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Bounds for the Lifespan of Solutions to Fourth-order Hyperbolic Equations with Initial Data at Arbitrary Energy Level

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Abstract. This paper deals with lower and upper bounds for the lifespan of solutions to a fourth-order nonlinear hyperbolic equation with strong damping:

$$u_{tt} + \Delta^2 u - \Delta u - \omega \Delta u_t + \alpha(t) u_t = |u|^{p-2} u.$$

First of all, the authors construct a new control function and apply the Sobolev embedding inequality to establish some qualitative relationships between initial energy value and the norm of the gradient of the solution for supercritical case ($2(N-2)/(N-4) < p < 2N/(N-4)$, $N \geq 5$). And then, the concavity argument is used to prove that the solution blows up in finite time for initial data at low energy level, at the same time, an estimate of the upper bound of blow-up time is also obtained.

Subsequently, for initial data at high energy level, the authors prove the monotonicity of the L^2 norm of the solution under suitable assumption of initial data, furthermore, we utilize the concavity argument and energy methods to prove that the solution also blows up in finite time for initial data at high energy level.

At last, for the supercritical case, a new control functional with a small dissipative term and an inverse Hölder inequality with correction constants are employed to overcome the difficulties caused by the failure of the embedding inequality ($H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow L^{2p-2}$ for $2(N-2)/(N-4) < p < 2N/(N-4)$) and then an explicit lower bound for blow-up time is obtained. Such results extend and improve those of [S. T. Wu, J. Dyn. Control Syst. 24 (2018), no. 2, 287–295].

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded simply connected domain. Consider the initial and boundary value problem of fourth-order wave equations with superlinear sources

$$\begin{cases} u_{tt} + \Delta^2 u - \Delta u - \omega \Delta u_t + \alpha(t) u_t = |u|^{p-2} u, & x \in \Omega, \ t \geq 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = \Delta u(x,t) = 0, & x \in \partial\Omega, \ t \geq 0. \end{cases} \tag{1.1}$$

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In what follows, we always assume that $\omega > 0$, $\alpha(t)\colon [0,\infty) \to (0,\infty)$ is a nonincreasing bounded differentiable function.

Problem (1.1) may describe some phenomena of granular materials such as the longitudinal motion of an elasto-plastic bar. In fact, the first identity of Problem (1.1) may be derived from the conservation of mass and momentum, for more details, the interested reader may refer to the derived process of Equation (3.6) of [4] and other references [2,3]. It is well known that the source term causes finite-time blow-up of solutions and drives the equation to possible instability while the damping term prevents finite-time blow-up of the solution and drives the equation toward stability. So, it is of interest to explore the mechanism of how the sources dominate the dissipation (strong damping term Δu_t and weak damping term u_t), which attracts considerable attention. For example, the author in [6] discussed the following equation

$$u_{tt} + \Delta^2 u - \alpha \Delta u \pm \beta \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \sigma_i(u_{x_i}) = f(u),$$

where $\alpha \geq 0$ and β is a constant. He defined and utilized a potential well (first introduced by Sattinger in [16]) to discuss some properties of solutions such as blow-up, boundedness and convergence to the steady-state solution as time variable goes to infinity. After then, Lin et. al. in [11] considered the fourth-order nonlinear evolution equation with strong damping

$$u_{tt} + \Delta^2 u - \Delta u - \omega \Delta u_t = f(u),$$

with $\omega > 0$. By using the classical potential well method and the concavity argument, they established the existence of global weak solutions and global strong solutions under additional assumptions about the initial date and nonlinearities. Later, Liu and Xu in [12,13] improved the above results. In 2018, Wu in [18] applied the potential well method to prove the solution of Problem (1.1) blew up in finite time for $2 < p < 2(N-2)/(N-4)$ $(N \geq 5)$, at the same time the estimate of an upper bound of blow-up time was also given. In fact, we all know that the upper bound guarantees blowing-up of the solution and the importance of the lower bound is that it may provide us a safe time interval for operation. In addition, Wu applied some methods used in [10,17] to obtain the estimate of a lower bound of blow-up time for the subcritical case of $2 < p < 2(N-2)/(N-4)$ $(N \geq 5)$ or $2 < p < \infty$ $(1 < N \leq 4)$. For more related works, the interested readers may refer to [5,14,15,19]. However, [18] leaves some unsolved problems:

(i) Whether does the solution of Problem (1.1) blow up in finite time when p lies in the supercritical internal $(2(N-2)/(N-4), 2N/(N-4))$?

(ii) Whether does the solution blow up in finite time for initial data at arbitrary energy level?

To the best of our knowledge, there are few related works for above problems. In fact, applying the classical potential well method and the methods used in [10,17] to study such problems, we have to face some difficulties:

- How to establish some qualitative relationships between initial energy value and the norm of the gradient of the solution for the supercritical case and high energy level?
- How to bypass the difficulty caused by the failure of the embedding $H^2(\Omega)\cap H_0^1(\Omega) \hookrightarrow L^{2p-2}(\Omega)$ for $2(N-2)/(N-4) < p < 2N/(N-4)$ and then give an estimate of lower bound for blow-up time?

In this paper, we first borrow some ideas of our previous works [8,9] to establish some qualitative relationships between initial energy value and the norm of the gradient of the solution for supercritical case and low energy level, and then we modify the functional constructed in [18] and apply the concavity method to prove that the solution blows up in finite time for initial data at the low energy level. Subsequently, in the case of initial data at high energy level, we construct a suitable function and apply energy inequalities to obtain the estimate of the lower bound of the L^2 norm of the solution. Moreover, the concavity method is used to prove that the solution blows up in finite time for initial data at high energy level. At the same time, we also give the upper bound for blow-up time. At the last part of this paper, we apply the interpolation inequality, Sobolev inequality and energy inequalities to establish an inverse Hölder inequality with the correction constant and then construct the new control functional with a small dissipative term to get some estimates of the lower bound of blow-up time for supercritical case.

This paper is organized as follows. In Section 2, we recall some useful lemmas and estimate of the lifespan of blow-up solutions for low initial level. Section 3 is devoted to discussing blow-up of solutions for high energy level. In last section, we will apply the interpolation inequality and energy inequalities to give the estimate of the lower bound for blow-up time.

2. Initial data at low energy level

For the sake of completeness, we first give some lemmas and notations which will be used later. Set

$$H = \{u \in H^2(\Omega) \cap H_0^1(\Omega) \mid u = \Delta u = 0 \text{ on } \partial\Omega\}$$

and

$$\|u\|_H = \sqrt{\|\Delta u\|_2^2 + \|\nabla u\|_2^2},$$

where $\|\cdot\|_p$ denotes the usual $L^p(\Omega)$ norm $\|\cdot\|_{L^p(\Omega)}$ for $1 \le p \le \infty$. From [1], we know that there exists a positive constant B_0 such that the following embedding inequality holds

$$\|u\|_p \le B_0 \|u\|_H \quad \text{for } u \in H. \tag{2.1}$$

That is, $B_0 = \sup_{u\in H\setminus\{0\}} \frac{\|u\|_p}{\|u\|_H}$. Here p satisfies

$$\begin{cases} 2 < p \le \frac{2N}{N-4} & \text{if } N \ge 5, \\ 2 < p < \infty & \text{if } 1 \le N \le 4. \end{cases} \tag{2.2}$$

Next, we first state, without the proof, the local existence which can be established by combining the argument of [18].

Theorem 2.1. [18] *Let $u_0 \in H$ and $u_1 \in L^2(\Omega)$. Then Problem* (1.1) *admits a unique solution $u(\,\cdot\,,t) \in L^\infty(0,T;H)$, $u_t \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ for $T > 0$ small enough. Moreover, the energy functional satisfies*

$$E'(t) = -\int_\Omega \alpha(t) u_t^2\,dx - \omega \int_\Omega |\nabla u_t|^2\,dx \le 0, \tag{2.3}$$

where the energy functional

$$E(t) = \frac{1}{2}\int_\Omega |u_t|^2\,dx + \frac{1}{2}\int_\Omega |\Delta u|^2\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{p}\int_\Omega |u|^p\,dx.$$

Before stating our main results, we give a useful lemma.

Lemma 2.2. *Let*

$$h(\lambda) = \frac{1}{2}\lambda - \frac{B_0^p}{p}\lambda^{p/2}, \quad \lambda > 0,\ p > 2,$$

then there exists $\lambda_1 = B_0^{2p/(2-p)}$ such that

(1) $\lim_{\lambda\to+\infty} h(\lambda) = -\infty$, $h(0) = 0$, $h'(\lambda_1) = 0$;

(2) $h(\lambda)$ *is increasing for* $0 < \lambda \le \lambda_1$; $h(\lambda)$ *is decreasing for* $\lambda \ge \lambda_1$.

The proof of this lemma is similar to these of [7–9], we omit it here. Next, we consider the case of low initial energy. Our main result is as follows:

Theorem 2.3. *Assume that the condition* (2.2) *and the following conditions are fulfilled:*

(H_1) $\int_\Omega |\nabla u_0|^2\,dx + \int_\Omega |\Delta u_0|^2\,dx > \lambda_1$;

(H_2) $E(0) < d = \frac{p-2}{2p} B_0^{2p/(2-p)}$.

Then the solution of Problem (1.1) *blows up in a finite time* $T^* > 0$*, that is,*

$$\lim_{t\to T^*} \|u\|_p^p = +\infty.$$

Moreover, T^* *satisfies*

$$T^* \le \frac{4p(d-E(0))\|u_0\|_2^2 + (p+1)\big[(\alpha(0)+1)\|u_0\|_2^2 + \omega\|\nabla u_0\|_2^2 + \|u_1\|_2^2\big]^2}{p(p-2)(d-E(0))\big[2\int_\Omega u_0u_1\,dx + (\alpha(0)+1)\|u_0\|_2^2 + \omega\|\nabla u_0\|_2^2 + \|u_1\|_2^2\big]}.$$

Proof. **Step 1.** We claim that there exists a $\lambda_2 > \lambda_1$ such that for $t > 0$,

$$\int_\Omega |\nabla u|^2\,dx + \int_\Omega |\Delta u|^2\,dx \ge \lambda_2. \tag{2.4}$$

In fact, we apply Inequality (2.1) to obtain

$$\begin{aligned} E(t) &\ge \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega |\Delta u|^2\,dx - \frac{B_0^p}{p}\left(\int_\Omega |\nabla u|^2\,dx + \int_\Omega |\Delta u|^2\,dx\right)^{p/2} \\ &\overset{\Delta}{=} \frac{1}{2}\lambda(t) - \frac{B_0^p}{p}\lambda^{p/2}(t) = h(\lambda(t)) \end{aligned}$$

with $\lambda(t) = \int_\Omega |\nabla u|^2\,dx + \int_\Omega |\Delta u|^2\,dx$.

According to $E(0) < d$ and Lemma 2.2, we know that there exists a $\lambda_2 > \lambda_1$ such that $h(\lambda_2) = E(0)$. Since $\lambda_0 \overset{\Delta}{=} \int_\Omega |\nabla u_0|^2\,dx + \int_\Omega |\Delta u_0|^2\,dx > \lambda_1$, we get $h(\lambda_0) \le E(0) = h(\lambda_2)$. Once again applying the monotonicity of $h(\lambda)$, we have that $\lambda_0 \ge \lambda_2 > \lambda_1$.

Next, we prove Inequality (2.4) by arguing by contradiction. Suppose that there exists a $t_0 > 0$ such that $\lambda(t_0) < \lambda_2$.

Case 1. If $\lambda_1 \le \lambda(t_0) < \lambda_2$, then we choose $t_1 = t_0$.

Case 2. If $\lambda(t_0) < \lambda_1$, then inequalities $\lambda_0 \ge \lambda_2 > \lambda_1$ and the continuity of $\lambda(t)$ imply that there exists $t_1 \in (0, t_0)$ such that $\lambda_1 \le \lambda(t_1) < \lambda_2$. And then, we apply the monotonicity of $h(t)$ and the definitions of $E(t)$ to obtain

$$E(t_1) \ge h(\lambda(t_1)) > h(\lambda_2) = E(0),$$

which contradicts Identity (2.3).

Step 2. Define

$$\begin{aligned} \varphi(t) &= \int_\Omega |u(\,\cdot\,,t)|^2\,dx + \int_0^t\!\int_\Omega \big(\alpha(s)u^2(\,\cdot\,,s) + \omega|\nabla u(\,\cdot\,,s)|^2\big)\,dxds \\ &\quad + \int_0^t\!\int_\Omega (s-t)\alpha_s(s)u^2(\,\cdot\,,s)\,dxds - t\int_\Omega \big(\alpha(0)u_0^2 + \omega|\nabla u_0|^2\big)\,dx + \beta(t+\sigma)^2, \end{aligned}$$

where β, σ will be determined later.

It is easy to check that

$$\varphi'(t) = 2\int_\Omega u_t u\,dx + 2\int_0^t\!\int_\Omega \big(\alpha(s)u_s(\,\cdot\,,s)u(\,\cdot\,,s) + \omega\nabla u(\,\cdot\,,s)\nabla u_s(\,\cdot\,,s)\big)\,dxds + 2\beta(t+\sigma),$$

and

$$
\begin{aligned}
\varphi''(t) &= 2\int_\Omega |u_t|^2\,dx + 2\int_\Omega u_{tt}u\,dx + 2\int_\Omega (\alpha(t)u_tu + \omega\nabla u\nabla u_t)\,dx + 2\beta \\
&= (p+2)\|u_t\|_2^2 + (p-2)\left(\int_\Omega |\Delta u|^2\,dx + \int_\Omega |\nabla u|^2\,dx\right) - 2pE(0) \\
&\quad + 2\beta + 2p\int_0^t\int_\Omega \big(\alpha(s)u_s^2(\,\cdot\,,s) + \omega|\nabla u_s(\,\cdot\,,s)|^2\big)\,dxds.
\end{aligned} \tag{2.5}
$$

By virtue of (2.4) and (2.5), we have

$$
\begin{aligned}
\varphi''(t) &= 2\int_\Omega |u_t|^2\,dx + 2\int_\Omega u_{tt}u\,dx + 2\int_\Omega (\alpha(t)u_tu + \omega\nabla u\nabla u_t)\,dx + 2\beta \\
&\geq (p+2)\|u_t\|_2^2 + 2p(d - E(0)) + 2\beta \\
&\quad + 2p\int_0^t\int_\Omega \big(\alpha(s)u_s^2(\,\cdot\,,s) + \omega|\nabla u_s(\,\cdot\,,s)|^2\big)\,dxds.
\end{aligned}
$$

And then, we choose $\beta = p(d - E(0))/(p+1) > 0$, $\sigma = \int_\Omega \big(\alpha(0)|u_0|^2 + \omega|\nabla u_0|^2 + |u_0|^2 + |u_1|^2\big)\,dx/(2\beta) > 0$ to obtain

$$
\begin{aligned}
\varphi''(t) &\geq (p+2)\|u_t\|_2^2 + 2(p+2)\beta \\
&\quad + 2p\int_0^t\int_\Omega \big(\alpha(s)u_s^2(\,\cdot\,,s) + \omega|\nabla u_s(\,\cdot\,,s)|^2\big)\,dxds.
\end{aligned} \tag{2.6}
$$

Subsequently, we estimate the term $(\varphi'(t))^2/4$:

$$
\begin{aligned}
&\frac{(\varphi'(t))^2}{4} \\
&\leq \|u_t\|_2^2\|u\|_2^2 + \beta^2(t+\sigma)^2 + 2\beta(t+\sigma)\|u_t\|_2\|u\|_2 \\
&\quad + 2\big(\beta(t+\sigma) + \|u_t\|_2\|u\|_2\big) \\
&\qquad \times \left(\int_0^t\int_\Omega \big(\alpha(s)u_s^2(\,\cdot\,,s) + \omega|\nabla u_s(\,\cdot\,,s)|^2\big)\,dxds \int_0^t\int_\Omega \big(\alpha(s)u^2(\,\cdot\,,s) + \omega|\nabla u(\,\cdot\,,s)|^2\big)\,dxds\right)^{1/2} \\
&\quad + \left(\int_0^t\int_\Omega \big(\alpha(s)u_s^2(\,\cdot\,,s) + \omega|\nabla u_s(\,\cdot\,,s)|^2\big)\,dxds \int_0^t\int_\Omega \big(\alpha(s)u^2(\,\cdot\,,s) + \omega|\nabla u(\,\cdot\,,s)|^2\big)\,dxds\right) \\
&\leq \left[\|u_t\|_2^2 + \int_0^t\int_\Omega \big(\alpha(s)u_s^2(\,\cdot\,,s) + \omega|\nabla u_s(\,\cdot\,,s)|^2\big)\,dxds + 2\beta\right]\varphi(t).
\end{aligned} \tag{2.7}
$$

Here we have used the following inequalities

$$
\begin{aligned}
2\|u\|_2\cdot\|u_t\|_2\beta(t+\sigma) &\leq \|u_t\|_2^2\Big(\beta(t+\sigma)^2 - t\int_\Omega \big(\alpha(0)u_0^2 + \omega|\nabla u_0|^2\big)\,dx\Big) \\
&\quad + \|u\|_2^2\beta^2(t+\sigma)^2\left(\beta(t+\sigma)^2 - t\int_\Omega \big(\alpha(0)u_0^2 + \omega|\nabla u_0|^2\big)\,dx\right)^{-1},
\end{aligned}
$$

and

$$\begin{aligned}
&2\beta(t+\sigma)\left(\int_0^t\int_\Omega \left(\alpha(s)u_s^2(\,\cdot\,,s)+\omega|\nabla u_s(\,\cdot\,,s)|^2\right)dxds\int_0^t\int_\Omega \left(\alpha(s)|u|^2+\omega|\nabla u|^2\right)dxds\right)^{1/2}\\
&\le \int_0^t\int_\Omega \left(\alpha(s)u_s^2(\,\cdot\,,s)+\omega|\nabla u_s(\,\cdot\,,s)|^2\right)dxds\left(\beta(t+\sigma)^2-t\int_\Omega \left(\alpha(0)u_0^2+\omega|\nabla u_0|^2\right)dx\right)\\
&\quad+\beta^2(t+\sigma)^2\int_0^t\int_\Omega \left(\alpha(s)|u|^2+\omega|\nabla u|^2\right)dxds\left(\beta(t+\sigma)^2-t\int_\Omega \left(\alpha(0)u_0^2+\omega|\nabla u_0|^2\right)dx\right)^{-1}.
\end{aligned}$$

By Inequalities (2.6) and (2.7), it is not difficulty to verify that

$$\varphi(t)\varphi''(t)-\frac{p+2}{4}(\varphi'(t))^2\ge 0,\quad p>2,$$

which implies

$$(\varphi^{1-(p+2)/4}(t))''\le 0\quad\text{for } t>0.$$

Noticing that $\varphi^{1-(p+2)/4}(0)>0$, $(\varphi^{1-(p+2)/4})'(0)<0$, then

$$\varphi^{1-(p+2)/4}(T^*)=0\quad\text{for some } T^*\in\left(0,\frac{-\varphi^{1-(p+2)/4}(0)}{(\varphi^{1-(p+2)/4})'(0)}\right).$$

Step 3. According to the analysis above, we have

$$\lim_{t\to T^*}\varphi(t)=\infty,$$

which implies that

$$\overline{\lim_{t\to T^*}}\|\nabla u(\,\cdot\,,t)\|_2=\infty. \tag{2.8}$$

Combining (2.3) with (2.8) and utilizing the definition of $E(t)$, we have

$$\lim_{t\to T^*}\|u(\,\cdot\,,t)\|_p=\infty.$$ □

Remark 2.4. In fact, the constant $d=\frac{p-2}{2p}B_0^{2p/(2-p)}$ coincides with the mountain pass level. i.e., we claim that

$$d=\inf_{u\in H\setminus\{0\}}\sup_{\alpha\ge0}J(\alpha u)=\inf_{u\in N}J(u),$$

where the functional

$$J(u)=\frac12\int_\Omega\left(|\nabla u|^2+|\Delta u|^2\right)dx-\frac1p\int_\Omega|u|^p\,dx,$$

and the Nehari manifold

$$N=\left\{u\in H\setminus\{0\},\int_\Omega|\nabla u|^2\,dx+\int_\Omega|\Delta u|^2\,dx=\int_\Omega|u|^p\,dx\right\}.$$

Proof. For any $\|u\|_p \neq 0$, a simple analysis shows that the maximum of $J(\alpha u)$ for $\alpha \geq 0$ may be attained at α_0 satisfying $\|u\|_H^2 = \alpha_0^{p-2}\|u\|_p^p$. And then, we have

$$\begin{aligned}\inf_{u\in H\setminus\{0\}} \sup_{\alpha\geq 0} J(\alpha u) &= \inf_{u\in H\setminus\{0\}} \left\{ \frac{\alpha_0^2}{2}\|u\|_H^2 - \frac{\alpha_0^p}{p}\|u\|_p^p \right\} \\ &= \frac{p-2}{2p} \inf_{u\in H\setminus\{0\}} \left(\frac{\|u\|_H}{\|u\|_p} \right)^{2p/(p-2)} = \frac{p-2}{2p} B_0^{2p/(2-p)}. \qquad \square\end{aligned}$$

3. Initial data at high energy level

In this section, we discuss properties of blowing-up solution to Problem (1.1) for initial data at the high energy level ($E(0) \geq d$). Let μ_1 be the first eigenvalue of the following problem

$$\begin{cases} \Delta^2 \psi = \mu\psi, & x \in \Omega, \\ \psi = \Delta\psi = 0, & x \in \partial\Omega, \end{cases}$$

and set $B_1 = \min\{\mu_1, (p+2)/(p-2)\}$. Our main results are as follows:

Theorem 3.1. *Assume that the following conditions are fulfilled:*

(H_3) $p > 2$, $u_0 \in H$ *and* $u_1 \in L^2(\Omega)$;

(H_4) $\displaystyle\int_\Omega u_0 u_1\,dx \geq \frac{\alpha(0) + \omega B_1 + 4p}{2(p-2)B_1} E(0) > 0$;

(H_5) $\displaystyle\int_\Omega |u_0|^2\,dx > \frac{pE(0)}{(p-2)B_1}$.

Then the solution of Problem (1.1) *blows up in a finite time* $T^* > 0$, *that is,*

$$\lim_{t\to T^*} \|u\|_p^p = +\infty.$$

Moreover, T^* *satisfies*

$$T^* \leq \frac{2\big[(p-2)B_1\|u_0\|_2^4 - pE(0)\|u_0\|_2^2 + (p+1)\big(\int_\Omega \big(\alpha(0)u_0^2 + \omega|\nabla u_0|^2\big)\,dx\big)^2\big]}{\big((p-2)^2 B_1\|u_0\|_2^2 - p(p-2)E(0)\big)\big(\int_\Omega \big(\alpha(0)u_0^2 + \omega|\nabla u_0|^2\big)\,dx\big)}.$$

Proof. **Step 1.** We first consider the monotonicity of the norm $\|u\|_2$. Define $F(t) = \int_\Omega u u_t\,dx$. Then

$$\begin{aligned} F'(t) &= \int_\Omega |u_t|^2\,dx + \int_\Omega u u_{tt}\,dx = \int_\Omega |u_t|^2\,dx - \int_\Omega |\Delta u|^2\,dx \\ &\quad - \int_\Omega |\nabla u|^2\,dx - \omega\int_\Omega \nabla u \nabla u_t\,dx - \int_\Omega \alpha(t) u u_t\,dx + \int_\Omega |u|^p\,dx \end{aligned}$$

$$\geq \frac{p+2}{2}\int_{\Omega}|u_t|^2\,dx + \left[\frac{p-2}{2} - \frac{\delta}{2}\left(\omega + \frac{\alpha(0)}{B_1}\right)\right] B_1\|u\|_2^2$$
$$- \frac{1}{2\delta}\left(\omega\int_{\Omega}|\nabla u_t|^2\,dx + \int_{\Omega}\alpha(t)u_t^2\,dx\right) - pE(t).$$

We choose $\delta = (p-2)B_1/(\alpha(0)+\omega B_1+4p)$, then

$$\begin{aligned}
&\frac{d}{dt}\left[F(t) - \frac{\alpha(0)+\omega B_1+4p}{2(p-2)B_1}E(t)\right]\\
&\geq \left[\frac{p-2}{2} - \frac{\delta(\omega B_1+\alpha(0))}{2B_1}\right] B_1\left[F(t) - \frac{\alpha(0)+\omega B_1+4p}{2(p-2)B_1}E(t)\right]\\
&= M_0\left[F(t) - \frac{\alpha(0)+\omega B_1+4p}{2(p-2)B_1}E(t)\right].
\end{aligned}$$

At last, Gronwall's inequality and the condition (H_4) show that

$$\begin{aligned}
\int_{\Omega} u(\,\cdot\,,t)u_t(\,\cdot\,,t)\,dx \geq e^{M_0 t}&\left[\int_{\Omega} u_0(\cdot)u_1(\cdot)\,dx - \frac{\alpha(0)+\omega B_1+4p}{2(p-2)B_1}E(0)\right]\\
&+ \frac{\alpha(0)+\omega B_1+4p}{2(p-2)B_1}E(t),
\end{aligned} \tag{3.1}$$

where $M_0 = 2p(p-2)B_1/(\omega B_1+\alpha(0)+4p) > 0$.

Step 2. Similar to the proof of Theorem 2.3, we also define

$$\begin{aligned}
\varphi(t) &= \int_{\Omega}|u(\,\cdot\,,t)|^2\,dx + \int_0^t\int_{\Omega}\left(\alpha(s)u^2(\,\cdot\,,s) + \omega|\nabla u(\,\cdot\,,s)|^2\right)dxds\\
&\quad + \int_0^t\int_{\Omega}(s-t)\alpha_s(s)u^2(\,\cdot\,,s)\,dxds - t\int_{\Omega}\left(\alpha(0)u_0^2 + \omega|\nabla u_0|^2\right)dx + \beta(t+t_0)^2,
\end{aligned}$$

where β, t_0 will be determined later.

It is easy to check that

$$\varphi'(t) = 2\int_{\Omega}u_t u\,dx + 2\int_0^t\int_{\Omega}\left(\alpha(s)u_s(\,\cdot\,,s)u(\,\cdot\,,s) + \omega\nabla u(\,\cdot\,,s)\nabla u_s(\,\cdot\,,s)\right)dxds + 2\beta(t+t_0),$$

and

$$\begin{aligned}
\varphi''(t) &= 2\int_{\Omega}|u_t|^2\,dx + 2\int_{\Omega}u_{tt}u\,dx + 2\int_{\Omega}(\alpha(t)u_t u + \omega\nabla u\nabla u_t)\,dx + 2\beta\\
&= (p+2)\|u_t\|_2^2 + (p-2)\left(\int_{\Omega}|\Delta u|^2\,dx + \int_{\Omega}|\nabla u|^2\,dx\right) - 2pE(0)\\
&\quad + 2\beta + 2p\int_0^t\int_{\Omega}\left(\alpha(s)u_s^2(\,\cdot\,,s) + \omega|\nabla u_s(\,\cdot\,,s)|^2\right)dxds.
\end{aligned} \tag{3.2}$$

Next, we will divide this proof into two cases.

Case 1. We assume that $E(t) \geq 0$. Then Inequality (3.1) and the condition (H_4) indicate that

$$\frac{d}{dt}\int_{\Omega}|u|^2\,dx \geq 0.$$

Moreover, we have

$$\|u\|_2^2 - \|u_0\|_2^2 = \int_0^t \frac{d}{ds}\|u\|_2^2\,ds = 2\int_0^t\int_\Omega u(\,\cdot\,,s)u_s(\,\cdot\,,s)\,dxds \geq 0,$$

which implies

$$\|u\|_2^2 \geq \|u_0\|_2^2. \tag{3.3}$$

Next, we estimate the term $\varphi''(t)$. Applying Identity (3.2) and Inequality (3.3), it is not hard to verify that

$$\begin{aligned}\varphi''(t) \geq{}& (p+2)\|u_t\|_2^2 + 2(p-2)B_1\|u_0\|_2^2 - 2pE(0) + 2\beta \\ &+ 2p\int_0^t\int_\Omega \big(\alpha(s)u_s^2(\,\cdot\,,s) + \omega|\nabla u_s(\,\cdot\,,s)|^2\big)\,dxds.\end{aligned}$$

Moreover, we choose $\beta = [(p-2)B_1\|u_0\|_2^2 - pE(0)]/(p+1) > 0$, $t_0 = \big[\int_\Omega(\alpha(0)u_0^2 + \omega|\nabla u_0|^2)\,dx\big]/\beta > 0$ to obtain

$$\varphi''(t) \geq (p+2)\|u_t\|_2^2 + 2(p+2)\beta + 2p\int_0^t\int_\Omega \big(\alpha(s)u_s^2(\,\cdot\,,s) + \omega|\nabla u_s(\,\cdot\,,s)|^2\big)\,dxds.$$

The rest of the proof is similar to those of Theorem 2.3, we omit it here.

Case 2. There exists $t_1 > 0$ such that $E(t_1) < 0$. This proof is similar to the argument of Theorem 2.3. We omit it here. □

It is worth pointing out that the principle significance of the condition (H_5) is that it allows us to establish an explicit upper bound of blow-up time. In fact, if this condition is removed, we also prove the nonexistence of solutions.

Theorem 3.2. *Assume the following conditions are satisfied*

(H_6) $p > 2$, $u_0 \in H$ *and* $u_1 \in L^2(\Omega)$*;*

(H_7) $$\int_\Omega u_0u_1\,dx > \frac{\alpha(0)+\omega B_1+4p}{2(p-2)B_1}E(0) > 0,$$

then the solution of Problem (1.1) *blows up in finite time.*

Proof. Case 1. For all $t \geq 0$, we first assume that $E(t) \geq 0$. According to (H_7) and Inequality (3.1), it is easy to check that

$$\frac{d}{dt}\int_\Omega |u|^2\,dx = 2\int_\Omega uu_t\,dx \geq 2e^{M_0t}\nu, \quad t \geq 0,$$

where $\nu = \int_\Omega u_0u_1\,dx - \frac{\alpha(0)+\omega B_1+4p}{2(p-2)B_1}E(0) > 0$.

Assume by contradiction that the solution u is global. Then, it is easily seen that

$$
\begin{aligned}
\|u(\,\cdot\,,t)\|_2 &= \|u_0\|_2 + 2\int_0^t\int_\Omega u(\,\cdot\,,\tau)u_\tau(\,\cdot\,,\tau)\,dxd\tau \geq \|u_0\|_2 + 2\int_0^t e^{M_0\tau}\nu\,d\tau \\
&= \|u_0\|_2 + \frac{2\nu}{M_0}\left(e^{M_0 t}-1\right).
\end{aligned} \tag{3.4}
$$

On the other hand, by Theorem 2.1, Minkowski inequality and Hölder inequality, we have

$$
\begin{aligned}
\|u(\,\cdot\,,t)\|_2 &\leq \|u_0\|_2 + \|u(\,\cdot\,,t)-u_0\|_2 \leq \|u_0\|_2 + \left\|\int_0^t u_\tau\,d\tau\right\|_2 \\
&\leq \|u_0\|_2 + \int_0^t \|u_\tau\|_2\,d\tau \leq \|u_0\|_2 + \frac{1}{\sqrt{B_1}}\int_0^t \|\nabla u_\tau\|_2\,d\tau \\
&\leq \|u_0\|_2 + \frac{\sqrt{t}}{\sqrt{B_1}}\left(\int_0^t\int_\Omega |\nabla u_\tau|^2\,dxd\tau\right)^{1/2} \\
&\leq \|u_0\|_2 + \frac{\sqrt{t}}{\sqrt{B_1\omega}}(E(0)-E(t))^{1/2} \leq \|u_0\|_2 + \sqrt{\frac{tE(0)}{B_1\omega}}.
\end{aligned} \tag{3.5}
$$

Applying the fact $\lim_{t\to+\infty}\left[2\nu/M_0 + \sqrt{tE(0)/(B_1\omega)}\right]e^{-M_0 t} = 0$ and inequalities (3.4) and (3.5), we have

$$
\frac{2\nu}{M_0} \leq 0,
$$

which contradicts $\nu > 0$ and $M_0 > 0$.

Case 2. There exists $t_1 > 0$ such that $E(t_1) < 0$. This proof is similar to the argument of the second case of Theorem 3.1. We omit it here. □

4. Lower bounds for lifespan time

In this section, we mainly give some estimates of lower bounds for the blow-up time T^* when $2(N-2)/(N-4) < p \leq 2N/(N-4)$ ($N \geq 5$). Especially, in this case, the methods used in [17, 18] are not applicable due to the failure of the embedding inequality $H^2 \cap H_0^1(\Omega) \hookrightarrow L^{2p-2}(\Omega)$.

Theorem 4.1. *If all the conditions of Theorem* 2.3 *and the following conditions are satisfied*

$$
\frac{2(N-2)}{N-4} < p \leq \frac{2(N-1)}{N-4}, \quad N \geq 5,
$$

then the lifespan T^* satisfies the following estimate

$$
T^* \geq \frac{M^{2-p}(u_0,u_1)}{(p-2)}\frac{C^2\omega}{(2B_0^2)^{p-1}},
$$

where

$$
C^2 = \frac{1}{\pi N(N-2)}\left[\frac{\Gamma(N)}{\Gamma(\frac{N}{2})}\right]^{2N}, \quad M^{2-p}(u_0,u_1) = \left[\int_\Omega \left(|u_1|^2 + |\nabla u_0|^2 + |\Delta u_0|^2\right)dx\right]^{2-p}.
$$

Proof. **Step 1.** Define

$$H(t)=\frac{1}{p}\int_\Omega |u|^p\,dx+\frac{1}{2}\int_\Omega \left(|u_t|^2+|\nabla u|^2+|\Delta u|^2\right)dx.$$

The fact $\lim_{t\to T^*}\|u\|_p=+\infty$ implies

$$\lim_{t\to T^*} H(t)=+\infty. \tag{4.1}$$

Step 2. A direct computation shows that

$$H'(t)=\int_\Omega |u|^{p-2}uu_t\,dx+\int_\Omega \left(u_tu_{tt}+\nabla u\nabla u_t+\Delta u\Delta u_t\right)dx.$$

Furthermore, we apply (2.3) and the definition of $E(t)$ to obtain

$$H'(t)=2\int_\Omega |u|^{p-2}uu_t\,dx-\int_\Omega \alpha(t)|u_t|^2\,dx-\omega\int_\Omega |\nabla u_t|^2\,dx. \tag{4.2}$$

On the other hand, applying Hölder inequality and Sobolev embedding inequality, we get

$$\begin{aligned}
&\left|2\int_\Omega |u|^{p-2}uu_t\,dx\right|\\
&\le 2\int_\Omega |u|^{p-1}|u_t|\,dx\\
&\le 2\left(\int_\Omega |u|^{2(p-1)N/(N+2)}\,dx\right)^{(N+2)/(2N)}\left(\int_\Omega |u_t|^{2N/(N-2)}\,dx\right)^{(N-2)/(2N)}\\
&\le 2C\left(\int_\Omega |u|^{2(p-1)N/(N+2)}\,dx\right)^{(N+2)/(2N)}\left(\int_\Omega |\nabla u_t|^2\,dx\right)^{1/2}\\
&\le (C\sqrt{\omega})^{-2}\left(\int_\Omega |u|^{2(p-1)N/(N+2)}\,dx\right)^{(N+2)/N}+\omega\int_\Omega |\nabla u_t|^2\,dx,
\end{aligned} \tag{4.3}$$

where $C^2=\frac{1}{\pi N(N-2)}\left[\Gamma(N)/\Gamma(N/2)\right]^{2N}$. According to $2(N-2)/(N-4)<p\le 2(N-1)/(N-4)$ $(N\ge 5)$, we apply Inequality (2.1) to get

$$\left(\int_\Omega |u|^{2(p-1)N/(N+2)}\,dx\right)^{(N+2)/N}\le B_0^{2(p-1)}\left[\int_\Omega \left(|\nabla u|^2+|\Delta u|^2\right)dx\right]^{p-1}. \tag{4.4}$$

Once again using (2.3) and the definition of $E(t)$, we have

$$\frac{1}{2}\int_\Omega \left(|\nabla u|^2+|\Delta u|^2\right)dx\le \frac{1}{p}\int_\Omega |u|^p\,dx+E(0). \tag{4.5}$$

By (4.2)–(4.5), we have

$$H'(t)\le \frac{(2B_0^2)^{p-1}}{C^2\omega}(H(t)+E(0))^{p-1}. \tag{4.6}$$

Finally, we combining (4.1) with (4.6) to get

$$\int_{H(0)+E(0)}^\infty \xi^{1-p}\,d\xi\le \frac{(2B_0^2)^{p-1}}{C^2\omega}T^* \quad\Longrightarrow\quad T^*\ge \frac{M^{2-p}(u_0,u_1)}{(p-2)}\frac{C^2\omega}{(2B_0^2)^{p-1}}. \qquad\square$$

From the process of the above proof, it is obvious to find that the embedding theorem $H \hookrightarrow L^{2N(p-1)/(N+2)}$ fails when $2(N-1)/(N-4) < p \leq 2N/(N-4)$, which leads to the above method is not applicable. However, we have to look for a new technique or construct a new control functional to bypass this difficulty. In what follows, we only consider the case when the spatial dimension N is bigger than four. First of all, we give a useful lemma.

Lemma 4.2. *Assume that u is the solution to Problem* (1.1)*. Then the following inequality remains true*

$$\frac{1}{p}\int_\Omega |u|^p\,dx \leq C_1\left(\int_\Omega |u|^{2(N-1)/(N-4)}\,dx\right)^\alpha + \frac{1}{p-1}E(0),$$

where α, C_1 are defined as follows:

$$\alpha = \frac{(N-4)[2N-(N-4)p]}{2N^2-8-N(N-4)p} > 1, \quad \frac{2(N-1)}{N-4} < p < \frac{2(N^2-4)}{N(N-4)},$$
$$C_1 = \frac{(2B_0^p)^{\alpha/[2N-(N-4)p]}}{2(p-1)}.$$

Proof. By the interpolation inequality, we know that the following inequality holds

$$\int_\Omega |u|^p\,dx \leq B_0^p\left(\int_\Omega |u|^k\,dx\right)^{(1-\theta)p/k}\left[\int_\Omega \left(|\Delta u|^2+|\nabla u|^2\right)dx\right]^{\theta p/2}, \tag{4.7}$$

where

$$\frac{1}{p} = \frac{1-\theta}{k}+\frac{\theta}{2^{**}}, \quad k = \frac{2(N-1)}{N-4}, \quad 2^{**} = \frac{2N}{N-4}, \quad N \geq 5.$$

Noticing that $2(N-1)/(N-4) < p < 2(N^2-4)/[N(N-4)]$ $(N \geq 5) \Longrightarrow 0 < \theta p/2 < 1$ and combining Young inequality with (4.7), we have

$$\int_\Omega |u|^p\,dx \leq B_0^p\varepsilon^{2/(\theta p-2)}\left(\int_\Omega |u|^k\,dx\right)^{2(1-\theta)p/[k(2-\theta p)]} + B_0^p\varepsilon^{2/(\theta p)}\int_\Omega \left(|\Delta u|^2+|\nabla u|^2\right)dx. \tag{4.8}$$

And then, we choose $B_0^p\varepsilon^{2/(\theta p)} = 1/2$ and utilize inequalities (4.5) and (4.8) to obtain the following inequality

$$\int_\Omega |u|^p\,dx \leq \frac{(2B_0^p)^{2/(2-\theta p)}}{2}\left(\int_\Omega |u|^k\,dx\right)^{2(1-\theta)p/[k(2-\theta p)]} + \frac{1}{p}\int_\Omega |u|^p\,dx + E(0).$$

This completes the proof of Lemma 4.2. □

Theorem 4.3. *If all the conditions of Theorem* 2.3 *and the following conditions are satisfied*

$$\frac{2(N-1)}{N-4} < p < \frac{2(N^2-4)}{N(N-4)}, \quad N \geq 5,$$

then the lifespan T^* satisfies that the following estimate

$$T^* \geq \frac{1}{M_0(\beta-1)}\left(Z(0)+\frac{2p\varepsilon_0 E(0)}{(p-1)\omega}\right)^{1-\beta},$$

where

$$M_0 = \frac{C^2\alpha^2(N-1)^2}{2^{2\alpha-4}(N+4)^2\varepsilon_0}\left(\frac{2\omega B_0^2}{\varepsilon_0}\right)^{(2N+4)/(N-4)}, \quad Z^{1/\alpha}(0)=\int_\Omega |u_0|^{(2N-2)/(N-4)}\,dx,$$

$$\varepsilon_0 = \frac{(p-1)\omega}{(2B_0^p)^{\alpha/[2N-(N-4)p]}}, \quad C^2 = \frac{1}{\pi N(N-2)}\left[\frac{\Gamma(N)}{\Gamma(\frac{N}{2})}\right]^{2N},$$

$$\beta = 2+\frac{8N+16-4(N-4)p}{(N-4)(2N+4p-Np)} > 2.$$

Proof. This proof will be divided into three steps.

Step 1: Equivalent Blowing-up. Define

$$Z(t) = \left(\int_\Omega |u(\,\cdot\,,t)|^k\,dx\right)^\alpha - \varepsilon_0\int_0^t\int_\Omega |\nabla u_\tau(\,\cdot\,,\tau)|^2\,dx d\tau,$$

where $k = 2(N-1)/(N-4)$, $\alpha = (N-4)[2N-(N-4)p]/[2N^2-8-N(N-4)p]$, $\varepsilon_0 = (p-1)\omega/(2B_0^p)^{\alpha/[2N-(N-4)p]}$.

By (2.3) and Lemma 4.2, we have

$$\omega\int_0^t\int_\Omega |\nabla u_\tau|^2\,dx d\tau \leq \frac{pE(0)}{p-1}+\frac{\omega}{2\varepsilon_0}\left(\int_\Omega |u|^k\,dx\right)^\alpha.$$

Moreover, we apply the above inequality to obtain

$$Z(t) \geq \frac{1}{2}\left(\int_\Omega |u|^k\,dx\right)^\alpha - \frac{p\varepsilon_0 E(0)}{(p-1)\omega}. \tag{4.9}$$

Combining the above inequality with the conclusion of Theorem 2.3, it is easy to prove that

$$\lim_{t\to T^*} Z(t) = +\infty.$$

Step 2. A direct computation shows that

$$Z'(t) = \alpha k\left(\int_\Omega |u|^k\,dx\right)^{\alpha-1}\int_\Omega |u|^{k-2}uu_t\,dx - \varepsilon_0\int_\Omega |\nabla u_t|^2\,dx. \tag{4.10}$$

Step 3. It is easily seen that Young's inequality, the Sobolev embedding inequality and Identity (2.3) yield

$$
\begin{aligned}
&\alpha k \left(\int_\Omega |u|^k\,dx\right)^{\alpha-1} \int_\Omega |u|^{k-2}uu_t\,dx \\
&\le \alpha k \left(\int_\Omega |u|^k\,dx\right)^{\alpha-1} \||u|^{k-1}\|_{2N/(N+2)}\|u_t\|_{2N/(N-2)} \\
&\le C\alpha k \left(\int_\Omega |u|^k\,dx\right)^{\alpha-1} \||u|^{k-1}\|_{2N/(N+2)}\|\nabla u_t\|_2 \\
&\le \frac{C^2\alpha^2k^2}{\varepsilon_0} \left(\int_\Omega |u|^k\,dx\right)^{2\alpha-2} \||u|^{k-1}\|_{2N/(N+2)}^2 + \varepsilon_0\|\nabla u_t\|_2^2 \\
&\le \frac{C^2\alpha^2k^2}{\varepsilon_0} \left(\int_\Omega |u|^k\,dx\right)^{2\alpha-2} \left(\frac{2B_0^2}{p}\int_\Omega |u|^p\,dx + 2B_0^2E(0)\right)^{(2N+4)/(N-4)} \\
&\quad + \varepsilon_0\|\nabla u_t\|_2^2.
\end{aligned}
\tag{4.11}
$$

Utilizing (4.11) and Lemma 4.2, we have

$$
\begin{aligned}
&\alpha k \left(\int_\Omega |u|^k\,dx\right)^{\alpha-1} \int_\Omega |u|^{k-2}uu_t\,dx \\
&\le \varepsilon_0 \int_\Omega |\nabla u_t|^2\,dx + M_0 \left(\int_\Omega |u|^k\,dx\right)^{2\alpha-2} \left(\frac{1}{2}\int_\Omega |u|^k\,dx + \frac{p\varepsilon_0E(0)}{(p-1)\omega}\right)^{(2N+4)/(N-4)}.
\end{aligned}
\tag{4.12}
$$

Furthermore, Inequalities (4.9) and (4.12) as well as Identity (4.10) show that

$$
Z'(t) \le M_0 \left[Z(t) + \frac{2p\varepsilon_0E(0)}{(p-1)\omega}\right]^\beta, \quad \beta = \frac{2N+4}{N-4} + 2 - \frac{2}{\alpha},
$$

which shows that

$$
T^* \ge \frac{1}{M_0(\beta-1)} \left(Z(0) + \frac{2p\varepsilon_0E(0)}{(p-1)\omega}\right)^{1-\beta}.
$$

This completes the proof of this theorem. □

Remark 4.4. Since the embedding relationship $H \hookrightarrow L^q(\Omega)$ ($1 \le q < \infty$, if $N = 1,2,3,4$) holds, we may follow our method used in this paper or [10,17,18] to obtain similar results. However, when $p \in \big[2(N^2-4)/[N(N-4)], 2N/(N-4)\big]$, it seems that we can not obtain similar results as Lemma 4.2. So, we need to develop a new method or technique to discuss this problem.

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