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Nonseparating Independent Sets of Cartesian Product Graphs

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Abstract. A set of vertices S of a connected graph G is a nonseparating independent set if S is independent and $G-S$ is connected. The nsis number $\mathcal{Z}(G)$ is the maximum cardinality of a nonseparating independent set of G. It is well known that computing the nsis number of graphs is NP-hard even when restricted to 4-regular graphs. In this paper, we first present a new sufficient and necessary condition to describe the nsis number. Then, we completely solve the problem of counting the nsis number of hypercubes Q_n and Cartesian product of two cycles $C_m \Box C_n$, respectively. We show that $\mathcal{Z}(Q_n) = 2^{n-2}$ for $n \geq 2$, and $\mathcal{Z}(C_m \Box C_n) = n + \lfloor (n+2)/4 \rfloor$ if $m = 4$, $m + \lfloor (m+2)/4 \rfloor$ if $n = 4$ and $\lfloor mn/3 \rfloor$ otherwise. Moreover, we find a maximum nonseparating independent set of Q_n and $C_m \Box C_n$, respectively.

1. Introduction

Graphs considered in this paper are connected and simple. Throughout the paper, the letter G denotes a graph, and the cycle with n vertices is denoted by C_n. For $W \subseteq V(G)$, by $G - W$ and $G[W]$ we mean the subgraphs induced by $V(G) - W$ and W, respectively.

It is expected that the reader is somewhat familiar with topological graph theory. For general background, see Gross and Tucker [4], or Mohar and Thomassen [8].

An *independent set* of a graph is a set of vertices in which no two of them are adjacent. A maximum independent set is an independent set of largest possible size for a given graph. This size is called the *independence number* of G and denoted $\alpha(G)$. We say that a set $S \subseteq V(G)$ is a *nonseparating independent set* (or *nsis* in short) of a graph G if S is independent and $G-S$ is connected. The maximum cardinality of a nsis of G is called the *nsis number* of G and is denoted by $\mathcal{Z}(G)$. Furthermore, we call a nsis containing exactly $\mathcal{Z}(G)$ vertices a $\mathcal{Z}$-set. Finding a $\mathcal{Z}$-set of graphs is called the nsis problem.

A set $S \subseteq V(G)$ is a *vertex cover* of G if for every edge uv of $E(G)$, $u \subseteq S$ or $v \subseteq S$. The *connected vertex cover* (or *cvc* in brief) problem is the variation of the vertex cover

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problem, where given a graph G, we find a vertex cover $S \subseteq V(G)$ of minimum cardinality such that the induced subgraph $G[S]$ is connected. In fact, the cvc problem is a classic problem in combinatorial optimization and operation research having many important applications in many fields. For example, in the field of wireless network design [9], the vertices and the edges represent the network nodes and transmission links, respectively. Some relay stations will be placed on some network nodes such that they form a connected subnetwork and every transmission link is incident to a relay station. People want to minimize the number of relay stations. This is exactly the cvc problem. In theory, the nsis problem is closely related with cvc problem. One may easily observe that S is a cvc if and only if $V(G) - S$ is a nsis.

In recent years, the nsis problem has been intensively studied from the algorithmic perspective because of their extensive applications. Garey and Johnson [3] had shown that this problem is NP-hard even for planar graphs with no degree exceeding 4. In particular, counting the nsis number of a 4-regular graph is also very hard [6]. Since then, researchers proved that it is also NP-complete in planar bipartite graphs of maximum degree 4 [2] and 3-connected graphs [12]. On the other hand, Ueno et al. [11] proved that this problem can be solved in polynomial time for graphs with no vertex degree exceeding 3. In addition, Escoffier et al. [1] showed that this problem is polynomial-time solvable in chordal graphs. From the literature, one may see that researchers only went an initial step towards the research of nsis problem. In fact, determining the nsis number of many certain graphs has been little studied. There is still much revolutionary work to do in the future.

In this paper, we shall determine the nsis numbers of hypercubes and Cartesian products of two cycles. As we will see in Sections 3 and 4, a maximum nsis of hypercubes and Cartesian products of two cycles is constructed, respectively. Therefore, a minimum cvc follows.

We first introduce a sufficient and necessary condition which is viewed to be a new way to describe the nsis number. Let T be a spanning tree of a graph G, we denote by $\alpha_1(T)$ the independence number of the subgraph induced by its leaves (i.e., those of degree 1) of T. Then, we have the following result.

1. $\mathcal{Z}(G) = \max_T\{\alpha_1(T) : T \text{ is a spanning tree of } G\}$, where the "max" is taken among all the possible spanning trees in G.

A spanning tree T attaining the "max" is called an *optimal tree* of G. It is easy to see that finding optimal trees for general graphs is NP-hard. Computationally, this implies that determining the nsis number is a very hard problem for general graphs. However, it may work well for some types of graphs such as cubic graphs and hypercubes Q_n. Based on the result above, we deduce that

2. $\mathcal{Z}(Q_n) = 2^{n-2}$ for $n \geq 2$.

Finally, by construction of a $\mathcal{Z}$-set of the Cartesian product of two cycles $C_m \square C_n$. We obtain that

3.

$$\mathcal{Z}(C_m \square C_n) = \begin{cases} n + \lfloor (n+2)/4 \rfloor & \text{if } m = 4, \\ m + \lfloor (m+2)/4 \rfloor & \text{if } n = 4, \\ \lfloor mn/3 \rfloor & \text{otherwise.} \end{cases}$$

2. Sufficient and necessary condition

In this section, we shall establish a new description for the nsis number of general graphs, as the following theorem shows.

Theorem 2.1. *For any graph G,*

$$\mathcal{Z}(G) = \max_T \{\alpha_1(T) : T \textit{ is a spanning tree of } G\}.$$

Proof. Let S be a $\mathcal{Z}$-set of G. Then $G - S$ is connected. Therefore, there is a spanning tree T_s of $G - S$. Since G is connected and S is independent, every vertex of S has a neighbor in T_s. Thus, we can construct a spanning tree of T' such that each vertex of S is a leaf of T'. It follows that

$$\mathcal{Z}(G) \leq \alpha_1(T') \leq \max_T \alpha_1(T).$$

Now we prove the converse inequality. Select an arbitrary spanning tree T_0 of G. Suppose that S_0 is a maximum independent set of the subgraph induced by leaves of T_0. One may easily verify that S_0 is a nsis of G. Thus, $\mathcal{Z}(G) \geq \alpha_1(T_0)$. Based on the arbitrariness of T_0, we conclude that

$$\mathcal{Z}(G) \geq \max_T \alpha_1(T).$$

This finishes the proof. □

Theorem 2.1 reveals a new relation between the nsis number and spanning trees. In other words, finding a spanning tree T of G such that $\alpha_1(T)$ achieves its maximum is crucial in computing the nsis number of G.

It is possible to find an optimal tree T (i.e., $\alpha_1(T) = \mathcal{Z}(G)$) for some types of graphs such as cubic graphs. Here, we have to introduce some notations and results about topological graphs.

A *surface* is a compact connected 2-dimensional manifold without boundary. Surfaces are partitioned into two classes: *orientable surfaces* and *nonorientable surfaces.* The orientable surface S_g can be obtained from the sphere with $2g$ pairwise disjoint holes attached with g tubes such that each tube welds two holes. The nonorientable surface N_k ($k \geq 1$) can be obtained from the sphere with k pairwise disjoint discs replaced by k Möbius bands. Recall that g and k are called the *genus* of S_g and N_k, respectively. A graph is said to be *embeddable* on a surface if it can be drawn on that surface in such a way that no two edges cross. Such a drawing is called an *embedding.* An embedding Π of G in a surface S is called a 2-*cell embedding* if each component of $S - \Pi$ is homeomorphic to an open disc. The *maximum genus* $\gamma_M(G)$ of G is defined to be the maximum integer k such that there exists a cellular embedding of G into an orientable surface of genus k.

Given a spanning tree T of a graph G, the subgraph $G - E(T)$ is called a *co-tree* of G. A component of a co-tree $G - E(T)$ is called *odd* if it contains odd number of edges. We use $w(T;G)$ to denote the number of odd components of $G - E(T)$. The *Betti deficiency* $\xi(G)$ is defined to be the minimum $w(T;G)$ over all spanning trees. A spanning tree T of G such that $w(T;G) = \xi(G)$ is said to be a *Xuong-tree* of G. The following results shows a relation between spanning tree and maximum genus.

(1) To compute the maximum genus of graphs, Xuong [13] gave the following edge-partition of co-trees. Let G be a connected graph with a Xuong-tree T_X. Then there exists an edge-partition of $G - E(T_X)$ as follows:

$$E(G) - E(T_X) = \{e_1, e_2\} \cup \{e_3, e_4\} \cup \cdots \cup \{e_{2m-1}, e_{2m}\} \cup \{f_1, f_2, \ldots, f_s\},$$

where (a) $m = \gamma_M(G)$, $s = \xi(G)$; (b) for any i with $1 \leq i \leq m$, $e_{2i-1} \cap e_{2i} \neq \emptyset$ and $\{f_1, f_2, \ldots, f_s\}$ is a matching of G.

An edge-partition of K_4 is shown in Figure 2.1.

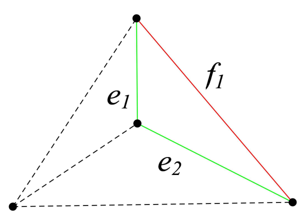

Figure 2.1: An edge-partition of K_4.

(2) Huang and Liu [5], and Ren and Long [7], respectively, proved that $\mathcal{Z}(G) = \gamma_M(G)$ holds for each cubic graph G.

Let T_X be a Xuong-tree of a cubic graph G with edge-partition as described in (1). Then the vertex set $\{u_i : u_i \in e_{2i-1} \cap e_{2i}, 1 \leq i \leq \gamma_M(G)\}$ is an independent set of G. Furthermore, for every i with $1 \leq i \leq \gamma_M(G)$, u_i is a leaf of T_X. Thereby, $S = \{u_1, u_2, \ldots, u_{\gamma_M(G)}\}$ is a nsis of G. Together with (2), these imply that S is a $\mathcal{Z}$-set. It follows that the Xuong-tree T_X is an optimal tree of G.

It follows from the above statement that, for a cubic graph G, computing its maximum genus, computing its nsis number $\mathcal{Z}(G)$ and fining a Xuong-tree are mutually equivalent. In Section 3, we will show spanning trees like Xuong-trees also play an important role in solving the nsis problem of hypercubes.

3. Hypercubes

In this section, we shall solve the nsis problem of hypercubes. Before proving our theorems, we need to introduce some basic terminologies and notations.

The *Cartesian product* $G\Box H$ of two disjoint graphs G and H is the graph with the vertex set $V(G)\times V(H)$ and for which $(x,u)(y,v)$ is an edge if $x=y$ and $uv\in E(H)$, or $xy\in E(G)$ and $u=v$. The hypercube, denoted by Q_n, of dimension n (≥ 1) is a graph obtained by taking Cartesian product of the complete graph K_2 with itself n times; that is, $Q_n = K_2\Box K_2\cdots\Box K_2$ (n times) (see Figure 3.1 for instance). Apparently, $Q_n = K_2\Box Q_{n-1}$ and Q_n is an n-regular, n-connected, bipartite graph with 2^n vertices. It is one of the most popular interconnection network topologies.

Figure 3.1: Q_2 and Q_3.

Before stating our result, we should calculate the independence number $\alpha(Q_n)$.

Lemma 3.1. $\alpha(Q_n) = 2^{n-1}$ *for* $n\geq 1$.

Proof. Since every hypercube is bipartite, $\alpha(Q_n)\geq |V(Q_n)|/2$. That is

$$\alpha(Q_n)\geq 2^n/2 = 2^{n-1}. \tag{3.1}$$

We prove the converse inequality by induction on n. There is nothing to prove for $n\leq 2$. Suppose that $n\geq 3$. Recall that Q_n is obtained from two copies of Q_{n-1}, say Q^1_{n-1}, Q^2_{n-1}. Let S be an independent set of Q_n. Then the sets $S\cap V(Q^1_{n-1})$ and $S\cap V(Q^2_{n-1})$ are independent sets of Q^1_{n-1} and Q^2_{n-1}, respectively. It follows that $|S\cap V(Q^1_{n-1})|\leq \alpha(Q^1_{n-1})$ and $|S\cap V(Q^2_{n-1})|\leq \alpha(Q^2_{n-1})$. By the induction hypothesis,

$$|S| = |S\cap V(Q^1_{n-1})| + |S\cap V(Q^2_{n-1})| \leq 2\cdot\alpha(Q_{n-1}) \leq 2\cdot 2^{n-2} = 2^{n-1}. \tag{3.2}$$

Using (3.1) and (3.2), we get that $|S| = \alpha(Q_n) = 2^{n-1}$. □

We now put all of the above together to count the value of $\mathcal{Z}(Q_n)$.

Theorem 3.2. *$\mathcal{Z}(Q_n) = \alpha(Q_{n-1}) = 2^{n-2}$ for $n \geq 2$.*

Proof. We use Q^1_{n-1} and Q^2_{n-1} to denote the two copies of Q_{n-1} which constitute Q_n. Let T_{n-1} be a spanning tree of Q^1_{n-1}. Then, we get a spanning tree T_n of Q_n by adding the edges between the corresponding vertices in Q^1_{n-1} and Q^2_{n-1} (see Figure 3.2 for Q_4 and T_4). Note that the leaves of T_n consist of the vertices of Q^1_{n-1}. Using Theorem 2.1, one may see that

$$\mathcal{Z}(Q_n) \geq \alpha_1(T) = \alpha(Q_{n-1}).$$

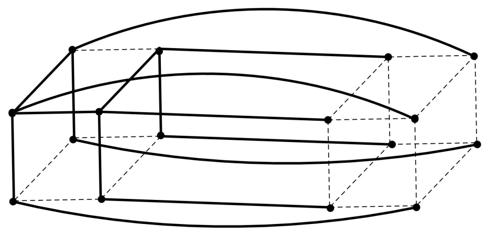

Figure 3.2: Q_4 and T_4.

To prove the converse inequality, we use induction n. The inequality is true for $n = 2$. So, assume that $n \geq 3$ and S a $\mathcal{Z}$-set of Q_n. Suppose that $S = A_1 \cup C_2$, where $A_1 \subseteq V(Q^1_{n-1})$ and $C_2 \subseteq V(Q^2_{n-1})$. Denote by C_1 the copy of C_2 in $V(Q^1_{n-1})$. Then $V(Q^1_{n-1})$ is divided into three parts A_1, B_1, C_1, in other words, $V(Q^1_{n-1}) = A_1 \cup B_1 \cup C_1$, where $B_1 = V(Q^1_{n-1}) - (A_1 \cup C_1)$. Analogously, $V(Q^2_{n-1}) = A_2 \cup B_2 \cup C_2$ (for an intuitive perception, see Figure 3.3).

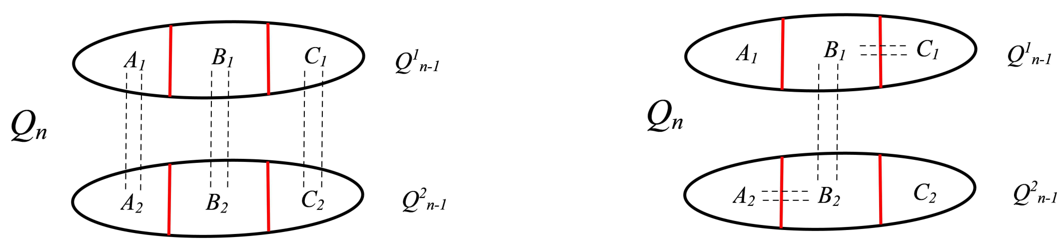

Figure 3.3: A partition of Q_n.

We claim that A_1 is a nsis of Q^1_{n-1}. To see its validity, it suffices to prove that $Q^1_{n-1}[B_1 \cup C_1]$ is a connected subgraph of Q^1_{n-1}. Since $Q_n[B_1 \cup C_1 \cup A_2 \cup B_2]$ is connected, and edges between $Q^1_{n-1}[B_1 \cup C_1]$ and $Q^2_{n-1}[A_2 \cup B_2]$ are those joining B_1 and B_2, $Q^1_{n-1}[B_1 \cup C_1]$ is connected. Similarly, C_2 is a nsis of Q^2_{n-1}. This means that

$$\mathcal{Z}(Q_n) = |S| = |A_1| + |C_2| \leq \mathcal{Z}(Q^1_{n-1}) + \mathcal{Z}(Q^2_{n-1}) = 2 \cdot \mathcal{Z}(Q_{n-1}).$$

By the induction hypothesis,

$$\mathcal{Z}(Q_n) \leq 2 \cdot \mathcal{Z}(Q_{n-1}) \leq 2 \cdot \alpha(Q_{n-2}).$$

Using Lemma 3.1, we derive that $\mathcal{Z}(Q_n) \leq \alpha(Q_{n-1})$. The proof is completed. □

Remark 3.3. By virtue of the proof of Theorem 3.2, we obtain that every maximum independent set of Q_{n-1} is a maximum nsis of Q_n. Note that Q_{n-1} is balanced bipartite, which together with Lemma 3.1 implies that each part of the bipartition of Q_{n-1} is a maximum independent set of Q_{n-1}, as well as a maximum nsis of Q_n.

Recalling Theorem 2.1, there exists a spanning tree T of Q_n such that $\alpha_1(T) = 2^{n-2}$. In fact, some Xuong-tree of Q_n could be chosen as such a tree T. In order to find the Xuong-tree more effectively, we need to character the value of $\xi(Q_n)$.

Proposition 3.4. $\xi(Q_n) = 1$ *for* $n \geq 2$.

Proof. We prove it by induction on n. Clearly, $\xi(Q_2) = 1$. Now, we assume that $n \geq 3$. Also, we use Q^1_{n-1} and Q^2_{n-1} to denote the two copies of Q_{n-1} which constitute Q_n. Let T_{n-1} be a Xuong-tree of Q^1_{n-1}, i.e., $w(T_{n-1}; Q^1_{n-1}) = 1$. Then, we could construct a spanning tree T_n of Q_n by adding the edges between the corresponding vertices in Q^1_{n-1} and Q^2_{n-1}. Since the number of edges in Q^2_{n-1} is even, $w(T_n; Q_n) = 1$. It means that T_n is a Xuong-tree of Q_n. Therefore, $\xi(Q_n) = 1$. We finish the proof. □

In Proposition 3.4, one may easily deduce that $\alpha_1(T_n) = 2^{n-2}$. That is to say, the Xuong-tree T_n is an optimal tree of Q_n.

4. Cartesian product of two cycles

In this section, we shall solve the nsis problem of $C_m \square C_n$. The general idea of the proof is as follows. First, we establish an upper bound on the nsis number in $C_m \square C_n$. Second, we construct nonseparating independent sets (**nsiss** for short) achieving this bound.

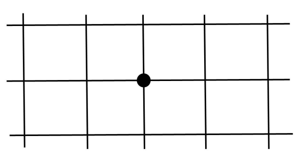

Figure 4.1: $C_3 \square C_5$.

We use the following standard labeling for the vertices of $C_m \square C_n$ and choose one that corresponds to matrix notation: the i-th vertex in the j-th copy of C_m will be denoted by

$u_{i,j}$. For example, in Figure 4.1 the vertex labelled by "•" is denoted by $u_{2,3}$. Carrying the matrix analogy further, we sometimes also speak of the copies of C_m and C_n as the columns and rows, respectively, of $C_m \Box C_n$. In order to recognize the nsis more easily in our figures, we only show the vertices to be explicitly removed.

Before going into details, we lay out a useful result, due to Pike and Zou [10], about the *decycling number* $\nabla(G)$ of a graph G, namely, the minimum number of vertices that have to be deleted in order to turn G into a forest.

Theorem 4.1. [10]

$$\nabla(C_m \Box C_n) = \begin{cases} \lceil 3n/2 \rceil & \textit{if } m = 4, \\ \lceil 3m/2 \rceil & \textit{if } n = 4, \\ \lceil (mn+2)/3 \rceil & \textit{otherwise.} \end{cases}$$

Based on the above theorem, we build an upper bound on the nsis number of $C_m \Box C_n$.

Lemma 4.2.

$$\mathcal{Z}(C_m \Box C_n) \leq \begin{cases} n + \lfloor (n+2)/4 \rfloor & \textit{if } m = 4, \\ m + \lfloor (m+2)/4 \rfloor & \textit{if } n = 4, \\ \lfloor mn/3 \rfloor & \textit{otherwise.} \end{cases}$$

Proof. Let S be a $\mathcal{Z}$-set of $C_m \Box C_n$. For brevity, suppose that $|S| = k$. Then,

$$4k + (mn - k - 1 + c) = 2mn,$$

where $4k$ is the number of edges incident to S, $2mn$ is the number of edges of $C_m \Box C_n$ and $mn - k - 1$ is the number of edges of a spanning tree in $C_m \Box C_n - S$, and $c \geq 0$ is a parameter. This implies that

$$3k = mn + 1 - c. \tag{4.1}$$

Notice that for any graph, its j (≥ 0) edges can be covered by at most j vertices. Let T be a spanning tree of $C_m \Box C_n - S$. Then, c is the number of edges in the co-tree $(C_m \Box C_n - S) - E(T)$. Thus, we can choose a set of vertices S_c of $C_m \Box C_n - S$ such that S_c covers the edges of $(C_m \Box C_n - S) - E(T)$ with $|S_c| \leq c$. It is straightforward to verify that the deletion S_c from $C_m \Box C_n - S$ leads to a forest. Now we deal with the following cases.

Case 1*:* $m = 4$. Applying the definition of the decycing number and Theorem 4.1, we deduce that

$$k + c \geq |S \cup S_c| \geq \left\lceil \frac{3n}{2} \right\rceil \geq \frac{3n}{2}. \tag{4.2}$$

Putting (4.1) and (4.2) together, we obtain that

$$(4n+1-c)+(k+c) \geq 3k+\frac{3n}{2}.$$

Therefore, $2k \leq 4n+1-3n/2$, and so $k \leq n+(n+2)/4$. Since k is a positive integer, $k \leq n+\lfloor (n+2)/4 \rfloor$.

Case 2: $n = 4$. By the symmetry of $C_m \square C_n$ and Case 1, it is easily seen that $k \leq m+\lfloor (m+2)/4 \rfloor$.

Case 3: $m \neq 4$ *and* $n \neq 4$. Under this case, we claim that $c \geq 1$. Suppose on the contrary that $c = 0$. Then S is decycling set with size $(mn+1)/3$. This is contradictory to Theorem 4.1. Hence, $k \leq \lfloor mn/3 \rfloor$. □

Observing Lemma 4.2, the result in $C_4 \square C_n$ is different from other cases. Therefore, we first deal with this case.

Lemma 4.3. $\mathcal{Z}(C_4 \square C_n) = n+\lfloor (n+2)/4 \rfloor$.

Proof. By Lemma 4.2, $\mathcal{Z}(C_4 \square C_n) \leq n+\lfloor (n+2)/4 \rfloor$. We now construct nsiss with that size. Let $r = \lfloor n/4 \rfloor$ and

$$M = \bigcup_{i=1}^{r} \{u_{1,4i-3}, u_{3,4i-3}, u_{2,4i-2}, u_{1,4i-1}, u_{3,4i-1}\}.$$

Then, M is a nsis of $C_4 \square C_n$, when $n \equiv 0 \pmod 4$; $M \cup \{u_{4,n-1}\}$ is a nsis of $C_4 \square C_n$, when $n \equiv 1 \pmod 4$; $M \cup \{u_{4,n-2}, u_{1,n-1}, u_{3,n-1}\}$ is a nsis of $C_4 \square C_n$, when $n \equiv 2 \pmod 4$; $M \cup \{u_{1,n-2}, u_{3,n-2}, u_{2,n-1}, u_{4,n-1}\}$ is a nsis of $C_4 \square C_n$, when $n \equiv 3 \pmod 4$ (as depicted in Figure 4.2 for $n = 15$).

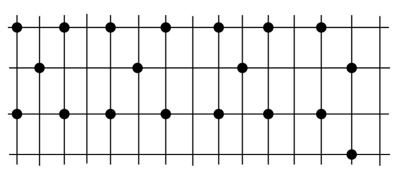

Figure 4.2: A $\mathcal{Z}$-set of $C_4 \square C_{15}$.

It is not hard to check that each nsis above has size $n+\lfloor (n+2)/4 \rfloor$. Thus, the proof is finished. □

In the rest part of this section, we devote to general cases, starting with several specific cases. By the symmetry of $C_m \square C_n$, from now on, we assume that $4 \notin \{m, n\}$.

First, we treat the cases $C_3 \square C_n$ and $C_8 \square C_n$.

Lemma 4.4. $\mathcal{Z}(C_3 \square C_n) = n$.

Proof. By Lemma 4.2, $\mathcal{Z}(C_3\Box C_n) \le n$. Let $k = \lfloor n/3 \rfloor$ and $M = \bigcup_{i=1}^{k}\{u_{1,3i-2}, u_{2,3i-1}, u_{3,3i}\}$. It is not hard to verify that $S = M$ is a nsis of $C_3\Box C_n$, where $n \equiv 0 \pmod 3$; $S = M \cup \{u_{2,n}\}$ is a nsis of $C_3\Box C_n$, where $n \equiv 1 \pmod 3$ (see Figure 4.3 for $C_3\Box C_{10}$); $S = M \cup \{u_{1,n-1}, u_{2,n}\}$ is a nsis of $C_3\Box C_n$, where $n \equiv 2 \pmod 3$.

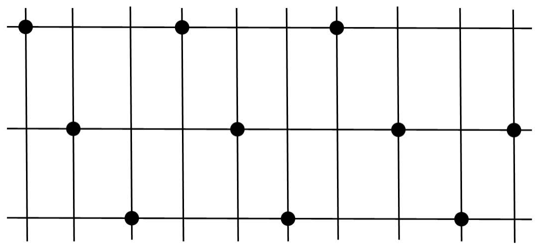

Figure 4.3: A $\mathcal{Z}$-set of $C_3\Box C_{10}$.

In each case, $|S| = n$. So, S is a $\mathcal{Z}$-set of $C_3\Box C_n$. This lemma is proved. □

Lemma 4.5. $\mathcal{Z}(C_8\Box C_n) = \lfloor 8n/3 \rfloor$.

Proof. Again by Lemma 4.2, $\mathcal{Z}(C_8\Box C_n) \le \lfloor 8n/3 \rfloor$. We further construct nsiss which achieve this bound. Let $k = \lfloor n/3 \rfloor$ and

$$M = \bigcup_{i=1}^{k}\{u_{1,3i-2}, u_{4,3i-2}, u_{7,3i-2}, u_{2,3i-1}, u_{5,3i-1}, u_{3,3i}, u_{6,3i}, u_{8,3i}\}.$$

For $n \equiv 0 \pmod 6$, $(M - \{u_{2,n-1}, u_{3,n}\}) \cup \{u_{3,n-1}, u_{2,n}\}$ is a nsis. For $n \equiv 1 \pmod 6$, $M \cup \{u_{2,n}, u_{5,n}\}$ is a nsis. For $n \equiv 2 \pmod 6$, $(M - \{u_{3,n-2}, u_{6,n-2}, u_{8,n-2}\}) \cup \{u_{1,n-2}, u_{4,n-2}, u_{7,n-2}, u_{2,n-1}, u_{6,n-1}, u_{8,n-1}, u_{3,n}, u_{5,n}\}$ is a nsis. For $n \equiv 3 \pmod 6$, M is a nsis. For $n \equiv 4 \pmod 6$, $(M - \{u_{3,n-1}, u_{6,n-1}, u_{8,n-1}\}) \cup \{u_{1,n-1}, u_{4,n-1}, u_{7,n-1}, u_{2,n}, u_{6,n}\}$ is a nsis. For $n \equiv 5 \pmod 6$, $M \cup \{u_{2,n-1}, u_{5,n-1}, u_{7,n-1}, u_{3,n}, u_{8,n}\}$ is a nsis.

Note that all of these nsiss have size $\lfloor 8n/3 \rfloor$. Thus, we build the lemma. □

Next, we give a result that will be frequently used later.

Lemma 4.6 (Double Expanding Lemma)**.** *Suppose that S is a nsis of $C_m\Box C_n$. Let $T = \{u_{2i,2j} : i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$ and $S' = \{u_{2i-1,2j-1} : u_{i,j} \in S\}$. Then $T \cup S'$ is a nsis of $C_{2m}\Box C_{2n}$.*

Proof. Obviously, $C_{2m}\Box C_{2n} - T$ is homeomorphic to a subdivision of $C_m\Box C_n$. Hence, $C_{2m}\Box C_{2n} - T - S'$ is connected. Note that $u_{2i,2j}$ is not adjacent to $u_{2k-1,2h-1}$ for any $i, j, k, h \ge 1$. It follows that $T \cup S'$ is independent. We conclude that $T \cup S'$ is a nsis of $C_{2m}\Box C_{2n}$. □

Figure 4.4 shows the expansion from $C_3 \Box C_3$ to $C_6 \Box C_6$.

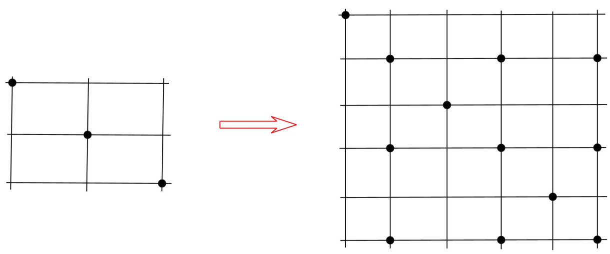

Figure 4.4: The expansion from $C_3 \Box C_3$ to $C_6 \Box C_6$.

Based on Lemmas 4.4, 4.5 and 4.6, the case $m \equiv 0 \pmod 3$ turns out to be easy.

Lemma 4.7. *$\mathcal{Z}(C_m \Box C_n) = rn$, where $m = 3r$.*

Proof. According to Lemma 4.2, $\mathcal{Z}(C_m \Box C_n) \le rn$. If n is odd, we define

$$M = \bigcup_{i=1}^{r} \left(\{u_{3i-2,1}\} \cup \bigcup_{j=1}^{k} \{u_{3i-1,2j}, u_{3i,2j+1}\} \right),$$

where $n = 2k+1$ (see Figure 4.5 for $C_6 \Box C_7$). Considering the subgraph $(C_{3r} \Box C_n) - M$, rows $3i-2$, $3i-1$ and $3i$ have a path from $u_{3i-2,2}$ to $u_{3i,2}$, for each $1 \le i \le r$. By joining these paths we have a cycle C. Each vertex beyond the cycle C and M has one neighbor in C. So, $(C_{3r} \Box C_n) - M$ is connected. It is clear that M is a nsis.

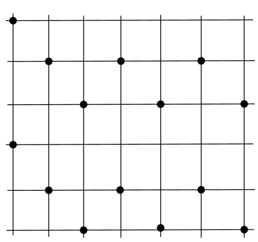

Figure 4.5: A $\mathcal{Z}$-set of $C_6 \Box C_7$.

If n is even and r is odd, set

$$M = \bigcup_{i=1}^{r} \left(\{u_{3i-2,1}, u_{3i-2,n-2}, u_{3i,n-1}, u_{3i-1,n}\} \cup \bigcup_{j=1}^{k} \{u_{3i,2j}, u_{3i-1,2j+1}\} \right),$$

where $n = 2k+4$. By an argument similar to above discussion, we have that M is a nsis. In both cases above, $|M| = rn$.

Now suppose that both of r and n are even and k the minimum nonnegative integer such that $r/2^k$ or $n/2^k$ is odd, or $n/2^k$ equals 8. Let $m_i = m/2^{k-i}$ and $n_i = n/2^{k-i}$ for each $i = 0, 1, \ldots, k$. By means of Lemma 4.5 and the discussion above, we may obtain a nsis with size $rn/2^{2k}$ in $C_{m_0}\square C_{n_0}$. Now for each $i = 0, 1, \ldots, k-1$, by using Lemma 4.6 we could construct a nsis with size $\mathcal{Z}(C_{m_0}\square C_{n_0}) + \frac{mn}{2^{2k}}\sum_{j=0}^{i} 4^j$ in $C_{m_{i+1}}\square C_{n_{i+1}}$. Finally, after a sequence of construction, we get a nsis with size rn of $C_m\square C_n$. As a consequence, $\mathcal{Z}(C_m\square C_n) = rn$. □

In the rest, we devote to the other cases. Since we have already handled the case $m \equiv 0 \pmod 3$, we only need to consider the cases $m \equiv i \pmod 6$, $i = 1, 2, 4, 5$. By Lemma 4.7 and the symmetry of $C_m\square C_n$, we don't have to consider the case $n \equiv 0 \pmod 3$ for any m.

Now, we start to deal with $C_{6r+1}\square C_n$, $r \geq 1$. First, we turn our attention to $C_7\square C_n$.

Lemma 4.8. $\mathcal{Z}(C_7\square C_n) = \lfloor 7n/3 \rfloor$.

Proof. By Lemma 4.2, $\mathcal{Z}(C_7\square C_n) \leq \lfloor 7n/3 \rfloor$. Let $k = \lfloor n/3 \rfloor$.

For $n \equiv 1 \pmod 3$, $S_1^1 = \bigcup_{i=1}^{k-1}\{u_{2,3i-2}, u_{6,3i-2}, u_{3,3i-1}, u_{5,3i-1}, u_{7,3i-1}, u_{1,3i}, u_{4,3i}\} \cup \{u_{3,n-3}, u_{7,n-3}, u_{2,n-2}, u_{5,n-2}, u_{1,n-1}, u_{3,n-1}, u_{6,n-1}, u_{4,n}, u_{7,n}\}$ is a nsis.

For $n \equiv 2 \pmod 3$, $S_1^2 = \bigcup_{i=1}^{k}\{u_{1,3i-2}, u_{5,3i-2}, u_{2,3i-1}, u_{4,3i-1}, u_{7,3i-1}, u_{3,3i}, u_{6,3i}\} \cup \{u_{1,n-1}, u_{4,n-1}, u_{3,n}, u_{6,n}\}$ is a nsis.

Furthermore, both of S_1^1 and S_1^2 have size $\lfloor 7n/3 \rfloor$. Thus, the proof is finished. □

In Figure 4.6, we depicts a nsis of $C_7\square C_7$ and $C_7\square C_8$, respectively.

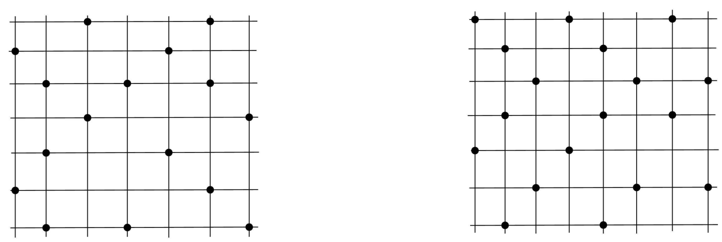

Figure 4.6: $\mathcal{Z}$-sets of $C_7\square C_7$ and $C_7\square C_8$.

Next, we construct a $\mathcal{Z}$-set of $C_{6r+1}\square C_n$ for $r \geq 2$.

Lemma 4.9. *If* $m = 6r+1$, *then* $\mathcal{Z}(C_m\square C_n) = 2rn + \lfloor n/3 \rfloor$.

Proof. If $r = 1$, then the result follows from Lemma 4.8. For $r > 1$, we construct a $\mathcal{Z}$-set by employing the idea as follows. We first choose the $\mathcal{Z}$-set of $C_7\square C_n$ as described in Lemma 4.8, and then add additional 6 new rows to $C_7\square C_n$ and select $2n$ vertices from these 6 new rows to add to the chosen $\mathcal{Z}$-set as a new $\mathcal{Z}$-set of $C_{7+6}\square C_n$. Repeat this operation until we get a $\mathcal{Z}$-set with size $2rn + \lfloor n/3 \rfloor$ in $C_m\square C_n$. The detailed operation is depicted as follows.

We further consider two cases.

(a) $n \equiv 1 \pmod 3$. Let $n = 3t+1$. We start with the $\mathcal{Z}$-set S_1^1 of $C_7 \Box C_n$ as described in Lemma 4.8. We say that a row is type-5 if its deleted vertices are in the same columns as those of the fifth row of $C_7 \Box C_n$ in Lemma 4.8. Type-6 and type-7 rows are defined analogously. Focusing on the three consecutive rows: type-5, 6, 7 in $C_7 \Box C_n$, we now illustrate how to insert six new rows and obtain a $\mathcal{Z}$-set of $C_{7+6} \Box C_n$. Following the row of type-5 in $C_7 \Box C_n$, we insert three new rows, the first two being of type-6 and type-7, respectively. For the third, we select the vertices in columns $3i$ $(i = 1, 2, \ldots, t-1)$ and $n-2$ to add to S_1^1. Now, following the original type-6 row, we insert another three new rows. For the first of these three new rows, we select the vertices in columns $3i$ $(i = 1, 2, \ldots, t-1)$, $n-2$ and n. For the second row, we select the vertices in columns $3i-1$ $(i = 1, 2, \ldots, t-1)$ and $n-3$ to add to S_1^1. We select the type-6 row as the third row. Thus, we have a nsis S_2 of $C_{7+6} \Box C_n$. Obviously, $|S_2| = 4n + \lfloor n/3 \rfloor$. Hence, S_2 is a $\mathcal{Z}$-set of $C_{7+6} \Box C_n$. Note that the new graph $C_{7+6} \Box C_n$ contains three consecutive rows that are of type-5, 6, 7 (Figure 4.7 shows the insertion process for $n = 13$).

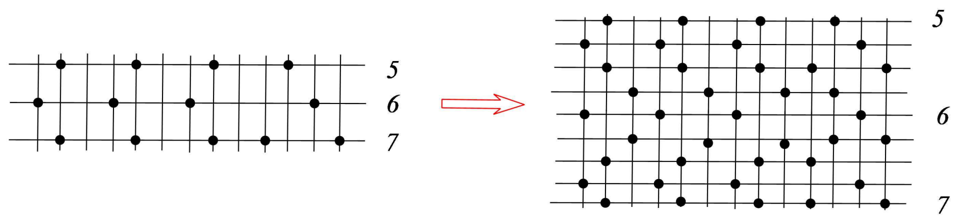

Figure 4.7: $n = 13$.

So, the insertion process may be repeated until we get a $\mathcal{Z}$-set of $C_m \Box C_n$ whose size is $2rn + \lfloor n/3 \rfloor$.

(b) $n \equiv 2 \pmod 3$. Let $n = 3t+2$. As before, we begin with the $\mathcal{Z}$-set S_1^2 of $C_7 \Box C_n$. A row is type-4 if its deleted vertices are in the same columns as those of the fourth row of $C_7 \Box C_n$. Similarly, a row is type-5 (resp. type-6) if its deleted vertices are in the same columns as those of the fifth (resp. sixth) row of $C_7 \Box C_n$.

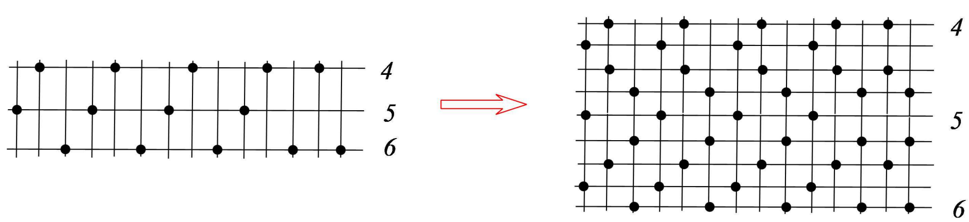

Figure 4.8: $n = 14$.

Focusing on the three consecutive rows: type-4, 5, 6 in $C_7\Box C_n$, we are ready to insert six new rows and obtain a $\mathcal{Z}$-set of $C_{7+6}\Box C_n$. Following the row of type-4 in $C_7\Box C_n$, we insert three new rows, being of type-5, type-4, and type-6 in that order. Following the original type-5 row, we insert another three new rows, being of type-6, type-4, and type-5 in that order. After the insertion, we obtain a nsis S_2 of $C_{7+6}\Box C_n$. Of course, $|S_2| = 4n + \lfloor n/3 \rfloor$ (see Figure 4.8 for an example of the case $n = 14$). That is to say, S_2 is a $\mathcal{Z}$-set of $C_{7+6}\Box C_n$. Note that the new graph, $C_{7+6}\Box C_n$ contains three consecutive rows that are of type-4, 5, 6 in that order. Hence we can repeat insertion procedure. Finally, we get a $\mathcal{Z}$-set of $C_m\Box C_n$ with size $2rn + \lfloor n/3 \rfloor$. □

A similar argument can be used to count $\mathcal{Z}(C_{6r+5}\Box C_n)$, $r \geq 1$. Also, we first treat $\mathcal{Z}(C_5\Box C_n)$.

Lemma 4.10. $\mathcal{Z}(C_5\Box C_n) = \lfloor 5n/3 \rfloor$.

Proof. Making use of Lemma 4.2, one may have that $\mathcal{Z}(C_5\Box C_n) \leq \lfloor 5n/3 \rfloor$. Let $k = \lfloor n/3 \rfloor$ and

$$M = \bigcup_{i=1}^{k} \{u_{1,3i-2}, u_{3,3i-2}, u_{2,3i-1}, u_{4,3i-1}, u_{5,3i}\}.$$

Then $S_1^1 = M \cup \{u_{4,n}\}$ is a nsis for $n \equiv 1 \pmod 3$ and $S_1^2 = M \cup \{u_{1,n-1}, u_{3,n-1}, v_{4,n}\}$ is a nsis for $n \equiv 2 \pmod 3$. Notice that both of the nsiss above have size $\lfloor 5n/3 \rfloor$. The proof is finished. □

Lemma 4.11. *If $m = 6r + 5$, then $\mathcal{Z}(C_m\Box C_n) = 2rn + \lfloor 5n/3 \rfloor$.*

Proof. The proof is similar to that of Lemma 4.9. We start with $C_5\Box C_n$ and repeatedly insert 6 new rows each time. There are two cases to be handled.

(a) $n \equiv 1 \pmod 3$. Let $n = 3t + 1$. We start from the $\mathcal{Z}$-set S_1^1 of $C_5\Box C_n$. A row is type-3 if its deleted vertices are in the same columns as those of the third row of $C_5\Box C_n$. Type-4 and type-5 rows are defined in a similar way. We now insert three new rows following the type-3 row in $C_5\Box C_n$, being of type-4, type-5 and type-3 in that order. Following the original type-4 row, we insert another three new rows. For the first of these rows, we select the vertices in columns $3i+1$ $(i = 1, 2, 3, \ldots, t-1)$ and $3t$. For the second row, we select the vertices in columns 1 and $3i$ $(i = 1, 2, 3, \ldots, t-1)$. For the third row, use the type-4 row. Thus, we get a nsis S_2 with size $2n + \lfloor 5n/3 \rfloor$ in $C_{5+6}\Box C_n$. In other words, S_2 is a $\mathcal{Z}$-set of $C_{5+6}\Box C_n$. Note that the new graph $C_{5+6}\Box C_n$ contains three consecutive rows that are of type-3, 4, 5 (Figure 4.9 depicts the insertion process for the case $n = 10$). Therefore we repeated the insertion process until we obtain a $\mathcal{Z}$-set of $C_m\Box C_n$ whose size is $2rn + \lfloor 5n/3 \rfloor$.

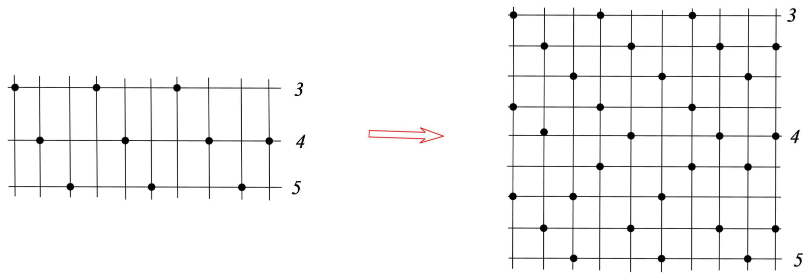

Figure 4.9: $n = 10$.

(b) $n \equiv 2 \pmod 3$. Let $n = 3t + 2$. As before, we begin with the $\mathcal{Z}$-set S_1^2 of $C_5 \Box C_n$. A row is type-2 if its deleted vertices are in the same columns as those of the second row of $C_5 \Box C_n$. Type-3 and type-4 rows are defined analogously. We now start to insert new rows. Following the type-2 row, we insert three new rows, the first two being type-3 and type-4, respectively. For the third, we select the vertices in columns $3i$ $(i = 1, 2, 3, \ldots, t)$ to add to S_1^2. Then, after the original type-3 row, we insert another three new rows. For the first, we select the vertices in columns $3i$ $(i = 1, 2, 3, \ldots, t)$ and n. The second and third are type-2 and type-3, respectively. We now have a $\mathcal{Z}$-set of $C_{5+6} \Box C_n$ (The insertion operation for $n = 11$ is illustrated in Figure 4.10).

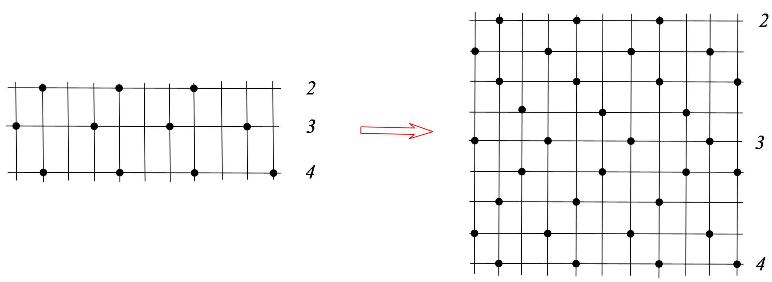

Figure 4.10: $n = 11$.

Here, the new graph $C_{5+6} \Box C_n$ contains three consecutive rows that are of type-2, 3, 4. Therefore we can repeatedly perform the insertion procedure to obtain a $\mathcal{Z}$-set of size $2rn + \lfloor 5n/3 \rfloor$ in $C_m \Box C_n$. □

For the remaining cases, both m and n are even. In such cases, we employ the Double Expanding Lemma (i.e., Lemma 4.6).

Lemma 4.12. *If* $m \equiv 2$ *or* $4 \pmod 6$*, and* $n \equiv 2$ *or* $4 \pmod 6$*, then* $\mathcal{Z}(C_m \Box C_n) = \lfloor mn/3 \rfloor$*.*

Proof. Let k be the minimum nonnegative integer such that $m/2^k$ or $n/2^k$ is odd, or equals 8 and let $m_i = m/2^{k-i}$, $n_i = n/2^{k-i}$ for each $i = 0, 1, \ldots, k$. Then we can find a nsis S_0 of cardinality $\lfloor m_0n_0/3\rfloor$ in $C_{m_0}\Box C_{n_0}$. Now, for each $i=0,1,\ldots,k-1$, applying Lemma 4.6 to $C_{m_i}\Box C_{n_i}$ to construct a nsis S_{i+1} of size $\mathcal{Z}(C_{m_0}\Box C_{n_0})+\frac{mn}{2^{2k}}\sum_{j=0}^{i}4^j$ in $C_{m_{i+1}}\Box C_{n_{i+1}}$. Consequently, we can construct a nsis of $C_m\Box C_n$ with size $\lfloor mn/3\rfloor$. □

Putting results above together, we are now in a position to state our main result in this section.

Theorem 4.13.

$$\mathcal{Z}(C_m\Box C_n)=\begin{cases} n+\lfloor (n+2)/4\rfloor & \text{if } m=4,\\ m+\lfloor (m+2)/4\rfloor & \text{if } n=4,\\ \lfloor mn/3\rfloor & \text{otherwise.}\end{cases}$$

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References

[1] B. Escoffier, L. Gourvès and J. Monnot, *Complexity and approximation results for the connected vertex cover problem in graphs and hypergraphs*, J. Discrete Algorithms **8** (2010), no. 1, 36–49.

[2] H. Fernau and D. F. Manlove, *Vertex and edge covers with clustering properties: complexity and algorithms*, J. Discrete Algorithms **7** (2009), no. 2, 149–167.

[3] M. R. Garey and D. S. Johnson, *The rectilinear Steiner tree problem is NP-complete*, SIAM J. Appl. Math. **32** (1977), no. 4, 826–834.

[4] J. L. Gross and T. W. Tucker, *Topological Graph Theory*, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, New York, 1987.

[5] Y. Huang and Y. Liu, *Maximum genus and maximum nonseparating independent set of a 3-regular graph*, Discrete Math. **176** (1997), no. 1-3, 149–158.

[6] Y. Li, Z. Yang and W. Wang, *Complexity and algorithms for the connected vertex cover problem in 4-regular graphs*, Appl. Math. Comput. **301** (2017), 107–114.

[7] S. Long and H. Ren, *The decycling number and maximum genus of cubic graphs*, J. Graph Theory **88** (2018), no. 3, 375–384.

[8] B. Mohar and C. Thomassen, *Graphs on Surfaces*, Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, MD, 2001.

[9] H. Moser, *Exact algorithms for generalizations of vertex cover*, Fakulätfür Mathematik und Informatik, Friedrich-Schiller-Universität Jena, 2005 Mas-ters thesis.

[10] D. A. Pike and Y. Zou, *Decycling Cartesian products of two cycles*, SIAM J. Discrete Math. **19** (2005), no. 3, 651–663.

[11] S. Ueno, Y. Kajitani and S. Gotoh, *On the nonseparating independent set problem and feedback set problem for graphs with no vertex degree exceeding three*, Discrete Math. **72** (1988), no. 1-3, 355–360.

[12] T. Wanatabe, S. Kajita and K. Onaga, *Vertex covers and connected vertex covers in* 3*-connected graphs*, in: *1991., IEEE International Sympoisum on Circuits and Systems*, (1991), 1017–1020.

[13] N. H. Xuong, *How to determine the maximum genus of a graph*, J. Combin. Theory Ser. B **26** (1979), no. 2, 217–225.

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