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Estimations for the Action Functional of the N-body Problem by Recursive Binary Decompositions

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In memory of Professor Hwai-Chiuan Wang (1938–2017)

Abstract. In this paper, we improve estimations in a previous work [3] (Arch. Rat. Mech. Anal. 2003) about estimations for the action functional of the N-body problem. Our method is based on repeated applications of binary decompositions for the N-body system, and is applicable to more general particle systems.

1. Introduction

The Newtonian N-body problem concerns the motion of N (≥ 2) mass points $m_1, m_2, \ldots, m_N$ moving in $\mathbb{R}^d$ in accordance with Newton's law of gravitation:

$$m_k \ddot{x}_k = \frac{\partial}{\partial x_k} U(x), \quad k = 1, \ldots, N, \tag{1.1}$$

where $x_k \in \mathbb{R}^d$ is the position of m_k and

$$U(x) = U(x_1, \ldots, x_N) = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|x_i - x_j|}$$

is the potential energy. The kinetic energy is given by

$$K(\dot{x}) = \sum_{k=1}^{N} \frac{1}{2} m_k |\dot{x}_k|^2.$$

Equations (1.1) are the Euler-Lagrange equations for the action functional $\mathcal{A}$ defined by

$$\mathcal{A}(x) = \int_0^T K(\dot{x}) + U(x)\, dt,$$

where T is a positive constant. In this article we will restrict $\mathcal{A}$ to the Sobolev space $H^1(\mathbb{R}/T\mathbb{Z}, (\mathbb{R}^d)^N)$.

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The discovery of the figure-8 orbit for the three-body problem by Chenciner-Montgomery [11] has attracted considerable attention. It inspires many follow-up works based on variational techniques. Many symmetric solutions of the N-body problem with equal masses were found by proper selection of the group action and careful estimation of the action functional $\mathcal{A}$. Analysis of the action functional near singularities has also cause growing attention to variational approaches for problems with fixed centers. We refer readers to [19, 23, 24] for a comprehensive introductions and bibliography.

In a previous work [3], we define binary decompositions for the N-body problem and use it to decompose U, K, and $\mathcal{A}$ into pairs. The purpose is to provide a flexible way to estimate $\mathcal{A}$, as one may put extra weights to colliding pairs to "penalize" the action functional. The decomposition makes no presumption on location of mass centers, and is therefore applicable to more general particle systems without conservation of linear momentum, such as systems with fixed centers. Such kind of decomposition in conjunction with Gordon's formula [17] (or its variants [3, Proposition 1]) allows us to provide effective lower bound estimates for the action value of collision paths in several spaces of symmetric loops.

The purpose of this paper is to improve estimates in [3]. The main idea is to treat geometric centers of some subsystems as real masses. Even though these "masses" are not there, certain parts of the action functional can be regarded as the action due to motions of such fictitious bodies. When masses are all equal, collisions of the original system has a one-to-one correspondence with collisions of geometric centers of subsystems with $N-1$ masses. We then recursively rewrite part of the action functional in terms of these geometric centers, and geometric centers of those geometric centers, and so on. Finer estimates were obtained by this iterative procedure.

As preparation, in Section 2 we begin with descriptions of symmetry constraints and some simple observations about geometric centers of subsystems. In Section 3 we quickly review the concept of binary decomposition. Details of the above stated recursive process are given in Section 4. We will use the figure-8 orbit [11] as example, and compare our improved estimates with previous estimates.

2. Symmetries and geometric centers of subsystems

Let $H = H^1(\mathbb{R}/T\mathbb{Z}, (\mathbb{R}^d)^N)$ be the space of T-periodic H^1-loops in $\mathbb{R}^d$. Given a finite group G, an orthogonal representation ρ of G on $\mathbb{R}^d$, an action τ of G on the circle $\mathbb{S}_T = \mathbb{R}/T\mathbb{Z}$ by isometries, and an action σ of G on symbols $\{1, 2, \ldots, N\}$. Consider the following action of G on H (see [9]) by orthogonal transformations: for any $g \in G$,

$$(g \cdot x)_k(t) = \rho(g) x_{\sigma(g^{-1})(k)}(\tau(g^{-1})t), \quad k = 1, \ldots, N.$$

The space of G-invariant loops

$$H^G = \{x \in H : g \cdot x = x \text{ for all } g \in G\}$$

is clearly a closed subspace of H. If $[0, T/m]$ is a fundamental domain of the G action, then the action value of any loop x in H^G over time interval $[0, T]$ is equal to m times its action value over $[0, T/m]$.

It is well-known that the action functional $\mathcal{A}$ is weakly sequentially lower semi-continuous, and if it is G-invariant, then regular critical points of $\mathcal{A}$ restricted to H^G are also critical points of $\mathcal{A}$ on the larger space H (see [21]). In practice, the requirement of being G-invariant often leads to the restriction of equal masses.

Many recent works along this direction have devoted their efforts into proper selection of the group action and careful estimation of the action functional $\mathcal{A}$. Estimation of the action functional $\mathcal{A}$ is meant to distinguish minimizing loops in H^G from collision loops in there. Namely, we hope the following inequality holds:

$$\inf_{x \in H^G} \mathcal{A}(x) < \inf_{\substack{x \in H^G \\ x \text{ has collision}}} \mathcal{A}(x). \tag{2.1}$$

As long as $\mathcal{A}$ is G-invariant and coercive (i.e., $\mathcal{A}(x) \to \infty$ as $\|x\| \to \infty$), a standard argument of variational calculus shows that the inequality guarantees existence of minimizers in H^G, and these minimizers are classical solutions of (1.1). This inequality are usually proven by local deformation arguments (see for instance, [4, 14, 16, 22, 25, 26]) or global estimates (see for instance, [5–7, 11, 27]). Here we focus on the later case.

Let x_k^* be the geometric center of the subsystem with x_k removed; that is,

$$x_k^* = \frac{1}{N-1} \sum_{i \neq k} x_i.$$

Let $x^* = (x_1^*, \ldots, x_N^*)$. Below is a simple observation which will be useful later.

Theorem 2.1. *If $x \in H^G$, then so is x^*. Moreover, $x_j(t) = x_k(t)$ if and only if $x_j^*(t) = x_k^*(t)$.*

Proof. Given $x \in H^G$ and $g \in G$. For any $k \in \{1, 2, \ldots, N\}$,

$$\begin{aligned}
(g \cdot x^*)_k(t) &= \rho(g) x^*_{\sigma(g^{-1})(k)}(\tau(g^{-1})t) \\
&= \frac{1}{N-1} \sum_{i \neq \sigma(g^{-1})(k)} \rho(g) x_i(\tau(g^{-1})t) \\
&= \frac{1}{N-1} \sum_{i:\, \sigma(g)(i) \neq k} \rho(g) x_{\sigma(g^{-1})\sigma(g)(i)}(\tau(g^{-1})t)
\end{aligned}$$

$$= \frac{1}{N-1}\sum_{j\neq k}\rho(g)x_{\sigma(g^{-1})(j)}(\tau(g^{-1})t)$$
$$= \frac{1}{N-1}\sum_{j\neq k}x_j(t) = x_k^*(t).$$

Thus x^* is also in H^G.

The other statement in the theorem is quite obvious:

$$x_j^*(t) - x_k^*(t) = \frac{1}{N-1}\sum_{i\neq j}x_i(t) - \frac{1}{N-1}\sum_{i\neq k}x_i(t) = \frac{1}{N-1}(x_k(t)-x_j(t)). \qquad \square$$

3. Binary decompositions of the N-body system

In [3] we define binary decompositions for system of masses $m_1,\ldots,m_N$ ($N\geq 2$), and use it to provide lower bounds for the action functional over collision paths. We recall the definition here. To make the idea as simple as possible, we omit gravitational constants appeared in [3] and will put emphasis on planar systems with equal masses.

A *binary decomposition* of the system of N masses $m_1,m_2,\ldots,m_N$ is a selection of two nonnegative $N\times N$ matrices $\mathbf{M}=[m_{ij}]$ and $\boldsymbol{\Lambda}=[\lambda_{ij}]$ satisfying

$$\begin{aligned}
m_{ii} = \lambda_{ii} = 0 \qquad & \text{for any } i,\\
m_{ij} > 0 \qquad & \text{for any } i\neq j,\\
0\leq \lambda_{ij} = \lambda_{ji} \leq 1 \qquad & \text{for any } i\neq j, \text{ and}\\
\sum_{j=1}^N m_{ij} = m_i \qquad & \text{for any } i.
\end{aligned}$$

As pointed out earlier, we do not assume the mass center of the loop is at the origin, so that same ideas are applicable to systems involving fixed centers. The idea of the decomposition is to treat $\{m_{ij}\}_{i\neq j}$ as a system of distinct $N(N-1)$ elementary particles, where m_{ij} and m_{ji} constitute a pair of particle-antiparticle which do not interact with any other particle. For fixed i, the subsystem $\{m_{ij} : j\neq i\}$ were binded to the same position x_i. Each binary decomposition corresponds to a decomposition of the action functional into Keplerian action functionals.

Consider the simplest case: $N=2$. The action functional $\mathcal{A}$ defined on $H^1(\mathbb{R}/T\mathbb{Z},\mathbb{C}^2)$ takes the form

$$\begin{aligned}
\mathcal{A}(x) &= \int_0^T \frac{1}{2}(m_1|\dot{x}_1|^2 + m_2|\dot{x}_2|^2) + \frac{m_1m_2}{|x_1-x_2|}\,dt\\
&= \mathcal{A}^0(x) + \mathcal{A}^1(x),
\end{aligned}$$

where

$$\mathcal{A}^0(x) = \int_0^T \frac{m_1 m_2}{2(m_1+m_2)}|\dot{x}_1 - \dot{x}_2|^2 + \frac{m_1 m_2}{|x_1 - x_2|}\,dt,$$
$$\mathcal{A}^1(x) = \int_0^T \frac{m_1+m_2}{2}|\dot{\overline{x}}_{12}|^2\,dt.$$

Here $\overline{x}_{12}$ is the mass center. Granting that linear momentum is an integral of motion, it is customary to drop the integral $\mathcal{A}^1$ and consider critical points of $\mathcal{A}^0$ (called the *Keplerian action functional*) over loops $\mathbf{r} = x_2 - x_1$ in $H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C})$.

Now, consider $N \geq 3$. Define $U_{ii} = K_{ii} = 0$ for each i, and

$$U_{ij}(x) = \frac{m_i m_j}{|x_i - x_j|}, \quad K_{ij}(\dot{x}) = \frac{1}{2}(m_{ij}|\dot{x}_i|^2 + m_{ji}|\dot{x}_j|^2), \quad \mathcal{A}_{ij}(x) = \int_0^T K_{ij}(\dot{x}) + U_{ij}(x)\,dt.$$

Then we have

$$U(x) = \sum_{\substack{(i,j)\\ i<j}} U_{ij}(x), \quad K(\dot{x}) = \sum_{\substack{(i,j)\\ i<j}} K_{ij}(\dot{x}), \quad \mathcal{A}(x) = \sum_{\substack{(i,j)\\ i<j}} \mathcal{A}_{ij}(x).$$

Let $\overline{x}_{ij} = \frac{1}{m_{ij}+m_{ji}}(m_{ij}x_i + m_{ji}x_j)$ be the mass center of the binary $\{m_{ij}, m_{ji}\}$. Then each K_{ij} can be further split into two parts:

$$K_{ij}(\dot{x}) = K^0_{ij}(\dot{x}) + K^1_{ij}(\dot{x}),$$
$$K^0_{ij}(\dot{x}) = \frac{m_{ij}m_{ji}}{2(m_{ij}+m_{ji})}|\dot{x}_i - \dot{x}_j|^2, \quad K^1_{ij}(\dot{x}) = \frac{m_{ij}+m_{ji}}{2}|\dot{\overline{x}}_{ij}|^2. \tag{3.1}$$

Likewise, each U_{ij} can be split into two parts:

$$U_{ij}(x) = U^0_{ij}(x) + U^1_{ij}(x),$$
$$U^0_{ij}(x) = \frac{\lambda_{ij} m_i m_j}{|x_i - x_j|}, \quad U^1_{ij}(x) = \frac{(1-\lambda_{ij}) m_i m_j}{|x_i - x_j|}. \tag{3.2}$$

Accordingly, each $\mathcal{A}_{ij}$ is split into two parts:

$$\mathcal{A}_{ij}(x) = \mathcal{A}^0_{ij}(x) + \mathcal{A}^1_{ij}(x),$$
$$\mathcal{A}^0_{ij}(x) = \int_0^T K^0_{ij}(\dot{x}) + U^0_{ij}(x)\,dt, \quad \mathcal{A}^1_{ij}(x) = \int_0^T K^1_{ij}(\dot{x}) + U^1_{ij}(x)\,dt. \tag{3.3}$$

Let

$$\mathcal{A}^0(x) = \sum_{\substack{(i,j)\\ i<j}} \mathcal{A}^0_{ij}, \quad \mathcal{A}^1(x) = \sum_{\substack{(i,j)\\ i<j}} \mathcal{A}^1_{ij}. \tag{3.4}$$

Then

$$\mathcal{A}(x) = \mathcal{A}^0(x) + \mathcal{A}^1(x).$$

The purpose of such decomposition is to find good lower bound estimates for the right-hand side of (2.1), by estimating each $\mathcal{A}^0_{ij}$, $\mathcal{A}^1_{ij}$. In [3] we provided lower bound estimates for both $\mathcal{A}^0$ and $\mathcal{A}^1$. In the next section we will show how estimates in [3] can be substantially improved.

With the presence of symmetry, colliding pairs often have more "contribution" to the action value, so the freedom of choosing m_{ij} and λ_{ij} allows us put extra weights to badly behaved (colliding) pairs, and make the lower bound estimate for $\mathcal{A}$ over collision paths sharper. In principle, this will yield good estimates when action values of colliding pairs are substantially higher than non-colliding pairs. In practice, it would be quite complicated if one has to optimize the selection of m_{ij} and λ_{ij} for each possible type of collision. Among applications in [3], the following convenient choice, called *standard binary decomposition*, were adopted:

$$\lambda_{ij} = \lambda, \quad m_{ij} = \frac{m_i}{N-1} \quad \text{for any } i \neq j.$$

Comparing with the relatively simple decomposition when confining to loops with center of mass at origin: (M is total mass)

$$\mathcal{A}(x) = \frac{1}{M}\sum_{i<j} m_i m_j \int_0^T \frac{1}{2}|\dot{x}_i - \dot{x}_j|^2 + \frac{M}{|x_i - x_j|}\, dt, \tag{3.5}$$

the decomposition (3.1)–(3.4) is complicated. In applications to well-known examples, such as the figure-8 orbit for the three-body problem, estimates obtained in [3] are not as sharp as the one which uses (3.5). In the next section, with improvement of estimates in [3], we will see that estimates using our binary decomposition are actually as sharp.

4. Improvement of estimates in [3]

Let us begin with a summary of the estimates in [3] for the case of standard binary decomposition. Although many arguments in [3] can be extended to non-planar loops, we only consider planar loops and identify $\mathbb{R}^2$ by $\mathbb{C}$.

We say a loop $x \in H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^N)$ has *d-fold rotation symmetry*, $d \geq 2$, if

$$x_k(t) = e^{2\pi i/d} x_k\left(t + \frac{T}{d}\right) \quad \text{for any } t \in \mathbb{R},\ k \in \{1, \dots, N\}.$$

We say a loop $x \in H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^N)$ has *mirror symmetry* if there is a straight line L in $\mathbb{C}$ such that $x(t)$ and $x(t+T/2)$ are symmetric with respect to L for any t. Many recent discoveries of simple or multiple choreographic solutions belong to either categories.

Given $x \in H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{C}^N)$. Following notations in [3], "$i \bowtie j$" means $x_i(t) = x_j(t)$ for some t, and the set I_x of *collision indexes* of x is given by

$$I_x = \{(i,j) : i < j \text{ and } i \bowtie j\}.$$

Theorem 4.1. [3] *Consider a systems of N equal masses $m_1 = \cdots = m_N = 1$ with positions $x_1, \ldots, x_N \in \mathbb{C}$. Let $x = (x_1, \ldots, x_N)$ and I_x be collision indexes of x. Let $\mathcal{A}^0$, $\mathcal{A}^1$ be as in (3.1)–(3.4), and $\lambda_{ij} = \lambda$, $m_{ij} = 1/(N-1)$ for any $i \neq j$. If x has d-fold rotation symmetry, then*

$$\mathcal{A}^0(x) \geq 3 \left(\frac{\lambda^2 \pi^2}{4(N-1)} \right)^{1/3} \left((d^{2/3} - 1)|I_x| + \binom{N}{2} \right) T^{1/3}, \tag{4.1}$$

$$\mathcal{A}^1(x) \geq 3 \left(\frac{(1-\lambda)^2 \pi^2}{16(N-1)} \right)^{1/3} \left((d^{2/3} - 1)|I_x| + \binom{N}{2} \right) T^{1/3}. \tag{4.2}$$

If x has mirror symmetry, then

$$\mathcal{A}^0(x) \geq 3 \left(\frac{\lambda^2 \pi^2}{4(N-1)} \right)^{1/3} \left((2^{2/3} - 1)|I_x| + \binom{N}{2} \right) T^{1/3}, \tag{4.3}$$

$$\mathcal{A}^1(x) \geq 3 \left(\frac{(1-\lambda)^2 \pi^2}{16(N-1)} \right)^{1/3} \left((2^{2/3} - 1)|I_x| + \binom{N}{2} \right) T^{1/3}. \tag{4.4}$$

Combination of (4.1), (4.2) provides a lower bound for the right-hand side of (2.1) if all loops in H^G have d-fold symmetry, and combination of (4.3), (4.4) offers a lower bound if loops in H^G must have mirror symmetry. This lower bound has a parameter λ. Maximize the bound over $\lambda \in [0,1]$, we conclude [3, Theorems 1 & 2]

$$\mathcal{A}(x) \geq 3 \left(\frac{5\pi^2}{16(N-1)} \right)^{1/3} \left((d^{2/3} - 1)|I_x| + \binom{N}{2} \right) T^{1/3}$$

when x has d-fold rotation symmetry, and

$$\mathcal{A}(x) \geq 3 \left(\frac{5\pi^2}{16(N-1)} \right)^{1/3} \left((2^{2/3} - 1)|I_x| + \binom{N}{2} \right) T^{1/3}$$

when x has mirror symmetry.

For the special case $N = 3$, we have

$$\mathcal{A}(x) \geq 3 \left(\frac{5\pi^2}{32} \right)^{1/3} \left((2^{2/3} - 1)|I_x| + 3 \right) T^{1/3}. \tag{4.5}$$

Now we show how this lower bound can be improved:

Theorem 4.2. *Let $N = 3$. Under the assumptions of Theorem 4.1, if x has d-fold rotation symmetry, then*

$$\mathcal{A}(x) \geq 2 \left(\frac{3\pi}{4} \right)^{2/3} \left((d^{2/3} - 1)|I_x| + 3 \right) T^{1/3}.$$

If x has mirror symmetry, then

$$\mathcal{A}(x) \geq 2 \left(\frac{3\pi}{4} \right)^{2/3} \left((2^{2/3} - 1)|I_x| + 3 \right) T^{1/3}. \tag{4.6}$$

Proof. We only prove the case with d-fold rotation symmetry, as the proof for the case with mirror symmetry is similar. The idea is to apply the decomposition $\mathcal{A} = \mathcal{A}^0 + \mathcal{A}^1$ to $\mathcal{A}^1$, writing $\mathcal{A}^1 = \mathcal{A}^{10} + \mathcal{A}^{11}$, then decompose $\mathcal{A}^{11}$ by writing $\mathcal{A}^{11} = \mathcal{A}^{110} + \mathcal{A}^{111}$, then $\mathcal{A}^{111} = \mathcal{A}^{1110} + \mathcal{A}^{1111}$, and so on.

The action functional is

$$\mathcal{A}(x) = \int_0^T \frac{1}{2}\sum_{k=1}^{3} |\dot{x}_k|^2 + \sum_{\substack{(i,j)\\ i<j}} \frac{1}{|x_i - x_j|}\, dt. \tag{4.7}$$

The first part of the decomposition $\mathcal{A} = \mathcal{A}^0 + \mathcal{A}^1$ is

$$\mathcal{A}^0(x) = \sum_{\substack{(i,j)\\ i<j}} \int_0^T \frac{1}{8}|\dot{x}_i - \dot{x}_j|^2 + \frac{\lambda}{|x_i - x_j|}\, dt. \tag{4.8}$$

Use the fact that $\overline{x}_{12} = x_3^*$, $\overline{x}_{23} = x_1^*$, $\overline{x}_{13} = x_2^*$, and $|x_i - x_j| = 2|x_i^* - x_j^*|$ for every i, j, we can write the second part of $\mathcal{A}$ as

$$\begin{aligned}\mathcal{A}^1(x) &= \sum_{\substack{(i,j)\\ i<j}} \int_0^T \frac{1}{2}|\dot{\overline{x}}_{ij}|^2 + \frac{1-\lambda}{|x_i - x_j|}\, dt \\ &= \int_0^T \frac{1}{2}\sum_{k=1}^{3} |\dot{x}_k^*|^2 + \frac{1-\lambda}{2}\sum_{\substack{(i,j)\\ i<j}} \frac{1}{|x_i^* - x_j^*|}\, dt.\end{aligned} \tag{4.9}$$

This is similar to the formulation (4.7). Imitate the decomposition (4.8), (4.9) for $\mathcal{A}$, consider the system $\{x_1^*, x_2^*, x_3^*\}$ and let x_k^{**} be the geometric center of subsystem with x_k^* removed, then we can write $\mathcal{A}^1 = \mathcal{A}^{10} + \mathcal{A}^{11}$, where

$$\begin{aligned}\mathcal{A}^{10}(x) &= \sum_{\substack{(i,j)\\ i<j}} \int_0^T \frac{1}{8}|\dot{x}_i^* - \dot{x}_j^*|^2 + \frac{1-\lambda}{2}\frac{\lambda}{|x_i^* - x_j^*|}\, dt, \\ \mathcal{A}^{11}(x) &= \int_0^T \frac{1}{2}\sum_{k=1}^{3} |\dot{x}_k^{**}|^2 + \left(\frac{1-\lambda}{2}\right)^2 \sum_{\substack{(i,j)\\ i<j}} \frac{1}{|x_i^{**} - x_j^{**}|}\, dt.\end{aligned}$$

Repeat this process, we obtain a sequence $\mathcal{A}^{(n)0} = \mathcal{A}^{1\cdots 10}$ with n 1's in the superscript, and a sequence $\mathcal{A}^{(n+1)} = \mathcal{A}^{1\cdots 1}$ with $n+1$ 1's in the superscript. Denote the geometric centers obtained from $x^* = (x_1^*, x_2^*, x_3^*)$ by $x^{(2*)} = (x_1^{(2*)}, x_2^{(2*)}, x_3^{(2*)})$, those obtained from $x^{(2*)}$ by $x^{(3*)}$, and so on. Then

$$\mathcal{A}^{(n)}(x) = \mathcal{A}^{(n)0}(x) + \mathcal{A}^{(n+1)}(x),$$

$$\mathcal{A}^{(n)0}(x) = \sum_{\substack{(i,j)\\ i<j}} \int_0^T \frac{1}{8} \left| \dot{x}_i^{(n*)} - \dot{x}_j^{(n*)} \right|^2 + \left(\frac{1-\lambda}{2} \right)^n \frac{\lambda}{\left| x_i^{(n*)} - x_j^{(n*)} \right|} \, dt,$$

$$\mathcal{A}^{(n+1)}(x) = \int_0^T \frac{1}{2} \sum_{k=1}^{3} \left| \dot{x}_k^{((n+1)*)} \right|^2 + \left(\frac{1-\lambda}{2} \right)^{n+1} \sum_{\substack{(i,j)\\ i<j}} \frac{1}{\left| x_i^{((n+1)*)} - x_j^{((n+1)*)} \right|} \, dt.$$

By Theorem 2.1, collision indexes $I_{x^{(n*)}}$ of $x^{(n*)}$ is the same as I_x for every $n \in \mathbb{N}$. Therefore, we can apply (4.1) to each $\mathcal{A}^{(n)0}(x)$:

$$\begin{aligned}
\mathcal{A}(x) &\geq \mathcal{A}^0(x) + \sum_{n=1}^{\infty} \mathcal{A}^{(n)0}(x) \\
&\geq 3 \left(\frac{\lambda^2 \pi^2}{8} \right)^{1/3} \left((d^{2/3} - 1)|I_x| + 3 \right) T^{1/3} \sum_{n=0}^{\infty} \left(\frac{1-\lambda}{2} \right)^{2n/3} \\
&= \frac{3}{2} \pi^{2/3} \left((d^{2/3} - 1)|I_x| + 3 \right) T^{1/3} \frac{(2\lambda)^{2/3}}{2^{2/3} - (1-\lambda)^{2/3}}.
\end{aligned}$$

This is valid for every $\lambda \in [0,1]$. The theorem follows by taking $\lambda = 3/4$. $\square$

We will show the application of this result to the famous figure-8 orbit with three equal masses. Estimates for several other examples in [3] can be also improved by following the same idea, but we shall skip discussions here.

Example 4.3. The figure-8 orbit for the three-body problem with equal masses was first numerically discovered by Moore [20]. In addition to Chenciner-Montgomery's first proof [11], it has received several proofs for its existence [2, 3, 8, 18, 27]. See [1, 10, 12, 15] for some interesting advances and open questions.

The figure-8 orbit is the minimizing solution on the space of G-invariant loops, where G is the group of orthogonal transformations on $H = H^1(\mathbb{R}/\mathbb{Z}, \mathbb{C}^3)$ generated by g_1, g_2:

$$\begin{aligned}
g_1 \cdot (x_1, x_2, x_3)(t) &= -(\overline{x}_3, \overline{x}_1, \overline{x}_2)(t + T/6), \\
g_2 \cdot (x_1, x_2, x_3)(t) &= -(x_2, x_1, x_3)(-t).
\end{aligned}$$

It is easy to check that G is isomorphic to the Dihedral group of order 12. The g_1^2-invariance implies $x_1(t) = x_2(t + T/3) = x_3(t + 2T/3)$ for all t. Solutions with this feature are known as *simple choreographic solutions*. In particular, this implies $|I_x|$ is either 0 or 3. The g_1^3-invariance implies loops in H^G have mirror symmetry.

The estimate (4.5) shows that, for any collision path $x \in H^G$,

$$\mathcal{A}(x) \geq \frac{9}{2} (5\pi^2)^{1/3} T^{1/3} \approx 16.5058 T^{1/3}.$$

The sharper estimate in [27] based on (3.5) reads

$$\mathcal{A}(x) \geq \frac{3}{2}(12\pi)^{2/3}T^{1/3} \approx 16.8647T^{1/3}.$$

It is exactly the same as the estimate (4.6) in Theorem 4.2. In any case, the estimate is well above the numerical value ($\approx 13.2078T^{1/3}$) of the action of T-periodic figure-8 orbit.

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