A}$, we denote the n-*sphere* on A by $S_n(A)$, and it is the complex consisting of A in degree n and 0 elsewhere. We denote the n-*disk* on A by $D_n(A)$ and, this is the complex consisting of $D_n(A)^k = A$ if $k = n$ or $n+1$, but $D_n(A)^k = 0$ for other values of k, and whose differential is the identity in degree n.

Assume that $\mathcal{P}$ has *enough κ-small projectives* in the sense of Proposition 4.2 in [3]. That is, there is a collection $\mathcal{P}'$ of objectives in $\mathcal{P}$ such that every $\mathcal{P}'$-deflation is a $\mathcal{P}$-deflation and, each object in $\mathcal{P}'$ is κ-small relative to split monomorphisms with cokernel in $\mathcal{P}$. Then $\{D_n(P) \mid P \in \mathcal{P}', n \in \mathbb{Z}\}$ is a class of projective generators of $\mathrm{Ch}(\mathcal{A}_{\mathcal{P}})$ and, there is a degreewise split conflation $S_{n+1}(P) \overset{\iota_n(P)}{\hookrightarrow} D_n(P) \twoheadrightarrow S_n(P)$ for each $P \in \mathcal{P}'$ and $n \in \mathbb{Z}$.

Let $\mathcal{I} = \big\{S_{n+1}(P) \overset{\iota_n(P)}{\hookrightarrow} D_n(P) \mid \forall\, P \in \mathcal{P}', n \in \mathbb{Z}\big\}$. Then $\mathcal{I}$-cell consists of degreewise split inflations. By Lemma 4.3 in [3], we know that $S_n(P)$ is κ-small relative to $\mathcal{I}$-cell for any $P \in \mathcal{P}'$ and $n \in \mathbb{Z}$.

Let $f\colon X \to Y$ be any morphism in $\mathrm{Ch}(\mathcal{A}_{\mathcal{P}})$. Then for any morphism $\iota_n(P)$ in $\mathcal{I}$, a morphism from $\iota_n(P)$ to f is in bijective correspondence with a commutative diagram

$$\begin{array}{ccc} P & \longrightarrow & \ker d_X^{n+1} \\ \big\downarrow & & \big\downarrow f^{n+1}|_{\ker d_X^{n+1}} \\ Y^n & \xrightarrow{d_Y^n} & Y^{n+1}. \end{array}$$

Let $W^n = \ker d_X^{n+1} \times_{Y^{n+1}} Y^n$ be the pullback. Then there is a $\mathcal{P}'$-deflation $P^n \twoheadrightarrow W^n$ with $P^n \in \mathcal{P}'$. Thus the above commutative diagram is in correspondence with a morphism $h^n(P)\colon P \to P^n$. Take $g = \coprod_{n\in\mathbb{Z}} \iota_n(P^n)$. Then g is in $\mathcal{I}$-cell by Proposition 10.2.7 in [8] and, there is a morphism $s\colon g \to f$ induced by the morphisms $P^n \to W^n$ for all $n \in \mathbb{Z}$. By the construction of g, for any morphism $t\colon \iota_n(P) \to f$, there is a morphism $p\colon \iota_n(P) \to g$ induced by $h_n(P)$ such that $s \circ p = t$. Therefore, $\mathcal{I}$ permits the generalized small object argument.

Moreover, for any object X in $\mathrm{Ch}(\mathcal{A}_{\mathcal{P}})$ and $\iota_n(P) \in \mathcal{I}$, since $D_n(P)$ is projective in $\mathrm{Ch}(\mathcal{A}_{\mathcal{P}})$, we have an exact sequence

$$\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A}_{\mathcal{P}})}(D_n(P), X) \xrightarrow{\iota_n^*} \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A}_{\mathcal{P}})}(S_{n+1}(P), X) \to \mathrm{Ext}^1_{\mathrm{Ch}(\mathcal{A}_{\mathcal{P}})}(S_n(P), X) \to 0$$

by applying $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A}_{\mathcal{P}})}(-, X)$ to the conflation $S_{n+1}(P) \overset{\iota_n(P)}{\hookrightarrow} D_n(P) \twoheadrightarrow S_n(P)$. Therefore, ι_n^* is surjective implies $X \in S_n^{\perp}$. Since $\mathrm{Cok}(\mathcal{I}) = \{S_n(P) \mid P \in \mathcal{P}', n \in \mathbb{Z}\}$, we know that $X \to 0$ is in $\mathcal{I}$-inj means $X \in \mathrm{Cok}(\mathcal{I})^{\perp}$. That is, $\mathcal{I}$ is homological. So $({}^{\perp}(\mathrm{Cok}(\mathcal{I})^{\perp}), \mathrm{Cok}(\mathcal{I})^{\perp})$ is a complete cotorsion pair in $\mathrm{Ch}(\mathcal{A}_{\mathcal{P}})$ by Theorem 1.1(1).

Remark 3.3. If we take $\mathcal{I} = \big\{0 \to D_n(P), S_{n+1}(P) \overset{\iota_n(P)}{\hookrightarrow} D_n(P) \mid \forall\, P \in \mathcal{P}', n \in \mathbb{Z}\big\}$ in Example 3.2, then ${}^{\perp}(\mathrm{Cok}(\mathcal{I})^{\perp}) = \mathrm{Cof}(\mathcal{I})$. In this case, it can be proved that $\mathcal{F}$ consists of acyclic chain complexes in $\mathrm{Ch}(\mathcal{A}_{\mathcal{P}})$ (a complex $\cdots \to X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \to \cdots$ is *acyclic* if each morphism d^n decomposes in $\mathcal{A}_{\mathcal{P}}$ as $X^n \xrightarrow{e_n} \ker d^{n+1} \xrightarrow{m_n} X^{n+1}$ where e_n is an deflation and m_n is a $\mathcal{P}$-inflation; furthermore, $\ker d^{n+1} \xrightarrow{m_n} X^{n+1} \xrightarrow{e_{n+1}} \ker d^{n+2}$ is a $\mathcal{P}$-conflation). By Theorem 2.2 in [10], $\mathrm{Cof}(\mathcal{I})$, $\mathcal{F}$ and $\mathrm{Ch}(\mathcal{A}_{\mathcal{P}})$ determine a projective closed model structure on $\mathrm{Ch}(\mathcal{A}_{\mathcal{P}})$. It is easy to see that this model structure is Quillen equivalent to the relative closed model structure on $\mathrm{Ch}(\mathcal{A})$ given in Theorem 2.2 in [3].

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