TAIWANESE JOURNAL OF MATHEMATICS
Vol. 23, No. 6, pp. 1389–1421, December 2019
DOI: 10.11650/tjm/181201

A Multiplicity Result for a Non-local Critical Problem

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Abstract. In this paper, we are interested in the multiple solutions of the following fractional critical problem

$$\begin{cases} (-\Delta)^s u = |u|^{2_s^*-2}u + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $s \in (0,1)$, $N > 4s$, $2_s^* = 2N/(N-2s)$, Ω is a smooth bounded domain in $\mathbb{R}^N$ and $(-\Delta)^s$ is the fractional Laplace operator. Because the nonlocal property of fractional Laplacian makes the variational functional of the fractional critical problem different from the one of local operator $-\Delta$. To the best of our knowledge, it is still unknown whether multiple solutions of the fractional critical problem exist for all $\lambda > 0$. In this paper, we give a partial answer. Precisely, by introducing some new ideas and careful estimates, we prove that for any $s \in (0,1)$, the fractional critical problem has at least $[(N+1)/2]$ pairs of nontrivial solutions if $0 < \lambda \neq \lambda_n$, and has $[(N+1-l)/2]$ pairs if $\lambda = \lambda_n$ with multiplicity number $0 < l < \min\{n, N+2\}$, via constraint method and Krasnoselskii genus. Here λ_n denotes the n-th eigenvalue of $(-\Delta)^s$ with zero Dirichlet boundary data in Ω and $[a]$ denotes the least positive integer k such that $k \geq a$.

1. Introduction

The critical problem is an important topic in the development of mathematics, which has been widely studied in the literature. One can see [3, 27] and references therein for more details. Recently, many fruitful results for the critical problem with nonlocal operators have appeared in the papers, especially for the ones with fractional Laplacian operator. This operator describes an anomalous diffusion phenomena, like flames propagation and chemical reactions of liquids, which appears in several fields such as physics, biology and probability. It can also be viewed as the infinitesimal generator of a stable Lévy process in probability theory (for details about backgrounds, please see [1, 4, 17, 19] and references therein). In differential geometry, the Yamabe problem is an important well-known critical problem and has been widely studied. Recently, as a nonlocal counterpart,

Received August 16, 2017; Accepted December 2, 2018.
Communicated by Jann-Long Chern.
2010 *Mathematics Subject Classification.* 35J20, 35J60, 35J67, 35R11.
Key words and phrases. multiple solutions, minimax methods, genus, critical growth, nonlocal operator.
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the fractional Yamabe problem and related critical problems have attracted much attention from mathematicians. For more details and applications, one can refer to [2, 19, 21] and references therein.

In this paper, we focus our attention on the following critical problem involving fractional Laplacian

$$\begin{cases} (-\Delta)^s u = |u|^{2_s^*-2}u + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $s \in (0,1)$ is a fixed parameter, $\lambda > 0$, $N > 4s$, $2_s^* = 2N/(N-2s)$ is the fractional critical Sobolev exponent and Ω is a smooth bounded domain. Here the fractional Laplacian $(-\Delta)^s$ is the nonlocal integro-differential operator defined by, up to a normalization number,

$$(-\Delta)^s u(x) := \mathrm{P.\,V.} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}}\, dy. \tag{1.2}$$

One can refer to [19] for more results and properties about fractional Laplacian. Note that the homogeneous Dirichlet datum in (1.1) is given on $\mathbb{R}^N \setminus \Omega$, not simply on $\partial\Omega$ as in the case of classical Laplacian, consistently with the nonlocal property of $(-\Delta)^s$. For the works related to the fractional Laplacian defined by (1.2), or to a more general integro-differential operator, one can see [21, 22, 24, 25] and references therein. In particular, it was proved in [21, 22, 24–26] that (1.1) admits a nontrivial solution, provided

- $N > 4s$ and $\lambda \in (0, +\infty)$;
- $N = 4s$ and $\lambda \in (0, +\infty)$ with $\lambda \neq \lambda_i$ for all $i = 1, 2, \ldots$;
- $2s < N < 4s$ and $\lambda \in (0, +\infty)$ is sufficiently large.

Here $(\lambda_i)_{i=1}^{\infty}$ denote the eigenvalues of $(-\Delta)^s$ with zero Dirichlet boundary data. Note that these results can be viewed as analog of that for Laplacian [6, 28], and there are many multiplicity results for (1.1) with $-\Delta$ operator in the literature. For instance, Devillanova and Solimini [12] proved that for any $\lambda > 0$, it has infinitely many solutions provided $N \geq 7$. Clapp and Weth [10] showed that for $N \geq 4$, if $\lambda \in (0, \mu_1)$, then it has at least $[(N+2)/2]$ pairs of nontrivial solutions; if $\lambda \in (\mu_i, \mu_{i+1})$, then it has at least $[(N+1)/2]$ pairs; if $\lambda = \mu_i$ is an eigenvalue of multiplicity $l < N+2$, then it has at least $[(N+1-l)/2]$ pairs. Here μ_i denotes the eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary condition in Ω. Later, Chen, Shioji and Zou [8] extended the multiplicity results to $\lambda = \mu_i$, that is, the problem has at least $[(N+1)/2]$ pairs for $N \geq 5$ and $\lambda \geq \mu_1$. For the history and more results about multiple solutions of (1.1) with $-\Delta$, one can refer to [3,8,12,13] and references therein. A natural question arises whether multiple solutions of (1.1) exist for $\lambda > 0$ like that for Laplacian?

Up to our knowledge, there are few results in the literature on the multiplicity of solutions to (1.1). In [14], the authors proved that the number of solutions to (1.1) is bounded below by the number of eigenvalues λ_i lying in the open interval $(\lambda, \lambda + S|\Omega|^{-2s/N})$, where $|\Omega|$ is the Lebesgue measure of Ω and S is the best constant defined in Section 2. Let us remark that it is not certain whether the interval $(\lambda, \lambda + S|\Omega|^{-2s/N})$ contain any eigenvalues or not.

In this paper, we give a partial answer. Let $[a]$ denote the least positive integer k such that $k \geq a$.

Theorem 1.1. *Let $s \in (0,1)$, $N > 4s$ and Ω be a smooth bounded domain in $\mathbb{R}^N$. Then*

(i) *if $\lambda > 0$ with $\lambda \neq \lambda_n$ for some $n \geq 1$, problem* (1.1) *possesses at least* $[(N+1)/2]$ *pairs of nontrivial solutions;*

(ii) *if $\lambda_{n-l} < \lambda_{n-l+1} = \cdots = \lambda_n = \lambda < \lambda_{n+1}$ with $0 < l < \min\{n, N+2\}$, problem* (1.1) *possesses at least* $[(N+1-l)/2]$ *pairs of nontrivial solutions.*

Moreover, these solutions satisfy $I(u) < \frac{2s}{N} S^{N/(2s)}$.

This result will be proved by developing the method of [8], via a constraint method and Krasnoselskii genus theory. Compared to Laplacian, the fractional Laplacian is nonlocal and there are two difficulties to obtain our results. The first difficulty lies in that problem (1.1) is a critical problem. As usual, we use $(\mathrm{PS})_c$ condition instead of (PS) condition. Here we say that I satisfies $(\mathrm{PS})_c$ (or (PS)) condition: if any sequence (u_m) satisfying $I(u_m) \to c$ (or $I(u_m)$ being bounded) and $I'(u_m) \to 0$ as $n \to \infty$ is relatively compact, (also see [27, Chapter II.2, Chapter III.2] for the definitions). Precisely, we shall apply the global compactness results of fractional Sobolev space in this paper, which were obtained in [20]. However, the nonexistence of nontrivial solutions to following limiting equation are unknown,

$$\begin{cases} (-\Delta)^s u = |u|^{2_s^*-2} u & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \mathbb{R}^N_+. \end{cases} \tag{1.3}$$

To overcome this difficulty, via the so-called "energy doubling" property, we show that (1.3) admits no solutions such that the energy is less than or equal to twice the least energy. Then the compactness is recovered for the functional between 0 and twice the least energy.

The second difficulty lies in that fractional Laplacian operator defined in (1.2) is nonlocal. This nonlocal property makes some calculations difficult, and a careful analysis is necessary in lots of estimates. See Lemma 2.1 and following sections for details. On the other hand, a fractional power of the Laplacian defined by using spectral decomposition is also nonlocal, but it can be transformed into a local problem via the extension technique introduced by Caffarelli and Silvestre in [5]. In this sense, many methods used for Laplacian can be applied for fractional power of the Laplacian. As in [9], it has been proved that the problem (1.1) with spectral fractional Laplacian admits infinitely many solutions for $N > 6s$ and $\lambda > 0$. In [15], it is showed that (1.1) has at least $[(N+1)/2]$ for $N > 2(1+\sqrt{2})s$ and $\lambda > 0$. However, this transformation is invalid here for fractional Laplacian defined in (1.2), so it make this problem interesting and challenging. That's why we are interested in multiplicity results of problem (1.1). For more multiplicity results and details about the equations involving the spectral fractional Laplacian, one can refer to [9, 15].

The paper is organized as follows. In Section 2, we introduce notations and some preliminary results for problem (1.1). In Section 3, we discuss the limiting problems of (1.1) and a compactness lemma. Some useful estimates are obtained in Section 4. Finally, we give a proof of Theorem 1.1 in Section 4.

2. Notations and preliminaries results

In this section, we introduce some notations and preliminary results.

- We denote by C the positive constants (possibly different), by $\mathbb{N}$ the set of all positive integers and by $\|u\|_p = \big(\int_\Omega |u|^p \, dx\big)^{1/p}$ the norm of $L^p(\Omega)$.

- The homogeneous fractional Sobolev space $D^s(\mathbb{R}^N)$, as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm $\|u\|_{D^s(\mathbb{R}^N)}$, is equivalently defined by

$$D^s(\mathbb{R}^N) := \left\{ u \in L^{2_s^*}(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \, dxdy < +\infty \right\}$$

with inner product

$$(u,v)_{D^s(\mathbb{R}^N)} := \int_{\mathbb{R}^{2N}} \frac{[u(x)-u(y)][v(x)-v(y)]}{|x-y|^{N+2s}} \, dxdy$$

and the corresponding norm

$$\|u\|^2_{D^s(\mathbb{R}^N)} = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \, dxdy.$$

For more details and related results, one can refer to [19].

- Denote by S the best fractional critical Sobolev constant from $D^s(\mathbb{R}^N)$ into $L^{2_s^*}(\mathbb{R}^N)$, i.e., $S = \inf_{u\in D^s(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|^2_{D^s(\mathbb{R}^N)}}{\|u\|^2_{L^{2_s^*}(\mathbb{R}^N)}}$. It follows from [11, Theorem 1.1] that S is attained at the functions

$$\widetilde{u}(x) = \kappa(\mu^2 + |x - x_0|^2)^{-(N-2s)/2}, \quad x \in \mathbb{R}^N$$

with $\kappa \in \mathbb{R} \setminus \{0\}$, $\mu > 0$ and $x_0 \in \mathbb{R}^N$. Equivalently, the function $\overline{u}(x) := \widetilde{u}(x)/\|\widetilde{u}\|_{L^{2_s^*}(\mathbb{R}^N)}$ satisfies $S = \|\overline{u}\|^2_{D^s(\mathbb{R}^N)}$ with $\|\overline{u}\|_{L^{2_s^*}(\mathbb{R}^N)} = 1$. In the following sections, we always assume $\mu = 1$ and let $u^*(x) := \overline{u}(x/S^{1/2s})$. According to [25], u^* is an explicit solution of the problem

$$\begin{cases} (-\Delta)^s u = |u|^{2_s^*-2}u & \text{in } \mathbb{R}^N, \\ u \in D^s(\mathbb{R}^N), \end{cases} \tag{2.1}$$

and $\|u^*\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^N)} = S^{N/(2s)}$. Moreover, the family of functions

$$U_{\epsilon,z}(x) = \epsilon^{-(N-2s)/2}u^*(x/\epsilon) = \widetilde{\kappa}\left(\frac{\epsilon}{\epsilon^2 + |x-z|^2}\right)^{(N-2s)/2}, \quad \epsilon > 0,\ z \in \mathbb{R}^N, \tag{2.2}$$

are solutions of (2.1) and verify

$$\|U_{\epsilon,z}\|^2_{D^s(\mathbb{R}^N)} = \|U_{\epsilon,z}\|^{2_s^*}_{L^{2_s^*}(\mathbb{R}^N)} = S^{N/(2s)}. \tag{2.3}$$

- We write $D^s(\mathbb{R}^N_+)$ as the completion of $C_0^\infty(\mathbb{R}^N_+)$ in $D^s(\mathbb{R}^N)$.

- The space $X_0^s(\Omega)$ is a Hilbert space defined by

$$X_0^s(\Omega) = \{u \in D^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

with inner product $(u,v)_{X_0^s} := (u,v)_{D^s(\mathbb{R}^N)}$ and norm $\|u\|_{X_0^s} := \|u\|_{D^s(\mathbb{R}^N)}$. From [24], we know that $X_0^s(\Omega)$ is a subspace of $D^s(\mathbb{R}^N)$. Moreover, $X_0^s(\Omega)$ is embedded continuously into $L^{2_s^*}(\Omega)$, and compactly into $L^t(\Omega)$ for any $t \in [1, 2_s^*)$ if Ω has a Lipschitz boundary.

- Let $(\lambda_i)_{i=1}^\infty$ be the eigenvalues of $(-\Delta)^s$ with zero Dirichlet boundary data and $(e_i)_{i=1}^\infty$ be the L^2-normalized orthogonal eigenfunctions corresponding to λ_i, that is,

$$\begin{cases} (-\Delta)^s e_i = \lambda_i e_i & \text{in } \Omega, \\ e_i = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \\ \int_\Omega |e_i|^2\, dx = 1. \end{cases} \tag{2.4}$$

Then

$$\begin{aligned} &\|e_i\|^2_{X^s_0(\Omega)} = \lambda_i && \text{for each } i \geq 1, \\ &(e_i, e_j)_{X^s_0(\Omega)} = 0 \text{ and } \int_\Omega e_i e_j \, dx = 0 && \text{for any } i, j \geq 1 \text{ with } i \neq j. \end{aligned} \tag{2.5}$$

According to [23, Proposition 9], there holds $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_i \leq \lambda_{i+1} \leq \cdots$ and $\lambda_i \to \infty$ as $i \to \infty$. For more information about the spectrum of fractional Laplacian, one can refer to [23] and references therein. If $\lambda_n \leq \lambda < \lambda_{n+1}$ for some $n \geq 1$, we set

$$H^- := \operatorname{span}\{e_1, \ldots, e_n\}, \quad H^+ := \overline{\operatorname{span}\{e_i : i \geq n+1\}}.$$

Before discussing problem (1.1), we first show some necessary lemmas.

Lemma 2.1. *For any $w, v \in D^s(\mathbb{R}^N)$ with $\operatorname{supp} w \cap \operatorname{supp} v = \emptyset$, we have*

$$(w, v)_{D^s(\mathbb{R}^N)} = -2 \int_{\mathbb{R}^{2N}} \frac{w(x)v(y)}{|x-y|^{N+2s}} \, dxdy \tag{2.6}$$

and

$$\|w+v\|^2_{D^s(\mathbb{R}^N)} = \|w\|^2_{D^s(\mathbb{R}^N)} + \|v\|^2_{D^s(\mathbb{R}^N)} - 4 \int_{\mathbb{R}^{2N}} \frac{w(x)v(y)}{|x-y|^{N+2s}} \, dxdy.$$

Proof. Set

$$\begin{aligned} A_1 &= \{(x,y) \in \mathbb{R}^{2N} : w(x) \neq 0, v(y) \neq 0\}, \quad A_2 = \{(x,y) \in \mathbb{R}^{2N} : w(x) \neq 0, v(y) = 0\}; \\ A_3 &= \{(x,y) \in \mathbb{R}^{2N} : w(x) = 0, v(y) \neq 0\}, \quad A_4 = \{(x,y) \in \mathbb{R}^{2N} : w(x) = 0, v(y) = 0\}. \end{aligned}$$

By direct computations, we have

$$\begin{aligned} &\int_{A_2} \frac{[w(x)-w(y)][v(x)-v(y)]}{|x-y|^{N+2s}} \, dxdy = \int_{A_3} \frac{[w(x)-w(y)][v(x)-v(y)]}{|x-y|^{N+2s}} \, dxdy = 0, \\ &\int_{A_1} \frac{[w(x)-w(y)][v(x)-v(y)]}{|x-y|^{N+2s}} \, dxdy = \int_{A_1} \frac{-w(x)v(y)}{|x-y|^{N+2s}} \, dxdy = -\int_{\mathbb{R}^{2N}} \frac{w(x)v(y)}{|x-y|^{N+2s}} \, dxdy, \\ &\int_{A_4} \frac{[w(x)-w(y)][v(x)-v(y)]}{|x-y|^{N+2s}} \, dxdy = \int_{A_4} \frac{-w(y)v(x)}{|x-y|^{N+2s}} \, dxdy = \int_{\mathbb{R}^{2N}} \frac{-w(y)v(x)}{|x-y|^{N+2s}} \, dxdy \\ &\qquad = -\int_{\mathbb{R}^{2N}} \frac{w(x)v(y)}{|x-y|^{N+2s}} \, dxdy. \end{aligned}$$

Then

$$\begin{aligned} (w, v)_{D^s(\mathbb{R}^N)} &= \int_{A_1 \cup A_2 \cup A_3 \cup A_4} \frac{[w(x)-w(y)][v(x)-v(y)]}{|x-y|^{N+2s}} \, dxdy \\ &= -2 \int_{\mathbb{R}^{2N}} \frac{w(x)v(y)}{|x-y|^{N+2s}} \, dxdy \end{aligned}$$

and

$$\begin{aligned}\|w+v\|^2_{D^s(\mathbb{R}^N)} &= \|w\|^2_{D^s(\mathbb{R}^N)} + \|v\|^2_{D^s(\mathbb{R}^N)} + 2(w,v)_{D^s(\mathbb{R}^N)} \\ &= \|w\|^2_{D^s(\mathbb{R}^N)} + \|v\|^2_{D^s(\mathbb{R}^N)} - 4\int_{\mathbb{R}^{2N}} \frac{w(x)v(y)}{|x-y|^{N+2s}}\,dxdy.\end{aligned}$$

The lemma follows. □

Remark 2.2. As a consequence of this lemma, the Lebesgue integral on the right-hand side of (2.6) is well defined, because $|(w,v)_{D^s(\mathbb{R}^N)}| \le \|w\|_{D^s(\mathbb{R}^N)}\|v\|_{D^s(\mathbb{R}^N)}$.

Set $u_\pm = \max\{\pm u, 0\}$. Then $u = u_+ - u_-$.

Lemma 2.3. *Suppose that $u \in D^s(\mathbb{R}^N)$ is a weak solution of* (2.1) *with $u_\pm \neq 0$, that is, for any $\phi \in D^s(\mathbb{R}^N)$, there holds*

$$(u,\phi)_{D^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |u|^{2^*_s-2}u\phi\,dx.$$

Then

$$\mathfrak{I}(u) > \frac{2s}{N}S^{N/(2s)},$$

where $\mathfrak{I}(u) = \frac{1}{2}\|u\|^2_{D^s(\mathbb{R}^N)} - \frac{1}{2^*_s}\int_{\mathbb{R}^N}|u|^{2^*_s}\,dx$ is the corresponding functional of (2.1).

Proof. Since $u_\pm \neq 0$, by (2.6), we have

$$\begin{aligned}(u_+,u_-)_{D^s(\mathbb{R}^N)} &= -2\int_{\mathbb{R}^{2N}} \frac{u_+(x)u_-(y)}{|x-y|^{N+2s}}\,dxdy \\ &= 2\int_{\{u>0\}}\int_{\{u<0\}} \frac{u(x)u(y)}{|x-y|^{N+2s}}\,dxdy < 0.\end{aligned} \tag{2.7}$$

Note that $0 = \mathfrak{I}'(u)u_\pm = (u,u_\pm)_{D^s(\mathbb{R}^N)} \mp \int_{\mathbb{R}^N}|u_\pm|^{2^*_s}\,dx$ and $(u,u_\pm)_{D^s(\mathbb{R}^N)} = \|u_\pm\|^2_{D^s(\mathbb{R}^N)} - (u_+,u_-)_{D^s(\mathbb{R}^N)}$. Then by (2.7), we deduce

$$\int_{\mathbb{R}^N}|u_\pm|^{2^*_s}\,dx = \|u_\pm\|^2_{D^s(\mathbb{R}^N)} - (u_+,u_-)_{D^s(\mathbb{R}^N)} > \|u_\pm\|^2_{D^s(\mathbb{R}^N)},$$

and there exists a unique positive number $t_\pm$ given by

$$t_\pm = \left(\frac{\|u\|^2_{D^s(\mathbb{R}^N)}}{\int_{\mathbb{R}^N}|u|^{2^*_s}\,dx}\right)^{\frac{1}{2^*_s-2}} = \left(\frac{\|u_\pm\|^2_{D^s(\mathbb{R}^N)}}{\|u_\pm\|^2_{D^s(\mathbb{R}^N)} - (u_+,u_-)_{D^s(\mathbb{R}^N)}}\right)^{\frac{1}{2^*_s-2}} < 1 \tag{2.8}$$

such that $t_\pm u_\pm \in \mathcal{N} := \{v \in D^s(\mathbb{R}^N)\setminus\{0\} : \mathfrak{I}'(v)v = 0\}$. Hence, $\|t_\pm u_\pm\|^2_{D^s(\mathbb{R}^N)} = \|t_\pm u_\pm\|^{2^*_s}_{L^{2^*_s}(\mathbb{R}^N)}$. This together with $\|t_\pm u_\pm\|^2_{D^s(\mathbb{R}^N)} \ge S\|t_\pm u_\pm\|^2_{L^{2^*_s}(\mathbb{R}^N)}$ gives that

$$\mathfrak{I}(t_\pm u_\pm) \ge \frac{s}{N}S^{N/(2s)}. \tag{2.9}$$

Consequently, by (2.8) and (2.9), we get from $t_\pm u_\pm \in \mathcal{N}$ that

$$\begin{aligned}\mathfrak{I}(u) &= \left(\frac{1}{2}-\frac{1}{2_s^*}\right)\int_{\mathbb{R}^N}|u|^{2_s^*}\,dx > \left(\frac{1}{2}-\frac{1}{2_s^*}\right)\left(\int_{\mathbb{R}^N}|t_+u_+|^{2_s^*}\,dx+\int_{\mathbb{R}^N}|t_-u_-|^{2_s^*}\,dx\right)\\ &= \mathfrak{I}(t_+u_+)+\mathfrak{I}(t_-u_-) \geq \frac{2s}{N}S^{N/(2s)}.\end{aligned}$$

The proof is completed. □

Lemma 2.4. *Suppose that $u \in D^s(\mathbb{R}^N_+)$ is a weak solution of* (1.3)*. Then*

(i) *if u has constant sign in $\mathbb{R}^N_+$, we have $u \equiv 0$.*

(ii) *if u is sign-changing in $\mathbb{R}^N_+$, we have*

$$\mathfrak{L}(u) > \frac{2s}{N}S^{N/(2s)}, \tag{2.10}$$

where $\mathfrak{L}(u) := \frac{1}{2}\|u\|^2_{D^s(\mathbb{R}^N)} - \frac{1}{2_s^*}\int_{\mathbb{R}^N_+}|u|^{2_s^*}\,dx$ is the corresponding functional of (1.3)*.*

Proof. (i) According to [18, Corollary 2.5], it follows that if $u \in D^s(\mathbb{R}^N_+)$ is a constant sign solution of (1.3), then $u \equiv 0$.

(ii) If u is a sign-changing solution of (1.3), by using the same argument as in Lemma 2.3, we can obtain (2.10) and the details are omitted. □

Now, we turn to problem (1.1). Recall that problem (1.1) is variational and its weak formulation is given by

$$\begin{cases}\int_{\mathbb{R}^{2N}}\frac{[u(x)-u(y)][\phi(x)-\phi(y)]}{|x-y|^{N+2s}}\,dxdy = \int_\Omega |u|^{2_s^*-2}u\phi\,dx + \lambda\int_\Omega u\phi\,dx \quad \text{for any } \phi \in X_0^s(\Omega),\\ u \in X_0^s(\Omega).\end{cases} \tag{2.11}$$

Then (2.11) is the Euler-Lagrange equation of functional $I\colon X_0^s(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\left(\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy - \lambda\int_\Omega |u|^2\,dx\right) - \frac{1}{2_s^*}\int_\Omega |u|^{2_s^*}\,dx.$$

Then $I \in C^1(X_0^s(\Omega),\mathbb{R})$ and the solutions of problem (1.1) correspond to critical points of I.

In order to obtain our results, following the idea of [8], we consider a new functional

$$J(u) := \frac{\|u\|^2_{X_0^s} - \lambda\|u\|_2^2}{\|u\|^2_{2_s^*}} = \|u\|^2_{X_0^s} - \lambda\|u\|_2^2$$

constraint on the manifold

$$M := \{u \in X_0^s(\Omega) : \|u\|_{2_s^*} = 1\}.$$

It is easy to see that $M \subset X_0^s(\Omega)$ is a complete manifold and invariant under the involution $u \to -u$. Moreover, $J \in C^1(M,\mathbb{R})$ and $u \in M$ is a critical point of J with $J(u) = \beta > 0$, if and only if $\widetilde{u} = \beta^{\frac{1}{2_s^*-2}} u$ is a critical point of I with $I(\widetilde{u}) = \frac{s}{N}\beta^{N/(2s)} > 0$. Evidently, $(u_m)_{m\geq 1} \subset M$ is a $(\mathrm{PS})_\beta$ sequence for J if and only if $(\widetilde{u}_m)_{m\geq 1} \subset X_0^s(\Omega)$ is a $(\mathrm{PS})_{\widetilde{\beta}}$ sequence for I, where $\widetilde{\beta} = \frac{s}{N}\beta^{N/(2s)}$ and $\widetilde{u}_m := \beta^{\frac{1}{2_s^*-2}} u_m$. Here, we say that a sequence $(u_m)_{m\geq 1} \subset M$ is a $(\mathrm{PS})_\beta$ sequence for J if

$$J(u_m) \to \beta, \quad \|J'(u_m)\| \to 0 \quad \text{as } m \to \infty;$$

and that $(\widetilde{u}_m)_{m\geq 1} \subset X_0^s(\Omega)$ is a $(\mathrm{PS})_{\widetilde{\beta}}$ sequence for I if

$$I(\widetilde{u}_m) \to \widetilde{\beta}, \quad \|I'(\widetilde{u}_m)\| \to 0 \quad \text{as } m \to \infty.$$

Since $J(u) = J(-u)$, it is well known that there exists an odd Lipschitz continuous map $\nu\colon \widehat{M} \to TM$ with $\nu(u) \in T_uM$ and $\|\nu(u)\| < 2\|J'(u)\|$, $\langle J'(u), \nu(u)\rangle > \|J'(u)\|^2$, where $\widehat{M} := \{u \in M : J'(u) \neq 0\}$ and TM is the tangent space of M. Moreover, similarly as [8, Lemma 2.1], the following deformation lemma on the manifold M follows without (PS) condition.

Denote

$$J^\beta := \{u \in M : J(u) \leq \beta\}, \quad K^\beta = \{u \in M : J'(u) = 0 \text{ and } J(u) = \beta\}.$$

Lemma 2.5. *Let $\epsilon, \delta > 0$, $\beta \in \mathbb{R}$ and $D \subset M$ with $D = -D$ such that $\|J'(u)\| \geq 4\epsilon/\delta$ for $u \in J^{-1}[\beta - 2\epsilon, \beta + 2\epsilon] \cap D_{2\delta}$, where $D_\delta = \{u \in M : \mathrm{dist}(u, D) \leq \delta\}$. Then there exists $\eta \in C^1([0,1] \times M, M)$ such that $\eta(t,\cdot)\colon M \to M$ is an odd homeomorphism map for any $t \in [0,1]$ and*

(i) *$\eta(0,u) = u$, $\forall\, u \in M$;*

(ii) *$\eta(t,u) = u$, $\forall\, u \notin J^{-1}[\beta - 2\epsilon, \beta + 2\epsilon] \cap D_{2\delta}$;*

(iii) *$\eta(1, J^{\beta+\epsilon} \cap D) \subset J^{\beta-\epsilon} \cap D_\delta$.*

Proof. The proof is similar to that of [8, Lemma 2.1]. But for the sake of completeness, we give a sketch of the proof below. Define

$$A := J^{-1}[\beta - 2\epsilon, \beta + 2\epsilon] \cap D_{2\delta}, \quad B := J^{-1}[\beta - \epsilon, \beta + \epsilon] \cap D_\delta$$

and $\psi\colon M \to \mathbb{R}$ by $\psi(u) = \frac{\mathrm{dist}(u, M\setminus A)}{\mathrm{dist}(u, M\setminus A) + \mathrm{dist}(u, B)}$. Then ψ is locally Lipschitz continuous, $\psi = 1$ on B and $\psi = 0$ on $M \setminus A$. Let us define the locally Lipschitz continuous vector field

$$f(u) = \begin{cases} -\psi(u)\frac{\nu(u)}{\|\nu(u)\|} & \text{if } u \in \widehat{M}, \\ 0 & \text{if } u \in M \setminus \widehat{M}, \end{cases}$$

and consider the Cauchy problem

$$\frac{d}{dt}\zeta = f(\zeta), \quad \zeta(0,u) = u.$$

Note that $\|f(u)\| \leq 1$ on M. Then the Cauchy problem has a unique solution $\zeta(t,u)$ for all $t \in \mathbb{R}$ such that $\zeta(t,u)$ is odd and homeomorphism with respect to $u \in M$. Now, let $\eta(t,u) = \zeta(\delta t, u)$. Then it is easy to see that (i) and (ii) hold.

Since

$$\frac{d}{dt}J(\eta(t,u)) = -\delta \left\langle J'(\eta), \psi(\eta)\frac{\nu(\eta)}{\|\nu(\eta)\|} \right\rangle \leq -\delta\psi(\eta)\frac{\|J'(\eta)\|^2}{\|\nu(\eta)\|} < 0,$$

$J(\eta(\,\cdot\,,u))$ is nonincreasing. For any $u \in J^{\beta+\epsilon} \cap D$,

$$\|\eta(t,u) - u\| = \left\| \int_0^t \frac{d}{ds}\eta(s,u)\,ds \right\| \leq \int_0^t \delta\|f(w)\|\,ds \leq \delta t,$$

which implies that $\eta(t,u) \in D_\delta$, $\forall\, t \in [0,1]$. Furthermore, if there is some $t_0 \in [0,1)$ such that $J(\eta(t_0,u)) \leq \beta - \epsilon$, then $J(\eta(1,u)) \leq J(\eta(t_0,u)) \leq \beta - \epsilon$. Otherwise, $J(\eta(t,u)) > \beta - \epsilon$ for all $t \in [0,1)$, and then $\eta(t,u) \in B$ and $\psi(\eta(t,u)) \equiv 1$. It follows that

$$J(\eta(1,u)) = J(u) - \delta \int_0^1 \left\langle J'(\eta), \frac{\nu(\eta)}{\|\nu(\eta)\|} \right\rangle dt \leq J(u) - \delta \int_0^1 \frac{\|\nu(\eta)\|}{2}\,dt \leq \beta - \epsilon,$$

that means $\eta(1,u) \in J^{\beta-\epsilon}$. Therefore, (iii) holds. □

Now, for any $j \in \mathbb{N}$, we define

$$\Sigma_j = \{A \subset \mathcal{E} : \gamma(A) \geq j\} \tag{2.12}$$

and consider

$$\beta_j := \inf_{A \in \Sigma_j} \sup_{u \in A} J(u),$$

where $\mathcal{E} = \{A \subset M : A \text{ is closed, symmetric}\}$ and γ denotes the usual Krasnoselskii genus. Note that if $0 < \lambda < \lambda_1$, then

$$\beta_1 \geq \inf_{u \in M} J(u) = \inf_{u \in X_0^s(\Omega)\setminus\{0\}} \|u\|_{X_0^s}^2 - \lambda \int_\Omega |u|^2\,dx \geq \inf_{u \in X_0^s(\Omega)\setminus\{0\}} \left(1 - \frac{\lambda}{\lambda_1}\right) \|u\|_{X_0^s}^2 > 0;$$

if $\lambda_n \leq \lambda < \lambda_{n+1}$ for some $n \geq 1$, then for any $j \geq n+1$ and $A \in \Sigma_j$, we have $A \cap \{u \in H^+ : \|u\|_{2_s^*} = 1\} \neq \emptyset$ and so

$$\beta_j > 0.$$

Furthermore, by using Lemma 2.5 and following the similar arguments as in the proof of [8, Lemmas 2.2 and 2.4], we have the following two lemmas.

Lemma 2.6. *The following statements are true:*

(i) *If $0 < \lambda < \lambda_1$, then for each $j \in \mathbb{N}$, there exists a $(\mathrm{PS})_{\beta_j}$ sequence $(u_k)_{k \geq 1}$ for J.*

(ii) *If $\lambda_n \leq \lambda < \lambda_{n+1}$ for some $n \geq 1$, then for each $j \geq n+1$, there exists a $(\mathrm{PS})_{\beta_j}$ sequence $(u_k)_{k \geq 1}$ for J.*

Proof. Let $j \in \mathbb{N}$ for $0 < \lambda < \lambda_1$, and $j \geq n+1$ for $\lambda_n \leq \lambda < \lambda_{n+1}$. If there is no $(\mathrm{PS})_{\beta_j}$ sequence (u_k) for J, then by Lemma 2.5, there exists $\eta \in C^1([0,1] \times M, M)$ such that $\eta(1, J^{\beta_j + \epsilon}) \subset J^{\beta_j - \epsilon}$. By the definition of β_j, there is a set $A \in \Sigma_j$ such that $\sup_A J(u) < \beta_j + \epsilon$. Since $\eta(1, \cdot\,)$ is odd, $\gamma(\eta(1,A)) \geq \gamma(A) \geq j$. Then $\eta(1,A) \in \Sigma_j$. However, $\eta(1,A) \subset \eta(1, J^{\beta_j+\epsilon}) \subset J^{\beta_j - \epsilon}$, which contradicts with the definition of β_j. □

Lemma 2.7. *If $0 < \beta_j = \beta_{j+1} < 2^{2s/N} S$, then K^{β_j} is infinite.*

Proof. The proof is similar to [8, Lemma 2.4] and the details are omitted here. □

In view of (2.2), set

$$E := \{U_{\epsilon,z} : \epsilon > 0, z \in \mathbb{R}^N\}.$$

According to [7] or [16], E contains all positive solutions of (2.1) in $D^s(\mathbb{R}^N)$. In addition, by using Lemmas 2.3, 2.4 and [20, Theorem 1.1], we have the following lemma.

Lemma 2.8. *Let $(u_m)_{m \geq 1}$ be a $(\mathrm{PS})_{\beta_j}$ sequence for functional J. Up to a subsequence, the following conclusions hold.*

(a) *If $\beta_j \in (0, S)$, then $(u_m)_{m \geq 1}$ converges in M and β_j is a critical value of J.*

(b) *If $\beta_j \in (S, 2^{2s/N} S)$, then one of the following cases holds true:*

(b1) *$(u_m)_{m \geq 1}$ converges in M and β_j is a critical value of J.*

(b2) *There exists a critical point u of J in M with $J(u) = \beta_* = (\beta_j^{N/(2s)} - S^{N/(2s)})^{2s/N} \in (0, S)$ such that*

(2.13)
$$\operatorname{dist}\left(\beta_j^{\frac{1}{2_s^*-2}} u_m - \beta_*^{\frac{1}{2_s^*-2}} u, E\right) \to 0 \quad \textit{or} \quad \operatorname{dist}\left(\beta_j^{\frac{1}{2_s^*-2}} u_m - \beta_*^{\frac{1}{2_s^*-2}} u, -E\right) \to 0.$$

(c) *If $\beta_j = S$, then one of the following cases holds true:*

(c1) *$(u_m)_{m \geq 1}$ converges in M and β_j is a critical value of J.*

(c2)

$$\operatorname{dist}\left(\beta_j^{\frac{1}{2_s^*-2}} u_m, E\right) \to 0 \quad \textit{or} \quad \operatorname{dist}\left(\beta_j^{\frac{1}{2_s^*-2}} u_m, -E\right) \to 0. \tag{2.14}$$

Proof. Let $(u_m)_{m\geq 1}$ be a $(\mathrm{PS})_{\beta_j}$ sequence for J. Then it follows that the sequence $(\widetilde{u}_m)$, given by $\widetilde{u}_m := \beta_j^{\frac{1}{2_s^*-2}} u_m$, is a $(\mathrm{PS})_{\widetilde{\beta}}$ sequence for I with $\widetilde{\beta} = \frac{s}{N}\beta_j^{N/(2s)}$. It is easy to see that $(\widetilde{u}_m)_{m\geq 1}$ are bounded in $X_0^s(\Omega)$ and thus there exists a (possibly trivial) solution $u^0 \in X_0^s(\Omega)$ of (1.1) such that

$$\widetilde{u}_m \rightharpoonup u^0 \quad \text{in } X_0^s(\Omega).$$

By applying splitting lemma (see [20, Theorem 1.1]) to problem (1.1), we get that either

$$\widetilde{u}_m \to u^0 \text{ strongly in } X_0^s(\Omega), \tag{2.15}$$

or there exists two finite sets $\mathcal{J}_1$, $\mathcal{J}_2$, nontrivial solutions $\{u^j\}_{j\in\mathcal{J}_1}$ to (2.1) in $D^s(\mathbb{R}^N)$ and solutions $\{u^j\}_{j\in\mathcal{J}_2}$ to (1.3) in $D^s(\mathbb{R}^N_+)$ such that

$$I(u_m) = I(u^0) + \sum_{\mathcal{J}_1}\mathfrak{I}(u^j) + \sum_{\mathcal{J}_2}\mathfrak{L}(u^j) + o(1) \tag{2.16}$$

and

$$\|u_m\|^2_{X_0^s(\Omega)} = \|u^0\|^2_{X_0^s} + \sum_{j\in\mathcal{J}_1}\|u^j\|^2_{D^s(\mathbb{R}^N)} + \sum_{j\in\mathcal{J}_2}\|u^j\|^2_{D^s(\mathbb{R}^N_+)} + o(1), \tag{2.17}$$

where $\mathfrak{I}$, $\mathfrak{L}$ are defined in Lemmas 2.3 and 2.4, respectively. If (2.15) happens, then we are done. Otherwise, for any $\widetilde{\beta}_j \in \big(0, \frac{2s}{N}S^{N/(2s)}\big)$, since $I(u^0), \mathfrak{I}(u^j), \mathfrak{L}(v^j) \geq 0$, it follows from (2.16) and Lemma 2.4 that $\mathcal{J}_2 = \emptyset$. Furthermore, it yields from (2.16) and Lemma 2.3 that either $\mathcal{J}_1 = \{1\}$ or $\mathcal{J}_1 = \emptyset$. Then, (2.16) and (2.17) are reduced to

$$I(\widetilde{u}_m) = I(u^0) + \sum_{j\in\mathcal{J}_1}\mathfrak{I}(u^j) + o(1) \tag{2.18}$$

and

$$\|\widetilde{u}_m\|^2_{X_0^s} = \|u^0\|^2_{X_0^s} + \sum_{j\in\mathcal{J}_1}\|u^j\|^2_{D^s(\mathbb{R}^N)} + o(1), \tag{2.19}$$

where $\mathcal{J}_1 = \{1\}$ or $\mathcal{J}_1 = \emptyset$.

Thus, it follows from (2.18) and (2.19) that if $\widetilde{\beta}_j \in \big(0, \frac{s}{N}S^{N/(2s)}\big)$, then $\mathcal{J}_1 = \emptyset$ and $\widetilde{\beta}_j$ is a critical value of I, so (a) holds. If $\widetilde{\beta}_j \in \big(\frac{s}{N}S^{N/(2s)}, \frac{2s}{N}S^{N/(2s)}\big)$, then $\mathcal{J}_1 = \{1\}$ and $\widetilde{\beta}_j$ is a critical value of I and thus (b1) holds. Otherwise, $\mathcal{J}_1 = \{1\}$ and there is a critical point $\widetilde{u}$ with $I(\widetilde{u}) = \widetilde{\beta}_j - \frac{s}{N}S^{N/(2s)} = \frac{s}{N}\beta_*^{N/(2s)}$ such that (2.13) holds, which implies (b2). Similarly, if $\widetilde{\beta}_j = \frac{s}{N}S^{N/(2s)}$, then $\widetilde{\beta}_j$ is a critical value of I and (c1) follows. Otherwise, (2.14) holds and (c2) follows.

The proof is completed. $\square$

3. Some useful estimates

For any point $z \in \Omega$, denote $B_r(z) = \{y \in \mathbb{R}^N : |y - z| < r\}$. Without loss of generality, we may assume that $0 \in \Omega$ and $B_{d_0}(0) \subset \Omega$ with some $d_0 > 0$. For any $0 < \eta < d_0$, we define

$$\xi_\eta(x) = \begin{cases} 0 & \text{if } x \in B_{\eta/2}(0), \\ \frac{2}{\eta}|x| - 1 & \text{if } x \in B_\eta(0) \setminus B_{\eta/2}(0), \\ 1 & \text{if } x \in \mathbb{R}^N \setminus B_\eta(0). \end{cases} \tag{3.1}$$

Clearly, $|\nabla \xi_\eta| \le 2/\eta$.

Lemma 3.1. *$\int_{\mathbb{R}^{2N}} \frac{|\xi_\eta(x) - \xi_\eta(y)|^2}{|x-y|^{N+2s}}\, dxdy \le C_0 \eta^{N-2s}$ for some $C_0 > 0$ independent of η.*

Proof. Set

$$\begin{aligned} \Gamma_1 &= \left\{(x,y) : |x-y| \ge \frac{\eta}{2}, x \in B_\eta(0), y \in \mathbb{R}^N \setminus B_\eta(0)\right\}, \\ \Gamma_2 &= \left\{(x,y) : |x-y| \le \frac{\eta}{2}, x \in B_\eta(0), y \in \mathbb{R}^N \setminus B_\eta(0)\right\}. \end{aligned}$$

According to the definition of ξ_η and mean value theorem, we have

$$\begin{aligned} \int_{\Gamma_1} \frac{|\xi_\eta(x) - \xi_\eta(y)|^2}{|x-y|^{N+2s}}\, dxdy &\le 2\int_{\Gamma_1} \frac{1}{|x-y|^{N+2s}}\, dxdy \\ &= 2\int_{B_\eta(0)} \int_{\{|x-y| \ge \eta/2\}} \frac{1}{|x-y|^{N+2s}}\, dydx \\ &\le C\eta^{N-2s} \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \int_{\Gamma_2} \frac{|\xi_\eta(x) - \xi_\eta(y)|^2}{|x-y|^{N+2s}}\, dxdy &\le \frac{4}{\eta^2}\int_{\Gamma_2} \frac{|x-y|^2}{|x-y|^{N+2s}}\, dydx \\ &= \frac{4}{\eta^2}\int_{B_\eta(0)} \int_{\{|x-y| \le \eta/2\}} \frac{1}{|x-y|^{N+2s-2}}\, dydx \\ &\le C\eta^{N-2s}. \end{aligned} \tag{3.3}$$

Note that

$$\begin{aligned} \int_{B_\eta(0)} \int_{B_\eta(0)} \frac{|\xi_\eta(x) - \xi_\eta(y)|^2}{|x-y|^{N+2s}}\, dxdy &\le \frac{4}{\eta^2}\int_{B_\eta(0)} \int_{B_\eta(0)} \frac{|x-y|^2}{|x-y|^{N+2s}}\, dydx \\ &= \frac{4}{\eta^2}\int_{B_\eta(0)} \int_{\{|x-y| \le 2\eta\}} \frac{1}{|x-y|^{N+2s-2}}\, dydx \\ &\le C\eta^{N-2s}. \end{aligned} \tag{3.4}$$

Then we conclude from (3.2)–(3.4) that

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{|\xi_\eta(x)-\xi_\eta(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&= \int_{B_\eta(0)}\int_{B_\eta(0)} \frac{|\xi_\eta(x)-\xi_\eta(y)|^2}{|x-y|^{N+2s}}\,dxdy + 2\int_{B_\eta(0)}\int_{\mathbb{R}^N\setminus B_\eta(0)} \frac{|\xi_\eta(x)-\xi_\eta(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&= \int_{B_\eta(0)}\int_{B_\eta(0)} \frac{|\xi_\eta(x)-\xi_\eta(y)|^2}{|x-y|^{N+2s}}\,dxdy + 2\int_{\Gamma_1} \frac{|\xi_\eta(x)-\xi_\eta(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&\quad + 2\int_{\Gamma_2} \frac{|\xi_\eta(x)-\xi_\eta(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&\le C_0\eta^{N-2s}
\end{aligned}$$

for some C_0 independent of η. The proof is completed. □

Now, for any $m \ge 1$ with $2/m < d_0$ and any integer $i \ge 1$, we set

$$e_i^m(x) = \xi_{2/m}(x)e_i(x) \quad \text{for all } x \in \mathbb{R}^N$$

and

$$H_{m,n}^-(\lambda) = \begin{cases} \operatorname{span}\{e_1^m,\dots,e_n^m\} & \text{if } \lambda_n < \lambda < \lambda_{n+1}; \\ \operatorname{span}\{e_1^m,\dots,e_{n-l}^m\} & \text{if } \lambda_{n-l} < \lambda_{n-l+1} = \cdots = \lambda_n = \lambda < \lambda_{n+1} \\ & \text{for some } 0 < l < \min\{n, N+2\}. \end{cases}$$

Lemma 3.2. *Suppose $\lambda_n \le \lambda < \lambda_{n+1}$ and $0 < l < \min\{n, N+2\}$, then there exists $m_0 > 1$ such that for any $m \ge m_0$,*

$$\max_{\substack{u\in H_{m,n}^-(\lambda)\\ \|u\|_2=1}} \|u\|_{X_0^s}^2 \le \begin{cases} \lambda_n + C_1 m^{2s-N} & \text{if } \lambda \in (\lambda_n,\lambda_{n+1}), \\ \lambda_{n-l} + C_1 m^{2s-N} & \text{if } \lambda_{n-l} < \lambda_{n-l+1} = \cdots = \lambda_n = \lambda < \lambda_{n+1} \end{cases} \tag{3.5}$$

for some $C_1 > 0$ independent of m.

Proof. We first prove the following estimates:

$$\|e_i^m\|_{X_0^s}^2 \le \lambda_i + Cm^{2s-N} \qquad \text{for any } i \in \mathbb{N}; \tag{3.6}$$

$$|(e_i^m, e_j^m)_{X_0^s}| \le Cm^{2s-N} \qquad \text{for any } i,j \in \mathbb{N},\ i \ne j; \tag{3.7}$$

$$|(e_i^m, e_j^m)_2| \le Cm^{-N} \qquad \text{for any } i,j \in \mathbb{N},\ i \ne j; \tag{3.8}$$

$$\|e_i^m\|_2^2 \ge 1 - Cm^{-N} \qquad \text{for any } i \in \mathbb{N}; \tag{3.9}$$

According to [21, Proposition 2.4], it follows that $e_i \in L^\infty(\Omega)$. Together with Lemma 3.1, this gives that

$$\begin{aligned}&\left|\int_{\mathbb{R}^{2N}} \frac{e_i(x)e_i(y)[\xi_{2/m}(x)-\xi_{2/m}(y)]^2}{|x-y|^{N+2s}}\,dxdy\right| \\ &\leq C\int_{\mathbb{R}^{2N}} \frac{[\xi_{2/m}(x)-\xi_{2/m}(y)]^2}{|x-y|^{N+2s}}\,dxdy \leq Cm^{2s-N}.\end{aligned} \tag{3.10}$$

In addition, multiplying (2.4) by $[\xi^2_{2/m}(x)-1]e_i$ and integrating by parts, we have

$$\begin{aligned}&\lambda_i\int_\Omega[\xi^2_{2/m}(x)-1]e_i^2\,dx \\ &= ([\xi^2_{2/m}(x)-1]e_i, e_i)_{X_0^s} \\ &= \int_{\mathbb{R}^{2N}} \frac{[\xi^2_{2/m}(x)-1][e_i(x)-e_i(y)]^2 + e_i(y)[\xi^2_{2/m}(x)-\xi^2_{2/m}(y)][e_i(x)-e_i(y)]}{|x-y|^{N+2s}}\,dxdy\end{aligned} \tag{3.11}$$

Then, it follows from (3.10) and (3.11) that

$$\begin{aligned}&\|e_i^m\|^2_{X_0^s} - \|e_i\|^2_{X_0^s} \\ &= \int_{\mathbb{R}^{2N}} \frac{[\xi_{2/m}(x)e_i(x)-\xi_{2/m}(y)e_i(y)]^2 - [e_i(x)-e_i(y)]^2}{|x-y|^{N+2s}}\,dxdy \\ &= \int_{\mathbb{R}^{2N}} \frac{[\xi^2_{2/m}(x)-1][e_i(x)-e_i(y)]^2 + e_i(y)[\xi^2_{2/m}(x)-\xi^2_{2/m}(y)][e_i(x)-e_i(y)]}{|x-y|^{N+2s}}\,dxdy \\ &\quad + \int_{\mathbb{R}^{2N}} \frac{e_i(x)e_i(y)[\xi_{2/m}(x)-\xi_{2/m}(y)]^2}{|x-y|^{N+2s}}\,dxdy \\ &= \lambda_i\int_\Omega[\xi^2_{2/m}(x)-1]e_i^2\,dx + \int_{\mathbb{R}^{2N}} \frac{e_i(x)e_i(y)[\xi_{2/m}(x)-\xi_{2/m}(y)]^2}{|x-y|^{N+2s}}\,dxdy \\ &\leq \left|\int_{\mathbb{R}^{2N}} \frac{e_i(x)e_i(y)[\xi_{2/m}(x)-\xi_{2/m}(y)]^2}{|x-y|^{N+2s}}\,dxdy\right| \\ &\leq Cm^{2s-N}.\end{aligned}$$

This combined with (2.4) yields (3.6).

Multiplying (2.4) by $[\xi^2_{2/m}(x)-1]e_j$ and integrating, we have

$$\begin{aligned}&\lambda_i\int_\Omega[\xi^2_{2/m}-1]e_ie_j\,dx \\ &= (e_i, (\xi^2_{2/m}-1)e_j)_{X_0^s} \\ &= \int_{\mathbb{R}^{2N}} \frac{[e_i(x)-e_i(y)]\big([\xi^2_{2/m}(x)-1]e_j(x)-[\xi^2_{2/m}(y)-1]e_j(y)\big)}{|x-y|^{N+2s}}\,dxdy \\ &= \int_{\mathbb{R}^{2N}} \frac{[\xi^2_{2/m}(x)-1][e_i(x)-e_i(y)][e_j(x)-e_j(y)] + [e_i(x)-e_i(y)][\xi^2_{2/m}(x)-\xi^2_{2/m}(y)]e_j(y)}{|x-y|^{N+2s}}\,dxdy.\end{aligned}$$

Similarly,

$$\begin{aligned}
&\lambda_j \int_\Omega [\xi_{2/m}^2 - 1] e_i e_j \, dx \\
&= (e_j, (\xi_{2/m}^2 - 1) e_i)_{X_0^s} \\
&= \int_{\mathbb{R}^{2N}} \frac{[e_j(x) - e_j(y)]\big([\xi_{2/m}^2(x) - 1] e_i(x) - (\xi_{2/m}^2(y) - 1) e_i(y)\big)}{|x-y|^{N+2s}} \, dxdy \\
&= \int_{\mathbb{R}^{2N}} \frac{[\xi_{2/m}^2(x) - 1][e_i(x) - e_i(y)][e_j(x) - e_j(y)] + [e_j(x) - e_j(y)][\xi_{2/m}^2(x) - \xi_{2/m}^2(y)] e_i(y)}{|x-y|^{N+2s}} \, dxdy.
\end{aligned}$$

Then it follows that

$$\begin{aligned}
&\frac{\lambda_i + \lambda_j}{2} \int_\Omega [\xi_{2/m}^2(x) - 1] e_i e_j \, dx \\
&= \int_{\mathbb{R}^{2N}} \frac{\frac{1}{2}[e_i(x)e_j(y) + e_i(y)e_j(x) - 2e_i(y)e_j(y)][\xi_{2/m}^2(x) - \xi_{2/m}^2(y)]}{|x-y|^{N+2s}} \\
&\qquad + \frac{[\xi_{2/m}^2(x) - 1][e_i(x) - e_i(y)][e_j(x) - e_j(y)]}{|x-y|^{N+2s}} \, dxdy.
\end{aligned}$$

This together with $e_i \in L^\infty(\mathbb{R}^N)$, (2.5) and Lemma 3.1 gives that for any $i \neq j$,

$$\begin{aligned}
&|(e_i^m, e_j^m)_{X_0^s}| \\
&= |(e_i^m, e_j^m)_{X_0^s} - (e_i, e_j)_{X_0^s}| \\
&= \bigg| \int_{\mathbb{R}^{2N}} \frac{[\xi_{2/m}(x)e_i(x) - \xi_{2/m}(y)e_i(y)][\xi_{2/m}(x)e_j(x) - \xi_{2/m}(y)e_j(y)]}{|x-y|^{N+2s}} \\
&\qquad - \frac{[e_i(x) - e_i(y)][e_j(x) - e_j(y)]}{|x-y|^{N+2s}} \, dxdy \bigg| \\
&= \bigg| \frac{\lambda_i + \lambda_j}{2} \int_\Omega (\xi_{2/m}^2 - 1) e_i e_j \, dx - \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{[e_i(x)e_j(y) + e_j(x)e_i(y)][\xi_{2/m}(x) - \xi_{2/m}(y)]^2}{|x-y|^{N+2s}} \, dxdy \bigg| \\
&\leq \frac{\lambda_i + \lambda_j}{2} \int_{B_{2/m}} |e_i e_j| \, dx + \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{|e_i(x)e_j(y) + e_j(x)e_i(y)|[\xi_{2/m}(x) - \xi_{2/m}(y)]^2}{|x-y|^{N+2s}} \, dxdy \\
&\leq C \int_{B_{2/m}} dx + C \int_{\mathbb{R}^{2N}} \frac{[\xi_{2/m}(x) - \xi_{2/m}(y)]^2}{|x-y|^{N+2s}} \, dxdy \\
&\leq C m^{-N} + C m^{2s-N} \\
&\leq C m^{2s-N}
\end{aligned}$$

if m is large enough. So (3.7) follows.

By (2.5) and (3.1), it follows that for $i \neq j$,

$$|(e_i^m, e_j^m)_2| = \left| \int_\Omega \xi_m^2(x) e_i e_j \, dx \right| = \left| \int_\Omega [\xi_m^2(x) - 1] e_i e_j \, dx \right| \leq C \int_{B_{2/m}} dx \leq C m^{-N}.$$

So (3.8) holds.

By (2.4) and (3.1), we get

$$\|e_i^m\|_2^2 = \int_\Omega e_i^2\,dx - \int_\Omega [1-\xi_{2/m}^2(x)]e_i^2\,dx \ge 1 - \int_{B_{2/m}} e_i^2\,dx \ge 1 - Cm^{-N},$$

which gives (3.9).

Now, we are ready to prove (3.5) by using the estimates (3.6)–(3.9). Let $u_m \in H_{m,n}^-(\lambda)$ with $\|u_m\|_2 = 1$ be such that

$$\|u_m\|_{X_0^s}^2 = \max_{u\in H_{m,n}^-(\lambda),\|u\|_2=1} \|u\|_{X_0^s}^2.$$

Thus, for the first case $\lambda_n < \lambda < \lambda_{n+1}$, there exist real numbers $a_1^m, \dots, a_n^m$ such that $u_m = \sum_{i=1}^n a_i^m e_i^m$. Then we have

$$\|u_m\|_{X_0^s}^2 = \sum_{i=1}^n (a_i^m)^2 \|e_i^m\|_{X_0^s}^2 + 2\sum_{1\le i<j\le n} a_i^m a_j^m (e_i^m, e_j^m)_{X_0^s}$$

and

$$1 = \|u_m\|_2^2 = \sum_{i=1}^n (a_i^m)^2 \|e_i^m\|_2^2 + 2\sum_{1\le i<j\le n} a_i^m a_j^m (e_i^m, e_j^m)_2. \tag{3.12}$$

Thanks to (3.8) and (3.9), there exists $m_0 > 1$ such that for any $m \ge m_0$,

$$\|e_i^m\|_2^2 \ge \frac{3}{4} \quad \text{and} \quad |(e_i^m, e_j^m)_{L^2(\Omega)}| \le \frac{1}{4} \quad \text{for any } i \ne j.$$

Hence it follows from (3.12) that

$$\begin{aligned}
1 &= \sum_{i=1}^n (a_i^m)^2 \|e_i^m\|_2^2 + 2\sum_{1\le i<j\le n} a_i^m a_j^m (e_i^m, e_j^m)_2 \\
&\ge \sum_{i=1}^n (a_i^m)^2 \|e_i^m\|_2^2 - 2\sum_{1\le i<j\le n} |a_i^m||a_j^m||(e_i^m, e_j^m)_2| \\
&\ge \frac{3}{4}\sum_{i=1}^n (a_i^m)^2 - \frac{1}{4}\sum_{1\le i<j\le n} (|a_i^m|^2 + |a_j^m|^2) \\
&\ge \frac{1}{4}\sum_{i=1}^n (a_i^m)^2,
\end{aligned}$$

which derives

$$|a_i^m| \le C \quad \text{for some constant } C \text{ independent of } m \text{ and } i. \tag{3.13}$$

Then by (3.9), (3.12)and (3.13), we conclude

$$
\begin{aligned}
1 &\geq \sum_{i=1}^{n}(a_i^m)^2\|e_i^m\|_2^2 - 2\sum_{1\leq i<j\leq n}|a_i^m a_j^m||(e_i^m, e_j^m)_2| \\
&\geq \sum_{i=1}^{n}(a_i^m)^2\|e_i^m\|_2^2 - C\sum_{1\leq i<j\leq n}|(e_i^m, e_j^m)_2| \geq \sum_{i=1}^{n}(a_i^m)^2\|e_i^m\|_2^2 - Cm^{-N} \\
&\geq \sum_{i=1}^{n}(a_i^m)^2 - C\sum_{i=1}^{n}(a_i^m)^2 m^{-N} - Cm^{-N} \geq \sum_{i=1}^{n}(a_i^m)^2 - Cm^{-N}.
\end{aligned}
\tag{3.14}
$$

This combined with (3.6), (3.7) and (3.13), (3.14), implies that

$$
\begin{aligned}
\|u_m\|_{X_0^s}^2 &= \sum_{i=1}^{n}(a_i^m)^2\|e_i^m\|_{X_0^s}^2 + 2\sum_{1\leq i<j\leq n} a_i^m a_j^m (e_i^m, e_j^m)_{X_0^s} \\
&\leq (1+Cm^{-N})(\lambda_n + Cm^{2s-N}) + Cm^{2s-N} \\
&\leq \lambda_n + C_1 m^{2s-N}
\end{aligned}
\tag{3.15}
$$

for some $C_1 > 0$. Therefore, (3.5) follows for $\lambda_n < \lambda < \lambda_{n+1}$.

For the second case $\lambda_{n-l} < \lambda_{n-l+1} = \cdots = \lambda_n = \lambda < \lambda_{n+1}$, by using similar argument as above, we can prove

$$
\|u_m\|_{X_0^s}^2 \leq \lambda_{n-l} + C_1 m^{2s-N} \tag{3.16}
$$

and the details are omitted.

Thus, the conclusion follows from (3.15) and (3.16). $\square$

As a consequence of Lemma 3.2, the following lemma holds true.

Lemma 3.3. *The following statements are true:*

(i) *If $\lambda_n < \lambda < \lambda_{n+1}$, then there exists $m_1 \geq m_0$ such that for any $m \geq m_1$,*

$$
\sup_{H_{m,n}^-(\lambda)}\left(I(u) + \frac{\lambda-\lambda_n}{2(\lambda+\lambda_n)}\|u\|_{X_0^s}^2\right) \leq 0.
$$

(ii) *If $\lambda_{n-l} < \lambda_{n-l+1} = \cdots = \lambda_n = \lambda < \lambda_{n+1}$ with $0 < l < \min\{n, N+2\}$, then there exists $\widetilde{m}_1 \geq m_0$ such that for any $m \geq \widetilde{m}_1$,*

$$
\sup_{H_{m,n}^-(\lambda)}\left(I(u) + \frac{\lambda-\lambda_{n-l}}{2(\lambda+\lambda_{n-l})}\|u\|_{X_0^s}^2\right) \leq 0.
$$

Proof. (i) For $\lambda_n < \lambda < \lambda_{n+1}$, by Lemma 3.2, there exists $m_1 \geq m_0$ such that for any $m \geq m_1$ and $u \in H_{m,n}^-(\lambda)$,

$$
\|u\|_{X_0^s}^2 \leq (\lambda_n + C_1 m^{2s-N})\|u\|_2^2 \leq \frac{\lambda+\lambda_n}{2}\|u\|_2^2.
$$

Then

$$\begin{aligned}I(u)+\frac{\lambda-\lambda_n}{2(\lambda+\lambda_n)}\|u\|_{X_0^s}^2&=\frac{\lambda}{\lambda+\lambda_n}\|u\|_{X_0^s}^2-\frac{\lambda}{2}\int_\Omega |u|^2\,dx-\frac{1}{2_s^*}\int_\Omega |u|^{2_s^*}\,dx\\&\le -\frac{1}{2_s^*}\int_\Omega |u|^{2_s^*}\,dx\\&\le 0.\end{aligned}$$

Thus, (i) follows immediately.

(ii) For $\lambda_{n-l}<\lambda_{n-l+1}=\cdots=\lambda_n=\lambda<\lambda_{n+1}$ with $0<l<\min\{n,N+2\}$, by using Lemma 3.2 and similar argument as above, there exists $\widetilde{m}_1\ge m_0$ such that for any $m\ge\widetilde{m}_1$,

$$I(u)+\frac{\lambda-\lambda_{n-l}}{2(\lambda+\lambda_{n-l})}\|u\|_{X_0^s}^2\le 0.$$

Then (ii) follows. $\square$

Let $U_\epsilon=U_{\epsilon,0}=\widetilde{\kappa}\big(\frac{\epsilon}{\epsilon^2+|x|^2}\big)^{(N-2s)/2}$ and $r_1=1/(6m)$. For any $r\in(0,r_1]$ and $\epsilon>0$, define a cut-off function U_ϵ^r by

$$U_\epsilon^r(x)=\begin{cases}U_\epsilon(x)-U_\epsilon(r) & \text{in } B_r(0),\\ 0 & \text{in } \mathbb{R}^N\setminus B_r(0).\end{cases}$$

Then

$$|U_\epsilon^r(x)|\le|U_\epsilon(x)|\le C\epsilon^{(2s-N)/2}\quad\text{and}\quad|\nabla U_\epsilon^r(x)|\le|\nabla U_\epsilon(x)|\le C\epsilon^{(2s-N)/2-1}\tag{3.17}$$

for $x\in\mathbb{R}^N$. Moreover, for any $0<\eta<\epsilon/2$, $z\in\Omega$ and $x\in\mathbb{R}^N$, we have

$$\begin{aligned}|\nabla(\xi_\eta(x-z)U_\epsilon^r(x))|&\le|\nabla\xi_\eta(x-z)||U_\epsilon^r(x)|+|\xi_\eta(x-z)||\nabla U_\epsilon^r(x)|\\&\le C\eta^{-1}|U_\epsilon^r|+|\nabla U_\epsilon^r|\le C\eta^{-1}\epsilon^{(2s-N)/2}.\end{aligned}\tag{3.18}$$

In the following, we denote $\xi_\eta\equiv 1$ for $\eta=0$. Let $0\le 2\eta<\epsilon<r$, then the following lemma holds true.

Lemma 3.4. *For any $z\in\Omega$, we have*

(i) $\displaystyle\int_{\mathbb{R}^{2N}}\frac{|\xi_\eta(x-z)U_\epsilon^r(x)-\xi_\eta(y-z)U_\epsilon^r(y)|^2}{|x-y|^{N+2s}}\,dxdy\le S^{N/(2s)}+C\left(\frac{\eta}{\epsilon}\right)^{N-2s}$,

(ii) $\displaystyle\int_\Omega|\xi_\eta(x-z)U_\epsilon^r(x)|^{2^*}\,dx\ge S^{N/(2s)}-C\left(\frac{\epsilon}{r}\right)^{N-2s}-C\left(\frac{\eta}{\epsilon}\right)^{N}$.

Proof. (i) Note that

$$\int_{\mathbb{R}^N\setminus B_r(0)}\int_{\mathbb{R}^N\setminus B_r(0)}\frac{|U_\epsilon^r(x)-U_\epsilon^r(y)|^2}{|x-y|^{N+2s}}\,dxdy=0$$

and

$$\int_{B_r(0)}\int_{\mathbb{R}^N\setminus B_r(0)} \frac{|U_\epsilon^r(x)-U_\epsilon^r(y)|^2}{|x-y|^{N+2s}}\,dydx = \int_{B_r(0)}\int_{\mathbb{R}^N\setminus B_r(0)} \frac{|U_\epsilon(x)-U_\epsilon(r)|^2}{|x-y|^{N+2s}}\,dydx \le \int_{B_r(0)}\int_{\mathbb{R}^N\setminus B_r(0)} \frac{|U_\epsilon(x)-U_\epsilon(y)|^2}{|x-y|^{N+2s}}\,dydx.$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{|U_\epsilon^r(x)-U_\epsilon^r(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&= \int_{B_r(0)}\int_{B_r(0)} \frac{|U_\epsilon^r(x)-U_\epsilon^r(y)|^2}{|x-y|^{N+2s}}\,dxdy + 2\int_{B_r(0)}\int_{\mathbb{R}^N\setminus B_r(0)} \frac{|U_\epsilon^r(x)-U_\epsilon^r(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&\quad + \int_{\mathbb{R}^N\setminus B_r(0)}\int_{\mathbb{R}^N\setminus B_r(0)} \frac{|U_\epsilon^r(x)-U_\epsilon^r(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&\le \int_{B_r(0)}\int_{B_r(0)} \frac{|U_\epsilon(x)-U_\epsilon(y)|^2}{|x-y|^{N+2s}}\,dxdy + 2\int_{B_r(0)}\int_{\mathbb{R}^N\setminus B_r(0)} \frac{|U_\epsilon(x)-U_\epsilon(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|U_\epsilon(x)-U_\epsilon(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&= S^{N/(2s)}.
\end{aligned} \tag{3.19}$$

In addition, set

$$\begin{aligned}
\Gamma_1 &= \left\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N : |x-y|\ge\frac{\eta}{2}, x\in B_\eta(z), y\in\mathbb{R}^N\setminus B_\eta(z)\right\}, \\
\Gamma_2 &= \left\{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N : |x-y|\le\frac{\eta}{2}, x\in B_\eta(z), y\in\mathbb{R}^N\setminus B_\eta(z)\right\}.
\end{aligned}$$

By (3.17) and (3.18), we have

$$\begin{aligned}
&\int_{\Gamma_1} \frac{|\xi_\eta(x-z)U_\epsilon^r(x)-\xi_\eta(y-z)U_\epsilon^r(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&\le 2\int_{\Gamma_1} \frac{|U_\epsilon^r(x)|^2+|U_\epsilon^r(y)|^2}{|x-y|^{N+2s}}\,dxdy \le C\epsilon^{2s-N}\int_{\Gamma_1}\frac{1}{|x-y|^{N+2s}}\,dxdy \\
&\le C\epsilon^{2s-N}\int_{B_\eta(z)}\int_{\{|x-y|>\eta/2\}}\frac{1}{|x-y|^{N+2s}}\,dydx \le C\left(\frac{\eta}{\epsilon}\right)^{N-2s}
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
&\int_{\Gamma_2} \frac{|\xi_\eta(x-z)U_\epsilon^r(x)-\xi_\eta(y-z)U_\epsilon^r(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&\le C\eta^{-2}\epsilon^{2s-N}\int_{\Gamma_2}\frac{|x-y|^2}{|x-y|^{N+2s}}\,dydx \\
&\le C\eta^{-2}\epsilon^{2s-N}\int_{B_\eta(z)}\int_{\{|x-y|>\eta/2\}}\frac{1}{|x-y|^{N+2s-2}}\,dydx \le C\left(\frac{\eta}{\epsilon}\right)^{N-2s}.
\end{aligned} \tag{3.21}$$

Moreover,

$$
\begin{aligned}
&\int_{B_\eta(z)}\int_{B_\eta(z)} \frac{|\xi_\eta(x-z)U_\epsilon^r(x) - \xi_\eta(y-z)U_\epsilon^r(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&\le C\eta^{-2}\epsilon^{2s-N}\int_{B_\eta(z)}\int_{B_\eta(z)} \frac{|x-y|^2}{|x-y|^{N+2s}}\,dydx \\
&\le C\eta^{-2}\epsilon^{2s-N}\int_{B_\eta(z)}\int_{\{|x-y|\le 2\eta\}} \frac{|x-y|^2}{|x-y|^{N+2s}}\,dydx \\
&\le C\left(\frac{\eta}{\epsilon}\right)^{N-2s}.
\end{aligned} \tag{3.22}
$$

Then we deduce from (3.20)–(3.22) and (3.19) that

$$
\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{|\xi_\eta(x-z)U_\epsilon^r(x) - \xi_\eta(y-z)U_\epsilon^r(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&= \int_{\mathbb{R}^N\setminus B_\eta(z)}\int_{\mathbb{R}^N\setminus B_\eta(z)} \frac{|U_\epsilon^r(x) - U_\epsilon^r(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&\quad + 2\int_{\Gamma_1\cup\Gamma_2} \frac{|\xi_\eta(x-z)U_\epsilon^r(x) - \xi_\eta(y-z)U_\epsilon^r(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&\quad + \int_{B_\eta(z)}\int_{B_\eta(z)} \frac{|\xi_\eta(x-z)U_\epsilon^r(x) - \xi_\eta(y-z)U_\epsilon^r(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|U_\epsilon^r(x) - U_\epsilon^r(y)|^2}{|x-y|^{N+2s}}\,dxdy + C\left(\frac{\eta}{\epsilon}\right)^{N-2s} \\
&\le S^{N/(2s)} + C\left(\frac{\eta}{\epsilon}\right)^{N-2s}.
\end{aligned}
$$

(ii) According to (2.2), we have

$$
\begin{aligned}
\int_{\mathbb{R}^N\setminus B_r(0)} |U_\epsilon(x)|^{2_s^*}\,dx &= C\int_r^\infty \left(\frac{\epsilon}{\epsilon^2+\rho^2}\right)^N \rho^{N-1}\,d_0\rho \\
&\le C\epsilon^N\int_r^\infty \rho^{-N-1}\,d_0\rho \le C\left(\frac{\epsilon}{r}\right)^N
\end{aligned} \tag{3.23}
$$

and

$$
\begin{aligned}
\int_{B_r(0)} |U_\epsilon(x)|^{2_s^*-1}U_\epsilon(r)\,dx &= \widetilde{\kappa}^{2_s^*}\left(\frac{\epsilon}{\epsilon^2+r^2}\right)^{(N-2s)/2}\int_{B_r(0)}\left(\frac{\epsilon}{\epsilon^2+|x|^2}\right)^{(N+2s)/2}\,dx \\
&\le \frac{C\epsilon^N}{r^{N-2s}}\int_0^r \frac{\rho^{N-1}}{(\epsilon^2+\rho^2)^{(N+2s)/2}}\,d\rho \\
&\le \frac{C\epsilon^N}{r^{N-2s}}\left(\int_0^\epsilon \frac{\rho^{N-1}}{\epsilon^{N+2s}}\,d\rho + \int_\epsilon^r \frac{\rho^{N-1}}{\rho^{N+2s}}\,d\rho\right) \\
&\le C\left(\frac{\epsilon}{r}\right)^{N-2s}.
\end{aligned} \tag{3.24}
$$

Then by (2.3), (3.23) and (3.24), we get

$$\begin{aligned}\int_\Omega |U_\epsilon^r|^{2_s^*}\,dx &= \int_{B_r(0)} |U_\epsilon(x)-U_\epsilon(r)|^{2_s^*}\,dx\\ &\geq \int_{B_r(0)} |U_\epsilon(x)|^{2_s^*}\,dx - 2_s^*\int_{B_r(0)} |U_\epsilon(x)|^{2_s^*-1}U_\epsilon(r)\,dx\\ &= \int_{\mathbb{R}^N} |U_\epsilon(x)|^{2_s^*}\,dx - \int_{\mathbb{R}^N\setminus B_r(0)} |U_\epsilon(x)|^{2_s^*} - 2_s^*\int_{B_r(0)} |U_\epsilon(x)|^{2_s^*-1}U_\epsilon(r)\,dx\\ &\geq \int_{\mathbb{R}^N} |U_\epsilon(x)|^{2_s^*}\,dx - C\epsilon^N r^{-N} - C\epsilon^{N-2s}r^{2s-N}\\ &\geq S^{N/(2s)} - C\left(\frac{\epsilon}{r}\right)^{N-2s}.\end{aligned}$$

This together with (3.17) shows that

$$\begin{aligned}\int_\Omega |\xi_\eta(x-z)U_\epsilon^r|^{2_s^*}\,dx &= \int_\Omega |U_\epsilon^r|^{2_s^*}\,dx - \int_\Omega [1-\xi_\eta(x-z)^2]|U_\epsilon^r|^{2_s^*}\,dx\\ &\geq \int_\Omega |U_\epsilon^r|^{2_s^*}\,dx - \int_{B_\eta(z)} |U_\epsilon^r|^{2_s^*}\,dx\\ &\geq S^{N/(2s)} - C\left(\frac{\epsilon}{r}\right)^{N-2s} - C\int_{B_\eta(z)} \epsilon^{-N}\,dx\\ &\geq S^{N/(2s)} - C\left(\frac{\epsilon}{r}\right)^{N-2s} - C\left(\frac{\eta}{\epsilon}\right)^{N}.\end{aligned}$$

The proof is completed. □

4. Proof of Theorem 1.1

Let $\overline{\theta} > 2s/(N-4s)$ and $\epsilon_r = r^{\overline{\theta}+1}, \eta_r = r^{2\overline{\theta}+1}$. Clearly, $U_{\epsilon_r}^r$ is continuous in $X_0^s(\Omega)$ with respect to $r \in (0, r_1] = (0, 1/(6m)]$. Moreover, let $r \in (0, r_1]$ and $\eta \in [0, \eta_r]$, then the following lemma holds true.

Proposition 4.1. *For each value $D > 0$, there exists $m_1^D \geq m_0$ such that for any $m \geq m_1^D$ and $z \in \Omega$, there holds*

$$\sup_{\tau\geq 0}\left(I(\tau\xi_\eta(\,\cdot - z)U_{\epsilon_r}^r) + Dm^{N+2s}\|\tau\xi_\eta(\,\cdot - z)U_{\epsilon_r}^r\|_{L^1(\Omega)}^2\right) < \frac{s}{N}S^{N/(2s)},$$

where m_0 is defined in Lemma 3.2. Moreover, m_1^D is increasing with respect to D.

Proof. It follows from Lemma 3.4 that

$$\begin{aligned}\int_{\mathbb{R}^{2N}} \frac{|\xi_\eta(x-z)U_{\epsilon_r}^r(x) - \xi_\eta(y-z)U_{\epsilon_r}^r(y)|^2}{|x-y|^{N+2s}}\,dxdy &\leq S^{N/(2s)} + C\left(\frac{\eta}{\epsilon_r}\right)^{N-2s}\\ &\leq S^{N/(2s)} + Cr^{\overline{\theta}(N-2s)}\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\int_\Omega |\xi_\eta(x-z)U_{\epsilon_r}^r(x)|^{2^*}\,dx &\geq S^{N/(2s)} - C\left(\frac{\epsilon_r}{r}\right)^{N-2s} - C\left(\frac{\eta}{\epsilon_r}\right)^{N-2s}\\ &\geq S^{N/(2s)} - Cr^{\overline{\theta}(N-2s)}.\end{aligned} \tag{4.2}$$

Moreover, by a direct computation,

$$\begin{aligned}\|\xi_\eta(x-z)U_{\epsilon_r}^r(x)\|_{L^1(\Omega)} &\leq \int_\Omega |U_{\epsilon_r}^r(x)|\,dx \leq \int_{B_r(0)} |U_{\epsilon_r}(x)|\,dx\\ &\leq C\int_0^r \left(\frac{\epsilon_r}{\epsilon_r^2+\rho^2}\right)^{(N-2s)/2}\rho^{N-1}\,d\rho\\ &\leq C\epsilon_r^{(N-2s)/2}\left(\int_0^{\epsilon_r}\frac{\rho^{N-1}}{\epsilon_r^{N-2s}}\,d\rho + \int_{\epsilon_r}^r \rho^{2s-1}\,d\rho\right)\\ &\leq C\epsilon_r^{(N-2s)/2}r^{2s} = Cr^{(\overline{\theta}+1)(N-2s)/2+2s}.\end{aligned} \tag{4.3}$$

Note that for r small enough, we have $\epsilon_r < r$ and $U_{\epsilon_r}^r(x) = U_{\epsilon_r}(x) - U_{\epsilon_r}(r) \geq \frac{1}{2}U_{\epsilon_r}(x)$ for $x \in B_{r/2}(0)$. Then if $\eta = 0$,

$$\begin{aligned}\int_\Omega |U_{\epsilon_r}^r|^2 &= \int_{B_r(0)} |U_{\epsilon_r}(x) - U_{\epsilon_r}(r)|^2 \geq \frac{1}{4}\int_{B_{r/2}(0)} |U_{\epsilon_r}(x)|^2\\ &\geq C\int_0^{r/2}\left(\frac{\epsilon_r}{\epsilon_r^2+\rho^2}\right)^{N-2s}\rho^{N-1}\,d\rho\\ &\geq C\epsilon_r^{N-2s}\left(\int_0^{\epsilon_r}\frac{\rho^{N-1}}{(2\epsilon_r{}^2)^{N-2s}} + \int_{\epsilon_r}^{r/2}\frac{\rho^{N-1}}{(2\rho^2)^{N-2s}}\right)\\ &\geq C\epsilon_r^{2s} = Cr^{2s(\overline{\theta}+1)},\end{aligned} \tag{4.4}$$

and if $\eta > 0$,

$$\begin{aligned}\int_\Omega |\xi_\eta(x-z)U_{\epsilon_r}^r|^2 &= \int_\Omega |U_{\epsilon_r}^r|^2 - \int_\Omega (1-\xi_\eta^2(x-z))|U_{\epsilon_r}^r|^2\\ &\geq \int_\Omega |U_{\epsilon_r}^r|^2 - \int_{B_\eta(z)} |U_{\epsilon_r}^r|^2 \geq C\epsilon_r^{2s} - C\eta^N\epsilon_r^{2s-N} \geq Cr^{2s(\overline{\theta}+1)}.\end{aligned} \tag{4.5}$$

Note that for any constants $B_1, B_2 > 0$,

$$\sup_{\tau\geq 0}\left(\frac{\tau^2}{2}B_1 - \frac{\tau^{2_s^*}}{2_s^*}B_2\right) = \frac{s}{N}B_1\left(\frac{B_1}{B_2}\right)^{(N-2s)/(2s)}. \tag{4.6}$$

Then by (4.1)–(4.6) and $m^{N+2s}r^{N+2s} \leq 6^{N+2s}$, we have

$$\begin{aligned}&I(\tau\xi_\eta(\,\cdot - z)U_{\epsilon_r}^r) + Dm^{N+2s}\|\tau\xi_\eta(\,\cdot - z)U_{\epsilon_r}^r\|_{L^1(\Omega)}^2\\ &= \frac{\tau^2}{2}\bigg(\int_{\mathbb{R}^{2N}}\frac{|\xi_\eta(x-z)U_{\epsilon_r}^r(x) - \xi_\eta(y-z)U_{\epsilon_r}^r(y)|^2}{|x-y|^{N+2s}}\,dxdy\end{aligned}$$

$$
\begin{aligned}
&\quad - \lambda \int_{\Omega} |\xi_\eta(x-z)U^r_{\epsilon_r}(x)|^2\,dx + 2Dm^{N+2s}\|\xi_\eta(\,\cdot-z)U^r_{\epsilon_r}\|^2_{L^1(\Omega)}\Big) \\
&\quad - \frac{\tau^{2^*_s}}{2^*_s}\int_{\Omega} |\xi_\eta(x-z)U^r_{\epsilon_r}(x)|^{2^*_s}\,dx \\
&\le \frac{\tau^2}{2}\Big(S^{N/(2s)} + Cr^{\overline{\theta}(N-2s)} - \lambda C r^{2s(\overline{\theta}+1)} + CDm^{N+2s}r^{(\overline{\theta}+1)(N-2s)+4s}\Big) \\
&\quad - \frac{\tau^{2^*_s}}{2^*_s}\Big(S^{N/(2s)} - Cr^{\overline{\theta}(N-2s)}\Big) \\
&\le \frac{\tau^2}{2}\Big(S^{N/(2s)} + (C+6^{N+2s}D)r^{\overline{\theta}(N-2s)} - \lambda C r^{2s(\overline{\theta}+1)}\Big) - \frac{\tau^{2^*_s}}{2^*_s}\Big(S^{N/(2s)} - Cr^{\overline{\theta}(N-2s)}\Big) \\
&\le \sup_{\tau\ge 0}\Big[\frac{\tau^2}{2}\Big(S^{N/(2s)} + (C+6^{N+2s}D)r^{\overline{\theta}(N-2s)} - \lambda C r^{2s(\overline{\theta}+1)}\Big) - \frac{\tau^{2^*_s}}{2^*_s}\Big(S^{N/(2s)} - Cr^{\overline{\theta}(N-2s)}\Big)\Big] \\
&\le \frac{s}{N}\Big(S^{N/(2s)} + (C+6^{N+2s}D)r^{\overline{\theta}(N-2s)} - \lambda C r^{2s(\overline{\theta}+1)}\Big) \\
&\quad \times \left(\frac{S^{N/(2s)} + (C+6^{N+2s}D)r^{\overline{\theta}(N-2s)} - \lambda C r^{2s(\overline{\theta}+1)}}{S^{N/(2s)} - Cr^{\overline{\theta}(N-2s)}}\right)^{(N-2s)/(2s)}.
\end{aligned}
$$

Since $2s(\overline{\theta}+1) < \overline{\theta}(N-2s)$, there exists $m_1^D \ge m_0$ increasing on D such that for any $m \ge m_1^D$ and $r \le 1/(6m)$,

$$
(C+6^{N+2s}D)r^{\overline{\theta}(N-2s)} - \lambda C r^{2s(\overline{\theta}+1)} < 0
$$

and

$$
\begin{aligned}
&\sup_{\tau\ge 0}\Big(I(\tau\xi_\eta(\,\cdot-\widetilde{z})U^r_{\epsilon_r}) + Dm^{N+2s}\|\tau\xi_\eta(\,\cdot-\widetilde{z})U^r_{\epsilon_r}\|^2_{L^1(\Omega)}\Big) \\
&\le \frac{s}{N}\Big(S^{N/(2s)} + (C+6^{N+2s}D)r^{\overline{\theta}(N-2s)} - \lambda C r^{2s(\overline{\theta}+1)}\Big) < \frac{s}{N}S^{N/(2s)}.
\end{aligned}
$$

Thus the lemma follows. □

For any integer $j \ge 1$, we write $\mathbb{B}^j = \{x \in \mathbb{R}^j : |x| \le 1\}$ and $\mathbb{S}^j = \{x \in \mathbb{R}^{j+1} : |x| = 1\}$. By using Proposition 4.1, the following lemma holds.

Lemma 4.2. *Let $D > 0$ and m_1^D be defined in Proposition 4.1. Then there exists $m_2^D \ge m_1^D$ such that for any $m \ge m_2^D$, there is an odd continuous map $h\colon \mathbb{R}^{N+2} \to X_0^s(B_{1/m}(0))$ satisfying*

$$
\sup_{u\in h(\mathbb{R}^{N+2})}\Big(I(u) + Dm^{N+2s}\|u\|^2_{L^1(\Omega)}\Big) < \frac{2s}{N}S^{N/(2s)}
$$

and $\lim_{|z|\to\infty}\big(I(h(z)) + Dm^{N+2s}\|h(z)\|^2_{L^1(\Omega)}\big) = -\infty$.

Proof. For any $\widetilde{z} \in \mathbb{B}^N$ and $m \ge m_1^D$, set $t = |\widetilde{z}|$, $\theta = \widetilde{z}/|\widetilde{z}|$ and define a continuous map $h_1\colon \mathbb{B}^N \to X_0^s(B_{1/(2m)}(0))$ as

$$
h_1(\widetilde{z})(\cdot) = \begin{cases} u_{\eta_{r_1}/2}(\cdot) - \xi_{\eta_{r_1}}(\cdot)u_{r_1}(\,\cdot+4tr_2\theta) & \text{if } 0 \le t \le 1/2, \\ u_{t(2r_1-\eta_{r_1})-r_1+\eta_{r_1}}(\,\cdot-2r_1(2t\theta-\theta)) - \xi_{\eta_{r_1}}(\cdot)u_{r_1}(\,\cdot+2r_1\theta) & \text{if } 1/2 \le t \le 1. \end{cases}
$$

Since $\xi_{\eta_{r_1}}(x) = 1$ for $|x| \geq \eta_{r_1}$, we have $\xi_{\eta_{r_1}}(\cdot)u_{r_1}(\cdot + 2r_1\theta) = u_{r_1}(\cdot + 2r_1\theta)$. Then, h_1 is odd on $\mathbb{S}^{N-1}$ and it induces an odd continuous map $h_2\colon \mathbb{S}^N \to X_0^s(B_{1/(2m)}(0))$ defined by

$$h_2(x_1, \dots, x_{N+1}) = \begin{cases} h_1(x_1, \dots, x_N) & \text{if } x_{N+1} \geq 0, \\ -h_1(-x_1, \dots, -x_N) & \text{if } x_{N+1} \leq 0. \end{cases}$$

According to Proposition 4.1, one sees that

$$\sup_{\tau \geq 0, \theta \in \mathbb{S}^N} \Big(I(\tau h_2(\theta)_\pm) + D m^{N+2s} \|\tau h_2(\theta)_\pm\|^2_{L^1(\Omega)} \Big) < \frac{s}{N} S^{N/(2s)}. \tag{4.7}$$

Now, let $\mathbb{Z}$ be a cylindric surface in $\mathbb{R}^{N+2}$ defined by

$$\mathbb{Z} := (\mathbb{S}^N \times [-1, 1]) \cup (\mathbb{B}^{N+1} \times \{-1, 1\}),$$

and denote the top by $\mathbb{Z}_1 = \mathbb{B}^{N+1} \times \{1\}$, the bottom by $\mathbb{Z}_2 = \mathbb{B}^{N+1} \times \{-1\}$ and the lateral surface by $\mathbb{Z}_3 = \mathbb{S}^N \times (-1, 1)$. Obviously $\mathbb{Z} = \mathbb{Z}_1 \cup \mathbb{Z}_2 \cup \mathbb{Z}_3$, and h_2 can be extended to an odd continuous map $h_3\colon \mathbb{Z} \to X_0^s(B_{1/m}(0))$ defined as follows: for $\theta \in \mathbb{S}^{N-1}$, $t_1 \in [0, 1]$, $t_2 \in [-1, 1]$, define

$$h_3(t_1\theta, t_2) := \begin{cases} (1 - t_2)[h_2(\theta)]_- + (1 + t_2)[h_2(\theta)]_+ & \text{if } t_1 = 1 \text{ (i.e., if } (t_1\theta, t_2) \in \mathbb{Z}_3), \\ 2t_1[h_2(\theta)]_+ + (1 - t_1)v_0 & \text{if } t_2 = 1 \text{ (i.e., if } (t_1\theta, t_2) \in \mathbb{Z}_1), \\ 2t_1[h_2(\theta)]_- - (1 - t_1)v_0 & \text{if } t_2 = -1 \text{ (i.e., if } (t_1\theta, t_2) \in \mathbb{Z}_2), \end{cases}$$

where $v_0(\cdot) := \xi_{\eta_{r_1}}(\cdot)u_{r_1}(\cdot + y_0) \in M_m^1$ and $y_0 \in \Omega$ is a fixed point with $|y_0| = 3/(4m)$. Furthermore, we extend h_3 to a map $h\colon \mathbb{R}^{N+2} \to X_0^s(B_{11/(12m)}(0))$ as follows:

$$h(\tau z) := \tau h_3(z) \quad \text{for } z \in \mathbb{Z},\ \tau \geq 0.$$

By construction, h is an odd continuous map and $\lim_{|z| \to +\infty} I(h(z)) = -\infty$.

Since $v_0 \in X_0^s(B_{11/(12m)}(0) \setminus B_{7/(12m)}(0))$, we see that $\operatorname{supp} h(z)_+ \cap \operatorname{supp} h(z)_- = \emptyset$. Then by Lemma 2.1, it follows that for any $\tau > 0$, if $z \in Z_3$,

$$\begin{aligned} I(\tau h_3(z)) &= I(\tau(1 - t_2)h_2(\theta)_+) + I(\tau(1 + t_2)h_2(\theta)_-) \\ &\quad - 4\tau^2(1 - t_2^2) \int_{\mathbb{R}^{2N}} \frac{h_2(\theta)_+(x)h_2(\theta)_-(y)}{|x - y|^{N+2s}}\, dxdy \\ &< I(\tau(1 - t_2)h_2(\theta)_+) + I(\tau(1 + t_2)h_2(\theta)_-), \end{aligned} \tag{4.8}$$

and if $z \in Z_1$,

$$\begin{aligned} I(\tau h_3(z)) &= I(2\tau t_1 h_2(\theta)_+) + I(\tau(1 - t_1)v_0) \\ &\quad - 8\tau^2 t_1(1 - t_1) \int_{\mathbb{R}^{2N}} \frac{h_2(\theta)_+(x)v_0(y)}{|x - y|^{N+2s}}\, dxdy \\ &< I(2\tau t_1 h_2(\theta)_+) + I(\tau(1 - t_1)v_0). \end{aligned} \tag{4.9}$$

Since Proposition 4.1 shows that $\sup_{\tau\ge 0, z\in Z_1}\left(I(\tau v_0)+Dm^{N+2s}\|\tau v_0\|^2_{L^1(\Omega)}\right)<\frac{s}{N}S^{N/(2s)}$, by (4.7)–(4.9), we have

$$
\begin{aligned}
&\sup_{\tau\ge 0, z\in Z_3}\Big(I(h(\tau z))+Dm^{N+2s}\|h(\tau z)\|^2_{L^1(\Omega)}\Big)\\
&=\sup_{\tau\ge 0, z\in Z_3}\Big(I(\tau h_3(z))+Dm^{N+2s}\tau^2\|h_3(z)\|^2_{L^1(\Omega)}\Big)\\
&<\sup_{\tau\ge 0, z\in Z_3}\Big(I(\tau(1-t_2)h_2(\theta)_+)+Dm^{N+2s}\tau^2(1-t_2)^2\|h_2(\theta)_+\|^2_{L^1(\Omega)}\\
&\qquad\qquad +I(\tau(1+t_2)h_2(\theta)_-)+Dm^{N+2s}\tau^2(1+t_2)^2\|h_2(\theta)_-\|^2_{L^1(\Omega)}\Big)\\
&=\sup_{\tau\ge 0, z\in Z_3}\Big(I(\tau h_2(\theta)_+)+Dm^{N+2s}\|\tau h_2(\theta)_+\|^2_{L^1(\Omega)}\Big)\\
&\quad+\sup_{\tau\ge 0, z\in Z_3}\Big(I(\tau h_2(\theta)_-)+Dm^{N+2s}\|\tau h_2(\theta)_-\|^2_{L^1(\Omega)}\Big)\\
&<\frac{2s}{N}S^{N/(2s)}
\end{aligned}
\tag{4.10}
$$

and

$$
\begin{aligned}
&\sup_{\tau\ge 0, z\in Z_1}\Big(I(h(\tau z))+Dm^{N+2s}\|h(\tau z)\|^2_{L^1(\Omega)}\Big)\\
&=\sup_{\tau\ge 0, z\in Z_1}\Big(I(\tau h_3(z))+Dm^{N+2s}\tau^2\|h_3(z)\|^2_{L^1(\Omega)}\Big)\\
&<\sup_{\tau\ge 0, \theta\in\mathbb{S}^{N-1}}\Big(I(2t_1\tau h_2(\theta)_+)+4Dm^{N+2s}\tau^2t_1^2\|h_2(\theta)_+\|^2_{L^1(\Omega)}\\
&\qquad\qquad +I(\tau(1-t_1)v_0)+Dm^{N+2s}\tau^2(1-t_1)^2\|v_0\|^2_{L^1(\Omega)}\Big)\\
&\le\sup_{\tau\ge 0, \theta\in\mathbb{S}^{N-1}}\Big(I(\tau h_2(\theta)_+)+Dm^{N+2s}\|\tau h_2(\theta)_+\|^2_{L^1(\Omega)}\Big)\\
&\quad+\sup_{\tau\ge 0, z\in Z_1}\Big(I(\tau v_0)+Dm^{N+2s}\|\tau v_0\|^2_{L^1(\Omega)}\Big)\\
&<\frac{2s}{N}S^{N/(2s)}.
\end{aligned}
\tag{4.11}
$$

In addition, since $\operatorname{dist}(\operatorname{supp} h_2(\theta)_-, \operatorname{supp} v_0)\ge r_1/2$, for any $z\in Z_2$, by Lemma 3.4(iii), we have

$$
\begin{aligned}
&8t_1(1-t_1)\int_{\mathbb{R}^{2N}}\frac{[h_2(\theta)]_-(x)v_0(y)}{|x-y|^{N+2s}}\,dxdy\\
&\le\frac{16t_1(1-t_1)}{r_1^{N+2s}}\int_\Omega[h_2(\theta)]_-(x)\,dx\int_\Omega v_0(y)\,dy\\
&\le\frac{Ct_1^2}{r_1^{N+2s}}\|h_2(\theta)_-\|^2_{L^1(\Omega)}+\frac{C(1-t_1)^2}{r_1^{N+2s}}\|v_0\|^2_{L^1(\Omega)}\\
&\le Cm^{N+2s}t_1^2\|h_2(\theta)_-\|^2_{L^1(\Omega)}+Cm^{N+2s}(1-t_1)^2\|v_0\|^2_{L^1(\Omega)}.
\end{aligned}
$$

Then

(4.12)

$$\begin{aligned}
&\sup_{\substack{\tau\geq 0\\ z\in\mathbb{Z}_2}}\left(I(h(\tau z))+Dm^{N+2s}\|h(\tau z)\|^2_{L^1(\Omega)}\right)\\
&=\sup_{\substack{\tau\geq 0\\ z\in\mathbb{Z}_2}}\Bigg(I(2t_1\tau[h_2(\theta)]_-)+I(\tau(1-t_1)v_0)+8\tau^2t_1(1-t_1)\int_{\mathbb{R}^{2N}}\frac{[h_2(\theta)]_-(x)v_0(y)}{|x-y|^{N+2s}}\,dxdy\\
&\qquad +Dm^{N+2s}\|2t_1\tau h_2(\theta)_-\|^2_{L^1(\Omega)}+Dm^{N+2s}\|\tau(1-t_1)v_0\|^2_{L^1(\Omega)}\Bigg)\\
&\leq \sup_{\substack{\tau\geq 0\\ z\in\mathbb{Z}_2}}\Big(I(2t_1\tau[h_2(\theta)]_-)+I(\tau(1-t_1)v_0)+(C+D)m^{N+2s}\tau^2t_1^2\|h(\theta)_-\|^2_{L^1(\Omega)}\\
&\qquad +(C+D)m^{N+2s}\tau^2(1-t_1)^2\|v_0\|^2_{L^1(\Omega)}\Big)\\
&\leq \sup_{\substack{\tau\geq 0\\ \theta\in\mathbb{S}^{N-1}}}\left(I(\tau h_2(\theta)_-)+(C+D)m^{N+2s}\|\tau h_2(\theta)_-\|^2_{L^1(\Omega)}\right)\\
&\quad +\sup_{\substack{\tau\geq 0\\ \theta\in\mathbb{S}^{N-1}}}\left(I(\tau v_0)+(C+D)m^{N+2s}\|\tau v_0\|^2_{L^1(\Omega)}\right),
\end{aligned}$$

where C is independent of m and D. By Proposition 4.1, there exists $m_1^{C+D}\geq m_1^D$ such that if $m\geq m_1^{C+D}$,

$$\sup_{\substack{\tau\geq 0\\ \theta\in\mathbb{S}^{N-1}}}\left(I(\tau h_2(\theta)_-)+(C+D)m^{N+2s}\|\tau h_2(\theta)_-\|^2_{L^1(\Omega)}\right)<\frac{s}{N}S^{N/(2s)} \tag{4.13}$$

and

$$\sup_{\substack{\tau\geq 0\\ \theta\in\mathbb{S}^{N-1}}}\left(I(\tau v_0)+(C+D)m^{N+2s}\|\tau v_0\|^2_{L^1(\Omega)}\right)<\frac{s}{N}S^{N/(2s)}. \tag{4.14}$$

Now, we take $m_2^D:=m_1^{C+D}$ and $m\geq m_2^D$ as in Lemma 4.2. Then by (4.13) and (4.14), it follows from (4.12) that

$$\sup_{\substack{\tau\geq 0\\ z\in\mathbb{Z}_2}}\left(I(h(\tau z))+Dm^{N+2s}\|h(\tau z)\|^2_{L^1(\Omega)}\right)<\frac{2s}{N}S^{N/(2s)}. \tag{4.15}$$

Thus, we conclude from (4.10), (4.11) and (4.15) that

$$\begin{aligned}
\sup_{u\in h(\mathbb{R}^{N+2})}\left(I(u)+Dm^{N+2s}\|u\|^2_{L^1(\Omega)}\right)&=\sup_{\substack{\tau\geq 0\\ z\in\mathbb{Z}_1\cup\mathbb{Z}_2\cup\mathbb{Z}_3}}\left(I(h(\tau z))+Dm^{N+2s}\|h(\tau z)\|^2_{L^1(\Omega)}\right)\\
&<\frac{2s}{N}S^{N/(2s)}.
\end{aligned}$$

The proof is completed. $\square$

From now on, we fix $m \geq m_2^D$, where m_2^D is defined in Lemma 4.2.

Lemma 4.3. (i) *Suppose* $0 < \lambda < \lambda_1$, *then there exists an odd continuous map* $\widehat{h}\colon \mathbb{R}^{N+2} \to X_0^s(\Omega)$ *such that* $\lim_{|x|\to\infty} I(\widehat{h}(x)) = -\infty$ *and*

$$\sup_{u\in\widehat{h}(\mathbb{R}^{N+2})} I(u) < \frac{2s}{N} S^{N/(2s)}.$$

(ii) *Suppose* $\lambda_n < \lambda < \lambda_{n+1}$ *for some* $n \geq 1$, *then there exists an odd continuous map* $\overline{h}\colon \mathbb{R}^{n+N+2} \to X_0^s(\Omega)$ *such that* $\lim_{|x|\to\infty} I(\overline{h}(x)) = -\infty$ *and*

$$\sup_{u\in\overline{h}(\mathbb{R}^{n+N+2})} I(u) < \frac{2s}{N} S^{N/(2s)}.$$

(iii) *Suppose* $\lambda_{n-l} < \lambda_{n-l+1} = \cdots = \lambda_n = \lambda < \lambda_{n+1}$ *with* $l < N+2$, *then there exists an odd continuous map* $\widetilde{h}\colon \mathbb{R}^{n+N+2-l} \to X_0^s(\Omega)$ *such that* $\lim_{|x|\to\infty} I(\widetilde{h}(x)) = -\infty$ *and*

$$\sup_{u\in\widetilde{h}(\mathbb{R}^{n+N+2-l})} I(u) < \frac{2s}{N} S^{N/(2s)}.$$

Proof. (i) Let $\widehat{h} = h$, where $h\colon \mathbb{R}^{N+2} \to X_0^s(\Omega)$ is defined as in Lemma 4.2, then (i) follows easily from Lemma 4.2.

(ii) Let $\lambda_n < \lambda < \lambda_{n+1}$. Noting that $e_i^m \in X_0^s(\Omega \setminus B_{1/m}(0))$, we define $h_4\colon \mathbb{R}^n \to X_0^s(\Omega \setminus B_{1/m}(0))$ by $h_4(a) = \sum_{i=1}^n a_i e_i^m$. Then according to Lemma 3.3, we have

$$\sup_{u\in h_4(\mathbb{R}^n)} \left(I(u) + \frac{\lambda - \lambda_n}{2(\lambda+\lambda_n)} \|u\|_{X_0^s}^2 \right) \leq 0.$$

Since all norms are equivalent in finite dimension space $H_{m,n}^-(\lambda)$, it is easy to see that $\lim_{|a|\to\infty} I(h(a)) = -\infty$.

Now, we define $\overline{h}\colon \mathbb{R}^{n+N+2} \to X_0^s(\Omega)$ by $\overline{h}(a,b) = h_4(a) + h(b)$ for all $a \in \mathbb{R}^n$, $b \in \mathbb{R}^{N+2}$. Then $\overline{h}$ is an odd continuous map, and satisfies

$$\lim_{|(a,b)|\to\infty} I(\overline{h}(a,b)) = -\infty.$$

Moreover, since $\operatorname{supp} h(b) \subset B_{11/(12m)(0)}$ and $\operatorname{supp} h(a) \subset B_{1/m(0)}$, by Young inequality and fractional Sobolev inequality, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{h_4(a)(x)h(b)(y)}{|x-y|^{N+2s}}\, dxdy \\
&= \int_{B_{11/(12m)}(0)} \left(\int_{\Omega\setminus B_{1/m}(0)} \frac{h_4(a)(x)}{|x-y|^{N+2s}}\, dx \right) h(b)(y)\, dy \\
&\leq \|h(a)\|_{L^{2_s^*}(\Omega\setminus B_{1/m(0)})} \||x-y|^{-N-2s}\|_{L^{2N/(N+2s)}(\Omega\setminus B_{1/m}(0))}
\end{aligned}$$

$$\begin{aligned}
(4.16)\quad &\le C\int_{B_{11/(12m)}(0)}\left(\int_{1/(12m)}^{\infty}\tau^{-N-1}\,d\tau\right)^{(N+2s)/(2N)}\|h_4(a)\|_{L^{2_s^*}(\Omega\setminus B_{1/m}(0))}h(b)(y)\,dy\\
&\le Cm^{(N+2s)/2}\|h(b)\|_{L^1(\Omega)}\|h_4(a)\|_{L^{2_s^*}(\Omega)}\\
&\le Cm^{(N+2s)/2}\|h(b)\|_{L^1(\Omega)}\|h_4(a)\|_{X_0^s}\\
&\le C_2m^{N+2s}\|h(b)\|^2_{L^1(\Omega)}+\frac{\lambda-\lambda_n}{2(\lambda+\lambda_n)}\|h_4(a)\|^2_{X_0^s}
\end{aligned}$$

for some constant $C_2>0$ independent of m. Then by (4.16), Lemmas 3.3 and 4.2, we have

$$\begin{aligned}
I(\overline{h}(a,b)) &= I(h_4(a))+I(h(b))+4\int_{\mathbb{R}^{2N}}\frac{h_4(a)(x)h(b)(y)}{|x-y|^{N+2s}}\,dxdy\\
&\le I(h_4(a))+\frac{\lambda-\lambda_n}{2(\lambda+\lambda_n)}\|h_4(a)\|^2_{X_0^s}+I(h(b))+Cm^{N+2s}\|h(b)\|^2_{L^1(\Omega)}\\
&< \frac{2s}{N}S^{N/(2s)}.
\end{aligned}$$

Thus, (ii) follows.

(iii) Letting $\lambda_{n-l}<\lambda_{n-l+1}=\cdots=\lambda_n=\lambda<\lambda_{n+1}$, we define

$$\widetilde{H}^1_m=\{e_1^m,\dots,e_{n-l}^m\}$$

and $\widetilde{h}_4\colon\mathbb{R}^{n-l}\to X_0^s(\Omega\setminus B_{1/m}(0))$ by $\widetilde{h}_4(a)=\sum_{i=1}^{n-l}a_ie_i^m$. Then by using similar argument as the proof of (i) above and Lemma 3.3(ii), we have

$$\sup_{u\in\widetilde{h}_4(\mathbb{R}^n)}\left(I(u)+\frac{\lambda-\lambda_{n-l}}{2(\lambda+\lambda_{n-l})}\|u\|^2_{X_0^s}\right)\le 0.$$

Moreover, $\lim_{|a|\to\infty}I(h(a))=-\infty$.

Define $\widetilde{h}\colon\mathbb{R}^{n+N+2-l}\to X_0^s(\Omega)$ by $\overline{h}(a,b)=\widetilde{h}_4(a)+h(b)$ for all $a\in\mathbb{R}^n$, $b\in\mathbb{R}^{N+2}$, where h is defined as in Lemma 4.2. Then $\widetilde{h}$ is an odd continuous map, and satisfies

$$\lim_{|(a,b)|\to\infty}I(\widetilde{h}(a,b))=-\infty.$$

Similar to (4.16), we can obtain that

$$\int_{\mathbb{R}^{2N}}\frac{\widetilde{h}_4(a)(x)h(b)(y)}{|x-y|^{N+2s}}\,dxdy\le\widetilde{C}_2m^{N+2s}\|h(b)\|^2_{L^1(\Omega)}+\frac{\lambda-\lambda_{n-l}}{2(\lambda+\lambda_{n-l})}\|\widetilde{h}_4(a)\|^2_{X_0^s}$$

for some constant $\widetilde{C}_2>0$ independent of m, and by using Lemmas 3.3 and 4.2 again,

$$\begin{aligned}
I(\overline{h}(a,b)) &= I(\widetilde{h}_4(a))+I(h(b))+4\int_{\mathbb{R}^{2N}}\frac{\widetilde{h}_4(a)(x)h(b)(y)}{|x-y|^{N+2s}}\,dxdy\\
&\le I(\widetilde{h}_4(a))+\frac{\lambda-\lambda_{n-l}}{2(\lambda+\lambda_{n-l})}\|\widetilde{h}_4(a)\|^2_{X_0^s}+I(h(b))+Cm^{N+2s}\|h(b)\|^2_{L^1(\Omega)}\\
&< \frac{2s}{N}S^{N/(2s)}.
\end{aligned}$$

The proof is finished. □

Furthermore, the following lemmas hold true.

Lemma 4.4. *The following statements are true:*

(i) *If* $0 < \lambda < \lambda_1$*, we have* $0 < \beta_1 \le \cdots \le \beta_{N+2} < 2^{2s/N} S$.

(ii) *If* $\lambda_n < \lambda < \lambda_{n+1}$ *for some* $n \ge 1$*, then* $0 < \beta_{n+1} \le \cdots \le \beta_{n+N+2} < 2^{2s/N} S$.

(iii) *If* $\lambda_{n-l} < \lambda_{n-l+1} = \cdots = \lambda_n = \lambda < \lambda_{n+1}$ *with* $0 < l < \min\{n, N+2\}$*, then* $0 < \beta_{n+1} \le \cdots \le \beta_{n+N+2-l} < 2^{2s/N} S$.

Proof. For any $k \ge 1$, let $A = \{u \in h(\mathbb{R}^k) : \|u\|_{2_s^*} = 1\}$. By using the same argument as in [8, Lemma 2.11], it is easy to see that $A \subset M$ and $\gamma(A) \ge k$. So $A \in \Sigma_k$, where Σ_k is defined as in (2.12).

(i) If $0 < \lambda < \lambda_1$, by taking $k = N+2$ and $v \in A$, we have

$$\frac{2s}{N} S^{N/(2s)} > \sup_{u \in h(\mathbb{R}^{N+2})} I(u) \ge \sup_{\tau \ge 0} I(\tau v) \ge \frac{s}{N} \left(\frac{\|v\|_{X_0^s}^2 - \lambda \|v\|_2^2}{\|v\|_{2_s^*}^2} \right)^{N/(2s)} = \frac{s}{N} J(v)^{N/(2s)}.$$

Thus, $\sup_{u \in A} J(u) < 2^{2s/N} S$. Then by the definition of β_{N+2}, we have

$$\beta_{N+2} < \frac{2s}{N} S^{N/(2s)}.$$

Since $\beta_1 > 0$ and $\beta_1 \le \cdots \le \beta_{N+1}$, (i) follows soon.

(ii) If $\lambda_n < \lambda < \lambda_{n+1}$ for some $n \ge 1$, by taking $k = n+N+2$ and similar argument as (i), we can prove $\sup_{u \in A} J(u) < 2^{2s/N} S$ and then $0 < \beta_{n+1} \le \cdots \le \beta_{n+N+2} < 2^{2s/N} S$. Thus (ii) follows.

(iii) If $\lambda_{n-l} < \lambda_{n-l+1} = \cdots = \lambda_n = \lambda < \lambda_{n+1}$ with $0 < l < \min\{n, N+2\}$, let $k = n+N+2-l$. By similar argument as (i), we can obtain $\sup_{u \in A} J(u) < 2^{2s/N} S$ and $\beta_{n+1} > 0$. Hence $0 < \beta_{n+1} \le \cdots \le \beta_{n+N+2-l} < 2^{2s/N} S$. Hence (iii) follows. The proof is completed. □

Proof of Theorem 1.1. If K^β is infinite for some $\beta \in (0, 2^{2s/N} S)$, then J possesses infinitely many critical points, and so do I. Thus, we may assume that K^β is finite for any $\beta \in (0, 2^{2s/N} S)$ and

$$\begin{aligned}
&0 < \beta_1 < \cdots < \beta_{N+2} < 2^{2s/N} S && \text{if } 0 < \lambda < \lambda_1; \text{ or} \\
&0 < \beta_n < \cdots < \beta_{n+N+2} < 2^{2s/N} S && \text{if } \lambda_n < \lambda < \lambda_{n+1} \text{ with some } n \ge 1; \text{ or} \\
&0 < \beta_{n+1} < \cdots < \beta_{n+N+2-l} < 2^{2s/N} S && \text{if } \lambda_{n-l} < \lambda_{n-l+1} = \cdots = \lambda_n = \lambda < \lambda_{n+1} \\
& && \text{with } 0 < l < \min\{n, N+2\},
\end{aligned}$$

due to Lemmas 2.7 and 4.4. Let $j_0 \in \mathbb{N}$ be the least integer such that $\beta_{j_0+1} \ge S$. Then it follows from Lemma 2.8 that

(i) if $0 < \lambda < \lambda_1$, then J admits at least $\max\{j_0, N+2-j_0\} \geq [(N+1)/2]$ pairs of nontrivial critical points;

(ii) if $\lambda_n < \lambda < \lambda_{n+1}$, then J has at least $\max\{j_0, N+2-j_0\} \geq [(N+1)/2]$ pairs of nontrivial critical points;

(iii) if $\lambda_{n-l} < \lambda_{n-l+1} = \cdots = \lambda_n = \lambda < \lambda_{n+1}$, then J possesses at least $\max\{j_0 - n, n + N + 2 - l - j_0\} \geq [(N+1-l)/2]$ pairs of nontrivial critical points. So do the functional I. The proof is completed. □

Acknowledgments

The research was supported partially by the Natural Science Foundation of Hunan Province (Grant Nos. 2016JJ2018 and 2018JJ3136); Scientific Research Fund of Hunan University of Science and Technology (Nos. E51794 and E51872).

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