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Ball Average Characterizations of Variable Besov-type Spaces

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Abstract. In this article, the authors characterize the variable Besov-type spaces $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, with $1/p(\cdot)$ and $1/q(\cdot)$ satisfying the globally log-Hölder continuous conditions, via Peetre maximal functions and averages on balls. The latter characterization, via averages on balls, gives one way to introduce these spaces on metric measure spaces.

1. Introduction

The main purpose of this article is to establish some new characterizations of the variable Besov-type space $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ with $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ as in (1.4) below. Recall that the variable Besov-type space $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ was first introduced in [46], however, the variable exponents $p(\cdot)$ and $q(\cdot)$ in [46] are required to belong to $C^{\log}(\mathbb{R}^n)$ as in (1.2) and (1.3) below, which is stronger than $\mathcal{P}^{\log}(\mathbb{R}^n)$ (see Remark 1.6(ii) below).

In the last decade, motivated by the articles [29] of Kováčik and Rákosník and [23] of Fan and Zhao as well as [11] of Cruz-Uribe and [16] of Diening, real-variable theories of function spaces with variable exponents, especially based on Besov and Triebel-Lizorkin spaces, have been rapidly developed (see, for instance, [4,6,18,20,22,32–35,39,40,44–46]). Precisely, in 2008, Xu [39,40] studied Besov spaces $B^s_{p(\cdot),q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $F^s_{p(\cdot),q}(\mathbb{R}^n)$ with variable exponent $p(\cdot)$ but with fixed q and s. The concept of function spaces with variable smoothness and variable integrability was firstly mixed up by Diening, Hästö and Roudenko in [18], in which the variable Triebel-Lizorkin spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ were introduced. Later, Almeida and Hästö introduced the variable Besov spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, where $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$. Moreover, it turns out that these spaces behave nicely with respect to the trace operator (see [18, Theorem 3.13], [7, Theorem 5.2] and [34, Theorem 5.1]). Here we point out that the vector-valued convolution inequalities, developed in [6, Lemma 4.7] and [18, Theorem 3.2], supply well remedy for the

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absence of the Fefferman-Stein vector-valued inequalities on the Hardy-Littlewood maximal operator, in the setting of the mixed Lebesgue sequence spaces $\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))$ and $L^{p(\cdot)}(\ell^{q(\cdot)}(\mathbb{R}^n))$, respectively, when studying Besov spaces and Triebel-Lizorkin spaces with variable smoothness and integrability.

Based on Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin-type spaces $F^{s,\tau}_{p,q}(\mathbb{R}^n)$ studied in [47], as generalizations of the Besov and the Triebel-Lizorkin spaces with variable smoothness and integrability, the Besov-type and the Triebel-Lizorkin-type spaces with variable exponents were also introduced in [45, 46]. However, as was mentioned above, when studying the variable Besov-type space in [46], the variable exponents $p(\cdot)$, $q(\cdot)$ are required to satisfy the globally log-Hölder continuous conditions $C^{\log}(\mathbb{R}^n)$, which is a little bit stronger than $\mathcal{P}^{\log}(\mathbb{R}^n)$ adopted in [46]. So it is a natural and interesting question to study the variable Besov-type space under the assumption that $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$. We should point out that variable function spaces have found their applications in fluid dynamics [1,2,36], image processing [10,26,38], partial differential equations and variational calculus [8,19,37] and harmonic analysis [5,12,17,41].

On another hand, motivated by a new characterization of Sobolev spaces obtained in [3], there exist some attempts to characterize Besov(-type) spaces and Triebel-Lizorkin(-type) spaces on $\mathbb{R}^n$ via ball averages (see, for instance, [9,13–15,25,42,43,48,49]). These ball averages, used in such new characterizations, only depend on the metric of $\mathbb{R}^n$ and the Lebesgue measure and hence these new characterizations provide some possible ways to introduce the corresponding function spaces with positive smoothness on metric measure spaces.

In this article, we aim to introduce and develop the Besov-type space with variable exponent $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ for any locally log-Hölder continuous functions $s(\cdot) \in L^\infty(\mathbb{R}^n)$ and measurable functions ϕ on $\mathbb{R}^{n+1}_+$, under the assumption $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, which is weaker than $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n)$ used in [46]. In this sense, the Besov-type spaces considered in this article have more generality than those in [46]. As the main result of this article, we characterize the spaces $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ by means of Peetre maximal functions and averages on balls, and the latter one is new even when $\phi \equiv 1$ and gives a way to introduce the variable Besov-type spaces on metric measure spaces. To limit the length of this article, we leave the study of characterizations of Triebel-Lizorkin-type spaces with variable exponents via averages on balls in a forthcoming article.

We begin with some basic notation and notions. Denote by $\mathcal{P}(\mathbb{R}^n)$ the *collection of all variable exponent functions* $p(\cdot)\colon \mathbb{R}^n \to [0,\infty]$ satisfying

$$0 < \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) =: p_- \le p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) \le \infty. \tag{1.1}$$

For any $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define the function $\varphi_{p(x)}$ by setting, for any $t \in [0, \infty)$,

$$\varphi_{p(x)}(t) := \begin{cases} t^{p(x)} & \text{if } p(x) \in (0, \infty), \\ 0 & \text{if } p(x) = \infty \text{ and } t \in [0, 1], \\ \infty & \text{if } p(x) = \infty \text{ and } t \in (1, \infty). \end{cases}$$

The *variable exponent modular*, with respect to $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, of a measurable function f on $\mathbb{R}^n$ is defined by setting

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} \varphi_{p(x)}(|f(x)|)\, dx.$$

Definition 1.1. Let $p \in \mathcal{P}(\mathbb{R}^n)$ and E be a measurable subset of $\mathbb{R}^n$. Then the *variable Lebesgue space* $L^{p(\cdot)}(E)$ is defined to be the set of all measurable functions f such that

$$\|f\|_{L^{p(\cdot)}(E)} := \inf\{\lambda \in (0, \infty) : \varrho_{p(\cdot)}(f\mathbf{1}_E/\lambda) \le 1\} < \infty,$$

here and hereafter, for any subset $E \subset \mathbb{R}^n$, $\mathbf{1}_E$ denotes its *characteristic function*.

Remark 1.2. Let $p \in \mathcal{P}(\mathbb{R}^n)$. If $p_- \in [1, \infty]$, then $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space (see [17, Theorem 3.2.7]). In particular, for any $\lambda \in \mathbb{C}$, $\|\lambda f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = |\lambda| \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ and, for any $f, g \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\|f + g\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \|g\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

For more properties of variable Lebesgue spaces, we refer the reader to the monographs [12, 17]. Next, we recall the mixed Lebesgue space $\ell^{q(\cdot)}(L^{p(\cdot)}(E))$ introduced by Almeida and Hästö in [6].

Definition 1.3. Let $p, q \in \mathcal{P}(\mathbb{R}^n)$ and E be a measurable subset of $\mathbb{R}^n$. Then the *mixed Lebesgue-sequence space* $\ell^{q(\cdot)}(L^{p(\cdot)}(E))$ is defined to be the set of all sequences $\{f_v\}_{v\in\mathbb{N}}$ of functions in $L^{p(\cdot)}(E)$ such that

$$\|\{f_v\}_{v\in\mathbb{N}}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(E))} := \inf\{\lambda \in (0, \infty) : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{f_v\mathbf{1}_E/\lambda\}_{v\in\mathbb{N}}) \le 1\} < \infty,$$

where, for any sequence $\{g_v\}_{v\in\mathbb{N}}$ of measurable functions on $\mathbb{R}^n$,

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}(\{g_v\}_{v\in\mathbb{N}}) := \sum_{v\in\mathbb{N}} \inf\{\mu_v \in (0, \infty) : \varrho_{p(\cdot)}(g_v/\mu_v^{1/q(\cdot)}) \le 1\}$$

with the convention $\lambda^{1/\infty} = 1$ for any $\lambda \in (0, \infty)$.

Remark 1.4. Let $p, q \in \mathcal{P}(\mathbb{R}^n)$.

(i) Let $\{g_v\}_{v\in\mathbb{N}}$ be a sequence of functions in $L^{p(\cdot)}(\mathbb{R}^n)$. By [6, Example 3.4], we know that, if, for any $v \in \{2, 3, \ldots\}$, $g_v \equiv 0$, then

$$\|\{g_v\}_{v\in\mathbb{N}}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))} = \|g_1\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

(ii) $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))}$ is a quasi-norm on $\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))$ (see [6, Theorem 3.8]); if either $1/p(x)+1/q(x)\le 1$ or q is a constant, then $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))}$ is a norm (see [6, Theorem 3.6]); if either $p(x)\ge 1$ and $q\in[1,\infty)$ is a constant almost everywhere or $1\le q(x)\le p(x)\le\infty$ for almost every $x\in\mathbb{R}^n$, then $\|\cdot\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))}$ is also a norm (see [28, Theorem 1]).

(iii) By [6, Proposition 3.3], we know that, if $q\in(0,\infty]$ is a constant, then

$$\|\{g_v\}_{v\in\mathbb{N}}\|_{\ell^q(L^{p(\cdot)}(\mathbb{R}^n))}=\left\{\sum_{v\in\mathbb{N}}\|g_v\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right\}^{1/q}$$

with the usual modification made when $q=\infty$.

(iv) It is easy to see that, for any sequence $\{g_v\}_{v\in\mathbb{N}}$ of measurable functions on $\mathbb{R}^n$ and $r\in(0,\infty)$,

$$\|\{g_v\}_{v\in\mathbb{N}}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))}=\|\{|g_v|^r\}_{v\in\mathbb{N}}\|^{1/r}_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r}(\mathbb{R}^n))}.$$

A function $g\colon\mathbb{R}^n\to\mathbb{R}$ is said to satisfy the *locally* log-*Hölder continuous condition*, denoted by $g\in C^{\log}_{\rm loc}(\mathbb{R}^n)$, if there exists a positive constant $C_{\log}(g)$ such that, for any $x,y\in\mathbb{R}^n$,

$$|g(x)-g(y)|\le\frac{C_{\log}(g)}{\log(e+1/|x-y|)};\tag{1.2}$$

moreover, g is said to satisfy the *globally* log-*Hölder continuous condition*, denoted by $g\in C^{\log}(\mathbb{R}^n)$, if $g\in C^{\log}_{\rm loc}(\mathbb{R}^n)$ and there exist constants $C_\infty(g)\in(0,\infty)$ and $g_\infty\in\mathbb{R}$ such that, for any $x\in\mathbb{R}^n$,

$$|g(x)-g_\infty|\le\frac{C_\infty(g)}{\log(e+|x|)}.\tag{1.3}$$

In what follows, we let

$$\mathcal{P}^{\log}(\mathbb{R}^n):=\left\{p(\cdot)\in\mathcal{P}(\mathbb{R}^n):\frac{1}{p(\cdot)}\in C^{\log}(\mathbb{R}^n)\right\}.\tag{1.4}$$

Here, it should be pointed out that, if $p_+\in(0,\infty)$, then it is easy to see that $p\in\mathcal{P}^{\log}(\mathbb{R}^n)$ if and only if $p\in C^{\log}(\mathbb{R}^n)$.

Let $\mathcal{G}(\mathbb{R}^{n+1}_+)$ be the set of all measurable functions $\phi\colon\mathbb{R}^{n+1}_+\to(0,\infty)$ having the following properties: there exist positive constants c_1 and c_2 such that, for any $x\in\mathbb{R}^n$ and $r\in(0,\infty)$,

$$c_1^{-1}\phi(x,2r)\le\phi(x,r)\le c_1\phi(x,2r)\tag{1.5}$$

and, for any $x,y\in\mathbb{R}^n$ and $r\in(0,\infty)$ with $|x-y|\le r$,

$$c_2^{-1}\phi(y,r)\le\phi(x,r)\le c_2\phi(y,r).\tag{1.6}$$

In what follows, for any $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$ and any cube $Q := Q(x,r) \subset \mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius $r \in (0,\infty)$, define $\phi(Q) := \phi(Q(x,r)) := \phi(x,r)$. Here we point out that (1.5) and (1.6) are called, respectively, the *doubling condition* and the *compatibility condition*, which have been used by Nakai [30, 31] and Nakai and Sawano [32] when they studied generalized Campanato spaces. There exist several examples of ϕ that satisfy (1.5) and (1.6); see [45, Remark 1.3].

Let $\mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz functions on $\mathbb{R}^n$ equipped with the well-known classical topology determined by a countable family of seminorms and $\mathcal{S}'(\mathbb{R}^n)$ its topological dual space equipped with the weak-$*$ topology. A pair of functions, (φ,Φ), is said to be *admissible* if $\varphi, \Phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$\operatorname{supp}\widehat{\varphi} \subset \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\} \quad \text{and} \quad |\widehat{\varphi}(\xi)| \ge \text{constant} > 0 \tag{1.7}$$

when $3/5 \le |\xi| \le 5/3$, and

$$\operatorname{supp}\widehat{\Phi} \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2\} \quad \text{and} \quad |\widehat{\Phi}(\xi)| \ge \text{constant} > 0 \quad \text{when } |\xi| \le 5/3, \tag{1.8}$$

where, for any $f \in \mathcal{S}(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$,

$$\widehat{f}(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\,dx$$

denotes its *Fourier transform*. For any $j \in \mathbb{N}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we put $\varphi_j(x) := 2^{jn}\varphi(2^j x)$. For any $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, denote by Q_{jk} the *dyadic cube* $2^{-j}([0,1)^n + k)$, by $x_{Q_{jk}} := 2^{-j}k$ its *lower left corner* and by $\ell(Q_{jk})$ its *side length*. Let

$$\mathcal{Q} := \{Q_{jk} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\} \tag{1.9}$$

and $j_Q := -\log_2 \ell(Q)$ for any $Q \in \mathcal{Q}$.

Now we introduce the definition of variable Besov-type spaces.

Definition 1.5. Let (φ,Φ) be a pair of admissible functions on $\mathbb{R}^n$. Let $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $s \in C^{\log}_{\rm loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$. Then the *Besov-type space with variable smoothness and integrability*, $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|^{\varphi}_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{2^{js(\cdot)}|\varphi_j * f|\right\}_{j\ge(j_P\vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} < \infty,$$

where $\mathcal{Q}$ is as in (1.9) and, when $j=0$, φ_0 is replaced by Φ, and the supremum is taken over all dyadic cubes P in $\mathbb{R}^n$.

Remark 1.6. (i) Since the quasi-norm $\|f\|^{\varphi}_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$ is independent of the choice of the admissible function pair (φ,Φ) satisfying (1.7) and (1.8), which will be proved in Theorem 2.1 below, we usually denote $\|f\|^{\varphi}_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$ simply by $\|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$.

(ii) We point out that, in [46], Yang et al. have introduced the variable Besov-type space $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. However, the variable exponents $p(\cdot)$ and $q(\cdot)$ in [46] are required to satisfy the globally log-Hölder continuous condition $C^{\log}(\mathbb{R}^n)$, which seems to be a little bit stronger than those used in Definition 1.5. Indeed, if $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $p_+ = \infty$, then $p(\cdot) \equiv \infty$, while if $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $p_+ = \infty$, then $p(\cdot)$ may not be equal to infinity almost everywhere, one of such examples is $p(x) := \log(e + |x|)$ for any $x \in \mathbb{R}^n$ (see [17, p. 103]). Moreover, by taking $\phi \equiv 1$, the space in Definition 1.5 completely goes back to the Besov space, with variable smoothness and integrability, $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ introduced by Almeida and Hästö in [6].

(iii) Let $\phi(Q) := \|\mathbf{1}_Q\|_{L^{\tau(\cdot)}(\mathbb{R}^n)}$ with $\tau \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $\tau_- \in (0, \infty)$, where τ_- is as in (1.1) with p replaced by τ. Then, from [50, Lemma 2.6], we deduce that $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$. In this case, the space $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is just the space $B^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ introduced and studied by Drihem [21, 22], which is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|_{L^{\tau(\cdot)}(\mathbb{R}^n)}} \left\| \left\{2^{js(\cdot)}|\varphi_j * f|\right\}_{j\geq(j_P\vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} < \infty.$$

This article is organized as follows.

In Section 2, we prove that the space $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is independent of the choice of admissible function pairs (φ, Φ) (see Theorem 2.1 below), via establishing a convolution-type vector-valued inequality in Lemma 2.2 below under the setting of this article. The Calderón reproducing formula also plays an important role in the proof of Theorem 2.1.

In Section 3, we first characterize the space $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ by means of Peetre maximal functions in Theorem 3.1 below via using the r-trick lemma obtained in [18, Lemma A.6] (see also Lemma 3.2 below) and Lemma 2.2. Secondly, we establish a new characterization of the space $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ in terms of averages on balls in Theorem 3.5 below. To prove Theorem 3.5, we obtain a key pointwise estimate for some operators via involving the decay function: for any given $v \in \mathbb{Z}_+$ and $m \in (0, \infty)$,

$$\eta_{v,m}(x) := \frac{2^{vn}}{(1 + 2^v|x|)^m}, \quad \forall\, x \in \mathbb{R}^n, \tag{1.10}$$

in Lemma 3.10 below. In the proofs of Theorems 3.1 and 3.5, the convolution-type vector-valued inequality in Lemma 2.2 is repeatedly used.

We point out that, recently, Drihem [22, Theorem 4.9(i)] established an equivalent characterization of the variable Besov-type space $B^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ in terms of ball means of differences via some different methods from the ones used in this article. Compared with [22, Theorem 4.9(i)], Theorem 3.5 below has an advantage that the smoothness index $s(\cdot)$ has an essentially wider range; see Remark 3.11 below for more details.

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ and $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$. We denote by C a *positive constant* which is independent of the main parameters, but may vary from line to line. The *symbol* $f \lesssim g$ means $f \le Cg$. If $f \lesssim g$ and $g \lesssim f$, we then write $f \sim g$. We also use the following convention: If $f \le Cg$ and $g = h$ or $g \le h$, we then write $f \lesssim g \sim h$ or $f \lesssim g \lesssim h$, *rather than* $f \lesssim g = h$ or $f \lesssim g \le h$. If E is a subset of $\mathbb{R}^n$, we denote by $\mathbf{1}_E$ its *characteristic function.* For any $x \in \mathbb{R}^n$ and $r \in (0,\infty)$, denote by $Q(x,r)$ the cube centered at x with side length r, whose sides are parallel to the axes of coordinates. For any $a, b \in \mathbb{R}$, let $a \vee b := \max\{a, b\}$. For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we use $\varphi^\vee$ to denote its *inverse Fourier transform*, which is defined by setting $\varphi^\vee(\xi) := \widehat{\varphi}(-\xi)$ for any $\xi \in \mathbb{R}^n$.

2. Independence of choices of (φ, Φ)

In this section, we show that the space $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ in Definition 1.5 is independent of the choice of the admissible function pairs (φ, Φ).

Theorem 2.1. *Let $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $s \in C^{\log}_{\rm loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$. Then the space $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ in Definition 1.5 is independent of the choice of the admissible function pairs (φ, Φ) as in (1.7) and (1.8).*

To prove Theorem 2.1, we begin with the following convolution-type vector-valued inequality, which generalizes [6, Lemma 4.7] (see also [27, Lemma 10]) by taking $\phi \equiv 1$. For any given $v \in \mathbb{Z}_+$, $m \in (0,\infty)$ and any $x \in \mathbb{R}^n$, let $\eta_{v,m}(x)$ be as in (1.10).

Lemma 2.2. *Let $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$ and $p, q \in \mathcal{P}^{\log}(\mathbb{R}^n)$. Suppose that $p_-, q_- \in [1,\infty]$ and $m \in (2n + C_{\log}(1/q) + 2\log_2 c_1, \infty)$, where p_- is as in (1.1), q_- as in (1.1) with p replaced by q, $C_{\log}(1/q)$ as in (1.2) with g replaced by $1/q$ and c_1 as in (1.5). Then there exists a positive constant C such that, for any sequence $\{f_v\}_{v\in\mathbb{Z}_+}$ of measurable functions,*

$$\sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \|\{\eta_{v,m} * f_v\}_{v\ge(j_P\vee 0)}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \le C \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \|\{f_v\}_{v\ge(j_P\vee 0)}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))},$$

where $\mathcal{Q}$ is as in (1.9).

Proof. For any given dyadic cube $Q \in \mathcal{Q}$, any $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$, we write

$$f_j(x) = \sum_{l\in\mathbb{Z}^n} [f_j(x)\mathbf{1}_{Q+l\ell(Q)}(x)].$$

Then, by choosing $r \in \left(0, \frac{1}{2}\min\{p_-, q_-, 2\}\right]$, together with (ii) and (iv) of Remark 1.4 and the well-known inequality that, for any $d \in (0,1]$ and $\{a_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$,

$$\left(\sum_{j\in\mathbb{N}} |a_j|\right)^d \le \sum_{j\in\mathbb{N}} |a_j|^d, \tag{2.1}$$

we find that

$$
\begin{aligned}
&\frac{1}{\phi(Q)}\left\|\{\eta_{j,m}*f_j\}_{j\geq(j_Q\vee 0)}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(Q))}\\
&\leq\frac{1}{\phi(Q)}\left\|\left\{\sum_{l\in\mathbb{Z}^n}[\eta_{j,m}*\{|f_j|\mathbf{1}_{Q+l\ell(Q)}\}]^r\right\}_{j\geq(j_Q\vee 0)}\right\|^{1/r}_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r}(Q))}\\
&\lesssim\frac{1}{\phi(Q)}\left\{\sum_{l\in\mathbb{Z}^n}\left\|\{\eta_{j,m}*[|f_j|\mathbf{1}_{Q+l\ell(Q)}]\}_{j\geq(j_Q\vee 0)}\right\|^r_{\ell^{q(\cdot)}(L^{p(\cdot)}(Q))}\right\}^{1/r}.
\end{aligned}\tag{2.2}
$$

Observe that, for any $l\in\mathbb{Z}^n$, $j\in\mathbb{Z}_+\cap[j_Q\vee 0,\infty)$, $x\in Q$ and $y\in Q+l\ell(Q)$, we have $1+2^j|x-y|\gtrsim 1+|l|$ and hence

$$
\begin{aligned}
\eta_{j,m}*[|f_j|\mathbf{1}_{Q+l\ell(Q)}](x)&=\int_{\mathbb{R}^n}\frac{2^{jn}}{(1+2^j|x-y|)^m}|f_j(y)|\mathbf{1}_{Q+l\ell(Q)}(y)\,dy\\
&\lesssim\frac{1}{(1+|l|)^\lambda}\eta_{j,m-\lambda}*[|f_j|\mathbf{1}_{Q+l\ell(Q)}](x),
\end{aligned}
$$

where λ is a constant such that $\lambda\in(n+2\log_2 c_1,\infty)$ and $m-\lambda>n+C_{\log}(1/q)$. By this, (2.2) and the convolution-type vector-valued inequality on $\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))$ in [6, Lemma 4.7] (see also [27, Lemma 10]), we know that

$$
\begin{aligned}
&\frac{1}{\phi(Q)}\left\|\{\eta_{j,m}*f_j\}_{j\geq(j_Q\vee 0)}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(Q))}\\
&\lesssim\left\{\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-\lambda r}\frac{1}{[\phi(Q)]^r}\left\|\{\eta_{j,m-\lambda}*[|f_j|\mathbf{1}_{Q+l\ell(Q)}]\}_{j\geq(j_Q\vee 0)}\right\|^r_{\ell^{q(\cdot)}(L^{p(\cdot)}(Q))}\right\}^{1/r}\\
&\lesssim\left\{\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-\lambda r}\frac{1}{[\phi(Q)]^r}\left\|\{|f_j|\mathbf{1}_{Q+l\ell(Q)}\}_{j\geq(j_Q\vee 0)}\right\|^r_{\ell^{q(\cdot)}(L^{p(\cdot)}(\mathbb{R}^n))}\right\}^{1/r}\\
&\lesssim\left\{\sum_{l\in\mathbb{Z}^n}\frac{[\phi(Q+l\ell(Q))]^r}{(1+|l|)^{\lambda r}[\phi(Q)]^r}\right\}^{1/r}\sup_{P\in\mathcal{Q}}\frac{1}{\phi(P)}\left\|\{f_j\}_{j\geq(j_P\vee 0)}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))}.
\end{aligned}\tag{2.3}
$$

On another hand, since, for any $Q\in\mathcal{Q}$ and $l\in\mathbb{Z}^n$,

$$\frac{\phi(Q+l\ell(Q))}{\phi(Q)}\lesssim(1+|l|)^{2\log_2 c_1}$$

due to [45, Lemma 2.6(ii)], it follows that, for any $Q\in\mathcal{Q}$,

$$\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-\lambda r}\frac{[\phi(Q+l\ell(Q))]^r}{[\phi(Q)]^r}\lesssim\sum_{l\in\mathbb{Z}^n}(1+|l|)^{-\lambda r}(1+|l|)^{2r\log_2 c_1}\lesssim 1,$$

where we used the fact that $\lambda>n+2\log_2 c_1$ in the last inequality. From this and (2.3), we deduce that, for any $Q\in\mathcal{Q}$,

$$\frac{1}{\phi(Q)}\left\|\{\eta_{j,m}*f_j\}_{j\geq(j_Q\vee 0)}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(Q))}\lesssim\sup_{P\in\mathcal{Q}}\frac{1}{\phi(P)}\left\|\{f_j\}_{j\geq(j_P\vee 0)}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))},$$

which, combined with the arbitrariness of $Q \in \mathcal{Q}$, implies that the conclusion of this lemma holds true. This finishes the proof of Lemma 2.2. $\square$

Remark 2.3. From the proof of Lemma 2.2, we deduce the following conclusion, the details being omitted. Under the same assumptions as in Lemma 2.2, there exists a positive constant C such that, for any sequence $\{f_v\}_{v\in\mathbb{N}}$ of measurable functions,

$$\sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \{\eta_{v,m} * f_v\}_{v\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \le C \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \{f_v\}_{v\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))},$$

where $\mathcal{Q}$ is as in (1.9).

The following Lemma 2.4 is just [27, Lemma 19], which is a variant of [18, Lemma 6.1].

Lemma 2.4. *Let $s \in C^{\log}_{\rm loc}(\mathbb{R}^n)$ and $d \in [C_{\log}(s), \infty)$, where $C_{\log}(s)$ denotes the constant as in* (1.2) *with g replaced by s. Then, for any $m \in (0,\infty)$, $v \in \mathbb{N}$ and $x, y \in \mathbb{R}^n$, $2^{vs(x)}\eta_{v,m+d}(x-y) \le C 2^{vs(y)}\eta_{v,m}(x-y)$ with C being a positive constant independent of x, y and v.*

Remark 2.5. Let all the notation be the same as in Lemma 2.4. Then, by Lemma 2.4, we conclude that there exists a positive constant C such that, for any non-negative measurable function f and $x \in \mathbb{R}^n$,

$$2^{vs(x)}\eta_{v,m+d} * f(x) \le C\eta_{v,m} * (2^{vs(\cdot)} f)(x).$$

Proof of Theorem 2.1. Let (φ, Φ) and (ψ, Ψ) be two pairs of admissible functions. To prove Theorem 2.1, by symmetry, it suffices to show that

$$\|f\|^{\varphi}_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|^{\psi}_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

By the Calderón reproducing formula (see, for instance, [47, (2.6)]), we know that there exists another admissible function pair (ψ^0, Ψ^0) such that

$$\widehat{\Psi}(\xi)\widehat{\Psi^0}(\xi) + \sum_{j=1}^{\infty} \widehat{\psi}(2^{-j}\xi)\widehat{\psi^0}(2^{-j}\xi) = 1, \quad \forall\, \xi \in \mathbb{R}^n.$$

Then it follows, from [24, (12.4)] (see also [47, Lemma 2.1]), that

$$f = \sum_{k=0}^{\infty} \psi_k * \psi_k^0 * f \quad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

which implies that, for any $j \in \mathbb{Z}_+$,

$$\varphi_j * f = \sum_{k=-1}^{1} \varphi_j * \psi^0_{j+k} * \psi_{j+k} * f,$$

where $\varphi_0 := \Phi$, $\psi_0 := \Psi$, $\psi_0^0 := \Psi^0$, $\psi_{-1} := 0$ and $\psi_{-1}^0 := 0$. For any $j \in \mathbb{Z}_+$ and $k \in \{-1, 0, 1\}$, by an argument similar to that used in the proof of [6, p. 1643], we find that, for any $r \in (0, \min\{1, p_-, q_-\})$ and $m \in (0, \infty)$ large enough,

$$|\varphi_j * \psi_{j+k}^0 * \psi_{j+k} * f| \lesssim [\eta_{j+k,3m} * (|\psi_{j+k} * f|^r)]^{1/r}.$$

From this, Lemma 2.2 and Remarks 1.4(ii) and 2.5, we deduce that

$$\begin{aligned}
\|f\|^{\varphi}_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} &= \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \{2^{js(\cdot)}\varphi_j * f\}_{j\geq(j_P\vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\
&\lesssim \sum_{k=-1}^{1} \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \{2^{jrs(\cdot)}\eta_{j+k,3m} * (|\psi_{j+k} * f|^r)\}_{j\geq(j_P\vee 0)} \right\|^{1/r}_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r}(P))} \\
&\lesssim \sum_{k=-1}^{1} \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \{\eta_{j+k,2m} * ([2^{js(\cdot)}|\psi_{j+k} * f|]^r)\}_{j\geq(j_P\vee 0)} \right\|^{1/r}_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r}(P))} \\
&\lesssim \sum_{k=-1}^{1} \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \{[2^{js(\cdot)}|\psi_{j+k} * f|]^r\}_{j\geq(j_P\vee 0)} \right\|^{1/r}_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r}(P))} \\
&\lesssim \|f\|^{\psi}_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)},
\end{aligned}$$

which completes the proof of Theorem 2.1. □

3. Equivalent characterizations of $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$

In this section, we establish equivalent characterizations of the space $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ via Peetre maximal functions and averages on balls.

3.1. Peetre maximal function characterizations

Let (φ, Φ) be a pair of admissible functions on $\mathbb{R}^n$. For any $a \in (0, \infty)$, $s\colon \mathbb{R}^n \to \mathbb{R}$, $j \in \mathbb{Z}_+$ and $f \in \mathcal{S}'(\mathbb{R}^n)$, the *Peetre maximal function* $\varphi_j^{*,a}(2^{js(\cdot)}f)$ of f is defined by setting, for any $x \in \mathbb{R}^n$,

$$\varphi_j^{*,a}(2^{js(\cdot)}f)(x) := \sup_{y\in\mathbb{R}^n} \frac{2^{js(y)}|\varphi_j * f(y)|}{(1 + 2^j|x - y|)^a},$$

where φ_0 is replaced by Φ.

Theorem 3.1. *Let p, q, s and ϕ be as in Definition* 1.5. *Assume that*

$$a \in (2n/\min\{1, p_-, q_-\} + c_{\log}(1/q) + 2\log_2 c_1, \infty),$$

where p_- and c_1 are, respectively, as in (1.1) *and* (1.5), *q_- is as in* (1.1) *with p replaced by q. Then $f \in B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\|f\|^*_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} < \infty$, where*

$$\|f\|^*_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \{\varphi_j^{*,a}(2^{js(\cdot)}f)\}_{j\geq(j_P\vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))}.$$

Moreover, $\|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \sim \|f\|^*_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$ *with the positive equivalence constants independent of* f.

To prove Theorem 3.1, we first recall the following r*-trick lemma*, which comes from [18, Lemma A.6] and its proof.

Lemma 3.2. *Let* $r \in (0,\infty)$, $v \in \mathbb{Z}_+$ *and* $m \in (n,\infty)$. *Then there exists a positive constant* C*, only depending on* r*,* m *and* n*, such that, for any* $x \in \mathbb{R}^n$ *and* $g \in \mathcal{S}'(\mathbb{R}^n)$ *with* $\operatorname{supp}\widehat{g} \subset \{\xi :\ |\xi| \le 2^{v+1}\}$,

$$\sup_{z\in Q} |g(z)| \le C[\eta_{v,m} * (|g|^r)(x)]^{1/r},$$

where $Q \in \mathcal{Q}$ *contains* x *and* $\ell(Q) = 2^{-v}$.

Proof of Theorem 3.1. If $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\|f\|^*_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$ is finite, then it is easy to see that

$$\|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} < \infty,$$

namely, $f \in B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Conversely, let $f \in B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Observe that, for any $j \in \mathbb{Z}_+$,

$$\operatorname{supp}\widehat{\varphi_j} \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2^{j+1}\}$$

due to (1.7) and (1.8). Then, by Lemma 3.2, we know that, for any $j \in \mathbb{Z}_+$, $t \in (0,\infty)$ and $y \in \mathbb{R}^n$,

$$|\varphi_j * f(y)| \lesssim [\eta_{j,2at} * (|\varphi_j * f|^t)(y)]^{1/t}. \tag{3.1}$$

Notice that, for any $x, y, z \in \mathbb{R}^n$,

$$(1 + 2^j|x-y|)^{-at} \le (1 + 2^j|x-z|)^{-at}(1 + 2^j|y-z|)^{at}.$$

Thus, by this, Lemma 2.4 and (3.1), we obtain, for any $j \in \mathbb{Z}_+$, $t \in (0,\infty)$ and $x \in \mathbb{R}^n$,

$$\begin{aligned}
[\varphi_j^{*,a}(2^{js(\cdot)}f)(x)]^t &\lesssim \sup_{y\in\mathbb{R}^n} \frac{2^{js(y)t}\eta_{j,2at} * (|\varphi_j * f|^t)(y)}{(1+2^j|x-y|)^{at}} \\
&\lesssim \sup_{y\in\mathbb{R}^n} \frac{\eta_{j,at} * (2^{js(\cdot)t}|\varphi_j * f|^t)(y)}{(1+2^j|x-y|)^{at}} \\
&\lesssim \sup_{y\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{2^{jn}2^{js(z)t}}{(1+2^j|y-z|)^{at}} |\varphi_j * f(z)|^t\,dz \frac{1}{(1+2^j|x-y|)^{at}} \\
&\lesssim \int_{\mathbb{R}^n} \frac{2^{jn}2^{js(z)t}}{(1+2^j|x-z|)^{at}} |\varphi_j * f(z)|^t\,dz \\
&\sim \eta_{j,at} * (2^{js(\cdot)t}|\varphi_j * f|^t)(x).
\end{aligned}$$

Now, if we choose $t \in (0, \min\{1, p_-, q_-\}]$ such that $at > 2n + tc_{\log(1/q)} + 2t\log_2 c_1$, then, by Remark 1.4(iv) and Lemma 2.2, we conclude that

$$\begin{aligned}
&\sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \{\varphi_j^{*,a}(2^{js(\cdot)}f)\}_{j\geq(j_P\vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\
&= \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \{[\varphi_j^{*,a}(2^{js(\cdot)}f)]^t\}_{j\geq(j_P\vee 0)} \right\|_{\ell^{q(\cdot)/t}(L^{p(\cdot)/t}(P))}^{1/t} \\
&\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \{\eta_{j,at} * (2^{js(\cdot)t}|\varphi_j * f|^t)\}_{j\geq(j_P\vee 0)} \right\|_{\ell^{q(\cdot)/t}(L^{p(\cdot)/t}(P))}^{1/t} \\
&\lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, $\|f\|^*_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$, which completes the proof of Theorem 3.1. □

By Theorem 3.1 and an argument similar to that used in the proof of [46, Proposition 5.6], we conclude the following embedding properties, the details being omitted.

Proposition 3.3. *Let p, q, s and ϕ be as in Definition* 1.5. *Then*

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

3.2. Characterizations via averages on balls

In this subsection, we establish a new characterization of the space $B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ via ball averages. To this end, we first recall some notation. In what follows, we always use the *symbol* $L^1_{\rm loc}(\mathbb{R}^n)$ to denote the set of all locally integrable functions. For any $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and ball $B(x,t) \subset \mathbb{R}^n$ with $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, the *ball average operator* B_t is defined by setting

$$B_t(f)(x) := \frac{1}{|B(x,t)|} \int_{B(x,t)} f(y)\,dy$$

and, for any $\ell \in \mathbb{N}$, the 2ℓ*-th order ball average operator* $B_{\ell,t}$ by setting

$$B_{\ell,t}(f)(x) := -\frac{2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell-j} B_{jt}(f)(x),$$

here and hereafter, for any $k, r \in \mathbb{N}$ with $k \geq r$, $\binom{k}{r}$ denotes the *binomial coefficient.*

In what follows, for any $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$, denote by the *symbol* $L^{p(\cdot)}_\phi(\mathbb{R}^n)$ the set of all measurable functions f on $\mathbb{R}^n$ satisfying

$$\|f\|_{L^{p(\cdot)}_\phi(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q},\ell(P)\geq 1} \frac{1}{\phi(P)} \|f\|_{L^{p(\cdot)}(P)} < \infty,$$

where $\mathcal{Q}$ is as in (1.9).

Remark 3.4. By Remark 1.4(i) and an argument similar to that used in the proof of Lemma 2.2, we conclude that, if $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$, $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $m \in (0,\infty)$ large enough, then, for any $f \in L^{p(\cdot)}_\phi(\mathbb{R}^n)$,

$$\|\eta_{0,m} * f\|_{L^{p(\cdot)}_\phi(\mathbb{R}^n)} \le C\|f\|_{L^{p(\cdot)}_\phi(\mathbb{R}^n)}$$

with C being a positive constant independent of f.

The main result of this subsection is stated as follows.

Theorem 3.5. *Let p, q, s and ϕ be as in Definition* 1.5 *with $p_-, q_- \in [1,\infty]$ and $c_1 \in (0, 2^{n/p_+})$, where p_-, p_+ and c_1 are, respectively, as in* (1.1) *and* (1.5)*, q_- is as in* (1.1) *with p replaced by q. If $\ell \in \mathbb{N}$ and $0 < s_- \le s_+ < 2\ell$, where s_- and s_+ are as in* (1.1) *with p replaced by s, then $f \in B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n) \cap L^{p(\cdot)}_\phi(\mathbb{R}^n)$ and*

$$|||f|||_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} := \|f\|_{L^{p(\cdot)}_\phi(\mathbb{R}^n)} + \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\|\{2^{ks(\cdot)}[f - B_{\ell,k}(f)]\}_{k\in\mathbb{N}}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} < \infty,$$

where $\mathcal{Q}$ is as in (1.9)*. Moreover, $\|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \sim |||f|||_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$ with the positive equivalence constants independent of f.*

Remark 3.6. (i) Let p and ϕ be as in Theorem 3.5. Then $L^{p(\cdot)}_\phi(\mathbb{R}^n) \subset L^1_{\rm loc}(\mathbb{R}^n)$. Indeed, if $f \in L^{p(\cdot)}_\phi(\mathbb{R}^n)$, then, by the Hölder inequality (see, for instance, [12, Theorem 2.26]), [45, Lemma 2.6] and [12, Lemma 2.39], we know that, for any dyadic cube $Q := Q_{jk} \in \mathcal{Q}$ with $j \in \mathbb{Z}\setminus\mathbb{N}$, $k \in \mathbb{Z}^n$ and $\mathcal{Q}$ as in (1.9),

$$\begin{aligned}\int_Q |f(x)|\,dx &\lesssim \|f\|_{L^{p(\cdot)}(Q)}\|\mathbf{1}_Q\|_{L^{p^*(\cdot)}(\mathbb{R}^n)}\\ &\lesssim \|f\|_{L^{p(\cdot)}_\phi(\mathbb{R}^n)}\phi(Q)\|\mathbf{1}_Q\|_{L^{p^*(\cdot)}(\mathbb{R}^n)}\\ &\lesssim \|f\|_{L^{p(\cdot)}_\phi(\mathbb{R}^n)}2^{j\log_2 c_1}(1+|k|)^{2\log_2 c_1}(|Q|+1) < \infty,\end{aligned}$$

where $1/p(\cdot) + 1/p^*(\cdot) = 1$ and c_1 is as in (1.5). Thus, $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and hence the above claim holds true. From this claim, we further deduce that Theorem 3.5 makes sense.

(ii) The conclusion of Theorem 3.5 is new even when $\phi \equiv 1$, namely, it is new even on the variable Besov space $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

To prove Theorem 3.5, we need some preparations. Let $\mathcal{C}(\mathbb{R}^n)$ be the set of all complex-valued uniformly continuous functions on $\mathbb{R}^n$ equipped with the sup-norm and $\mathcal{C}^\infty(\mathbb{R}^n)$ the set of all smooth functions on $\mathbb{R}^n$. Following [47, Section 1.3.3], let $\Psi \in \mathcal{C}^\infty(\mathbb{R}^n)$ be a radial function with compact support such that, when $|x| \le 1$, $\Psi(x) = 1$ and, when $|x| \ge 3/2$, $\Psi(x) = 0$. If we let $\Psi^0 := \Psi$ and, for any $j \in \mathbb{N}$, $\Psi^j(\cdot) := \Psi(2^{-j}\cdot) - \Psi(2^{-j+1}\cdot)$, then we obtain a smooth decomposition of unity, namely, for any $x \in \mathbb{R}^n$, $\sum_{j=0}^\infty \Psi^j(x) = 1$. Let, for any $x \in \mathbb{R}^n$, $\varphi(x) := \widehat{\Psi(2\,\cdot)}(-x)$,

$$\varphi_0(x) := \widehat{\Psi}(-x) \quad \text{and} \quad \varphi_j(x) := 2^{jn}\varphi(2^j x), \quad \forall\, j \in \mathbb{N}. \tag{3.2}$$

Proposition 3.7. *Let p, q, s and ϕ be as in Definition* 1.5. *If $s_- \in (0,\infty)$, then*

$$B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow L^{p(\cdot)}_{\phi}(\mathbb{R}^n).$$

Proof. Let $f \in B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $\{\varphi_j\}_{j=0}^{\infty}$ be a smooth decomposition of unity as in (3.2). Then $f = \sum_{j=0}^{\infty} \varphi_j * f$ in $\mathcal{S}'(\mathbb{R}^n)$. By an argument similar to that used in the proof of [6, Theorem 6.1(ii)], we know that

$$B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \hookrightarrow B^{0,\phi}_{p(\cdot),\underline{p}}(\mathbb{R}^n),$$

where $\underline{p} := \min\{1, p_-\}$. Thus, by Remark 1.2 and (2.1), we find that, for any $P \in \mathcal{Q}$ with $\ell(P) \geq 1$,

$$\begin{aligned}\left\|\sum_{j=0}^{\infty} \varphi_j * f\right\|_{L^{p(\cdot)}(P)} &\leq \left\{\sum_{j=0}^{\infty} \left\||\varphi_j * f|^{\underline{p}}\right\|_{L^{p(\cdot)/\underline{p}}(P)}\right\}^{1/\underline{p}} \\ &\lesssim \phi(P)\|f\|_{B^{0,\phi}_{p(\cdot),\underline{p}}(\mathbb{R}^n)} \\ &\lesssim \phi(P)\|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)},\end{aligned} \tag{3.3}$$

which implies that $\sum_{j=0}^{\infty} \varphi_j * f$ converges in $L^{p(\cdot)}_{\phi}(\mathbb{R}^n)$. In this sense, we regard f as a function in $L^{p(\cdot)}_{\phi}(\mathbb{R}^n)$. Moreover, by (3.3), we know that

$$\|f\|_{L^{p(\cdot)}_{\phi}(\mathbb{R}^n)} = \sup_{P\in\mathcal{Q},\ell(P)\geq 1} \frac{1}{\phi(P)}\left\|\sum_{j=0}^{\infty} \varphi_j * f\right\|_{L^{p(\cdot)}(P)} \lesssim \|f\|_{B^{0,\phi}_{p(\cdot),\underline{p}}(\mathbb{R}^n)} \lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

This finishes the proof of Proposition 3.7. □

Lemma 3.8. *Let p, q, s, ϕ be as in Definition* 1.5. *Assume that $c_1 \in (0, 2^{n/p_+})$. Then $f \in B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\|f|B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\| < \infty$; moreover, there exists a positive constant C, independent of f, such that*

$$\|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \leq \left\|f|B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\right\| \leq C\|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}, \tag{3.4}$$

where, for any $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\left\|f|B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\right\| := \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)}\left\|\left\{2^{js(\cdot)}|\varphi_j * f|\right\}_{j\in\mathbb{Z}_+}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))}.$$

Proof. To show this lemma, we only need to prove that, for any $f \in B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, the second inequality of (3.4) holds true. Let $P \subset \mathbb{R}^n$ be any given dyadic cube and, for any $j \in \mathbb{Z}_+$ and $x \in \mathbb{R}^n$, $f_j(x) := 2^{js(x)}|\varphi_j * f(x)|$. Then, by Remark 1.4(ii), we have

$$\begin{aligned}\frac{1}{\phi(P)}\left\|\{f_j\}_{j\in\mathbb{Z}_+}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} &\lesssim \frac{1}{\phi(P)}\left\|\{f_j\}_{j=0}^{(j_P\vee 0)-1}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\ &\quad + \frac{1}{\phi(P)}\left\|\{f_j\}_{j=(j_P\vee 0)}^{\infty}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\ &=: \mathrm{I}_{P,1} + \mathrm{I}_{P,2},\end{aligned} \tag{3.5}$$

where $\mathrm{I}_{P,1} := 0$ if $j_P \le 0$. Obviously, $\mathrm{I}_{P,2} \lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$.

To estimate $\mathrm{I}_{P,1}$, we only need to consider the case that $j_P > 0$. It is easy to see that, for any $j \in \mathbb{Z}_+$ with $j \le j_P - 1$, there exists a unique dyadic cube P_j such that $P \subset P_j$ and $\ell(P_j) = 2^{-j}$ and hence, by an argument similar to that used in the proof of [46, p. 1873], we find that

$$\frac{1}{\phi(P)}\|f_j\|_{L^{p(\cdot)}(P)} \lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \frac{\|\mathbf{1}_P\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\mathbf{1}_{P_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \frac{\phi(P_j)}{\phi(P)}.$$

Observe that, for any cube $Q \subset \mathbb{R}^n$, when $|Q| \le 2^n$, then, for any $x \in Q$, $\|\mathbf{1}_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim |Q|^{1/p(x)}$ and, when $|Q| \ge 1$, $\|\mathbf{1}_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim |Q|^{1/p_\infty}$ (see [17, Corollary 4.5.9]), where p_∞ is as in (1.3) with g replaced by p. Then it follows, from an argument similar to that used in the proof of [50, Lemma 2.6], that

$$\frac{\|\mathbf{1}_P\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\mathbf{1}_{P_j}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \lesssim 2^{(j-j_P)n/p_+},$$

which, combined with the fact that $\phi(P_j)/\phi(P) \lesssim 2^{-(j-j_P)\log_2 c_1}$ thanks to the assumptions of ϕ, further implies that

$$\frac{1}{\phi(P)}\|f_j\|_{L^{p(\cdot)}(P)} \lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} 2^{(j-j_P)(n/p_+ - \log_2 c_1)}.$$

Now, by choosing $r \in (0, \frac{1}{2}\min\{p_-, q_-, 2\}]$, (i), (ii) and (iv) of Remark 1.4 and the assumption $c_1 \in (0, 2^{n/p_+})$, we conclude that

$$\begin{aligned}
\mathrm{I}_{P,1} &\sim \frac{1}{\phi(P)}\left\|\{f_j^r\}_{j=0}^{j_P-1}\right\|^{1/r}_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r}(P))} \\
&\lesssim \left\{\sum_{j=0}^{j_P-1} \frac{1}{[\phi(P)]^r}\|f_j^r\|_{L^{p(\cdot)/r}(P)}\right\}^{1/r} \sim \left\{\sum_{j=0}^{j_P-1} \frac{1}{[\phi(P)]^r}\|f_j\|^r_{L^{p(\cdot)}(P)}\right\}^{1/r} \\
&\lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \left\{\sum_{j=0}^{j_P-1} 2^{(j-j_P)(n/p_+ - \log_2 c_1)r}\right\}^{1/r} \lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore, by (3.5), we find that

$$\left\|f|B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)\right\| \lesssim \sup_{P\in\mathcal{Q}}(\mathrm{I}_{P,1} + \mathrm{I}_{P,2}) \lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)},$$

which completes the proof of Lemma 3.8. □

The following conclusion is a variant of [27, Lemma 8].

Lemma 3.9. *Let $p, q \in \mathcal{P}(\mathbb{R}^n)$, $\phi \in \mathcal{G}(\mathbb{R}^{n+1}_+)$ and $\delta \in (0, \infty)$. Assume that $\{g_k\}_{k\in\mathbb{Z}}$ is a sequence of non-negative measurable functions on $\mathbb{R}^n$ and*

$$G_v(x) := \sum_{k=v}^{\infty} 2^{(v-k)\delta} g_k(x), \quad \forall\, x \in \mathbb{R}^n,\ \forall\, v \in \mathbb{Z}.$$

Then there exist positive constants C_1 *and* C_2 *such that*

$$\sup_{P\in\mathcal{Q}}\frac{1}{\phi(P)}\left\|\{G_v\}_{v\ge(j_P\vee 0)}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))}\le C_1\sup_{P\in\mathcal{Q}}\frac{1}{\phi(P)}\left\|\{g_k\}_{k\ge(j_P\vee 0)}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \tag{3.6}$$

and

$$\sup_{P\in\mathcal{Q}}\frac{1}{\phi(P)}\left\|\{G_v\}_{v\in\mathbb{N}}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))}\le C_2\sup_{P\in\mathcal{Q}}\frac{1}{\phi(P)}\left\|\{g_k\}_{k\in\mathbb{N}}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))},$$

where $\mathcal{Q}$ *is as in* (1.9).

Proof. To show this lemma, we only prove (3.6) by similarity. Let $r\in\left(0,\frac{1}{2}\min\{p_-,q_-,2\}\right]$. Then, by (ii) and (iv) of Remark 1.4, we know that

$$\begin{aligned}
&\sup_{P\in\mathcal{Q}}\frac{1}{\phi(P)}\left\|\{G_v\}_{v\ge(j_P\vee 0)}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))}\\
&=\sup_{P\in\mathcal{Q}}\frac{1}{\phi(P)}\left\|\left\{\sum_{k=v}^{\infty}2^{-(k-v)\delta}g_k\right\}_{v\ge(j_P\vee 0)}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))}\\
&=\sup_{P\in\mathcal{Q}}\frac{1}{\phi(P)}\left\|\left\{\sum_{l=0}^{\infty}[2^{-l\delta}g_{l+v}]^r\right\}_{v\ge(j_P\vee 0)}\right\|^{1/r}_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r}(P))}\\
&\lesssim\left\{\sum_{l=0}^{\infty}2^{-lr\delta}\left[\sup_{P\in\mathcal{Q}}\frac{1}{\phi(P)}\left\|\{g_{l+v}\}_{v\ge(j_P\vee 0)}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))}\right]^r\right\}^{1/r}\\
&\lesssim\sup_{P\in\mathcal{Q}}\frac{1}{\phi(P)}\left\|\{g_k\}_{k\ge(j_P\vee 0)}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))},
\end{aligned}$$

which implies that (3.6) holds true. This finishes the proof of Lemma 3.9. □

By a subtle modification of the proof of [14, Lemma 2.2], we obtain the following conclusion.

Lemma 3.10. *Let* $M\in(0,\infty)$ *and* $\{T_t\}_{t\in(0,\infty)}$ *be a family of operators given by*

$$T_tf(x):=([m(t\,\cdot)]^{\vee})*f(x),\quad\forall\,x\in\mathbb{R}^n,\ \forall\,f\in\mathcal{S}'(\mathbb{R}^n)\cap L^1_{\rm loc}(\mathbb{R}^n),\ \forall\,t\in(0,\infty)$$

for some $m\in\mathcal{S}(\mathbb{R}^n)$. *Then there exists a positive constant* C *such that, for any* $t\in(0,\infty)$, $f\in\mathcal{S}'(\mathbb{R}^n)\cap L^1_{\rm loc}(\mathbb{R}^n)$ *and* $x\in\mathbb{R}^n$,

$$|T_tf(x)|\le C\left[\|\nabla^M m\|_{L^1(\mathbb{R}^n)}+\|m\|_{L^1(\mathbb{R}^n)}\right](\eta_{t,M}*|f|)(x). \tag{3.7}$$

Proof. For any $t\in(0,\infty)$, $f\in\mathcal{S}'(\mathbb{R}^n)\cap L^1_{\rm loc}(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$, by the definition of the Fourier transform, we find that

$$\begin{aligned}
|T_t(f)(x)|&\sim\left|\int_{\mathbb{R}^n}f(y)\int_{\mathbb{R}^n}m(t\xi)e^{i(x-y)\cdot\xi}\,d\xi dy\right|\\
&\lesssim\left|\int_{|x-y|<t}f(y)\int_{\mathbb{R}^n}m(t\xi)e^{i(x-y)\cdot\xi}\,d\xi dy\right|+\left|\int_{|x-y|\ge t}\cdots\right|\\
&=:\mathrm{I}+\mathrm{II}.
\end{aligned} \tag{3.8}$$

For I, it is easy to see that

$$
\begin{aligned}
\mathrm{I} &\lesssim \int_{|x-y|<t} |f(y)| t^{-n} \int_{\mathbb{R}^n} |m(z)|\, dz dy \\
&\sim \int_{|x-y|<t} \frac{t^{-n}}{(1+t^{-1}|x-y|)^M} |f(y)|\, dy\, \|m\|_{L^1(\mathbb{R}^n)} \\
&\lesssim (\eta_{t,M} * |f|)(x).
\end{aligned} \tag{3.9}
$$

For II, via the Fubini theorem and the integration by parts, we conclude that

$$
\begin{aligned}
\mathrm{II} &\lesssim \int_{|x-y|\ge t} \frac{|f(y)|}{|x-y|^M} \int_{\mathbb{R}^n} t^M |\nabla^M m(t\xi)|\, d\xi dy \\
&\sim \int_{|x-y|\ge t} \frac{t^{-n}|f(y)|}{(1+t^{-1}|x-y|)^M} \int_{\mathbb{R}^n} |\nabla^M m(\xi)|\, d\xi dy \\
&\lesssim (\eta_{t,M} * |f|)(x) \|\nabla^M m\|_{L^1(\mathbb{R}^n)}.
\end{aligned} \tag{3.10}
$$

Combining (3.8), (3.9) and (3.10), we conclude that (3.7) holds true, which completes the proof of Lemma 3.10. □

Now we prove Theorem 3.5.

Proof of Theorem 3.5. Let $f \in B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $\{\varphi_j\}_{j=0}^\infty$ be a smooth decomposition of unity as in (3.2). Then, by the proof of Proposition 3.7, we find that $f = \sum_{j=0}^\infty \varphi_j * f$ converges in $L^{p(\cdot)}_\phi(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, where $\{\varphi_j\}_{j=0}^\infty$ is as in (3.2). Thus, by this and Remark 3.6(i), we know that, for any $k \in \mathbb{Z}_+$,

$$
f - B_{\ell,2^{-k}}(f) = \left(\sum_{j=0}^{k-1} + \sum_{j=k}^{\infty}\right) (I - B_{\ell,2^{-k}})(\varphi_j * f). \tag{3.11}
$$

Since $\operatorname{supp}\widehat{\varphi_j} \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2^{j+1}\}$ for any $j \in \mathbb{Z}_+$, it follows that

$$
\operatorname{supp}\widehat{\varphi_j * f} \subset \{\xi \in \mathbb{R}^n : |\xi| \le 2^{j+1}\},
$$

which, together with the r-trick lemma (see Lemma 3.2), implies that, for any $M \in (0,\infty)$ large enough and $x \in \mathbb{R}^n$,

$$
|\varphi_j * f(x)| \le (\eta_{j,M} * |\varphi_j * f|)(x). \tag{3.12}
$$

When $i \in \{1,\dots,\ell\}$, it is easy to see that

$$
\begin{aligned}
|B_{i2^{-k}}(\varphi_j * f)(x)| &= \frac{1}{|B(x,i2^{-k})|} \int_{B(x,i2^{-k})} |f * \varphi_j(y)|\, dy \\
&\sim \int_{B(x,i2^{-k})} \frac{2^{kn}}{(1+2^k|x-y|)^M} |f * \varphi_j(y)|\, dy \\
&\lesssim \eta_{k,M} * (|f * \varphi_j|)(x).
\end{aligned}
$$

By this, (3.12) and Remark 1.4(ii), we conclude that, for any $P \in \mathcal{Q}$,

$$\begin{aligned} \mathrm{I} &:= \left\| \left\{ 2^{ks(\cdot)} \sum_{j=k}^{\infty} |(I - B_{\ell,2^{-k}})(\varphi_j * f)| \right\}_{k\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\ &\lesssim \left\| \left\{ 2^{ks(\cdot)} \sum_{j=k}^{\infty} \eta_{j,M} * |\varphi_j * f| \right\}_{k\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\ &\quad + \left\| \left\{ 2^{ks(\cdot)} \sum_{j=k}^{\infty} \eta_{k,M} * |\varphi_j * f| \right\}_{k\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\ &=: \mathrm{I}_1 + \mathrm{I}_2. \end{aligned} \tag{3.13}$$

For I_1, by Lemmas 3.9 with δ replaced by s_- (hence we need $s_- > 0$ here), 2.4 and 2.2 (here we need $p_-, q_- \in [1, \infty]$), we know that

$$\begin{aligned} \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \mathrm{I}_1 &\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ \sum_{j=k}^{\infty} 2^{(k-j)s_-} 2^{js(\cdot)} \eta_{j,M} * |\varphi_j * f| \right\}_{k\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\ &\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ 2^{js(\cdot)} \eta_{j,M} * |\varphi_j * f| \right\}_{j\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\ &\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ \eta_{j,M/2} * (2^{js(\cdot)} |\varphi_j * f|) \right\}_{j\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\ &\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ 2^{js(\cdot)} |\varphi_j * f| \right\}_{j\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\ &\lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.14}$$

Similarly, for I_2, we also have

$$\begin{aligned} \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \mathrm{I}_2 &\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ \eta_{k,M/2} * \left[2^{ks(\cdot)} \sum_{j=k}^{\infty} |\varphi_j * f| \right] \right\}_{k\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\ &\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ 2^{ks(\cdot)} \sum_{j=k}^{\infty} |\varphi_j * f| \right\}_{k\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\ &\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ \sum_{j=k}^{\infty} 2^{(k-j)s_-} 2^{js(\cdot)} |\varphi_j * f| \right\}_{k\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\ &\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ 2^{js(\cdot)} |\varphi_j * f| \right\}_{j\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\ &\lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.15}$$

Combining (3.13), (3.14) and (3.15), we conclude that

$$\sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \mathrm{I} \le \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \mathrm{I}_1 + \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \mathrm{I}_2 \lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}. \tag{3.16}$$

Next, we estimate the term $\sum_{j=0}^{k-1}\cdots$ in (3.11). To this end, let

$$T^{\ell}_{k,j}(f) := (I - B_{\ell,2^{-k}})(\varphi_j * f).$$

By the Calderón reproducing formula (see, for instance, [47, (2.6)]), we find that there exists another admissible function pair (ψ, Ψ) such that, for any $\xi \in \mathbb{R}^n$,

$$\Phi(\xi)\Psi(\xi) + \sum_{j=1}^{\infty} \widehat{\varphi}(2^{-j}\xi)\widehat{\psi}(2^{-j}\xi) = 1.$$

Let $m^{\ell}_{k,j} \in \mathcal{S}(\mathbb{R}^n)$ be as in [49, (2.11)]. Then, by an argument similar to that used in the proof of [49, (2.11)], we conclude that, for any $k \in \mathbb{Z}_+$ and $j \in \{0, 1, \ldots, k\}$,

$$T^{\ell}_{k,j}(f) = ([m_{k,j}(2^{-j}\,\cdot)]^{\vee}) * (\varphi_{j-1} + \varphi_j + \varphi_{j+1}) * f,$$

where $\varphi_0 := \Phi$ and $\varphi_{-1} := 0$. Moreover, by the proof of [49, Theorem 1.3], we find that, for any $M \in \mathbb{N}$,

$$\|m^{\ell}_{k,j}\|_{L^1(\mathbb{R}^n)} + \|\nabla^M m^{\ell}_{k,j}\|_{L^1(\mathbb{R}^n)} \lesssim 2^{2\ell(j-k)}.$$

Thus, by Lemma 3.10, we find that

$$|(I - B_{\ell,2^{-k}})(\varphi_j * f)| \lesssim 2^{2\ell(j-k)}\eta_{j,M} * |f_j|,$$

where $f_j := (\varphi_{j-1} + \varphi_j + \varphi_{j+1}) * f$. Therefore, by this estimate, (ii) and (iv) of Remark 1.4 and (2.1), we find that, for any $r \in \left(0, \frac{1}{2}\min\{p_-, q_-, 2\}\right]$,

$$\begin{aligned}
\mathrm{II} &:= \left\| \left\{ 2^{ks(\cdot)} \sum_{j=0}^{k-1} |(I - B_{\ell,2^{-k}})(\varphi_j * f)| \right\}_{k\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\
&\lesssim \left\| \left\{ \sum_{j=0}^{k} 2^{(k-j)s_+} 2^{2\ell(j-k)} \eta_{j,M/2} * [2^{js(\cdot)}|f_j|] \right\}_{k\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\
&\sim \left\| \left\{ \sum_{v=-k}^{0} 2^{v(2\ell - s_+)} \eta_{k+v,M/2} * [2^{(k+v)s(\cdot)}|f_{k+v}|] \right\}_{k\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\
&\lesssim \left\| \left\{ \sum_{v=-k}^{0} [2^{v(2\ell - s_+)} \eta_{k+v,M/2} * \{2^{(k+v)s(\cdot)}|f_{k+v}|\}]^r \right\}_{k\in\mathbb{N}} \right\|^{1/r}_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r}(P))} \\
&\lesssim \left[\sum_{v=-\infty}^{0} 2^{vr(2\ell - s_+)} \left\| \left\{ [\eta_{k+v,M/2} * \{2^{(k+v)s(\cdot)}|f_{k+v}|\gamma_{v,k}\}]^r \right\}_{k\in\mathbb{N}} \right\|_{\ell^{q(\cdot)/r}(L^{p(\cdot)/r}(P))} \right]^{1/r} \\
&\sim \left[\sum_{v=-\infty}^{0} 2^{vr(2\ell - s_+)} \left\| \left\{ \eta_{k+v,M/2} * [2^{(k+v)s(\cdot)}|f_{k+v}|\gamma_{v,k}] \right\}_{k\in\mathbb{N}} \right\|^r_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \right]^{1/r},
\end{aligned}$$

where $\gamma_{v,k} := 1$ when $v \geq -k$ and, otherwise, $\gamma_{v,k} := 0$. From this, $s_+ < 2\ell$, Lemmas 2.2 and 3.8, we deduce that

$$\begin{aligned}\sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \mathrm{II} &\lesssim \left[\sum_{v=-\infty}^{0} 2^{vr(2\ell-s_+)} \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\|\left\{2^{ks(\cdot)}|f_k|\right\}_{k\in\mathbb{Z}_+}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))}^r\right]^{1/r} \\ &\lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{3.17}$$

By (3.17) and (3.16), we conclude that

$$\sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\|\left\{2^{ks(\cdot)}[f - B_{\ell,k}(f)]\right\}_{k\in\mathbb{N}}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)},$$

which, together with Proposition 3.7, implies that $|||f|||_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$.

Conversely, by the proof of [14, (2.16)], we find that, for any $j \in \mathbb{N}$ and $x \in \mathbb{R}^n \setminus \{\vec{0}_n\}$,

$$f * \varphi_j(x) = (h(2^{-j}\cdot))^\vee * [f - B_{\ell,2^{-j}}(f)](x),$$

where $\vec{0}_n$ denotes the *origin* of $\mathbb{R}^n$ and h is a function in $\mathcal{C}^\infty(\mathbb{R}^n)$ with $\operatorname{supp} h \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 2\}$. Then, by Lemma 3.10, we know that, for any $M \in \mathbb{N}$,

$$|f * \varphi_j(x)| \lesssim \eta_{j,M} * [|f - B_{\ell,2^{-j}}(f)|](x), \quad \forall\, x \in \mathbb{R}^n,\ \forall\, j \in \mathbb{N}. \tag{3.18}$$

For any given $P \in \mathcal{Q}$, if $j_P > 0$, then, by (3.18) and Lemmas 2.4 and 2.2, we obtain

$$\begin{aligned} \mathrm{S}_P &:= \frac{1}{\phi(P)} \left\|\left\{2^{js(\cdot)}|\varphi_j * f|\right\}_{j\geq(j_P\vee 0)}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\ &\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\|\left\{2^{js(\cdot)}\eta_{j,M} * [|f - B_{\ell,2^{-j}}(f)|]\right\}_{j\in\mathbb{N}}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\ &\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\|\left\{\eta_{j,M/2} * [2^{js(\cdot)}|f - B_{\ell,2^{-j}}(f)|]\right\}_{j\in\mathbb{N}}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\ &\lesssim \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\|\left\{2^{js(\cdot)}[f - B_{\ell,2^{-j}}(f)]\right\}_{j\in\mathbb{N}}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\ &\lesssim |||f|||_{B^{s(\cdot),\phi}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}, \end{aligned} \tag{3.19}$$

where $M \in \mathbb{N}$ is chosen large enough. If $j_P \leq 0$, then, from Remarks 1.4(ii) and 3.4, (3.18), Lemmas 2.4 and 2.2 and the fact that $|\varphi_0 * f| \lesssim \eta_{0,M} * (|f|)$, we deduce that

$$\begin{aligned} \mathrm{S}_P &= \frac{1}{\phi(P)} \left\|\left\{2^{js(\cdot)}|\varphi_j * f|\right\}_{j\geq 0}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\ &\lesssim \sup_{P\in\mathcal{Q},\ell(P)\geq 1} \frac{1}{\phi(P)} \|\varphi_0 * f\|_{L^{p(\cdot)}(P)} \\ &\quad + \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\|\left\{2^{js(\cdot)}\eta_{j,M} * [|f - B_{\ell,2^{-j}}(f)|]\right\}_{j\in\mathbb{N}}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\eta_{0,M} * f\|_{L_\phi^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad + \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ \eta_{j,M/2} * [2^{js(\cdot)}|f - B_{\ell,2^{-j}}(f)|] \right\}_{j\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\
&\lesssim \|f\|_{L_\phi^{p(\cdot)}(\mathbb{R}^n)} + \sup_{P\in\mathcal{Q}} \frac{1}{\phi(P)} \left\| \left\{ 2^{js(\cdot)}[f - B_{\ell,2^{-j}}(f)] \right\}_{j\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} \\
&\sim |||f|||_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)},
\end{aligned}
$$

where $M \in \mathbb{N}$ is chosen large enough. Therefore, by this and (3.19), we further conclude that

$$
\|f\|_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)} = \sup_{P\in\mathcal{Q}} \mathrm{S}_P \lesssim |||f|||_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\phi}(\mathbb{R}^n)}.
$$

This finishes the proof of Theorem 3.5. ☐

Remark 3.11. (i) In particular, when $\ell = 1$ and ϕ is as in Remark 1.6(iii), we obtain the following conclusion. Let p, q, s be as in Definition 1.5 with $p_-, q_- \in [1, \infty]$, $\tau_- \in (p_+, \infty)$ and $0 < s_- \le s_+ < 2$. Then, by Theorem 3.5, we know that $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\tau(\cdot)}(\mathbb{R}^n)$ if and only if $f \in L_{\tau(\cdot)}^{p(\cdot)}(\mathbb{R}^n)$ and

$$
\begin{aligned}
&|||f|||_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\tau(\cdot)}(\mathbb{R}^n)} \\
&:= \|f\|_{L_{\tau(\cdot)}^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad + \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|_{L^{\tau(\cdot)}(\mathbb{R}^n)}} \left\| \left\{ 2^{k[s(\cdot)+n]} \int_{B(\cdot,2^{-k})} [f(\cdot) - f(y)]\, dy \right\}_{k\in\mathbb{N}} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} < \infty,
\end{aligned}
$$

where $\mathcal{Q}$ is as in (1.9) and

$$
\|f\|_{L_{\tau(\cdot)}^{p(\cdot)}(\mathbb{R}^n)} := \sup_{p\in\mathcal{Q}, \ell(P)\ge 1} \frac{1}{\|\mathbf{1}_P\|_{L^{\tau}(\mathbb{R}^n)}} \|f\|_{L^{p(\cdot)}(P)},
$$

and, by [22, Proposition 4.1], we know that $L_{\tau(\cdot)}^{p(\cdot)}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$.

(ii) Let ϕ be as in Remark 1.6(iii). Then, in [22, Theorem 4.9(i)], Drihem established an equivalent characterization of the space $B_{p(\cdot),q(\cdot)}^{s(\cdot),\tau(\cdot)}(\mathbb{R}^n)$ via ball means of differences. Precisely, let $\tau \in \mathcal{P}^{\log}(\mathbb{R}^n)$, p, q, s be as in Definition 1.5 with $p_- \in (1, \infty)$, $0 < q_- \le q_+ < \infty$ and

$$
0 < s_- \le s_+ < 1 + n \min\left\{0, \left(\frac{1}{p} - \frac{1}{\tau}\right)_-\right\}.
$$

Then $f \in B_{p(\cdot),q(\cdot)}^{s(\cdot),\tau(\cdot)}(\mathbb{R}^n)$ if and only if $f \in L_{\tau(\cdot)}^{p(\cdot)}(\mathbb{R}^n)$ and

$$
\begin{aligned}
&\widetilde{\|f\|}_{B_{p(\cdot),q(\cdot)}^{s(\cdot),\tau(\cdot)}(\mathbb{R}^n)} \\
&:= \|f\|_{L_{\tau(\cdot)}^{p(\cdot)}(\mathbb{R}^n)} \\
&\quad + \sup_{P\in\mathcal{Q}} \frac{1}{\|\mathbf{1}_P\|_{L^{\tau}(\mathbb{R}^n)}} \left\| \left\{ 2^{k[s(\cdot)+n]} \int_{B(\cdot,2^{-k})} |f(\cdot) - f(y)|\, dy \right\}_{k\ge(j_P\vee 0)} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(P))} < \infty,
\end{aligned}
$$

where $\mathcal{Q}$ is as in (1.9).

(iii) Compared (ii) and (iii) of this remark, we find that the main difference of those two quasi-norms exists in that the absolute value $|f(\cdot)-f(y)|$ in $\widetilde{\|f\|}_{B^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$ is replaced by $[f(\cdot)-f(y)]$ in $|||f|||_{B^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$. However, this slight change induces a quite different behavior between $\widetilde{\|f\|}_{B^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$ and $|||f|||_{B^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$. The former characterizes the space $B^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ only with smoothness order less than 1 even when $(1/p-1/\tau)_->0$, while the later characterizes the space $B^{s(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ with smoothness order less than 2. Therefore, in this sense, Theorem 3.5 essentially improves the corresponding result obtained in [22, Theorem 4.9(i)].

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