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Localized Front Structures in FitzHugh-Nagumo Equations

Chao-Nien Chen*, Che-Hao Lin and Shyuh-yaur Tzeng

In memory of Professor Hwai-Chiuan Wang

Abstract. We are interested in various types of localized waves in FitzHugh-Nagumo equations. Variational methods have been successfully worked out to establish the existence of traveling and standing waves. Starting with a simple planar traveling front, an ordered method is employed to demonstrate different front propagation between two stable equilibria. If these two stable equilibria are in the same energy level, a saddle-focus condition ensures that there are infinite number of standing waves with multiple fronts.

1. Introduction

Activator-inhibitor systems are of great interest to the scientific community as breeding grounds for the investigation of pattern formation and wave propagation. As shown by Turing [35], such structures often arise from the destabilization of a stable equilibrium in the presence of diffusion. To get better understanding the dynamics in activator-inhibitor systems, we investigate the structure of wave solutions in FitzHugh-Nagumo equations:

$$\frac{\partial u_1}{\partial t} = \Delta u_1 + \frac{1}{d}(f(u_1) - u_2), \quad \frac{\partial u_2}{\partial t} = \Delta u_2 + u_1 - \gamma u_2. \tag{1.1}$$

Here $f(u) = -u(u-\beta)(u-1)$, $0 < \beta < 1/2$ and $\gamma > 0$. System (1.1) has been extensively studied [11,12,21,25,30,38,39] for diffusion-induced instability and emergence of patterns; in which u_1 can be viewed as an activator while u_2 acts to be an inhibitor.

For (1.1), the simplest traveling wave is of the form $(u_1(\xi - ct), u_2(\xi - ct))$; that is, along the moving frame $x = \xi - ct$,

$$du_1'' + dcu_1' + f(u_1) - u_2 = 0, \quad u_2'' + cu_2' + u_1 - \gamma u_2 = 0.$$

The case of $c = 0$ is called standing wave.

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*Corresponding author.

Fronts and pulses are the most well-known one-dimensional waves in reaction-diffusion systems. Fronts are generic structures connecting two different states of a system having a bi-stable nonlinearity, while the profile of a pulse is in close proximity to a trivial background state except for one or several localized spatial regions where changes are substantial. Such localized waves represent states which are far away from the homogeneous equilibrium. Depending on the system parameters and initial conditions, localized dissipative structures may stay at rest or propagate with a dynamically stabilized velocity. The transition from a stationary object to a traveling wave breaks the symmetry of the structures. For the scalar reaction-diffusion equation, there are many interesting results [2, 4, 20, 24, 26–29, 36] on the subject of traveling wave solutions. In recent years, a number of existence results [5–11, 22, 23, 32] for the standing and traveling waves of (1.1) have been established.

Localized waves in (1.1) often result from the balance between dissipation and nonlinearity, and their structures usually depending on the values of γ and d. There are certain properties for the null-clines $v = u/\gamma$ and $v = f(u)$ when γ takes up different values. If $\gamma < 4/(1-\beta)^2$, the null-clines intersect at $(0,0)$ only and (1.1) is referred to as a excitable system. Let $\gamma_* \equiv 9/[(1-2\beta)(2-\beta)]$. When $\gamma = \gamma_*$, two regions enclosed by the line $v = u/\gamma_*$ and the curve $v = f(u)$ are equal in area with opposing signs. The following hypotheses will be under consideration:

(H1) $\gamma > 9/[(1-2\beta)(2-\beta)]$ and $d > \gamma^{-2}$.

(H2) $d > \beta/\gamma$ and $(\beta - d\gamma)^2 - 4d > 0$.

(H3) $d > (2-\beta)/(\gamma-1)$.

It has been shown [5] that if (H1) is satisfied then (1.1) has a traveling front solution (c, u_1, u_2) such that $c > 0$, $\lim_{x\to\infty}(u_1, u_2) = (0,0)$ and $\lim_{x\to-\infty}(u_1, u_2) = (\rho, \rho/\gamma)$. Here ρ is the largest root of $f(s) - s/\gamma$. We may replace $x = \xi - ct$ by the ansatz $x = c(\xi - ct)$ as proposed in [24]. Then

$$dc^2u_1'' + dc^2u_1' + f(u_1) - u_2 = 0, \quad c^2u_2'' + c^2u_2' + u_1 - \gamma u_2 = 0. \tag{1.2}$$

The existence of a planar traveling front of (1.1) has been studied by variational method. Let $L^2_{\mathrm{ex}} = L^2_{\mathrm{ex}}(\mathbb{R}) \equiv \left\{u : \int_{-\infty}^{\infty} e^x (u(x))^2\,dx < \infty\right\}$. For a given $u \in L^2_{\mathrm{ex}}(\mathbb{R})$, we define

$$\mathcal{L}_c u(x) \equiv \int_{-\infty}^{\infty} G(x,\eta)u(\eta)\,d\eta,$$

where G is a Green's function for the differential operator $(\gamma - c^2\frac{d^2}{dx^2} - c^2\frac{d}{dx})$. It is known [5, 7] that $\mathcal{L}_c \colon L^2_{\mathrm{ex}} \to L^2_{\mathrm{ex}}$ is self-adjoint with respect to the L^2_{ex} inner product. Set

$F(q) = -\int_0^q f(\eta)\,d\eta = q^4/4 - (1+\beta)q^3/3 + \beta q^2/2$. Consider

$$J_c(u) \equiv \int_{\mathbb{R}} e^x \left\{ \frac{dc^2}{2} u_x^2 + \frac{1}{2} u \mathcal{L}_c u + F(u) \right\} dx$$

on the Hilbert space $H^1_{\text{ex}} = H^1_{\text{ex}}(\mathbb{R})$ with the norm

$$\|u\|_{H^1_{\text{ex}}} = \sqrt{\int_{\mathbb{R}} e^x u_x^2\,dx + \int_{\mathbb{R}} e^x u^2\,dx}.$$

If (H1) is satisfied and u_1 is a minimizer of J_c, then $(c, u_1, \mathcal{L}_c u_1)$ represents a traveling front solution of (1.1); here the first step is to determine the value of c by continuity argument. For the existence of a traveling pulse solution [7] when (1.1) is a excitable system, we studied a local minimizer of J_c.

Some earlier results on the pattern formation can be found in [14,16,21,25,30,38] and the references therein, including the existence of standing waves [6,11,17,19,32] of (1.1). More recently a nonlocal Lyapunov-Schmidt reduction was utilized [18] to demonstrate plenty of standing waves with multiple fronts. The stability of patterns and waves has also been investigated [14–18, 21, 30, 39]. In addition to the maximum principle, index method [13–15] provides a new tool for studying stability questions.

It has been shown [5] that among the traveling waves in a multi-dimensional cylinder the planar wave pattern is selected as a global minimizer. When (1.1) possesses certain monotonicity properties, an ordered method [1,3,21,27–29,31–34] provides a way to construct traveling waves with more complicated front structures; for instance with a curved front. This gives a simple approach to illustrate that certain delicate wave propagation are possibly inherited from simple ones.

Section 2 starts with asymptotic estimates for a planar traveling front solution of (1.1). With the aide of a quasi-monotone system introduced in Section 3, higher dimensional front propagation will be studied in Sections 4 and 5. As many stable waves are known, including the multiple front standing waves stated in Section 6, it seems reasonable to expect that in higher dimensional spaces the wave structure of (1.1) is quite rich.

2. Asymptotic estimates

Let (c, u_1, u_2) be a planar traveling front solution of (1.1) with $c > 0$, $\lim_{x\to\infty}(u_1, u_2) = (0, 0)$ and $\lim_{x\to-\infty}(u_1, u_2) = (\rho, \rho/\gamma)$. We now analyze its asymptotical behavior at ∞ and $-\infty$. Define $g_1(u) \equiv f(u) + \beta u = u^2(1 + \beta - u)$ and rewrite (1.2) as

$$c^2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{xx} + c^2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_x - A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -\frac{1}{d} \begin{pmatrix} g_1(u_1) \\ 0 \end{pmatrix}, \tag{2.1}$$

where

$$A = \begin{pmatrix} \beta/d & 1/d \\ -1 & \gamma \end{pmatrix}.$$

We begin with the calculation on the eigenvalues, eigenvectors and left eigenvectors of A.

The eigenvalues λ_1, λ_2 of A satisfy

$$d\lambda^2 - (\beta + d\gamma)\lambda + 1 + \gamma\beta = 0.$$

Suppose (H2) is satisfied, then λ_1, λ_2 are distinct positive numbers. Since $\beta < d\gamma$ and

$$\lambda_1 + \lambda_2 = \mathrm{trace}(A) = \gamma + \frac{\beta}{d},$$

it follows that

$$\lambda_1 < \frac{1}{2}\left(\gamma + \frac{\beta}{d}\right) < \lambda_2.$$

Hence $\lambda_2 > \beta/d$ and $\lambda_1 < \gamma$.

For each eigenvalue λ_i, denoted by $\mathbf{a}_i$ an associated eigenvector for A and $\mathbf{1}_i = (1, -\eta_i)$ its left eigenvector. As $\lambda_1 \neq \lambda_2$ it is known that $\mathbf{1}_1 \cdot \mathbf{a}_2 = \mathbf{1}_2 \cdot \mathbf{a}_1 = 0$, we may take $\mathbf{a}_1 = (\eta_2, 1)^T$ and $\mathbf{a}_2 = (\eta_1, 1)^T$. Since the first row of the matrix $A - \lambda_2 I$ is $(\beta/d - \lambda_2, 1/d)$, η_2 must be positive. The second row of the matrix $A - \lambda_1 I$ is $(-1, \gamma - \lambda_1)$, which gives $\eta_1 > 0$. Thus

$$0 < \frac{\beta}{d} < \lambda_1 < \frac{1}{2}\left(\gamma + \frac{\beta}{d}\right) < \lambda_2 < \gamma.$$

Moreover a direct calculation from the first row of

$$(A^T - \lambda_i I)\begin{pmatrix} 1 \\ -\eta_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

gives

$$\lambda_i = \eta_i + \frac{\beta}{d}.$$

Hence

$$\mathbf{1}_1 \cdot \mathbf{a}_1 = \eta_2 - \eta_1 = \lambda_2 - \lambda_1 > 0 \quad \text{and} \quad \mathbf{1}_2 \cdot \mathbf{a}_2 = \eta_1 - \eta_2 > 0.$$

With (u_1, u_2) satisfying (1.2) and $(u_1, u_2) \to (0, 0)$ as $x \to \infty$, the dominant behavior of (u_1, u_2) for large x can be analyzed by linearizing (2.1) about $(0, 0)$. Denoted by s_2, s_3, the roots of $c^2 s^2 + c^2 s - \lambda_1 = 0$ and s_1, s_4, those of $c^2 s^2 + c^2 s - \lambda_2 = 0$. It is easy to check that

$$s_1 < s_2 < -1 < 0 < s_3 < s_4,$$

since $\lambda_2 > \lambda_1 > 0$. Moreover as $x \to \infty$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \sim \widehat{C}_1 e^{s_2 x}\mathbf{a}_1 + \widehat{C}_2 e^{s_1 x}\mathbf{a}_2$$

with $\widehat{C}_1$, $\widehat{C}_2$ being constant. A similar calculation shows that as $x \to -\infty$

$$\begin{pmatrix} \rho - u_1 \\ \rho/\gamma - u_2 \end{pmatrix} \sim \widehat{C}_3 e^{s_5 x} \mathbf{a}_3 + \widehat{C}_4 e^{s_6 x} \mathbf{a}_4$$

for some $\widehat{C}_3, \widehat{C}_4 > 0$ and $0 < s_5 < s_6$.

3. Quasi-monotone system

Consider a system of parabolic partial differential equations

$$\frac{\partial \mathbf{v}}{\partial t} = \Delta \mathbf{v} + \mathbf{G}(\mathbf{v}), \tag{3.1}$$

where $\mathbf{v} \in \mathbb{R}^n$ and $\mathbf{G}(\mathbf{v}) = (G_1(\mathbf{v}), G_2(\mathbf{v}), \ldots, G_n(\mathbf{v}))$. A vector-valued function $\mathbf{G}$ is quasi-monotone increasing if

$$\frac{\partial G_i}{\partial v_j} > 0 \quad \text{for } i, j = 1, 2, \ldots, n,\ i \neq j. \tag{3.2}$$

When condition (3.2) holds, (3.1) is referred to as a quasi-monotone system. To use an ordered method, we introduce following notation.

Definition 3.1. (i) $\mathbf{w} \leq \mathbf{v}$ if $w_i \leq v_i$ for all i.

(ii) $\mathbf{w} < \mathbf{v}$ if $w_i < v_i$ for all i.

(iii) $\mathbf{u} \in [\mathbf{w}, \mathbf{v}]$ if $\mathbf{w} \leq \mathbf{u} \leq \mathbf{v}$ and $\mathbf{u} \in (\mathbf{w}, \mathbf{v})$ if $\mathbf{w} < \mathbf{u} < \mathbf{v}$.

Although the FitzHugh-Nagumo equation is not in the form of quasi-monotone; however under suitable conditions, we may introduce a quasi-monotone system, by using a truncation argument, to study the solutions of (1.1). Set $b = (1-\rho)/2$. Let $\overline{f} = f$ on $[-b, 1]$, and outside the interval $[-b, 1]$, modify this function f to $\overline{f}$ such that $0 > \overline{f}' \geq \beta - 1$. Then replace f by $\overline{f}$ in (1.1).

Next set $\mathbf{v} = (v_1, v_2) = (u_1, u_1 - u_2)$. Substituting into (1.1) yields (3.1) with

$$\begin{aligned} \mathbf{G}(\mathbf{v}) &= (G_1(v_1, v_2), G_2(v_1, v_2)) \\ &= \left(\frac{1}{d}[\overline{f}(v_1) - v_1 + v_2], \frac{1}{d}\overline{f}(v_1) + \left(\gamma - 1 - \frac{1}{d}\right) v_1 - \left(\gamma - \frac{1}{d}\right) v_2 \right). \end{aligned}$$

Clearly

$$\frac{\partial G_1}{\partial v_2} = \frac{1}{d}.$$

Moreover if $v_1 \in [-b, 1]$ then

$$\frac{\partial G_2}{\partial v_1} = \frac{1}{d}(\overline{f}'(v_1) - 1) + \gamma - 1 \geq \min_{s \in [-2b, 1]} \{f'(s) - 1\} \frac{1}{d} + \gamma - 1 = \frac{1}{d}(\beta - 2) + \gamma - 1.$$

Thus (1.1) is converted to a quasi-monotone system (3.1), provided that (H3) is satisfied.

For a quasi-monotone system, the following existence result is known (see e.g., [37]).

Proposition 3.2. *Assume that* $\mathbf{G}$ *is vanished at only three points* $\mathbf{b}_i$*,* $i = 1, 2, 3$ *and* $\mathbf{b}_1 < \mathbf{b}_2 < \mathbf{b}_3$*. Suppose for* $\nabla\mathbf{G}(\mathbf{b}_1)$ *and* $\nabla\mathbf{G}(\mathbf{b}_3)$*, all the eigenvalues lie in the left half-plane, and* $\nabla\mathbf{G}(\mathbf{b}_2)$ *has at least one eigenvalue with positive real part, then the quasi-monotone system* (3.1) *has a unique monotone decreasing traveling wave such that*

$$\lim_{z\to-\infty} \mathbf{\Psi}(z) = \mathbf{b}_1, \quad \lim_{z\to\infty} \mathbf{\Psi}(z) = \mathbf{b}_3.$$

It easy to check that

$$\det(\nabla\mathbf{G}(\mathbf{b}_i)) > 0 \quad \text{and} \quad \operatorname{tr}(\nabla\mathbf{G}(\mathbf{b}_i)) < 0$$

for $i = 1, 3$, while

$$\det(\nabla\mathbf{G}(\mathbf{b}_2)) < 0.$$

Let $\mathbf{\Psi} = (\Psi_1, \Psi_2)$ be the monotone decreasing traveling front of (3.1); that is,

$$\mathbf{\Psi}'' + c\mathbf{\Psi}' + \mathbf{G}(\mathbf{\Psi}) = \mathbf{0}.$$

Then $(\Psi_1, \Psi_2) = (u_1, u_1 - u_2)$, where $(u_1(\xi - ct), u_2(\xi - ct))$ is the planar traveling front mentioned in Section 2. It is easily seen that Ψ_1, Ψ_2 satisfy a number of properties stated in the following proposition.

Proposition 3.3. *There exist* $C_i > 0$ *and* $\sigma_i > 0$ *such that*

$$\begin{aligned}
C_1 e^{-\sigma_1 z} \le \Psi_1 \le C_2 e^{-\sigma_1 z} \quad &\textit{if } z \ge 0,\\
C_3 e^{\sigma_2 z} \le \rho - \Psi_1 \le C_4 e^{\sigma_2 z} \quad &\textit{if } z \le 0,\\
C_5 e^{-\sigma_1 z} \le \Psi_2 \le C_6 e^{-\sigma_1 z} \quad &\textit{if } z \ge 0,\\
C_7 e^{\sigma_2 z} \le \left(1 - \frac{1}{\gamma}\right)\rho - \Psi_2 \le C_8 e^{\sigma_2 z} \quad &\textit{if } z \le 0,\\
\max\{|\Psi_i'(z)|, |\Psi_i''(z)|, |z\Psi_i'(z)|\} \le C_9 e^{-\sigma_3|z|},&\\
\inf_{z\ge 0} -\frac{\Psi_1'(z)}{\Psi_1(z)} = \sigma_4, \quad \inf_{z\le 0} -\frac{\Psi_1'(z)}{\rho - \Psi_1(z)} = \sigma_5,&\\
\inf_{z\ge 0} -\frac{\Psi_2'(z)}{\Psi_2(z)} = \sigma_6, \quad \inf_{z\le 0} -\frac{\Psi_2'(z)}{(1 - \frac{1}{\gamma}\rho) - \Psi_2(z)} = \sigma_7.&
\end{aligned} \tag{3.3}$$

4. Front propagation on a V-shape curve

In the section we aim at a different wave structure for the front propagation; that is, a traveling front solution of the form $(v_1(x, z), v_2(x, z))$ with $z = y - kt$ and $k > c$. Note that c is the speed of planar front. The choice of the wave speed k is determined by the shape of the wave front in propagation.

Let $\mathbf{v} = (v_1, v_2)$ and define

$$Q_i[\mathbf{v}] = -\frac{\partial^2 v_i}{\partial x^2} - \frac{\partial^2 v_i}{\partial z^2} - k\frac{\partial v_i}{\partial z} - G_i(\mathbf{v}). \tag{4.1}$$

Theorem 4.1. *Suppose $k > c$, then there exists a $\mathbf{v}^*(x, z)$ such that $Q_i[\mathbf{v}^*] = 0$.*

Remark 4.2. For every $k > c$ there exists a traveling front with speed k.

An ordered method can be applied to (4.1), since $\mathbf{G}$ is quasi-monotone increasing. A function $\mathbf{v}^-$ is a sub-solution of (4.1) if $Q_i[\mathbf{v}^-] \leq 0$ or

$$\int_{\mathbb{R}^2} \mathbf{v}^- \left(-\frac{\partial^2 \phi_0}{\partial x^2} - \frac{\partial^2 \phi_0}{\partial z^2} + k\frac{\partial \phi_0}{\partial x_n}\right) \leq \int_{\mathbb{R}^2} \mathbf{G}(\mathbf{v}^-)\phi_0 \tag{4.2}$$

for $\phi_0 \in \mathcal{D}^+(\mathbb{R}^2)$, the set of nonnegative functions in $C_0^\infty(\mathbb{R}^2)$. A super-solution $\mathbf{v}^+$ satisfies the reversed inequalities. We establish Theorem 4.1 by making use of comparison arguments [31]. The construction of sub-solution and super-solution will follow from an idea from [29].

Set

$$m = \frac{\sqrt{k^2 - c^2}}{c}.$$

It is clear that both $\mathbf{\Psi}(c(z + mx)/k)$ and $\mathbf{\Psi}(c(z - mx)/k)$ satisfy (4.1) if $k > c$.

Lemma 4.3. *If $\mathbf{v}^-(x, z) = \mathbf{\Psi}(\frac{c}{k}(z - m|x|)$ then $\mathbf{v}^-$ is a sub-solution.*

Proof. Note that

$$\mathbf{v}^-(x, z) = \max\left\{\mathbf{\Psi}\left(\frac{c}{k}(z - mx)\right), \mathbf{\Psi}\left(\frac{c}{k}(z + mx)\right)\right\},$$

and (4.2) follows from direct calculation. $\square$

As in [29], the construction of a super-solution is more technically involved. Let v_i^+ be the i-th component of $\mathbf{v}^+$, and

$$v_i^+(x, z; \varepsilon, \alpha) = \Psi_i\left(\frac{z - \varphi(\alpha x)/\alpha}{\sqrt{1 + \varphi'(\alpha x)^2}}\right) + \varepsilon\,\mathrm{sech}(km\alpha x),$$

where ε and $\alpha = \alpha(\varepsilon)$ are positive numbers to be determined later. A way to make $\mathbf{v}^+$ to be a super-solution is to find a suitable function $\varphi(s)$. This can be achieved if $\varphi(s)$ satisfy

$$-\frac{\varphi_{ss}}{(1 + \varphi_s^2)^{3/2}} + \frac{k}{(1 + \varphi_s^2)^{1/2}} = c.$$

A number of properties of φ stated in the next lemma follows from direct calculation. We refer to [29] for a detailed proof.

Lemma 4.4. *For $k > c$, let $\nu(s) = \frac{k(\varphi(s)-m|s|)}{k-c\sqrt{1+\varphi'(s)^2}}$.*

(i) *$\varphi(s) > m|s|$ and*

$$0 < \nu_- \le \nu(s) \le \nu_+.$$

(ii) *There exist $K_i > 0$ such that*

$$\max\{|\varphi''(s)|, |\varphi'''(s)|\} \le K_1 \operatorname{sech}(kms), \tag{4.3}$$

$$K_2 \operatorname{sech}(kms) \le \frac{k}{\sqrt{1+\varphi'(s)^2}} - c \le K_3 \operatorname{sech}(kms). \tag{4.4}$$

Remark 4.5. $\varphi(x) = m|x| + O(|x|e^{-km|x|})$ as $x \to \pm\infty$.

Proposition 4.6. *There exist $\varepsilon_0 > 0$ and $\alpha_0 = \alpha_0(\varepsilon)$ such that if $\varepsilon \in (0, \varepsilon_0)$ and $\alpha \in (0, \alpha_0(\varepsilon))$ then $Q_i[\mathbf{v}^+] \ge 0$.*

Proof. Let $s(x) = \alpha x$, $\sigma(s) = \varepsilon \operatorname{sech}(kms)$ and

$$\zeta = \frac{z - \varphi(s)/\alpha}{\sqrt{1+\varphi'(s)^2}}.$$

Direct calculation yields

$$\zeta_x = -\frac{\alpha\varphi'\varphi''}{1+\varphi'^2}\zeta - \frac{\varphi'}{\sqrt{1+\varphi'^2}}, \tag{4.5}$$

$$\zeta_{xx} = -\frac{\alpha^2\varphi''^2 + \alpha^2\varphi'\varphi'''}{1+\varphi'^2}\zeta + \frac{3\alpha^2\varphi'^2\varphi''^2}{(1+\varphi'^2)^2}\zeta + \frac{\alpha(\varphi'^2-1)\varphi''}{(1+\varphi'^2)^{3/2}}. \tag{4.6}$$

Then

$$\begin{aligned}
Q_1[\mathbf{v}^+] &= -\frac{\partial^2 v_1^+}{\partial x^2} - \frac{\partial^2 v_1^+}{\partial z^2} - k\frac{\partial v_1^+}{\partial z} - \frac{1}{d}[f(v_1^+) + v_2^+ - v_1^+] \\
&= -\Psi_1''(\zeta)\zeta_x^2 - \Psi_1'(\zeta)\zeta_{xx} - \frac{\Psi_1''(\zeta)}{1+\varphi'^2} - \frac{k\Psi_1'(\zeta)}{\sqrt{1+\varphi'^2}} \\
&\quad - \frac{1}{d}\left[f(\Psi_1(\zeta)+\sigma) + dk^2m^2\alpha^2\sigma'' + \Psi_2(\zeta) - \Psi_1(\zeta)\right] \\
&= \left(1 - \frac{1}{1+\varphi'^2} - \zeta_x^2\right)\Psi_1''(\zeta) - \zeta_{xx}\Psi_1'(\zeta) + \left(c - \frac{k}{\sqrt{1+\varphi'^2}}\right)\Psi_1'(\zeta) \\
&\quad + \frac{1}{d}[f(\Psi_1(\zeta)) - f(\Psi_1(\zeta)+\sigma) - d\alpha^2\sigma''(s)]
\end{aligned}$$

and

$$Q_2[\mathbf{v}^+] = -\frac{\partial^2 v_2^+}{\partial x^2} - \frac{\partial^2 v_2^+}{\partial z^2} - k\frac{\partial v_2^+}{\partial z} - \frac{1}{d}f(v_1^+) - \left(\gamma - 1 - \frac{1}{d}\right)v_1^+ + \left(\gamma - \frac{1}{d}\right)v_2^+$$

$$
\begin{aligned}
&= -\Psi_2''(\zeta)\zeta_x^2 - \Psi_2'(\zeta)\zeta_{xx} - \frac{\Psi_2''(\zeta)}{1+\varphi'^2} - \frac{k\Psi_2'(\zeta)}{\sqrt{1+\varphi'^2}} - \frac{1}{d}f(\Psi_1(\zeta)+\sigma) \\
&\quad - \alpha^2\sigma'' - \left(\gamma - 1 - \frac{1}{d}\right)(\Psi_1(\zeta)+\sigma) + \left(\gamma - \frac{1}{d}\right)\Psi_2(\zeta) + \left(\gamma - \frac{1}{d}\right)\sigma \\
&= \left(1 - \frac{1}{1+\varphi'^2} - \zeta_x^2\right)\Psi_2''(\zeta) - \zeta_{xx}\Psi_2'(\zeta) + \left(c - \frac{k}{\sqrt{1+\varphi'^2}}\right)\Psi_2'(\zeta) \\
&\quad - \frac{1}{d}f(\Psi_1(\zeta)+\sigma) + \frac{1}{d}f(\Psi_1(\zeta)) - \alpha^2\sigma'' + \sigma \\
&\geq \left(1 - \frac{1}{1+\varphi'^2} - \zeta_x^2\right)\Psi_2''(\zeta) - \zeta_{xx}\Psi_2'(\zeta) + \left(c - \frac{k}{\sqrt{1+\varphi'^2}}\right)\Psi_2'(\zeta) \\
&\quad + \frac{1}{d}[f(\Psi_1(\zeta)) - f(\Psi_1(\zeta)+\sigma) - d\alpha^2\sigma''] + \sigma.
\end{aligned}
$$

Set

$$
I = \frac{1}{d}[f(\Psi_1(\zeta)) - f(\Psi_1(\zeta)+\sigma) - d\alpha^2\sigma''].
$$

Invoking (3.3), (4.3), (4.4), (4.5) and (4.6), we get

$$
\begin{aligned}
&\left(1 - \frac{1}{1+\varphi'^2} - \zeta_x^2\right)\Psi_1''(\zeta) - \zeta_{xx}\Psi_1'(\zeta) \\
&= -\alpha\left\{\left(\frac{\varphi'\varphi''}{1+\varphi'^2}\right)^2\alpha\zeta^2 + \frac{2\varphi'^2\varphi''}{(1+\varphi'^2)^{3/2}}\zeta\right\}\Psi_1''(\zeta) \\
&\quad + \alpha\left\{-\frac{\varphi''^2+\varphi'\varphi'''}{1+\varphi'^2}\alpha\zeta - \frac{3\varphi'^2\varphi''^2}{(1+\varphi'^2)^2}\alpha\zeta + \frac{(\varphi'^2-1)\varphi''}{(1+\varphi'^2)^{3/2}}\right\}\Psi_1'(\zeta) \\
&\geq -K_4\alpha\,\mathrm{sech}(kms)
\end{aligned} \tag{4.7}
$$

and

$$
-\left(\frac{k}{\sqrt{1+\varphi'^2}} - c\right)\Psi_1'(\zeta) \geq -K_2\Psi_1'(\zeta)\,\mathrm{sech}(kms) > 0. \tag{4.8}
$$

Since $f'(0) < 0$ and $f'(\rho) < 0$, there exists a $\delta_1 \in (0, \rho/8)$ such that

$$
-f'(\eta) \geq \kappa_1 \quad \text{if } \eta \in [0, 2\delta_1] \cup [\rho - 2\delta_1, \rho].
$$

Recall that Ψ_1 is monotone decreasing, $\lim_{z\to-\infty}\Psi_1(z) = \rho$ and $\lim_{z\to\infty}\Psi_1(z) = 0$. Without loss of generality, we may assume that

$$
\Psi_1(0) = \rho - \frac{\delta_1}{2} \quad \text{and} \quad \Psi_1(\overline{z}) = \frac{\delta_1}{2}.
$$

Thus

$$
\frac{\delta_1}{2} < \Psi_1(z) < \rho - \frac{\delta_1}{2} \quad \text{if } 0 < z < \overline{z}.
$$

Let $\kappa_2 = \min_{0\le z\le \overline{z}}(-\Psi_1'(z))$, $\kappa_3 = \sup\{\sigma''/\sigma\}$ and $\kappa_4 = \max_{0\le z\le\rho}|f'(z)|$. Define

$$\varepsilon_0 = \min\left\{1, \frac{\delta_1}{2}, \frac{K_2\kappa_2 d}{4\kappa_4}\right\}$$

and

$$\alpha_0(\varepsilon) = \min\left\{1, \sqrt{\frac{\kappa_1}{4d\kappa_3}}, \frac{K_2\kappa_2}{4(K_4+\kappa_3)}, \frac{\kappa_1\varepsilon}{4dK_4}, \frac{e^2c^2\sigma_3^2\varepsilon\nu_-}{4C_9K_3k}\right\}.$$

We claim:

$$Q_1[\mathbf{v}^+] \ge 0 \quad \text{if } \varepsilon < \varepsilon_0 \text{ and } \alpha < \alpha_0(\varepsilon). \tag{4.9}$$

We first consider $\zeta \in [0, \overline{z}]$. Applying the Mean Value Theorem yields

$$|I| \le \frac{1}{d}(\kappa_4\sigma + d\kappa_3\alpha^2\sigma) \le \frac{1}{d}(\kappa_4 + d\kappa_3\alpha^2)\varepsilon\,\mathrm{sech}(kms). \tag{4.10}$$

Also, (4.8) implies that

$$-\left(\frac{k}{\sqrt{1+\varphi'^2}} - c\right)\Psi_1'(\zeta) \ge \kappa_2K_2\,\mathrm{sech}(kms). \tag{4.11}$$

Putting (4.7), (4.10), (4.11) all together, we obtain

$$\begin{aligned} Q_1[\mathbf{v}^+] &\ge \left(-K_4\alpha - \frac{\kappa_4\varepsilon}{d} - \kappa_3\alpha^2\varepsilon + \kappa_2K_2\right)\mathrm{sech}(kms) \\ &\ge \left(-(K_4+\kappa_3\varepsilon)\alpha - \frac{\kappa_4\varepsilon}{d} + \kappa_2K_2\right)\mathrm{sech}(kms) \\ &\ge \frac{\kappa_2K_2}{2}\,\mathrm{sech}(kms) \ge 0. \end{aligned}$$

Next we treat the case $\zeta \in (-\infty, 0) \cap (\overline{z}, \infty)$. This together with $\varepsilon < \delta_1/2$ implies that $\Psi_1(\zeta) + \sigma \in [0, \delta_1] \cup [\rho - \delta_1, \rho]$. Applying the Mean Value Theorem, we get

$$I \ge \left(\frac{\kappa_1}{d} - \kappa_3\alpha^2\right)\varepsilon\,\mathrm{sech}(kms).$$

This together with (4.7) and (4.8) gives

$$Q_1[\mathbf{v}^+] \ge \left(-K_4\alpha - \kappa_3\alpha^2\varepsilon + \frac{\kappa_1\varepsilon}{d}\right)\mathrm{sech}(kms) \ge \frac{\kappa_1\varepsilon}{2d}\,\mathrm{sech}(kms) \ge 0,$$

which completes the proof of (4.9).

The proof of $Q_2[\mathbf{v}^+] \ge 0$ is similar. We thus conclude that $\mathbf{v}^+$ is a super-solution. □

Proposition 4.7.

$$\mathbf{v}^- < \mathbf{v}^+ \tag{4.12}$$

and

$$\lim_{R\to\infty} \sup_{x^2+z^2>R^2} |v_i^+(x,z) - v_i^-(x,z)| \le 2\varepsilon \tag{4.13}$$

for any $\varepsilon > 0$.

Proof. To simplify the notation, we let $\zeta_1 = \frac{c}{k}(z - m|x|) > 0$ and $\zeta_2 = \frac{z-m|x|}{\sqrt{1+\varphi'(\alpha x)^2}}$ in the following calculation. Recall that

$$v_i^+ - v_i^- = \Psi_i(\zeta) - \Psi_i(\zeta_1) + \sigma(s).$$

Since $\sigma \geq 0$ and Ψ_i is a decreasing function, (4.12) immediately follows if $\zeta \leq \zeta_1$.

It remains to treat the case of $\zeta > \zeta_1$. By direct calculation

$$0 < \zeta - \zeta_1 = \left(\frac{1}{\sqrt{1+\varphi'(s)^2}} - \frac{c}{k} \right) (z - m|x|) - \frac{\varphi(s) - m|s|}{\alpha\sqrt{1+\varphi'(s)^2}},$$

which implies

$$z - m|x| > \frac{\nu(s)}{\alpha}.$$

Invoking Lemma 4.4 yields $z - m|x| \geq \nu_-/\alpha$ and consequently

$$\zeta_1 \geq \frac{c\nu_-}{k\alpha}. \tag{4.14}$$

Also, Lemma 4.4 gives

$$m|x| \leq \frac{1}{\alpha}\varphi(\alpha x) \quad \text{if } \alpha > 0,$$

from which we know $\zeta \leq \zeta_2$. Since Ψ_1 is a decreasing function,

$$\begin{aligned} v_1^+ - v_1^- &= \Psi_1(\zeta) - \Psi_1(\zeta_2) + \Psi_1(\zeta_2) - \Psi_1(\zeta_1) + \sigma(s) \\ &\geq \Psi_1(\zeta_2) - \Psi_1(\zeta_1) + \sigma(s). \end{aligned}$$

Applying the Mean Value Theorem yields

$$v_1^+ - v_1^- \geq - \left(\frac{1}{\sqrt{1+\varphi'(s)^2}} - \frac{c}{k} \right) (z - m|x|)|\Psi_1'(\theta\zeta_2 + (1-\theta)\zeta_1)| + \sigma(s)$$

for some $\theta \in (0, 1)$. In view of (3.3) and $\zeta_1 < \zeta_2$,

$$|\Psi_1'(\theta\zeta_2 + (1-\theta)\zeta_1)| \leq C_9 \exp(-\sigma_3|\theta\zeta_2 + (1-\theta)\zeta_1|) \leq C_9 \exp(-\sigma_3|\zeta_1|).$$

Hence

$$v_1^+ - v_1^- \geq - \left(\frac{1}{\sqrt{1+\varphi'(s)^2}} - \frac{c}{k} \right) \frac{k\zeta_1 C_9}{c} \exp(-\sigma_3|\zeta_1|) + \varepsilon \operatorname{sech}(kms).$$

Since $s^2 \exp(-s) \leq 4/e^2$ for all $s \geq 0$, using (4.4) and (4.14), we obtain

$$v_1^+ - v_1^- \geq -\frac{4C_9K_3k}{e^2c^2\sigma_3^2\nu_-}\alpha \operatorname{sech}(kms) + \varepsilon \operatorname{sech}(kms) \geq 0,$$

provided that $0 < \varepsilon \leq \varepsilon_0$ and $0 < \alpha \leq \alpha_0(\varepsilon)$.

By taking suitably small ε_0 and α_0, the proof of $v_2^+ \geq v_2^-$ can be done by the same manner.

It remains to show (4.13). By the Mean Value Theorem

$$v_i^+ - v_i^- = \Psi_i'(\zeta_3)(\zeta - \zeta_1) + \sigma(s)$$

where

$$\zeta_3 = \widehat{\theta}\zeta + (1-\widehat{\theta})\zeta_1 \tag{4.15}$$

with $\widehat{\theta} \in (0,1)$. Since $0 < \sigma(s) \leq \varepsilon$, it suffices to show that there exist $R_1, R_2 > 0$ such that

$$|\Psi_i'(\zeta_3)(\zeta - \zeta_1)| \leq \varepsilon \tag{4.16}$$

if $|x| \geq R_1/\alpha$ and $|z| \geq R_2$. In view of Remark 4.5,

$$\zeta = \frac{z - \frac{1}{\alpha}\varphi(\alpha x)}{\sqrt{1+\varphi'(\alpha x)^2}} \to \frac{c}{k}(z - m|x|) = \zeta_1 \quad \text{as } |x| \to \infty.$$

This together with (3.3) implies that (4.16) holds if $|s| \geq R_1$ with R_1 large enough.

If $|s| < R_1$, we know from (4.15) that $\zeta_3 > \min\{\zeta, \zeta_1\}$ and

$$\liminf_{z\to\infty} \sup_{|s|\leq R_1} |\zeta_3| = \infty.$$

Then by (3.3),

$$\lim_{|z|\to\infty} \sup_{|s|\leq R_1} |\Psi_i'(\zeta_3)(\zeta - \zeta_1)| \leq \lim_{|z|\to\infty} \sup_{|s|\leq R_1} |\Psi_i'(\zeta_3)\zeta_3| = 0.$$

Hence there is an $R_2 > 0$ such that

$$\sup_{|z|\geq R_2, |s|\leq R_1} |\Psi_i'(\zeta_3)(\zeta - \zeta_1)| < \varepsilon.$$

The proof of (4.16) is complete, so is the proposition. □

Remark 4.8. For fixed $k > c$, the traveling front solution $\mathbf{v}^*(x,z)$ of (1.1), obtained by Theorem 4.1, has the following properties:

(i) $\lim_{z\to-\infty} \mathbf{v}^*(x,z) = (\rho, \rho/\gamma)$, $\lim_{z\to\infty} \mathbf{v}^*(x,z) = (0,0)$.

(ii) $\frac{\partial}{\partial z}\mathbf{v}^*(x,z) < 0$.

(iii) $\lim_{|x|\to\infty} \mathbf{v}^*(x,z) = (\rho, \rho/\gamma)$, $\lim_{|x|\to\infty} \frac{\partial}{\partial x}\mathbf{v}^*(x,z) = (0,0)$.

5. Higher dimensional spaces

With the traveling front $\mathbf{v}^*(x,z)$ established in Theorem 4.1, the ordered method can be further used to construct a traveling front solution of the form $\mathbf{w}(x_1,x_2,z)$ with $z=y-k_1t$. Here $k_1>k$ and k is the wave speed of $\mathbf{v}^*$. Let $m_1=\sqrt{k_1^2-k^2}/k$, $\mathbf{w}=(w_1,w_2)$ and define

$$\overline{Q}_i[\mathbf{w}] = -\frac{\partial^2 w_i}{\partial x_1^2} - \frac{\partial^2 w_i}{\partial x_2^2} - \frac{\partial^2 w_i}{\partial z^2} - k_1\frac{\partial w_i}{\partial} z - G_i(\mathbf{w}).$$

Take $\mathbf{w}^-(x_1,x_2,z)=\mathbf{v}^*\big(x_2,\frac{k}{k_1}(z-m_1|x_1|)\big)$ to be a sub-solution and

$$\mathbf{w}^+(x_1,x_2,z;\varepsilon,\alpha) = \mathbf{v}^*\left(x_2, \frac{z-\varphi(\alpha x_1)/\alpha}{\sqrt{1+\varphi'(\alpha x_1)^2}}\right) + \varepsilon\,\mathrm{sech}(k_1 m_1 \alpha x_1)$$

to be a super-solution. Then a traveling front solution $\mathbf{w}^*(x_1,x_2,z)$ of (1.1) follows from comparison argument. It is clear that

$$\lim_{z\to-\infty}\mathbf{w}^*(x_1,x_2,z)=(\rho,\rho/\gamma), \quad \lim_{z\to\infty}\mathbf{w}^*(x_1,x_2,z)=(0,0).$$

6. Further remarks

A standing wave solution of (1.1) satisfies

$$u_1''+f(u_1)-u_2=0, \quad u_2''+u_1-\gamma u_2=0. \tag{6.1}$$

In this section, we set $\gamma=9(2\beta^2-5\beta+2)^{-1}$. This is the only situation that two constant solutions $(0,0)$ and $(\rho,\rho/\gamma)$ are in the same energy level. Like what is known in the Allen-Cahn equation, standing fronts appear in reaction-diffusion system with a balanced double-well potential. Through a minimization argument, a basic type standing front solution of (1.1) has been established [17].

Theorem 6.1. *If $d>\gamma^{-2}$, there exists a standing front solution $\mathbf{u}^*$ of (1.1) such that $\mathbf{u}^*(x)\to(0,0)$ as $x\to-\infty$ and $\mathbf{u}^*(x)\to(\rho,\rho/\gamma)$ as $x\to\infty$.*

Clearly $\mathbf{u}_*$ is also a standing front solution of (1.1) if we define $\mathbf{u}_*(x)=\mathbf{u}^*(-x)$. The multiple front solutions can be construction under the following conditions:

(H4) $\beta\in(0,(7-\sqrt{45})/2)$ and $1/\gamma<\sqrt{d}<2/\gamma$.

Note that (6.1) is a second order Hamiltonian system. If (H4) holds, the equilibria $(0,0)$ and $(\rho,\rho/\gamma)$ are of saddle-focus type. A nonlocal Lyapunov-Schmidt reduction was employed [18] to obtain large amount of standing waves with an arbitrary number of fronts located far enough from each other:

Theorem 6.2. *Assume that* (H4) *is satisfied. Then there are two real numbers* κ_+, κ_-, *and, for each sufficiently small* $\tau > 0$, *a large constant* $D_\tau > 0$, *such that for any positive integer* N *and any sequence of positive integers* $\mathbf{n} = (n_i)_{1 \le i \le N}$ *with* $n_i \ge D_\tau$ *for every* i, *there exist positive numbers* $X_1, \dots, X_{N-1}$ *and a solution* $\mathbf{u_n}$ *of* (6.1) *satisfying the following properties:*

(i) $\|\mathbf{u_n} - \mathbf{u}^*\|_{H^1(-\infty, A_1)} \le \tau$.

(ii) *For* i *odd in* $[1, N]$,

$$\|\mathbf{u_n}(\,\cdot + C_i) - \mathbf{u}_*\|_{H^1(-A_i, A_{i+1})} \le \tau, \quad |X_i - 2\pi n_i/\omega - \kappa_+| < \tau.$$

(iii) *For* i *even in* $[2, N]$,

$$\|(\mathbf{u_n}(\,\cdot + C_i) - \mathbf{u}^*\|_{H^1(-A_i, A_{i+1})} \le \tau, \quad |X_i - 2\pi n_i/\omega - \kappa_-| < \tau.$$

Here, $C_1 = X_1$, $C_i = C_{i-1} + X_i$, $A_i = X_i/2$ *for* $1 \le i \le N$, *and* $A_{N+1} = +\infty$.

The existence of infinite number of standing waves stated in Theorem 6.2 has been established in [18]; in addition, it was shown [18] that such standing wave solutions are local minimizers. We expect that there are different types of front propagation between such standing waves.

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Chao-Nien Chen
Department of Mathematics, National Tsing Hua University, Hsinchu, Taiwan
E-mail address: `chen@math.nthu.edu.tw`

Che-Hao Lin
Department of Applied Mathematics, Tunghai University, Taichung, Taiwan
E-mail address: `linch@thu.edu.tw`

Shyuh-yaur Tzeng
Department of Mathematics, National Changhua University of Education, Changhua, Taiwan
E-mail address: `sytzeng@cc.ncue.edu.tw`