TAIWANESE JOURNAL OF MATHEMATICS
Vol. 23, No. 2, pp. 409–425, April 2019
DOI: 10.11650/tjm/181108

Exact Bounds and Approximating Solutions to the Fredholm Integral Equations of Chandrasekhar Type

Sheng-Ya Feng* and Der-Chen Chang

In memory of Professor Hwai-Chiuan Wang

Abstract. In this paper, we study the L^p solutions of the Fredholm integral equations with Chandrasekhar kernels. The Hilbert type inequality is resorted to establish an existence and uniqueness result for the Fredholm integral equation associated with Chandrasekhar kernel. A couple of examples well support the condition and extend the classical results in the literature with one generalizing the classical Chandrasekhar kernel. In order to approximate the original solution, a truncated operator is introduced to overcome the non-compactness of the integral operator. An error estimate of the convergence is made in terms of the truncated parameter, the upper bounds of the symbolic function constituting the integral kernel and initial data to the equation.

1. Introduction

I. Fredholm (1900) proposed a new theory for integral equations containing a parameter of the form

$$x(t) + \lambda \int_a^b k(t,s)x(s)\,ds = y(t).$$

Fredholm's theory generalizes the solvability of a linear system $x + \lambda Ax = b$ in finite-dimensional vector space to the infinite-dimensional one, which sets up the framework of linear functional analysis with the most valuable sources. At the beginning of last century, a number of celebrated mathematicians such as E. Picard (1906), E. Schmidt (1907), H. Poincaré (1909), H. Weyl (1909), M. Fréchet (1912) and D. Hilbert (1912), substantially developed Fredholm's theory and extensively connected to other disciplines of sciences. Evidently, the theory of equations with symmetric kernel was considerably enriched, and systematic method of the orthogonal series were profoundly developed. Following the remarkable mathematical notion of Banach space, F. Riesz (1952) proceeds to the extension of Fredholm's theory to linear operators in abstract normed spaces.

Received June 27, 2018; Accepted November 14, 2018.
Communicated by Tai-Chia Lin.
2010 *Mathematics Subject Classification*. Primary: 45B05; Secondary: 26D15, 47H10.
Key words and phrases. Chandrasekhar kernel, Hilbert-type inequality, Fredholm integral equation, L^p norm, approximating solution.
*Corresponding author.

The Chandrasekhar's integral equation

$$x(t) = 1 + x(t) \int_0^1 \frac{t}{t+s} \omega(s) x(s)\, ds$$

initiated the study of radiative transfer theory in a plane-parallel atmosphere. Chandrasekhar's remarkable work [10] in 1950s formulated the radiative transferring process and derived the integral equation for the scattering function and transmitted functions, which was promptly developed as a major scientific subject across astrophysics and mathematics. Besides the radiative transfer theory, the so-called quadratic integral equations are extensively applicable to many research areas: the kinetic theory of gases, neutron transport, traffic model and the queuing theory [2]. Readers may consult more recent references [9,18,19,26,28] for excellent exposition on this topic. It is instructive to notice that Chandrasekar's H-function is closely related to the angular pattern or single scattering, which formulates the solution to the Chandrasekar's integral equation as the characteristic function ω is an even polynomial in s. The restriction that $\int_0^1 \omega(s)\, ds \leq 1/2$ is treated as a necessary condition in astrophysical applications. The theory of nonlinear quadratic integral equations was substantially developed by exploiting the techniques on the measure of noncompactness, see e.g., [3–5, 9, 10, 15–17] as well as fixed point theorem [24].

Fredholm's solution is in the form of a ratio of two infinite series, each term containing the multiple integrals of determinants of higher order than the previous one. Tricomi [27] pointed out that the solution were so complicated for numerical calculations due to the multiple integrals in various terms. Even computing power is greatly enhanced nowadays, Prem et al. [21] described no computational method managing to evaluate Fredholm's solution numerically. The effort in finding the approximation solutions to Chandrasekhar's integral equation was made by several efficient methods, see e.g., [6, 20] for an updated discussion on this topic. The common approach is to use iterative method such as the classical Newton's method [13, 23]. However, an iteration turns to be expensive since it requires to compute and store the Jacobian matrix, as well as solving Newton's system which is a linear system in each iteration (see e.g., [8, 13, 22]). Moreover, the existence of solutions to Chandrasekhar's integral equation is still concerned, especially in some cases where the solutions are not unique [7].

In virtue of the essential difficulties in the existence theory and numerical accessibility of the nonlinear integral equations of Chandrasekhar type, we consider the Fredholm integral equation defined on $(0, \infty)$

$$\varphi(y) = \psi(y) + \lambda \int_0^\infty k(x,y) \varphi(x)\, dx \tag{1.1}$$

with φ an unknown function, whereas ψ is an initial data in some certain function spaces. By special forms of the integral kernel, i.e., $k(x,y) = \mu(x,y)/(x+y)$, the above equation covers the linearized Chandrasekhar's integral equation. In recent research on the Fredholm integral equation [11], the authors discuss the continuous and L^2 solutions. The aim of this paper is to continue our study on the L^p solutions to the Fredholm integral equation. In Section 2, we review classical results on Hilbert type inequalities and establish an existence and uniqueness result of the L^p solutions to the Fredholm equation with the integral kernels fulfilling some certain limit conditions. A couple of examples listed in Section 3 support the assumed conditions in Section 2 quite well and extend the integral kernels to various classes. In Section 4, we introduce a bilateral truncated operator to approximate the original operator and find the solution to the approximating equation. An error bound estimate is carried out as long as the symbolic function and initial data are polynomial decay at infinity. As a conclusion in Section 5, we make final remarks for this paper.

2. L^p solutions $(1 < p < \infty)$

2.1. Hilbert inequalities

In this section, we recall some basic facts on Hilbert inequalities. The work originates from D. Hilbert, G. Hardy and followed by many other mathematicians during the last 110 years. We commend interested readers consult [12, 31] and the references therein for an excellent exposition of this direction.

Given nonnegative real sequences $\{a_n\}$ and $\{b_n\}$ such that $\sum_{n=1}^{\infty} a_n^2 < \infty$ and $\sum_{n=1}^{\infty} b_n^2 < \infty$, D. Hilbert (cf. [29]) established a well-known inequality

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \pi \left(\sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2 \right)^{1/2}, \tag{2.1}$$

where the bound π is the best constant, and it was called *Hilbert inequality.* I. Schur [25] proved the optimality of Hilbert inequality and obtained a functional analog. Specifically, given $f(x)$ and $g(x)$ nonnegative square integrable functions on half line $(0,\infty)$, it follows that

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{x+y}\, dxdy < \pi \left(\int_0^{\infty} f^2(x)\, dx \int_0^{\infty} g^2(x)\, dx \right)^{1/2}, \tag{2.2}$$

where the bound π remains the best, and it was called *Hilbert integral inequality.*

For any $a = \{a_m\}_{m=1}^{\infty} \in l^2$ with the norm $\|a\|_2 = \left(\sum_{n=1}^{\infty} a_n^2\right)^{1/2}$, the *Hilbert operator* $L\colon l^2 \to l^2$ is defined by $c = \{c_n\}_{n=1}^{\infty} \in l^2$ with

$$c_n = (La)(n) = \sum_{m=1}^{\infty} \frac{a_m}{m+n}, \quad n \in \mathbb{N}.$$

For $b = \{b_n\}_{n=1}^{\infty} \in l^2$, the inner product is given by

$$(La, b) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{a_m}{m+n} \right) b_n.$$

Equation (2.1) rewritten as

$$|(La, b)| < \pi \|a\|_2 \|b\|_2$$

with $\|a\|_2 > 0$ and $\|b\|_2 > 0$, asserts that L is bounded on l^2 and its operator norm is $\|L\|_2 = \pi$. Equivalently, $\|La\|_2 < \pi \|a\|_2$, namely

$$\left\{ \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{a_m}{m+n} \right)^2 \right\}^{1/2} < \pi \left(\sum_{n=1}^{\infty} a_n^2 \right)^{1/2}.$$

Similarly, for $f \in L^2(0, \infty)$ with norm $\|f\|_2 = \left(\int_0^{\infty} |f^2(x)|^2 \, dx \right)^{\frac{1}{2}}$ in real Hilbert space $L^2(0, \infty)$, one defines *Hilbert integral operator* T on $L^2(0, \infty)$ into itself by

$$(Tf)(y) = \int_0^{\infty} \frac{f(x)}{x+y} \, dx, \quad y \in (0, \infty).$$

For $g \in L^2(0, \infty)$, one also defines inner product

$$(Tf, g) = \int_0^{\infty} \left(\int_0^{\infty} \frac{f(x)}{x+y} \, dx \right) g(y) \, dy = \int_0^{\infty} \int_0^{\infty} \frac{f(x) g(y)}{x+y} \, dx dy.$$

Given $\|f\|_2 > 0$ and $\|g\|_2 > 0$, (2.2) would read as

$$|(Tf, g)| < \pi \|f\|_2 \|g\|_2,$$

or $\|Tf\|_2 < \pi \|f\|_2$, namely

$$\left\{ \int_0^{\infty} \left(\int_0^{\infty} \frac{f(x)}{x+y} \, dx \right)^2 dy \right\}^{1/2} < \pi \left(\int_0^{\infty} |f(x)|^2 \, dx \right)^{1/2},$$

and the operator norm $\|T\|_2 = \pi$ is the best bound for the above Hilbert integral inequality.

Remark 2.1. The norms of Hilbert-type integral operators could be exactly estimated, casting a light on the sharp bound of the parameter λ in the linear Fredholm integral equation to ensure the existence of the solutions. Moreover, one may extend the basic models to a relatively wide class of integral kernels.

2.2. An existence result

We consider the L^p solutions ($1 < p < \infty$) of the linear Fredholm integral equation of the second kind (1.1), where $\psi \in L^p(0, \infty)$. In order to achieve this goal, we resort to the following Lemma 2.2 which was established in [30].

For $1 < p < \infty$, $1/p + 1/p' = 1$, the integral kernel $k(x,y) = k(y,x)$ is symmetric and nonnegative almost every on $(0,\infty)\times(0,\infty)$. Given $f \in L^p(0,\infty)$ and $g \in L^{p'}(0,\infty)$, one defines

$$Kf(y) := \int_0^\infty k(x,y) f(x)\, dx, \quad y \in (0,\infty)$$

and

$$Kg(x) := \int_0^\infty k(x,y) g(y)\, dy, \quad x \in (0,\infty).$$

For any $\varepsilon > 0$ and $x > 0$, we define

$$k_\varepsilon(r,x) := \int_0^\infty k(x,y) \left(\frac{x}{y}\right)^{(1+\varepsilon)/r} dy, \quad r = p \text{ or } p'$$

and

$$k_0(r,x) := \int_0^\infty k(x,y) \left(\frac{x}{y}\right)^{1/r} dy, \quad r = p \text{ or } p'.$$

Lemma 2.2. *The following statements hold.*

(1) *If $k_0(p) = k_0(r,x)$ ($r = p$ or p') is independent of $x > 0$, then $K\colon L^p(0,\infty) \to L^p(0,\infty)$ is a continuous linear operator and $\|K\|_{L^p\to L^p} \le k_0(p)$.*

(2) *If $k_\varepsilon(p) = k_\varepsilon(r,x)$ ($r = p$ or p') is independent of $x > 0$ and $k_\varepsilon(p) = k_0(p) + o(1)$ ($\varepsilon \to 0^+$), then $\|K\|_{L^p\to L^p} = k_0(p)$.*

(3) *Moreover, if the conditions in (2) are fulfilled, for any $f \in L^p(0,\infty)$ and $\|f\|_{L^p} > 0$, then the strict inequality $\|Kf\|_{L^p} < \|K\|_{L^p\to L^p}\|f\|_{L^p}$ holds.*

We put the linear Fredholm integral equation (1.1) in the previous notations as

$$\varphi = \psi + \lambda K\varphi$$

and define an operator T from $L^p(0,\infty)$ into itself by

$$T\varphi = \psi + \lambda K\varphi.$$

Noticing that

$$\|T\varphi_1 - T\varphi_2\|_{L^p} = |\lambda| \|K(\varphi_1 - \varphi_2)\|_{L^p} \le |\lambda| k_0(p) \|\varphi_1 - \varphi_2\|_{L^p},$$

one claims that T is a contraction operator if $|\lambda| k_0(p) < 1$ and thus by means of contraction mapping principle, one reaches the following

Theorem 2.3. *For the linear Fredholm integral equation (1.1) if the kernel $k(x,y)$ is symmetric and nonnegative almost every on $(0,\infty)\times(0,\infty)$ and fulfils the condition in Lemma 2.2, i.e., $k_\varepsilon(p) = k_\varepsilon(r,x)$ ($r = p$ or p') is independent of $x > 0$ and $k_\varepsilon(p) = k_0(p) + o(1)$ ($\varepsilon \to 0^+$), then the linear Fredholm integral equation (1.1) exists the unique solution in $\overline{\varphi} \in L^p(0,\infty)$ as long as $|\lambda| < 1/k_0(p)$.*

Remark 2.4. If $f \in L^p(0,\infty)$, $g \in L^{p'}(0,\infty)$, as an equivalent statement of Lemma 2.2, one has

$$\left|\int_0^\infty\int_0^\infty k(x,y)f(x)g(y)\,dxdy\right| \le k_0(p)\left\{\int_0^\infty |f(x)|^p\,dx\right\}^{1/p}\left\{\int_0^\infty |g(y)|^{p'}\,dy\right\}^{1/p'}.$$

One may consider $K\colon L^{p'}(0,\infty)\to L^{p'}(0,\infty)$ by the same way to yield

$$\|K\|_{L^p\to L^p} = \|K\|_{L^{p'}\to L^{p'}} = k_0(p).$$

In Lemma 2.2, the condition $k_0(p) = k_0(p')$ ensures that the linear operator K is bounded from $L^r(0,\infty)$ into itself ($r = p$ or p'). In next section, we would make several examples to support such condition.

3. Sharp bounds

For the linear Fredholm integral equation of the second kind (1.1), taking $k(x,y) = \mu(x,y)/\nu(x,y)$, we call $\mu(x,y)$ characteristic function, $\nu(x,y)$ symbolic function and (1.1) *Chandrasekhar integral equation* or *Fredholm integral equation with Chandrasekhar kernel.* The prototype of Chandrasekhar's equation corresponds to $\mu(x,y) = 1$ and $\nu(x,y) = x+y$.

In this section, we illustrate several examples to support the conditions in Lemma 2.2. For more applications, one needs to handle two kinds of Eulerian integrals, beta function (Eulerian integral of first kind) and gamma function (Eulerian integral of second kind). For $p > 0$, $q > 0$ and $s > 0$, one defines the beta function

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx$$

and the gamma function

$$\Gamma(s) = \int_0^{+\infty} x^{s-1}e^{-x}\,dx.$$

The two kinds of Eulerian integrals are related by

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p>0,\ q>0 \tag{3.1}$$

and satisfy some useful equalities

$$B(p,q) = \int_0^{+\infty}\frac{t^{p-1}}{(1+t)^{p+q}}\,dt = \int_0^1 \frac{t^{p-1}+t^{q-1}}{(1+t)^{p+q}}\,dt = B(q,p), \quad p>0,\ q>0, \tag{3.2}$$

$$\frac{\Gamma(a+1)}{(b+1)^{a+1}} = \int_0^1 (-\ln u)^a u^b\,du, \quad a>-1,\ b>-1,$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(s\pi)}, \quad 0<s<1. \tag{3.3}$$

The proofs and more properties of Eulerian integrals could be found in classical analysis [32].

Example 3.1. Consider $k(x,y) = \frac{1}{(x^\alpha + y^\alpha)^{1/\alpha}}$ with $\alpha > 0$.

Putting $u = y/x$, $v = u^\alpha$ and making use of (3.2), one observes that

$$\begin{aligned} k_\varepsilon(p,x) &= \int_0^\infty \frac{1}{(x^\alpha + y^\alpha)^{1/\alpha}} \left(\frac{x}{y}\right)^{(1+\varepsilon)/p} dy = \int_0^\infty \frac{1}{[1 + (\frac{y}{x})^\alpha]^{1/\alpha} x} \left(\frac{x}{y}\right)^{(1+\varepsilon)/p} dy \\ &= \int_0^\infty \frac{1}{(1+u^\alpha)^{1/\alpha}} u^{-(1+\varepsilon)/p}\, du \xrightarrow{\varepsilon \to 0^+} \int_0^\infty \frac{1}{(1+u^\alpha)^{1/\alpha}} u^{-1/p}\, du \\ &= \frac{1}{\alpha} \int_0^\infty \frac{v^{1/(p'\alpha)-1}}{(1+v)^{1/\alpha}}\, dv = \frac{1}{\alpha} B\left(\frac{1}{p\alpha}, \frac{1}{p'\alpha}\right) = k_0(p) = k_0(p'). \end{aligned}$$

In virtue of Lemma 2.2 and Theorem 2.3, one has

$$\left\{\int_0^\infty \left| \int_0^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^{1/\alpha}}\, dx \right|^p dy \right\}^{1/p} \le \frac{1}{\alpha} B\left(\frac{1}{p\alpha}, \frac{1}{p'\alpha}\right) \left\{\int_0^\infty |f(x)|^p\, dx\right\}^{1/p},$$

and the linear Chandrasekhar integral equation

$$\varphi(y) = \psi(y) + \lambda \int_0^\infty \frac{\varphi(x)}{(x^\alpha + y^\alpha)^{1/\alpha}}\, dx$$

has a unique solution in $\overline{\varphi} \in L^p(0,\infty)$ as long as $|\lambda| < \left\{\frac{1}{\alpha} B\left(\frac{1}{p\alpha}, \frac{1}{p'\alpha}\right)\right\}^{-1}$.

Taking $\alpha = 1$, by means of (3.1) and (3.3), one has

$$k_0(p) = B\left(\frac{1}{p}, \frac{1}{p'}\right) = \Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{p'}\right) = \frac{\pi}{\sin(\pi/p)}.$$

Consequently, one recovers the well-known *Hardy-Hilbert inequality*

$$\left\{\int_0^\infty \left| \int_0^\infty \frac{f(x)}{x+y}\, dx \right|^p dy \right\}^{1/p} \le \frac{\pi}{\sin(\pi/p)} \left\{\int_0^\infty |f(x)|^p\, dx\right\}^{1/p},$$

and the linear Chandrasekhar integral equation

$$\varphi(y) = \psi(y) + \lambda \int_0^\infty \frac{\varphi(x)}{x+y}\, dx$$

has a unique solution in $\overline{\varphi} \in L^p(0,\infty)$ as long as $|\lambda| < \left\{\frac{\pi}{\sin(\pi/p)}\right\}^{-1}$.

In particular when $p = p' = 2$, it leads to

$$\left\{\int_0^\infty \left| \int_0^\infty \frac{f(x)}{(x^\alpha + y^\alpha)^{1/\alpha}}\, dx \right|^2 dy \right\}^{1/2} \le \frac{1}{\alpha} B\left(\frac{1}{2\alpha}, \frac{1}{2\alpha}\right) \left\{\int_0^\infty |f(x)|^2\, dx\right\}^{1/2},$$

and the linear Chandrasekhar integral equation

$$\varphi(y) = \psi(y) + \lambda \int_0^\infty \frac{\varphi(x)}{(x^\alpha + y^\alpha)^{1/\alpha}}\, dx$$

has a unique solution in $\overline{\varphi} \in L^2(0,\infty)$ as long as $|\lambda| < \left\{\frac{1}{\alpha} B\left(\frac{1}{2\alpha}, \frac{1}{2\alpha}\right)\right\}^{-1}$. Moreover, taking $\alpha = 1$, one yields

$$\left\{\int_0^\infty \left|\int_0^\infty \frac{f(x)}{x+y}\,dx\right|^2 dy\right\}^{1/2} \le \pi \left\{\int_0^\infty |f(x)|^2\,dx\right\}^{1/2},$$

and the linear Chandrasekhar integral equation

$$\varphi(y) = \psi(y) + \lambda \int_0^\infty \frac{\varphi(x)}{x+y}\,dx$$

has a unique solution in $\overline{\varphi} \in L^2(0,\infty)$ as long as $|\lambda| < 1/\pi$.

Example 3.2. Consider $k(x,y) = \frac{1}{|x-y|^\alpha (\max\{x,y\})^{1-\alpha}}$ with $\alpha < 1$.

Putting $u = y/x$ and $u = x/y$, one observes that

$$\begin{aligned}
k_\varepsilon(p,x) &= \int_0^\infty \frac{1}{|x-y|^\alpha (\max\{x,y\})^{1-\alpha}} \left(\frac{x}{y}\right)^{(1+\varepsilon)/p} dy \\
&= \int_0^x \frac{1}{(x-y)^\alpha x^{1-\alpha}} \left(\frac{x}{y}\right)^{(1+\varepsilon)/p} dy + \int_x^\infty \frac{1}{(y-x)^\alpha y^{1-\alpha}} \left(\frac{x}{y}\right)^{(1+\varepsilon)/p} dy \\
&= \int_0^1 \frac{1}{(1-u)^\alpha} u^{-(1+\varepsilon)/p}\,du + \int_0^1 \frac{1}{(1-u)^\alpha} u^{(1+\varepsilon)/p-1}\,du \\
&\xrightarrow{\varepsilon \to 0^+} \int_0^1 \frac{1}{(1-u)^\alpha} (u^{-1/p} + u^{-1/p'})\,du \\
&= B\left(\frac{1}{p}, 1-\alpha\right) + B\left(\frac{1}{p'}, 1-\alpha\right) = k_0(p) = k_0(p').
\end{aligned}$$

In virtue of Lemma 2.2 and Theorem 2.3, one has

$$\begin{aligned}
&\left\{\int_0^\infty \left|\int_0^\infty \frac{f(x)}{|x-y|^\alpha (\max\{x,y\})^{1-\alpha}}\,dx\right|^p dy\right\}^{1/p} \\
&\le \left[B\left(\frac{1}{p}, 1-\alpha\right) + B\left(\frac{1}{p'}, 1-\alpha\right)\right] \left\{\int_0^\infty |f(x)|^p\,dx\right\}^{1/p},
\end{aligned}$$

and the linear Chandrasekhar integral equation

$$\varphi(y) = \psi(y) + \lambda \int_0^\infty \frac{\varphi(x)}{|x-y|^\alpha (\max\{x,y\})^{1-\alpha}}\,dx$$

has a unique solution in $\overline{\varphi} \in L^p(0,\infty)$ as long as

$$|\lambda| < \left\{\left[B\left(\frac{1}{p}, 1-\alpha\right) + B\left(\frac{1}{p'}, 1-\alpha\right)\right]\right\}^{-1}.$$

Taking $\alpha = 0$, one has

$$k_0(p) = B\left(\frac{1}{p}, 1\right) + B\left(\frac{1}{p'}, 1\right) = pp',$$

and hence,

$$\left\{\int_0^\infty \left|\int_0^\infty \frac{f(x)}{\max\{x,y\}}\,dx\right|^p dy\right\}^{1/p} \le pp'\left\{\int_0^\infty |f(x)|^p\,dx\right\}^{1/p},$$

and the linear Chandrasekhar integral equation

$$\varphi(y)=\psi(y)+\lambda\int_0^\infty \frac{\varphi(x)}{\max\{x,y\}}\,dx$$

has a unique solution in $\overline{\varphi}\in L^p(0,\infty)$ as long as $|\lambda|<1/(pp')$.

In particular when $p=p'=2$ and $\alpha=0$, we have

$$\left\{\int_0^\infty \left|\int_0^\infty \frac{f(x)}{\max\{x,y\}}\,dx\right|^2 dy\right\}^{1/2} \le 4\left\{\int_0^\infty |f(x)|^2\,dx\right\}^{1/2},$$

and the linear Chandrasekhar integral equation

$$\varphi(y)=\psi(y)+\lambda\int_0^\infty \frac{\varphi(x)}{\max\{x,y\}}\,dx$$

has a unique solution in $\overline{\varphi}\in L^2(0,\infty)$ as long as $|\lambda|<1/4$.

Example 3.3. Consider $k(x,y)=\frac{1}{(x+y)^{1-\alpha}(\max\{x,y\})^\alpha}$ with $0<\alpha<1$.

Putting $u=y/x$ and $u=x/y$, one observes that

$$\begin{aligned}
k_\varepsilon(p,x) &= \int_0^\infty \frac{1}{(x+y)^{1-\alpha}(\max\{x,y\})^\alpha}\left(\frac{x}{y}\right)^{(1+\varepsilon)/p} dy\\
&= \int_0^x \frac{1}{(x+y)^{1-\alpha}x^\alpha}\left(\frac{x}{y}\right)^{(1+\varepsilon)/p} dy + \int_x^\infty \frac{1}{(x+y)^{1-\alpha}y^\alpha}\left(\frac{x}{y}\right)^{(1+\varepsilon)/p} dy\\
&= \int_0^1 \frac{1}{(1+u)^{1-\alpha}}u^{-(1+\varepsilon)/p}\,du + \int_0^1 \frac{1}{(1+u)^{1-\alpha}}u^{(1+\varepsilon)/p-1}\,du\\
&\xrightarrow{\varepsilon\to 0^+} \int_0^1 (1+u)^{\alpha-1}(u^{-1/p}+u^{-1/p'})\,du\\
&= \int_0^1 \sum_{k=0}^\infty C_{\alpha-1}^k (u^{k-1/p}+u^{k-1/p'})\,du = \sum_{k=0}^\infty C_{\alpha-1}^k \int_0^1 (u^{k-1/p}+u^{k-1/p'})\,du\\
&= \sum_{k=0}^\infty C_{\alpha-1}^k \frac{(2k+1)pp'}{(pk+1)(p'k+1)} = k_0(p)=k_0(p').
\end{aligned}$$

In virtue of Lemma 2.2 and Theorem 2.3, one has

$$\begin{aligned}
&\left\{\int_0^\infty \left|\int_0^\infty \frac{f(x)}{(x+y)^{1-\alpha}(\max\{x,y\})^\alpha}\,dx\right|^p dy\right\}^{1/p}\\
&\le \sum_{k=0}^\infty C_{\alpha-1}^k \frac{(2k+1)pp'}{(pk+1)(p'k+1)}\left\{\int_0^\infty |f(x)|^p\,dx\right\}^{1/p},
\end{aligned}$$

and the linear Chandrasekhar integral equation

$$\varphi(y) = \psi(y) + \lambda \int_0^\infty \frac{\varphi(x)}{(x+y)^{1-\alpha}(\max\{x,y\})^\alpha}\, dx$$

has a unique solution in $\overline{\varphi} \in L^p(0,\infty)$ as long as

$$|\lambda| < \left\{ \sum_{k=0}^\infty C_{\alpha-1}^k \frac{(2k+1)pp'}{(pk+1)(p'k+1)} \right\}^{-1}.$$

Example 3.4. Consider $k(x,y) = \frac{1}{(x^{1-\alpha}+y^{1-\alpha})(\min\{x,y\})^\alpha}$ with $\alpha < \min\{1/p, 1/p'\}$.

Putting $u = y/x$ and $u = x/y$, one observes that

$$\begin{aligned}
k_\varepsilon(p,x) &= \int_0^\infty \frac{1}{(x^{1-\alpha}+y^{1-\alpha})(\min\{x,y\})^\alpha} \left(\frac{x}{y}\right)^{(1+\varepsilon)/p} dy \\
&= \int_0^x \frac{1}{(x^{1-\alpha}+y^{1-\alpha})y^\alpha} \left(\frac{x}{y}\right)^{(1+\varepsilon)/p} dy + \int_x^\infty \frac{1}{(x^{1-\alpha}+y^{1-\alpha})x^\alpha} \left(\frac{x}{y}\right)^{(1+\varepsilon)/p} dy \\
&= \int_0^1 \frac{1}{1+u^{1-\alpha}} u^{-\alpha-(1+\varepsilon)/p}\, du + \int_0^1 \frac{1}{1+u^{1-\alpha}} u^{-\alpha+(1+\varepsilon)/p-1}\, du \\
&\xrightarrow{\varepsilon \to 0^+} \int_0^1 \frac{1}{1+u^{1-\alpha}} (u^{-\alpha-1/p} + u^{-\alpha-1/p'})\, du \\
&= \int_0^1 \sum_{k=0}^\infty (-1)^k (u^{(1-\alpha)k-\alpha-1/p} + u^{(1-\alpha)k-\alpha-1/p'})\, du \\
&= \sum_{k=0}^\infty (-1)^k \int_0^1 (u^{(1-\alpha)k-\alpha-1/p} + u^{(1-\alpha)k-\alpha-1/p'})\, du \\
&= \sum_{k=0}^\infty (-1)^k \frac{[(2k+1)-2(k+1)\alpha]pp'}{[(1-\alpha)k+1/p-\alpha][(1-\alpha)k+1/p'-\alpha]} \\
&= k_0(p) = k_0(p').
\end{aligned}$$

In virtue of Lemma 2.2 and Theorem 2.3, one has

$$\begin{aligned}
&\left\{ \int_0^\infty \left| \int_0^\infty \frac{f(x)}{(x^{1-\alpha}+y^{1-\alpha})(\min\{x,y\})^\alpha}\, dx \right|^p dy \right\}^{1/p} \\
&\le \sum_{k=0}^\infty (-1)^k \frac{[(2k+1)-2(k+1)\alpha]pp'}{[(1-\alpha)k+1/p-\alpha][(1-\alpha)k+1/p'-\alpha]} \left\{ \int_0^\infty |f(x)|^p\, dx \right\}^{1/p},
\end{aligned}$$

and the linear Chandrasekhar integral equation

$$\varphi(y) = \psi(y) + \lambda \int_0^\infty \frac{\varphi(x)}{(x^{1-\alpha}+y^{1-\alpha})(\min\{x,y\})^\alpha}\, dx$$

has a unique solution in $\overline{\varphi} \in L^p(0,\infty)$ as long as

$$|\lambda| < \left\{ \sum_{k=0}^\infty (-1)^k \frac{[(2k+1)-2(k+1)\alpha]pp'}{[(1-\alpha)k+1/p-\alpha][(1-\alpha)k+1/p'-\alpha]} \right\}^{-1}.$$

Remark 3.5. Example 3.1 generalizes the classical Chandrasekhar kernel, replacing $(x+y)^{-1}$ by $[(x^\alpha+y^\alpha)^{1/\alpha}]^{-1}$. The norm of the integral operator

$$Kf(y)=\int_0^\infty \frac{f(x)}{(x^\alpha+y^\alpha)^{1/\alpha}}\,dx$$

defined on $L^p(0,\infty)$ equals to

$$\|K\|_{L^p\to L^p}=\frac{1}{\alpha}B\left(\frac{1}{p\alpha},\frac{1}{p'\alpha}\right).$$

4. Approximating solutions

In this section, we consider the approximating solution to (1.1), where we shall not stress that the integral kernel $k(x,y)$ is symmetric or the parameter λ is sufficiently small to ensure the existence of the resolvent $(I-\lambda K)^{-1}$. The main difficulty arises from the non-compactness of the operator K. An efficient way is to construct a compact approximation operator as in [14], where the converging properties of the related operator are concerned. We follow this approach and pay particular attention to the case where the characteristic and symbolic function are polynomials since it serves as an original model to the linearized Chandrasekhar equation.

4.1. Truncation operator

In $L^2[0,\infty)$, we consider the approximating solutions to the following integral equation

$$\varphi(x)=\psi(x)+\int_0^\infty k(x,y)\varphi(y)\,dy \tag{4.1}$$

shortly rewritten by

$$\varphi=\psi+K\varphi, \tag{4.2}$$

where

$$K\varphi(x):=\int_0^\infty k(x,y)\varphi(y)\,dy$$

with $k(x,y)$ being a measurable real-valued function on $[0,\infty)\times[0,\infty)$. Apparently, if the kernel $k(x,y)$ is symmetric, (4.1) is in the exact form of (1.1).

One introduces a bilateral truncation operator

$$K_T=M_{[0,T]}KM_{[0,T]},$$

where M_D is defined by

$$M_D\varphi(x)=\chi_D(x)\varphi(x)$$

for any $\varphi \in L^2[0,\infty)$ and χ_D is the characteristic function for a subset D. The integral equation associated to the truncation operator K_T follows that

$$\varphi = \psi + K_T\varphi. \tag{4.3}$$

If $(I-K)^{-1}$ and $(I-K_T)^{-1}$ exist for any $T>0$, the solutions to (4.2) and (4.3) are respectively denoted by

$$\varphi = (I-K)^{-1}\psi, \quad \varphi_T = (I-K_T)^{-1}\psi.$$

For $k(x,y) = k_1(x,y)k_2(x,y)$, setting

$$A := \left\{\sup_{x\geq 0}\int_0^\infty k_1^2(x,y)\,dy\right\}^{1/2},\ B := \left\{\sup_{y\geq 0}\int_0^\infty k_2^2(x,y)\,dx\right\}^{1/2},\ C := \sup_{x\geq 0, y\geq 0}|k_2(x,y)|,$$

one notices that [14] establishes the following

Proposition 4.1. *Suppose that $A<\infty$, $B<\infty$ and $C<\infty$, the integral operator K in* (4.2) *and K_T in* (4.3) *hold.*

(1) *K is bounded on $L^2[0,\infty)$.*

(2) *For any $T>0$, K_T is a compact operator on $L^2[0,\infty)$.*

(3) *If $(I-K_T)^{-1}$ exists and is uniformly bounded in T, then $(I-K)^{-1}$ exists and $\varphi_T \to \varphi$ as $T\to\infty$.*

4.2. Error estimates

One needs a general result from [1] to estimate the bound of the resolvent of the original operator and the truncated operator in terms of the integral kernel.

Lemma 4.2. *Let K and L be linear bounded operators on $L^2[0,\infty)$, $(I-K)^{-1}$ exist and $\Lambda := \|(I-K)^{-1}(L-K)L\| < 1$, then $(I-L)^{-1}$ exists, bounded on $L^2[0,\infty)$ with norm*

$$\|(I-L)^{-1}\| \leq \frac{1+\|(I-K)^{-1}\|\|L\|}{1-\Lambda}$$

and for any $f\in L^2[0,\infty)$, it follows that

$$\|(I-L)^{-1}f - (I-K)^{-1}f\| \leq \frac{\|(I-K)^{-1}\|\|Lf-Kf\| + \Lambda\|(I-K)^{-1}f\|}{1-\Lambda}.$$

Set

$$\Omega(T) := \left\{\int_T^\infty\int_0^T k_2^2(x,y)\,dydx\right\}^{1/2},$$

and for any $f \in L^2[0,\infty)$

$$\Omega_f(T) := \left\{ \int_T^\infty \int_0^T k_2^2(x,y) f^2(y)\, dydx \right\}^{1/2}, \quad \omega_f(T) := \left\{ \int_T^\infty f^2(y)\, dy \right\}^{1/2}.$$

The integral representation of the twist operator $(I-K)^{-1}(L-K)L$ in Lemma 4.2 attributes the bound to behavior of the integral kernel. In particular, we are concerned with the case that the symbolic function and initial data are functions of polynomial growth at infinity, where our conditions improve the order of convergence in [14].

Theorem 4.3. *For integral equation $\varphi = \psi + K\varphi$, suppose that $(I-K)^{-1}$ exists in $L^2[0,\infty)$, $A < \infty$, $B < \infty$ and $C < \infty$. If*

$$|k_2(x,y)|^2 \le \frac{1}{(1+x)^p(1+y)^q}, \quad |\psi(y)|^2 \le \frac{1}{(1+y)^r}, \quad x \ge 0,\ y \ge 0 \tag{4.4}$$

for some $p > 1$, $q > 1$ and $r > 1$, then for sufficiently large $T > 0$, it follows that

$$\begin{aligned} \|\varphi - \varphi_T\| \le\ & \frac{A\sqrt{(p-1)(q-1)}(1+T)^{(p-1)/2}\|(I-K)^{-1}\|}{\sqrt{(p-1)(q-1)}(1+T)^{(p-1)/2} - A^2C\|(I-K)^{-1}\|} \\ & \times \left[\frac{1 + AC\|\varphi\|}{\sqrt{(p-1)(q-1)}(1+T)^{(p-1)/2}} + \frac{B}{\sqrt{r-1}(1+T)^{(r-1)/2}} \right]. \end{aligned}$$

Proof. A direct computation yields

$$(K_T - K)\psi(x) = -\chi_{[T,\infty)}(x) \int_0^T k(x,y)\psi(y)\, dy - \int_T^\infty k(x,y)\psi(y)\, dy,$$

thus

$$\begin{aligned} & \|(K_T - K)\psi\| \\ & \le \left\{ \int_T^\infty \left| \int_0^T k(x,y)\psi(y)\, dy \right|^2 dx \right\}^{1/2} + \left\{ \int_0^\infty \left| \int_T^\infty k(x,y)\psi(y)\, dy \right|^2 dx \right\}^{1/2} \\ & =: I_1 + I_2. \end{aligned}$$

Noticing that

$$I_1^2 \le A^2 \int_T^\infty \int_0^T |k_2(x,y)|^2 |\psi(y)|^2\, dydx = A^2 \Omega_\psi^2(T)$$

and

$$\begin{aligned} I_2^2 & \le \int_0^\infty \left(\int_T^\infty |k_1(x,y)||k_2(x,y)||\psi(y)|\, dy \right)^2 dx \\ & \le A^2 \int_0^\infty \int_T^\infty |k_2(x,y)|^2 |\psi(y)|^2\, dydx \\ & \le A^2 B^2 \int_T^\infty |\psi(y)|^2\, dy = A^2 B^2 \omega_\psi^2(T), \end{aligned}$$

one immediately has

$$\|(K_T - K)\psi\| \leq A(\Omega_\psi(T) + B\omega_\psi(T)).$$

On the other hand,

$$\begin{aligned}(K_T - K)K_T\varphi &= (M_{[0,T]}KM_{[0,T]} - K)\chi_{[0,T]} \int_0^T k(x,y)\varphi(y)\,dy \\ &= (M_{[0,T]} - I)\int_0^T k(z,x)\left[\int_0^T k(x,y)\varphi(y)\,dy\right] dx,\end{aligned}$$

hence

$$\begin{aligned}&\|(I-K)^{-1}(K_T - K)K_T\varphi\| \\ &\leq \|(I-K)^{-1}\|\|(K_T-K)K_T\varphi\| \\ &\leq \|(I-K)^{-1}\|\left\{\int_T^\infty \left[\int_0^T |k(z,x)|\left(\int_0^T |k_1(x,y)||k_2(x,y)||\varphi(y)|\,dy\right) dx\right]^2 dz\right\}^{1/2} \\ &\leq \|(I-K)^{-1}\|\left[\int_T^\infty \left(\int_0^T |k_1(x,y)||k_2(x,y)|\,dx\right)^2 (AC\|\varphi\|)^2\,dz\right]^{1/2} \\ &\leq A^2C\|\varphi\|\|(I-K)^{-1}\|\Omega(T).\end{aligned}$$

As long as $\Omega(T) \ll 1$ for sufficiently large T, by taking $L = K_T$ and making use of Lemma 4.2, one reaches that

$$\|\varphi - \varphi_T\| \leq \frac{A\|(I-K)^{-1}\|}{1 - A^2C\|(I-K)^{-1}\|\Omega(T)}[AC\|\varphi\|\Omega(T) + \Omega_\psi(T) + B\omega_\psi(T)]. \tag{4.5}$$

The polynomial decay in (4.4) implies that

$$\begin{aligned}\Omega^2(T) &\leq \int_T^\infty\int_0^T \frac{dydx}{(1+x)^p(1+y)^q} \leq \frac{1}{(p-1)(q-1)(1+T)^{p-1}}, \\ \Omega_\psi^2(T) &\leq \int_T^\infty\int_0^T \frac{dydx}{(1+x)^p(1+y)^{q+r}} \leq \frac{1}{(p-1)(q+r-1)(1+T)^{p-1}}, \\ \omega_\psi^2(T) &\leq \int_T^\infty \frac{dy}{(1+y)^r} \leq \frac{1}{(r-1)(1+T)^{r-1}},\end{aligned}$$

and a substitution the above bounds into (4.5) completes the proof. □

5. Conclusion

The Fredholm integral equations generalize the linearized Chandrasekhar integral equation to associate with a much wider class of kernels, which attain their sharp bounds of norms. The real symmetric integral kernel $k(x,y)$ ensures that the associated operator K

is symmetric and $\|K\|_{L^p \to L^p} = \|K\|_{L^{p'} \to L^{p'}}$. A couple of examples well fulfill the condition that $k_0(p) = k_0(p')$ with one generalizing the classical Chandrasekhar kernel. In L^2 sense, the approximation solution is constructed by the associated truncation operator and the error bound is shown to have polynomial decay as long as the symbolic function and initial data are of polynomial decay at infinity.

Acknowledgments

Feng is partially supported by the National Natural Science Foundation of China (Grant No. 11501203 and No. 11426109), the China Postdoctoral Science Foundation (Grant No. 2016M600278) as well as the State Scholarship Foundation of China (Grant No. 201806745028). Chang is partially supported by an NSF grant DMS-1408839 and a McDevitt Endowment Fund at Georgetown University.

References

[1] P. M. Anselone, *Collectively Compact Operator Approximation Theory and Applications to Integral Equations*, Prentice-Hall, Englewood Cliffs, N.J., 1971.

[2] I. K. Argyros, *Quadratic equations and applications to Chandrasekhar's and related equations*, Bull. Austral. Math. Soc. **32** (1985), no. 2, 275–292.

[3] ———, *On a class of quadratic integral equations with perturbation*, Funct. Approx. Comment. Math. **20** (1992), 51–63.

[4] J. Banaś, M. Lecko and W. G. El-Sayed, *Existence theorems for some quadratic integral equations*, J. Math. Anal. Appl. **222** (1998), no. 1, 276–285.

[5] J. Banaś and B. Rzepka, *Monotonic solutions of a quadratic integral equation of fractional order*, J. Math. Anal. Appl. **332** (2007), no. 2, 1371–1379.

[6] P. B. Bosma and W. A. de Rooij, *Efficient methods to calculate Chandrasekhar's H-functions*, Astron. Astrophys. **126** (1983), 283–292.

[7] I. W. Busbridge, *The Mathematics of Radiative Transfer*, Cambridge University Press, Cambridge, 1960.

[8] ———, *On solutions of Chandrasekhar's integral equation*, Trans. Amer. Math. Soc. **105** (1962), 112–117.

[9] J. Caballero, A. B. Mingarelli and K. Sadarangani, *Existence of solutions of an integral equation of Chandrasekhar type in the theory of radiative transfer*, Electron. J. Differential Equations **2006** (2006), no. 57, 11 pp.

[10] S. Chandrasekhar, *Radiative Transfer*, Oxford University Press, London, 1950.

[11] D.-C. Chang and S.-Y. Feng, *On integral equations of Chandrasekhar type*, J. Nonlinear Convex Anal. **19** (2018), no. 3, 525–541.

[12] Q. Chen and B. Yang, *A survey on the study of Hilbert-type inequalities*, J. Inequal Appl. **2015**, 2015:302, 29 pp.

[13] J. E. Dennis and R. B. Schnabel, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Prentice Hall Series in Computational Mathematics, Prentice Hall, Englewood Cliffs, NJ, 1983.

[14] J. Di and Z. Shi, *Approximate solution of the integral equation on the half line*, J. Zhe Jiang Univ. Tech. **37** (2009), 326–331.

[15] A. M. A. El-Sayed and H. H. G. Hashem, *Integrable solution for quadratic Hammerstein and quadratic Urysohn functional integral equations*, Comment. Math. **48** (2008), no. 2, 199–207.

[16] ———, *A coupled system of fractional order integral equations in reflexive Banach spaces*, Comment. Math. **52** (2012), no. 1, 21–28.

[17] A. M. A. El-Sayed, H. H. G. Hashem and E. A. A. Ziada, *Picard and Adomian decomposition methods for a coupled system of quadratic integral equations of fractional order*, J. Nonlinear Anal. Optim. **3** (2012), no. 2, 171–183.

[18] J. A. Ezquerro and M. A. Hernández, *On the application of a fourth-order two-point method to Chandrasekhar's integral equation*, Aequationes Math. **62** (2001), no. 1-2, 39–47.

[19] H. H. G. Hashem and A. M. A. El-Sayed, *Stabilization of coupled systems of quadratic integral equations of Chandrasekhar type*, Math. Nachr. **290** (2017), no. 2-3, 341–348.

[20] J. Juang, K.-Y. Lin and W.-W. Lin, *Spectral analysis of some iterations in the Chandrasekhar's H-function*, Numer. Funct. Anal. Optim. **24** (2003), no. 5-6, 575–586.

[21] P. K. Kythe and P. Puri, *Computational Methods for Linear Integral Equations*, Birkhäuser Boston, Boston, MA, 2002.

[22] R. W. Leggett, *A new approach to the H-equation of Chandrasekhar*, SIAM J. Math. Anal. **7** (1976), no. 4, 542–550.

[23] J. M. Ortega and W. C. Rheinboldt, *On discretization and differentiation of operators with application to Newton's method*, SIAM J. Numer. Anal. **3** (1966), no. 1, 143–156.

[24] H. A. H. Salem, *On the quadratic integral equations and their applications*, Comput. Math. Appl. **62** (2011), no. 8, 2931–2943.

[25] I. Schur, *Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen*, J. Reine Angew. Math. **140** (1911), 1–28.

[26] T. Tanaka, *Integration of Chandrasekhar's integral equation*, Journal of Quantitative Spectroscopy and Radiative Transfer **76** (2003), no. 2, 121–144.

[27] F. G. Tricomi, *Integral Equations*, Pure and Applied Mathematics **V**, Interscience Publishers, New York, 1957.

[28] M. Y. Waziri, W. J. Leong, M. A. Hassan and M. Monsi, *A low memory solver for integral equations of Chandrasekhar type in the radiative transfer problems*, Math. Probl. Eng. **2011** (2011), Art. ID 467017, 12 pp.

[29] H. Weyl, *Singulare integral Gleichungen mit besonderer Berücksichtigung des Fourierschen integral theorems*, Inaugeral-Dissertation, Gottingen, 1908.

[30] B. Yang, *On the norm of an integral operator and applications*, J. Math. Anal. Appl. **321** (2006), no. 1, 182–192.

[31] ______, *On the Norm of Operator and Hilbert-type Inequalities*, Science Press, Beijing, 2009.

[32] V. A. Zorich, *Mathematical Analysis II*, Universitext, Springer-Verlag, Berlin, 2004.

Sheng-Ya Feng
School of Mathematical Sciences and Key Lab of Mathematics for Nonlinear Science, Fudan University, Shanghai 200433, P. R. China
and
Department of Mathematics, East China University of Science and Technology, Shanghai 200237, P. R. China
E-mail address: `syfeng@fudan.edu.cn`

Der-Chen Chang
Department of Mathematics and Statistics, Georgetown University, Washington D.C. 20057, USA
and
Graduate Institute of Business Administration, College of Management, Fu Jen Catholic University, Taipei 242, Taiwan
E-mail address: `chang@georgetown.edu`