On Orthogonality of Elementary Operators in Norm-attainable Classes

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Abstract. Various notions of orthogonality of elementary operators have been characterized by many mathematicians in different classes. In this paper, we characterize orthogonality of these operators in norm-attainable classes. We first give necessary and sufficient conditions for norm-attainability of Hilbert space operators then we give results on orthogonality of the range and the kernel of elementary operators when they are implemented by norm-attainable operators in norm-attainable classes.

1. Introduction

Norm-attainability is one of the aspects which has been given attention in studies on Hilbert space operators for along period of time with nice results obtained [5]. Let H be an infinite dimensional separable complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. An operator $S \in B(H)$ is said to be norm-attainable if there exists a unit vector $x_0 \in H$ such that $\|Sx_0\| = \|S\|$. The class of all norm-attainable operators is denoted by $NA(H)$. For an operator $S \in B(H)$ we define a numerical range by $W(S) = \{\langle Sx, x \rangle : x \in H, \|x\| = 1\}$ and the maximal numerical range by $W_0(S) = \{\beta \in \mathbb{C} : \langle Sx_n, x_n \rangle \to \beta, \text{ where } \|x_n\| = 1, \|Sx_n\| \to \|S\|\}$. The second aspect in consideration is orthogonality which is a concept that has been analyzed for quite a period of time (see [13, 36–38]). Benitez [4] described several types of orthogonality which have been studied in real normed spaces namely: Robert's orthogonality, Birkhoff's orthogonality, Orthogonality in the sense of James, Isoceles, Pythagoras, Carlsson, Diminnie, Area among others. Some of these orthogonalities are described as follows. For $x \in \mathcal{M}$ and $y \in \mathcal{N}$ where $\mathcal{M}$ and $\mathcal{N}$ are subspaces of E which is a normed linear space, we have

  (i) Roberts: $\|x - \lambda y\| = \|x + \lambda y\|, \forall \lambda \in \mathbb{R}$;

 (ii) Birkhoff: $\|x + \lambda y\| \geq \|x\|$;

(iii) Isosceles: $\|x - y\| = \|x + y\|$;

(iv) Pythagorean: $\|x - y\|^2 = \|x\|^2 + \|y\|^2$;

(v) *a*-Pythagorean: $\|x - ay\|^2 = \|x\|^2 + a^2\|y\|^2$, $a \neq 0$;

(vi) Diminnie: $\sup\{f(x)g(y) - f(y)g(x) : f, g \in S'\} = \|x\|\|y\|$ where S' denotes the unit sphere of the topological dual of E;

(vii) Area: $\|x\|\|y\| = 0$ or they are linearly independent and such that $x, -x, y, -y$ divide the unit ball of their own plane (identified by $\mathbb{R}^2$) in four equal areas.

In this paper we will consider the orthogonality of elementary operators when they are implemented by norm-attainable operators. Consider a normed algebra $\mathcal{A}$ and let $T_{A,B} \colon \mathcal{A} \to \mathcal{A}$. T is called an *elementary operator* if it has the following representation: $T(X) = \sum_{i=1}^{n} A_i X B_i$, $\forall X \in \mathcal{A}$, where A_i, B_i are fixed in $\mathcal{A}$. Let $\mathcal{A} = B(H)$. For $A, B \in B(H)$ we define the particular elementary operators: The left multiplication operator $L_A \colon B(H) \to B(H)$ by $L_A(X) = AX$, $\forall X \in B(H)$; the right multiplication operator $R_B \colon B(H) \to B(H)$ by $R_B(X) = XB$, $\forall X \in B(H)$; the generalized derivation (implemented by A, B) by $\delta_{A,B} = L_A - R_B$; the basic elementary operator (implemented by A, B) by $M_{A,B}(X) = AXB$, $\forall X \in B(H)$; the Jordan elementary operator (implemented by A, B) by $\mathcal{U}_{A,B}(X) = AXB + BXA$, $\forall X \in B(H)$.

Regarding orthogonality involving elementary operators, Anderson [1] established the orthogonality of the range and kernel of normal derivations. Others who have also worked on orthogonality include: Kittaneh [21], Mecheri [29] among others. For details see [1–27, 30–35]. We shall investigate the orthogonality of the range and the kernel of several types of important elementary operators in Banach spaces. Anderson [1] in his investigations proved that if N and S are operators in $B(H)$ such that N is normal and $NS = SN$ then for all $X \in B(H)$, $\|\delta_N(X) + S\| \geq \|S\|$. If S (above) is a Hilbert-Schmidt operator then Kittaneh [21] (see also the references therein) showed that $\|\delta_N(X) + S\|_2^2 = \|\delta_N(X)\|_2^2 + \|S\|_2^2$. We extend this study to norm-attainable classes. Very similar researches have been made by Chmieliski, Mal, Paul, Sain, Stypula and Wójcik for Birkhoff orthogonality (see [37, 38]) and for approximate Birkhoff orthogonality (see [13, 33, 36]).

2. Preliminaries

In this section, we give some preliminary results. We begin by the following proposition.

Proposition 2.1. *Let H be an infinite dimensional separable complex Hilbert space. Let $S \in B(H)$, $\beta \in W_0(S)$ and $\alpha > 0$. Then the following conditions hold:*

  (i) *There exists $Z \in B(H)$ such that $\|S\| = \|Z\|$, with $\|S - Z\| < \alpha$.*

  (ii) *There exists a vector $\eta \in H$ such that $\|\eta\| = 1$, $\|Z\eta\| = \|Z\|$ with $\langle Z\eta, \eta \rangle = \beta$.*

Proof. Let $\|S\| = 1$ and also that $0 < \alpha < 2$. Let $x_n \in H$ $(n = 1, 2, \ldots)$ be such that $\|x_n\| = 1$, $\|Sx_n\| \to 1$ and also $\lim_{n\to\infty}\langle Sx_n, x_n\rangle = \beta$. Let $S = GL$ be the polar decomposition of S. Here G is a partial isometry and we write $L = \int_0^1 \beta\, dE_\beta$, the spectral decomposition of $L = (S^*S)^{1/2}$. Since L is a positive operator with norm 1, for any $x \in H$ we have that $\|Lx_n\| \to 1$ as n tends to ∞ and $\lim_{n\to\infty}\langle Sx_n, x_n\rangle = \lim_{n\to\infty}\langle GLx_n, x_n\rangle = \lim_{n\to\infty}\langle Lx_n, G^*x_n\rangle$. Now for $H = \overline{\mathrm{Ran}(L)} \oplus \mathrm{Ker}(L)$, we can choose x_n such that $x_n \in \overline{\mathrm{Ran}(L)}$ for large n. Indeed, let $x_n = x_n^{(1)} \oplus x_n^{(2)}$, $n = 1, 2, \ldots$. Then we have that $Lx_n = Lx_n^{(1)} \oplus Lx_n^{(2)} = Lx_n^{(1)}$ and that $\lim_{n\to\infty}\|x_n^{(1)}\| = 1$, $\lim_{n\to\infty}\|x_n^{(2)}\| = 0$ since $\lim_{n\to\infty}\|Lx_n\| = 1$. Replacing x_n with $\frac{x_n^{(1)}}{\|x_n^{(1)}\|}$, we get $\lim_{n\to\infty}\big\|L\frac{1}{\|x_n^{(1)}\|}x_n^{(1)}\big\| = \lim_{n\to\infty}\big\|S\frac{1}{\|x_n^{(1)}\|}x_n^{(1)}\big\| = 1$, $\lim_{n\to\infty}\big\langle S\frac{1}{\|x_n^{(1)}\|}x_n^{(1)}, \frac{1}{\|x_n^{(1)}\|}x_n^{(1)}\big\rangle = \beta$. Next let $x_n \in \overline{\mathrm{Ran}(L)}$. Since G is a partial isometry from $\overline{\mathrm{Ran}(L)}$ onto $\overline{\mathrm{Ran}(S)}$, we have that $\|Gx_n\| = 1$ and $\lim_{n\to\infty}\langle Lx_n, G^*x_n\rangle = \beta$. Since L is a positive operator, $\|L\| = 1$ and for any $x \in H$, $\langle Lx, x\rangle \le \langle x, x\rangle = \|x\|^2$. Replacing x with $L^{1/2}x$, we get that $\langle L^2x, x\rangle \le \langle Lx, x\rangle$, where $L^{1/2}$ is the positive square root of L. Therefore we have that $\|Lx\|^2 = \langle Lx, Lx\rangle \le \langle Lx, x\rangle$. It is obvious that $\lim_{n\to\infty}\|Lx_n\| = 1$ and that $\|Lx_n\|^2 \le \langle Lx_n, x_n\rangle \le \|Lx_n\|^2 = 1$. Hence, $\lim_{n\to\infty}\langle Lx_n, x_n\rangle = 1 = \|L\|$. Moreover, Since $I - L \ge 0$, we have $\lim_{n\to\infty}\langle (I-L)x_n, x_n\rangle = 0$, thus $\lim_{n\to\infty}\|(I-L)^{1/2}x_n\| = 0$. Indeed, $\lim_{n\to\infty}\|(I-L)x_n\| \le \lim_{n\to\infty}\|(I-L)^{1/2}\| \cdot \|(I-L)^{1/2}x_n\| = 0$. For $\alpha > 0$, let $\gamma = [0, 1-\alpha/2]$ and let $\rho = (1-\alpha/2, 1]$. We have $L = \int_\gamma \mu\, dE_\mu + \int_\rho \mu\, dE_\mu = LE(\gamma) \oplus LE(\rho)$. Next we show that $\lim_{n\to\infty}\|E(\gamma)x_n\| = 0$. If there exists a subsequence x_{n_i}, $(i = 1, 2, \ldots)$ such that $\|E(\gamma)x_{n_i}\| \ge \epsilon > 0$ $(i = 1, 2, \ldots)$, then since $\lim_{i\to\infty}\|x_{n_i} - Lx_{n_i}\| = 0$, it follows that $\lim_{i\to\infty}\|x_{n_i} - Lx_{n_i}\|^2 = \lim_{i\to\infty}\big(\|E(\gamma)x_{n_i} - LE(\gamma)x_{n_i}\|^2 + \|E(\rho)x_{n_i} - LE(\rho)x_{n_i}\|^2\big) = 0$. Hence, we have that $\lim_{i\to\infty}\|E(\gamma)x_{n_i} - LE(\gamma)x_{n_i}\|^2 = 0$. Now, it is clear that $\|E(\gamma)x_{n_i} - LE(\gamma)x_{n_i}\| \ge \|E(\gamma)x_{n_i}\| - \|LE(\gamma)\| \cdot \|E(\gamma)x_{n_i}\| \ge (I - \|LE(\gamma)\|)\|E(\gamma)x_{n_i}\| \ge \frac{\alpha}{2}\epsilon > 0$. This is a contradiction. Therefore, $\lim_{n\to\infty}\|E(\gamma)x_n\| = 0$. Since $\lim_{n\to\infty}\langle Lx_n, x_n\rangle = 1$, we have that $\lim_{n\to\infty}\langle LE(\rho)x_n, E(\rho)x_n\rangle = 1$ and $\lim_{n\to\infty}\langle E(\rho)x_n, G^*E(\rho)x_n\rangle = \beta$. It is easy to see that $\lim_{n\to\infty}\|E(\rho)x_n\| = 1$, $\lim_{n\to\infty}\big\langle L\frac{E(\rho)x_n}{\|E(\rho)x_n\|}, \frac{E(\rho)x_n}{\|E(\rho)x_n\|}\big\rangle = 1$ and $\lim_{n\to\infty}\big\langle L\frac{E(\rho)x_n}{\|E(\rho)x_n\|}, G^*\frac{E(\rho)x_n}{\|E(\rho)x_n\|}\big\rangle = \beta$ Replacing x with $\frac{E(\rho)x_n}{\|E(\rho)x_n\|}$, we can assume that $x_n \in E(\rho)H$ for each n and $\|x_n\| = 1$. Let $J = \int_\gamma \mu\, dE_\mu + \int_\rho \mu\, dE_\mu = J_1 \oplus E(\rho)$. Then it is evident that $\|J\| = \|S\| = \|L\| = 1$, $Jx_n = x_n$, and $\|J - L\| \le \alpha/2$. If we can find a contraction V such that $\|V - G\| \le \alpha/2$ and $\|Vx_n\| = 1$ and $\langle Vx_n, x_n\rangle = \beta$, for a large n then letting $Z = VJ$, we have that $\|Zx_n\| = \|VJx_n\| = 1$, and that $\langle Zx_n, x_n\rangle = \langle VJx_n, x_n\rangle = \langle Vx_n, x_n\rangle = \beta$, $\|S - Z\| = \|GL - VJ\| \le \|GL - GJ\| + \|GJ - VJ\| \le \|G\| \cdot \|L - J\| + \|G - V\| \cdot \|J\| \le \alpha/2 + \alpha/2 = \alpha$. Lastly, we now construct the desired contraction V. Clearly, $\lim_{n\to\infty}\langle x_n, G^*x_n\rangle = \beta$, because $\lim_{n\to\infty}\langle Lx_n, G^*x_n\rangle = \beta$ and $\lim_{n\to\infty}\|x_n - Lx_n\| = 0$. Let $Gx_n = \phi_n x_n + \varphi_n y_n$, $(y_n \perp x_n, \|y_n\| = 1)$ then $\lim_{n\to\infty}\phi_n = \beta$, because $\lim_{n\to\infty}\langle Gx_n, x_n\rangle = \lim_{n\to\infty}\langle x_n, G^*x_n\rangle = \beta$ but $\|Gx_n\|^2 = |\phi_n|^2 + |\varphi_n|^2 = 1$, so

we have that $\lim_{n\to\infty} |\varphi_n| = \sqrt{1-|\beta|^2}$. Now without loss of generality, there exists an integer M such that $|\phi_M - \beta| < \alpha/8$. Choose φ_M^0 such that $|\varphi_M^0| = \sqrt{1-|\beta|^2}$, $|\varphi_M - \varphi_M^0| < \alpha/8$. We have that $Gx_M = \phi_M x_M + \varphi_M y_M - \beta x_M + \beta x_M - \varphi_M^0 y_M + \varphi_M^0 y_M = (\phi - \beta)x_M + (\varphi_M - \varphi_M^0)y_M + \beta x_M + \varphi_M^0 y_M$. Let $q_M = \beta x_M + \varphi_M^0 y_M$, $Gx_M = (\phi - \beta)x_M + (\varphi_M - \varphi_M^0)y_M + q_M$. Suppose that $y \perp x_M$, then $\langle Gx_M, Gy \rangle = (\phi - \beta)\langle x_M, Gy \rangle + (\varphi_M - \varphi_M^0)\langle y_M, Gy \rangle + \langle q_M, Gy \rangle = 0$, because G^*G is a projection from H to $\text{Ran}(L)$. It follows that $|\langle q_M, Gy \rangle| \leq |\phi_M - \beta| \cdot \|y\| + |\varphi_M - \varphi_M^0| \cdot \|y\| \leq \frac{\alpha}{4}\|y\|$. If we suppose that $Gy = \phi q_M + y^0$, $(y^0 \perp q_M)$ then y^0 is uniquely determined by y. Hence we can define V as follows: $V: x_M \to q_M, y \to y^0, \phi x_M + \varphi_M y \to \phi q_M + \varphi_M y^0$, with both ϕ, φ being complex numbers. V is a linear operator. We prove that V is a contraction. Now, $\|Vx_M\|^2 = \|q_M\|^2 = |\beta|^2 = |\varphi_M^0|^2 = 1$, $\|Vy\|^2 = \|Gy\|^2 - |\phi y|^2 \leq \|Gy\|^2 \leq \|y\|^2$. It follows that $\|V\phi\|^2 = \|\phi\|^2\|Vx_M\|^2 + |\varphi|^2\|Vy\|^2 \leq |\phi|^2 + |\varphi|^2 = 1$, for each $x \in H$ satisfying that $x = \phi x_M + \varphi_M y$, $\|x\| = 1$, $x_M \perp y$, which is equivalent to that V is a contraction. From the definition of V, we can show that $\|Gx_M - Vx_M\|^2 = |\phi - \beta|^2 + |\varphi_M - \varphi_M^0|^2 \leq \frac{2\alpha^2}{16} = \frac{1}{8}\alpha^2$. If $y \perp x_M$, $\|y\| \leq 1$ then obtain $\|Gy - Vy\| = |\phi|\|Vx_M\| = |\langle Gy, Vx_M \rangle| = |\langle q_M, Gy \rangle| < \alpha/4$. Hence for any $x \in H$, $x = \phi x_M + \varphi_M y$, $\|x\| = 1$, $\|Gx - Vx\|^2 = \|\phi(G-V)x_M + \varphi(G-V)y\|^2 = |\phi|^2\|(G-V)x_M\|^2 + |\varphi|^2\|(G-V)y\|^2 < |\phi|^2 \cdot \frac{\alpha^2}{16} + |\varphi|^2 \cdot \frac{\alpha^2}{16} < \alpha^2/8$, which implies that $\|(G-V)x\| < \alpha/2$, $\|x\| = 1$, and hence $\|(G-V)\| < \alpha/2$. Let $Z = VJ$. Then Z is what we desire and this completes the proof. $\qquad\square$

The next result gives the conditions for norm-attainability of an inner derivation. We give the following proposition.

Proposition 2.2. *Let H be an infinite dimensional separable complex Hilbert space and $S \in B(H)$. δ_S is norm-attainable if there exists a vector $\zeta \in H$ such that $\|\zeta\| = 1$, $\|S\zeta\| = \|S\|$, $\langle S\zeta, \zeta \rangle = 0$.*

Proof. Define X by setting $X: \zeta \to \zeta$, $S\zeta \to -S\zeta$, $x \to 0$, whenever $x \perp \{\zeta, S\zeta\}$. Since X is a bounded operator on H and $\|X\zeta\| = \|X\| = 1$, $\|SX\zeta - XS\zeta\| = \|S\zeta - (-S\zeta)\| = 2\|S\zeta\| = 2\|S\|$. It follows that $\|\delta_S\| = 2\|S\|$ via the result in [28, Theorem 1], because $\langle S\zeta, \zeta \rangle = 0 \in W_0(S)$. Hence we have that $\|SX - XS\| = 2\|S\| = \|\delta_S\|$. Therefore, δ_S is norm-attainable. $\qquad\square$

The next result gives the conditions for norm-attainability of a generalized derivation. We give the following proposition.

Proposition 2.3. *Let H be an infinite dimensional separable complex Hilbert space. Let $S, T \in B(H)$. If there exist vectors $\zeta, \eta \in H$ such that $\|\zeta\| = \|\eta\| = 1$, $\|S\zeta\| = \|S\|$, $\|T\eta\| = \|T\|$ and $\frac{1}{\|S\|}\langle S\zeta, \zeta \rangle = -\frac{1}{\|T\|}\langle T\eta, \eta \rangle$, then $\delta_{S,T}$ is norm-attainable.*

Proof. By linear dependence of vectors, if η and $T\eta$ are linearly dependent, i.e., $T\eta = \phi\|T\|\eta$, then it is true that $|\phi| = 1$ and $|\langle T\eta, \eta\rangle| = \|T\|$. It follows that $|\langle S\zeta, \zeta\rangle| = \|S\|$ which implies that $S\zeta = \varphi\|S\|\zeta$ and $|\varphi| = 1$. Hence $\langle \frac{S\zeta}{\|S\|}, \zeta\rangle = \varphi = -\langle \frac{T\eta}{\|T\|}, \eta\rangle = -\phi$. Defining X as $X\colon \eta \to \zeta$, $\{\eta\}^\perp \to 0$, we have $\|X\| = 1$ and $(SX - XT)\eta = \varphi(\|S\| + \|T\|)\zeta$, which implies that $\|SX - XT\| = \|(SX - XT)\eta\| = \|S\| + \|T\|$. By [3], it follows that $\|SX - XT\| = \|S\| + \|T\| = \|\delta_{S,T}\|$. That is $\delta_{S,T}$ is norm-attainable. If η and $T\eta$ are linearly independent, then $|\langle \frac{T\eta}{\|T\|}, \eta\rangle| < 1$, which implies that $|\langle \frac{S\zeta}{\|S\|}, \zeta\rangle| < 1$. Hence ζ and $S\zeta$ are also linearly independent. Let us redefine X as follows: $X\colon \eta \to \zeta$, $\frac{T\eta}{\|T\|} \to -\frac{S\zeta}{\|S\|}$, $x \to 0$, where $x \in \{\eta, T\eta\}^\perp$. We show that X is a partial isometry. Let $\frac{T\eta}{\|T\|} = \langle \frac{T\eta}{\|T\|}, \eta\rangle\eta + \tau h$, $\|h\| = 1$, $h \perp \eta$. Since η and $T\eta$ are linearly independent, $\tau \neq 0$. So we have that $X\frac{T\eta}{\|T\|} = \langle \frac{T\eta}{\|T\|}, \eta\rangle X\eta + \tau Xh = -\langle \frac{S\zeta}{\|S\|}, \zeta\rangle\zeta + \tau Xh$, which implies that $\langle X\frac{T\eta}{\|T\|}, \zeta\rangle = -\langle \frac{S\zeta}{\|S\|}, \zeta\rangle + \tau\langle Xh, \zeta\rangle = -\langle \frac{S\zeta}{\|S\|}, \zeta\rangle$.

It follows that $\langle Xh, \zeta\rangle = 0$, i.e., $Xh \perp \zeta$ ($\zeta = X\eta$). Hence we have that $\left\|\langle \frac{S\zeta}{\|S\|}, \zeta\rangle\zeta\right\|^2 + \|\tau Xh\|^2 = \left\|X\frac{T\eta}{\|T\|}\right\|^2 = \left|\langle \frac{T\eta}{\|T\|}, \eta\rangle\right|^2 + |\tau|^2 = 1$, which implies that $\|Xh\| = 1$. Now it is evident that X a partial isometry and $\|(SX - XT)\zeta\| = \|SX - XT\| = \|S\| + \|T\|$, which is equivalent to $\|\delta_{S,T}(X)\| = \|S\| + \|T\|$. By Proposition 2.2 and [31], $\|\delta_{S,T}\| = \|S\| + \|T\|$. Hence $\delta_{S,T}$ is norm-attainable. $\square$

The next result is a consequence of Propositions 2.2 and 2.3. It gives the necessary and sufficient conditions for norm-attainability of a basic elementary operator.

Corollary 2.4. *Let $S, T \in B(H)$. If both S and T are norm-attainable then the basic elementary operator $M_{S,T}$ is also norm-attainable.*

Proof. For any pair (S, T) it is known that $\|M_{S,T}\| = \|S\|\|T\|$. We can assume that $\|S\| = \|T\| = 1$. If both S and T are norm-attainable, then there exist unit vectors ζ and η with $\|S\zeta\| = \|T\eta\| = 1$. We can therefore define an operator X by $X = \langle \cdot, T\eta\rangle\zeta$. Clearly, $\|X\| = 1$. Therefore, we have $\|SXT\| \geq \|SXT\eta\| = \|\|T\eta\|^2 S\zeta\| = 1$. Hence, $\|M_{S,T}(X)\| = \|SXT\| = 1$, that is $M_{S,T}$ is also norm-attainable. $\square$

In the next section, we dedicate our work to orthogonality of elementary operators on norm-attainable classes. From this point henceforth, all the elementary operators are implemented by norm-attainable operators unless otherwise stated. First we note that Ω denotes the algebra of all norm-attainable operators. In fact Ω is a Banach algebra. Let $T\colon \Omega \to \Omega$ be defined by $T(X) = \sum_{i=1}^n A_i X B_i$, $\forall X \in \Omega$, where A_i, B_i are fixed in Ω. We define the range of T by $\mathrm{Ran}(T) = \{Y \in \Omega : Y = T(X), \forall X \in \Omega\}$, and the Kernel of T by $\mathrm{Ker}(T) = \{X \in \Omega : T(X) = 0, \forall X \in \Omega\}$. It is known [29] that for any of the examples of the elementary operators defined in Section 1 (inner derivation, generalized derivation, basic elementary operator, Jordan elementary operator), the following implications hold for

a general bounded linear operator T on a normed linear space W, i.e., $\mathrm{Ran}(T) \perp \mathrm{Ker}(T) \Rightarrow$ $\overline{\mathrm{Ran}(T)} \cap \mathrm{Ker}(T) = \{0\} \Rightarrow \mathrm{Ran}(T) \cap \mathrm{Ker}(T) = \{0\}$. Here $\overline{\mathrm{Ran}(T)}$ denotes the closure of the range of T and $\mathrm{Ker}(T)$ denotes the kernel of T and $\mathrm{Ran}(T) \perp \mathrm{Ker}(T)$ means $\mathrm{Ran}(T)$ is orthogonal to the Kernel of T in the sense of Birkhoff. Let $A \in \Omega$. The algebraic numerical range $V(A)$ of A is defined by $V(A) = \{f(A) : f \in \Omega' \text{ and } \|f\| = f(I) = 1\}$ where Ω' is the dual space of Ω and I is the identity element in Ω. If $V(A) \subseteq \mathbb{R}$, then A is called a Hermitian element. Given two Hermitian elements S and R such that $SR = RS$, then $D = S + Ri$ is called normal [32].

3. Main results

Proposition 3.1. *Let $A, B, C \in \Omega$ with $CB = I$ (I is an identity element of Ω). Then for a generalized derivation $\delta_{A,B} = AX - XB$ and an elementary operator $\Theta_{A,B}(X) = AXB - X$, $R_B(\overline{\mathrm{Ran}(\delta_{A,C})} \cap \mathrm{Ker}(\delta_{A,C})) = \overline{\mathrm{Ran}(\Theta_{A,B})} \cap \mathrm{Ker}(\Theta_{A,B})$. Moreover, if $\overline{\mathrm{Ran}(\delta_{A,C})} \cap \mathrm{Ker}(\delta_{A,C}) = \{0\}$ then $\overline{\mathrm{Ran}(\Theta_{A,B})} \cap \mathrm{Ker}(\Theta_{A,B}) = \{0\}$.*

Proof. First, we prove that if $CB = I$ then $R_B \delta_{A,C} = \Theta_{A,B}$. To see this, $\forall X \in \Omega$, $R_B \delta_{A,C}(X) = AXB - XCB = AXB - X = \Theta_{A,B}$. Suppose that $P \in R_B(\overline{\mathrm{Ran}(\delta_{A,C})} \cap \mathrm{Ker}(\delta_{A,C}))$. Now, it is a fact that the uniform norm assigns to real- or complex-valued continuous bounded operator R_B defined on any set Ω the nonnegative number $\|R_B\|_\infty = \sup\{\|R_B(X)\| : X \in \Omega\}$. Since $R_B \delta_{A,C} = \Theta_{A,B}$ and R_B is continuous for the uniform norm, then $P \in \overline{\mathrm{Ran}(\Theta_{A,B})} \cap \mathrm{Ker}(\Theta_{A,B})$. Conversely, since R_C is continuous for the uniform norm, then by the same argument we prove that if $P \in R_B(\overline{\mathrm{Ran}(\Theta_{A,B})} \cap \mathrm{Ker}(\Theta_{A,B}))$ then $P \in R_B(\overline{\mathrm{Ran}(\delta_{A,C})} \cap \mathrm{Ker}(\delta_{A,C}))$. $\square$

It is important to note the following. Let $A, B, C \in \Omega$ with $CB = I$ (I is an identity element of Ω). Then $R_B(\overline{\mathrm{Ran}(\delta_{A,C})} \cap \mathrm{Ker}(\delta_{A,C})) = \mathrm{Ran}(\Theta_{A,B}) \cap \mathrm{Ker}(\Theta_{A,B})$. Indeed, since $\mathrm{Ran}(\Theta_{A,B}) \subseteq \overline{\mathrm{Ran}(\Theta_{A,B})}$, then by Proposition 3.1, the equality holds.

Proposition 3.2. *Let S and R be norm-attainable Hermitian elements. Then $\delta_{S,R}$ is also norm-attainable Hermitian.*

Proof. From [23], it is known that if X is a norm-attainable class then $V(\delta_{S,R}) = V(S) - V(R)$ for all $S, R \in B(X)$. Therefore, $V(\delta_{S,R}) \subseteq V(L_S) - V(L_R) = V(S) - V(R) \subseteq \mathbb{R}$. $\square$

Corollary 3.3. *If D and E are norm-attainable and normal elements in Ω then $\delta_{D,E}$ is also norm-attainable and normal.*

Proof. Assume $D = S + Ri$ and $E = T + Ui$ where S, R, T, U are Hermitian elements in Ω such that $SR = RS$ and $TU = UT$. Then $\delta_{D,E} = \delta_{S,T} + i\delta_{R,U}$ with $\delta_{S,T}\delta_{R,U} = \delta_{R,U}\delta_{S,T}$. Since S, R, T, U are Hermitian, then by Proposition 3.2 $\delta_{R,U}$ and $\delta_{S,T}$ are Hermitian and so is $\delta_{D,E}$. $\square$

Remark 3.4. [23] Let X be a norm-attainable class and $T \in B(X)$. If T is a norm-attainable normal operator, then $\text{Ran}(T) \perp \text{Ker}(T)$. Moreover, if D and E are norm-attainable normal elements in Ω then $\text{Ran}(\delta_{D,E}) \perp \text{Ker}(\delta_{D,E})$. Indeed, assume that D and E are norm-attainable normal elements in Ω. Then by Corollary 3.3, $\delta_{D,E}$ is norm-attainable normal and by Proposition 3.2 $\text{Ran}(\delta_{D,E}) \perp \text{Ker}(\delta_{D,E})$.

Corollary 3.5. *If $A, B \in \Omega$ are norm-attainable normal and there exists $C \in \Omega$ such that $BC = I$ then $\overline{\text{Ran}(\Theta_{A,C})} \cap \text{Ker}(\Theta_{A,C}) = \{0\}$.*

Proof. If $A, B \in \Omega$ are norm-attainable normal and self-adjoint elements, then by Corollary 3.3, $\text{Ran}(\delta_{A,B}) \perp \text{Ker}(\delta_{A,B})$. This implies that $\overline{\text{Ran}(\delta_{A,B})} \cap \text{Ker}(\delta_{A,B}) = \{0\}$. Using Proposition 2.2, we conclude that $\overline{\text{Ran}(\Theta_{A,C})} \cap \text{Ker}(\Theta_{A,C}) = \{0\}$. $\qquad\square$

The next theorem gives a stronger result on power sequences of operators $A^n, B^n \in \Omega$ for all $n \in \mathbb{N}$.

Theorem 3.6. *Let $A, B \in \Omega$ be norm-attainable normal and self-adjoint with $C \in \Omega$ such that $BC = I$ and $\|C\| \leq 1$. If $\|A\| \leq 1$ and $\|B\| \leq 1$ for all $n \in \mathbb{N}$ then $\text{Ran}(\delta_{A,B}) \perp \text{Ker}(\delta_{A,B})$.*

Proof. It is well known [1] that $A^n X - X B^n = \sum_{i=0}^{n-1} A^{n-i-1}(AX - XB)B^i$ and

$$A^n X - X B^n - \sum_{i=0}^{n-1} A^{n-i-1}(AX - XB - Y)B^i = nYB^{n-1},$$

where $Y \in \text{Ker}(\delta_{A,B})$. Multiplying this equality by C^{n-1} we obtain

$$A^n X C^{n-1} - XB - \sum_{i=0}^{n-1} A^{n-i-1}(AX - XB - Y)B^i C^{n-1} = nYB^{n-1}C^{n-1}$$

which is equivalent to

$$nY = A^n X C^{n-1} - XB - \sum_{i=0}^{n-1} A^{n-i-1}(AX - XB - Y)B^i C^{n-1}.$$

Now, the assumption that $BC = I$ with $\|C\| \leq 1$ and $\|B\| \leq 1$ implies that $\|C^n\| = \|B^n\| = 1$ for all $n \in \mathbb{N}$. This shows that dividing both sides by n and taking norms we obtain

$$\|Y\| \leq \frac{1}{n}\{\|A^n\|\|X\|\|C\|^{n-1} + \|X\|\|B\|\}$$
$$+ \frac{1}{n}\sum_{i=0}^{n-1} \|A\|^{n-i-1}\|AX - XB - Y\|\|B\|^i\|C\|^{n-1}$$
$$= \frac{1}{n}\{\|A^n\|\|X\| + \|X\|\} + \frac{1}{n}\sum_{i=0}^{n-1}\|A\|^{n-i-1}\|AX - XB - Y\|.$$

Hence, $\|Y\| \leq \frac{2}{n}\|X\| + \frac{1}{n}\sum_{i=0}^{n-1}\|AX - XB - Y\|$. Taking limits as $n \to \infty$, we obtain that $\|Y\| \leq \|AX - XB - Y\|$. Therefore, $\mathrm{Ran}(\delta_{A,B}) \perp \mathrm{Ker}(\delta_{A,B})$. $\qquad\square$

The following theorem from Kittaneh [21] gives a general orthogonality condition for linear operators. The proof is omitted.

Theorem 3.7. [21] *Let Ω be a normed algebra with the norm $\|\cdot\|$ satisfying $\|XY\| \leq \|X\|\|Y\|$ for all $X, Y \in \Omega$ and let $\delta\colon \Omega \to \Omega$ be a linear map with $\|\delta\| \leq 1$. If $\delta(Y) = Y$ for some $Y \in \Omega$, then $\|\delta(X) - X + Y\| \geq \|Y\|$ for all $X \in \Omega$.*

We utilize the Theorem 3.7 to prove some results for general elementary operators. Let $T\colon \Omega \to \Omega$ be an elementary operator defined by $T(X) = \sum_{i=1}^{n} A_i X B_i$, $\forall X \in \Omega$. Now suppose that $T(Y) = Y$ for some $Y \in \Omega$. If $\|T\| \leq 1$, then $\|T(X) - X + Y\| \geq \|Y\|$ for all $X \in \Omega$. The following theorem follows immediately.

Theorem 3.8. *Suppose that $T(Y) = Y$ for some norm-attainable normal self-adjoint $Y \in \Omega$. If $\left\|\sum_{i=1}^{n} A_i A_i^*\right\|^{1/2}\left\|\sum_{i=1}^{n} B_i^* B_i\right\|^{1/2} \leq 1$, then $\|T(X) - X + Y\| \geq \|Y\|$ for all $X \in \Omega$.*

Proof. We only need to show that $\|T\| \leq 1$. Let $Z_1 = [A_1, \ldots, A_n]$ and $Z_2 = [B_1, \ldots, B_n]^T$. Taking $Z_1 Z_1^*$ and $Z_2^* Z_2$ shows that $\|Z_1\| = \left\|\sum_{i=1}^{n} A_i A_i^*\right\|^{1/2}$ and $\|Z_2\| = \left\|\sum_{i=1}^{n} B_i^* B_i\right\|^{1/2}$. From [10], it is known that $T(X) = Z_1(X \otimes I_n)Z_2$, where I_n is the identity of $M_n(\mathbb{C})$. Therefore it follows that $\|T(X)\| \leq \|Z_1\|\|Z_2\|\|X\|$. Hence $\|T\| \leq 1$. $\qquad\square$

Now, we consider the orthogonality of Jordan elementary operators. We later consider the necessary and sufficient conditions for their normality. At this juncture a type of norm, called the unitarily invariant norm comes in handy. A unitarily invariant norm is any norm defined on some two-sided ideal of $NA(H)$ and $NA(H)$ itself which satisfies the following two conditions. For unitary operators $U, V \in NA(H)$ the equality $\|\|UXV\|\| = \|\|X\|\|$ holds, and $\|\|X\|\| = s_1(X)$ for all rank one operators X. It is proved that any unitarily invariant norm depends only on the sequence of singular values. Also, it is known that the maximal ideal, on which $\|\|UXV\|\|$ has sense, is a Banach space with respect to that unitarily invariant norm. Among all unitarily invariant norms there are few important special cases. The first is the Schatten p-norm ($p \geq 1$) defined by $\|X\|_p = \left(\sum_{j=1}^{+\infty} s_j(X)^p\right)^{1/p}$ on the set $\mathcal{C}_p = \{X \in NA(H) : \|X\|_p < +\infty\}$. For $p = 1, 2$ this norm is known as the nuclear norm (Hilbert-Schmidt norm) and the corresponding ideal is known as the ideal of nuclear (Hilbert-Schmidt) operators. The ideal $\mathcal{C}_2$ is also interesting for another reason. Namely, it is a Hilbert space with respect to the $\|\cdot\|_2$ norm. The other important special case is the set of so-called Ky Fan norms $\|X\|_k = \sum_{j=1}^{k} s_j(X)$. The well-known Ky Fan dominance property asserts that the condition $\|X\|_k \leq \|Y\|$ for all $k \geq 1$ is necessary and sufficient for the validity of the inequality $\|\|X\|\| \leq \|\|Y\|\|$ in all

unitarily invariant norms. For further details refer to [19].We state the following theorem from [19] on orthogonality.

Theorem 3.9. *Let $A, B \in NA(H)$ be normal operators such that $AB = BA$, and let $\mathcal{U}(X) = AXB - BXA$. Furthermore, suppose that $A^*A + B^*B > 0$. If $S \in \mathrm{Ker}(\mathcal{U})$, then $|||\mathcal{U}(X) + S||| \geq |||S|||$.*

We extend Theorem 3.9 to distinct operators $A, B, C, D \in NA(H)$ in the theorem below.

Theorem 3.10. *Let $A, B, C, D \in NA(H)$ be normal operators such that $AC = CA$, $BD = DB$, $AA^* \leq CC^*$, $B^*B \leq D^*D$. For an elementary operator $\mathcal{U}(X) = AXB - CXD$ and $S \in NA(H)$ satisfying $ASB = CSD$, then $\|\mathcal{U}(X) + S\| \geq \|S\|$ for all $X \in NA(H)$.*

Proof. From $AA^* \leq CC^*$ and $B^*B \leq D^*D$, let $A = CU$, and $B = VD$, where U, V are contractions. So we have $AXB - CXD = CUXVD - CXD = C(UXV - X)D$. Assume C and D^* are injective, $ASB = CSD$ if and only if $USV = S$. Moreover, C and U commute. Indeed from $A = CU$ we obtain $AC = CUC$. Therefore, $C(A - UC) = 0$. Thus since C is injective $A = CU$. Similarly, D and V commute. So, $\|\mathcal{U}(X) + S\| = \|[AXB - CXD] + S\| = \|[U(CXD)V - CXD] + S\| \geq \|S\|, \forall X \in NA(H)$. Now, under the condition of Theorem 3.10, A and C have operator matrices $A = \begin{pmatrix} 0 & 0 \\ 0 & A_0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 \\ 0 & C_0 \end{pmatrix}$ with respect to the space decomposition $H = \overline{\mathcal{R}(C)} \oplus \mathcal{N}(C)$, respectively. Here, A_0 is a normal operator on $\overline{\mathcal{R}(C)}$ and C_0 is an injective and normal operator on $\overline{\mathcal{R}(C)}$. B and D have operator matrices $B = \begin{pmatrix} B_0 & 0 \\ 0 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} D_0 & 0 \\ 0 & 0 \end{pmatrix}$ with respect to the space decomposition $H = \overline{\mathcal{R}(D)} \oplus \mathcal{N}(D)$, respectively. Here, B_0 is a normal operator on $\overline{\mathcal{R}(D)}$ and D_0 is an injective and normal operator on $\overline{\mathcal{R}(D)}$. X and S have operator matrices $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ and $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$ which are as operator from the space decomposition $H = \overline{\mathcal{R}(D)} \oplus \mathcal{N}(D)$ into the space decomposition $H = \overline{\mathcal{R}(C)} \oplus \mathcal{N}(C)$, respectively.

In this case, $\mathcal{U}(X) = AXB - CXD = \begin{pmatrix} A_0 X_{11} B_0 - C_0 X_{11} D_0 & 0 \\ 0 & 0 \end{pmatrix}$ and $A_0 S_{11} B_0 - C_0 S_{11} D_0 = 0$. Therefore, $\|A_0 X_{11} B_0 - C_0 X_{11} D_0 + S_{11}\| \geq \|S_{11}\|$. Hence,

$$
\begin{aligned}
\|\mathcal{U}(X) + S\| &= \left\| \begin{pmatrix} A_0 X_{11} B_0 - C_0 X_{11} D_0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} A_0 X_{11} B_0 - C_0 X_{11} D_0 + S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \right\| \\
&\geq \left\| \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \right\|. \qquad \square
\end{aligned}
$$

The result in Theorem 3.10 can be generalized in Banach algebras and other complex spaces like operator spaces and function spaces.

4. Conclusion

We conclude this paper by remarking that these results can be extended to give more results on generalized finite operators in terms of orthogonality and norm-attainability in C^*-algebras.

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