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Critical Points Theorems via the Generalized Ekeland Variational Principle and its Application to Equations of $p(x)$-Laplace Type in $\mathbb{R}^N$

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Abstract. In this paper, we investigate abstract critical point theorems for continuously Gâteaux differentiable functionals satisfying the Cerami condition via the generalized Ekeland variational principle developed by C.-K. Zhong. As applications of our results, under certain assumptions, we show the existence of at least one or two weak solutions for nonlinear elliptic equations with variable exponents

$$-\operatorname{div}(\varphi(x,\nabla u)) + V(x)|u|^{p(x)-2}u = \lambda f(x,u) \quad \text{in } \mathbb{R}^N,$$

where the function $\varphi(x,v)$ is of type $|v|^{p(x)-2}v$ with a continuous function $p\colon \mathbb{R}^N \to (1,\infty)$, $V\colon \mathbb{R}^N \to (0,\infty)$ is a continuous potential function, λ is a real parameter, and $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Especially, we localize precisely the intervals of λ for which the above equation admits at least one or two nontrivial weak solutions by applying our critical points results.

1. Introduction

Variational methods have emerged as one of the most effective analytic tools in the study of nonlinear equations. The idea behind them is attempting to solve a given problem by looking for critical points of an "energy" functional. The classical mountain pass theorem is a "phenomenal result" which has become the beginning of a new approach to critical point theory. The celebrated mountain pass theorem of A. Ambrosetti and P. H. Rabinowitz [2] provided the existence of at least one critical point for a C^1-functional satisfying the Palais-Smale condition (PS) and an appropriate geometry, called mountain pass geometry. This critical point theory has become one of the forceful tools for solving ordinary and partial differentiable equations of variational type; see [4, 5, 10, 16] and the

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references therein. Since the seminal work of Ambrosetti and Rabinowitz, such theory has by now been widely investigated by many researchers and some authors in [26, 40] also have attempted to weaken, in a fruitful way, the crucial assumption of the mountain pass theorem as mentioned. Especially, we focus on some kind of compactness property for the functional, one of which is the Palais-Smale condition ((PS)-condition for short). A notion which is slightly weaker than (PS)-condition has been introduced by G. Cerami [17] and it is called the Cerami condition ((C)-condition for short).

In all aspects, the first purpose of this paper is to investigate abstract critical point theorems for the existence of at least one or two critical points for functionals satisfying (C)-condition where the method is to use the generalized Ekeland variational principle developed by C.-K. Zhong [43]. These results lead to extensions of Bonanno's results for functionals satisfying (PS)-condition given in [9, 10]. Recently, the abstract critical point result which ensures the existence of at least one nontrivial local minimum has been extended and generalized in different directions and in different settings. In particular, multiple critical points theorems initiated by works of Ricceri [37, 38], which can be seen as a starting point in that direction, have been developed in the papers of S. A. Marano and D. Motreanu [34, 35], G. Bonanno and P. Candito [12]; see also [14]. In [9], the author gave an inventive approach about the existence of a critical point for a functional of local boundedness from below, by using the Ekeland variational principle and a novel Palais-Smale condition (see Theorem 3.1 of [9]). Combining this result with the fact that the mountain pass geometry is equivalent to the existence of one local minimum that is not strictly global, existence results of two critical points for functionals unbounded from below satisfying (PS)-condition were established in [10]. It is well known that the mountain pass theorem, when the functional is bounded from below on every bounded set of a real Banach space, actually is a result of multiple critical points. Especially the condition of local boundedness from below really is a natural condition in the setting of nonlinear differential problems. In this direction, we first present existence results of at least one critical point for an energy functional satisfying (C)-condition as a variant of Theorem 5.2 in [9] and Theorem 2.3 in [10]. Next, we point out two consequences for the existence of two critical points for functionals unbounded from below satisfying (C)-condition which is based on the work of Bonanno [10]. To do this, we observe the relation between the mountain pass geometry and local minima by using Zhong's Ekeland variational principle.

As applications of our critical points results, the other purpose is to study the existence of weak solutions for the nonlinear elliptic equations of the $p(x)$-Laplace type

$$-\operatorname{div}(\varphi(x, \nabla u)) + V(x)|u|^{p(x)-2}u = \lambda f(x, u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where the function $\varphi(x, v)$ is of type $|v|^{p(x)-2}v$ with a continuous function $p\colon \mathbb{R}^N \to (1, \infty)$,

$V\colon \mathbb{R}^N \to (0,\infty)$ is a continuous potential function, and $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition that will be specified later. The study on quasilinear elliptic problems of the $p(x)$-Laplace type operator $\operatorname{div}(\varphi(x,\nabla u))$ which is a perturbation of the $p(x)$-Laplacian $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is a very interesting topic in recent years. There have been a large number of research activities dealing with this kind of equations as well as their corresponding functional spaces setting; see [3, 15, 18, 21, 25, 28, 29, 31, 32] and the references therein. The investigation on problems of differential equations and variational problems involving $p(x)$-growth conditions has been a considerable topic because they can be presented as a model for many physical phenomena which arise in the research of elastic mechanics, electro-rheological fluid ("smart fluids") and image processing, etc. We refer the readers to [21, 39] and the references therein.

Taking into account our purpose that is to find one or two nontrivial weak solutions for (1.1), Bonanno's critical point theorems in [9, 10] which have been studied by many researchers are important to derive the existence of nontrivial weak solutions for nonlinear elliptic equations; see [6, 7, 12, 13]. Based on the paper [9], the existence of a nontrivial solution to a parametric Neumann problem for a class of nonlinear elliptic equations involving the $p(x)$-Laplacian and a discontinuous nonlinear term was established in [7]. In [12], G. Bonanno and A. Chinnì obtained the existence of at least one or two distinct weak solutions for the $p(x)$-Laplacian Dirichlet problems whenever the parameter λ belongs to a precise positive interval with aid of the works [9,10]; see Theorems 4.1 and 5.1 in [12]. In the present paper, as applications of our critical points theorems introduced in Section 2, we establish the existence of at least one or two weak solutions for the problem (1.1) under conditions slightly different from those of Theorems 4.1 and 5.1 in [12] in some sense.

When dealing with the existence of nontrivial weak solutions for some problems of variational type, one of crucial ingredients is that the nonlinear term f satisfies the A. Ambrosetti and P. H. Rabinowitz condition:

(AR) There exist positive constants M and θ such that $\theta > p_+$ and

$$0 < \theta F(x,t) \le f(x,t)t \quad \text{for } x \in \Omega \text{ and } |t| \ge M,$$

where $p_+ = \sup_{x\in\Omega} p(x)$, $F(x,t) = \int_0^t f(x,s)\,ds$, and Ω is a bounded domain in $\mathbb{R}^N$.

However, many authors in [1, 18, 27, 33, 36, 41] have tried to drop (AR)-condition that is crucial to guarantee the boundedness of Palais-Smale sequence of the Euler-Lagrange functional which plays a decisive role in applying the critical point theory. They achieved the existence of at least one nontrivial solution for nonlinear elliptic problems with the nonlinear term f satisfying a weaker condition than (AR)-condition. As seen before, by applying an abstract result in [9], the existence of at least one nontrivial weak solution for $p(x)$-Laplacian Dirichlet problem without assuming (AR)-condition as well as some conditions given in [1, 27, 33, 36, 41] which complement with (AR)-condition is discussed in [12]; see also [7]. As in [7, 12] we determine the precise positive interval of λ's for which (1.1) admits at least one solution without assuming the conditions on the nonlinear term f which are needed for ensuring the compactness condition of the energy functional associated with (1.1); see Theorems 3.13 and 3.14. It is worth noticing that we prove the existence result when the nonlinear term f has a subcritical growth condition although the given domain is the whole spaces $\mathbb{R}^N$.

On the other hand, (AR)-condition is needed to ensure the existence of two distinct weak solutions; see also [6, 12]. In that sense, we show that the problem (1.1) admits two distinct weak solutions provided that f satisfies a weaker condition than (AR)-condition which will be specified later in Subsection 3.3. Roughly speaking, using abstract critical points results for an energy functional satisfying (C)-condition, we derive the existence of at least two distinct weak solutions for the problem (1.1) provided that λ is suitable; see Theorems 3.18 and 3.19.

The novelty of this paper is twofold: The first one is that we give abstract critical points theorems for the existence of at least one or two critical points for an energy functional satisfying (C)-condition instead of (PS)-condition as a variant of the work [10]. The second is that these theorems are applied to find at least two weak solutions of (1.1) under weaker condition than (AR)-condition in comparison with that of the papers [6, 12]. As far as we are aware there were no such multiplicity results in this situation.

This paper is organized as follows. In Section 2, we study abstract results on the existence of at least one or two critical points for continuously Gâteaux differentiable functionals with (C)-condition. In addition, we present the existence of nontrivial critical points. In Section 3, we show the existence of at least one or two weak solutions for the problem by observing the boundedness of the Cerami sequence of the energy functional corresponding to the problem (1.1). In Subsection 3.2, we employ our critical points theorems to investigate that the problem (1.1) admits at least one nontrivial weak solution. In Subsection 3.3, the existence of at least two nontrivial weak solutions for the problem is achieved by applying our result in Section 2. In Appendix, we investigate the Pucci-Serrin theorem for a Gâteaux differentiable functional satisfying (C)-condition in place of (PS)-condition.

2. Abstract critical point theorems

Let $(X, \|\cdot\|_X)$ be a real reflexive Banach space. We denote the dual space of X by X^*, while $\langle \cdot , \cdot \rangle$ stands for the duality pairing between X^* and X.

A functional $J\colon X \to \mathbb{R}$ is called locally Lipschitz when, for every $u \in X$, there correspond a neighborhood U of u and a constant $L \geq 0$ such that

$$|J(v) - J(w)| \leq L\|v - w\|_X \quad \text{for all } v, w \in U.$$

If $u, v \in X$, the symbol $J^\circ(u; v)$ indicates the generalized directional derivative of J at point u along direction v, namely

$$J^\circ(u; v) := \limsup_{w \to u, t \to 0^+} \frac{J(w + tv) - J(w)}{t}.$$

The generalized gradient of the function J at u, denoted by $\partial J(u)$, is the set

$$\partial J(u) := \{u^* \in X : \langle u^*, v\rangle \leq J^\circ(u; v) \text{ for all } v \in X\}.$$

It is clear that $\partial J(u) \neq \emptyset$ by Hahn-Banach theorem. A functional $J\colon X \to \mathbb{R}$ is called Gâteaux differentiable at $u \in X$ if there is $\varphi \in X^*$ (denoted by $J'(u)$) such that

$$\lim_{t \to 0^+} \frac{J(u + tv) - J(u)}{t} = J'(u)(v)$$

for all $v \in X$. It is called continuously Gâteaux differentiable if it is Gâteaux differentiable for any $u \in X$ and the function $u \to J'(u)$ is a continuous map from X to its dual X^*. We recall that if J is continuously Gâteaux differentiable then it is locally Lipschitz and one has $J^\circ(u; v) = J'(u)(v)$ for all $u, v \in X$.

A Gâteaux differentiable functional $J\colon X \to \mathbb{R}$ satisfies (C)-condition, if any Cerami sequence $\{u_n\} \subset X$ for J, i.e., $\{J(u_n)\}$ is bounded and $\|J'(u_n)\|_{X^*}(1 + \|u_n\|_X) \to 0$ as $n \to \infty$, has a convergent subsequence.

As a key tool, recall the following lemma which is a generalization of Ekeland's variational principle [23] due to Zhong in [43, Theorem 2.1].

Lemma 2.1. *Let $\phi\colon [0, \infty) \to [0, \infty)$ be a continuous nondecreasing function such that $\int_0^\infty 1/(1 + \phi(r))\, dr = +\infty$. Let $\mathcal{M}$ be a complete metric space and x_0 be a fixed point of $\mathcal{M}$. Suppose that $f\colon \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function, not identically $+\infty$, bounded from below. Then, for every $\varepsilon > 0$ and $y \in \mathcal{M}$ such that*

$$f(y) < \inf_{\mathcal{M}} f + \varepsilon,$$

and every $\lambda > 0$, there exists some point $z \in \mathcal{M}$ such that

$$f(z) \leq f(y), \quad d(z, x_0) \leq \overline{r} + r_0$$

and

$$f(x) \geq f(z) - \frac{\varepsilon}{\lambda(1 + \phi(d(x_0, z)))} d(x, z) \quad \text{for all } x \in \mathcal{M},$$

where $d(x, y)$ is the distance of two points $x, y \in \mathcal{M}$, $r_0 = d(x_0, y)$, and $\overline{r}$ is such that

$$\int_{r_0}^{r_0 + \overline{r}} \frac{1}{1 + \phi(r)}\, dr \geq \lambda. \tag{2.1}$$

For our critical point theorems with (C)-condition, we use the following corollary which is a consequence of Zhong's Ekeland variational principle.

Corollary 2.2. *Let X be a Banach space and x_0 be a fixed point of X. Suppose that $f\colon X \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function, not identically $+\infty$, bounded from below. Then, for every $\varepsilon > 0$ and $y \in X$ such that*

$$f(y) < \inf_X f + \varepsilon,$$

and every $\lambda > 0$, there exists some point $z \in X$ such that

$$f(z) \le f(y), \quad \|z - x_0\|_X \le (1 + \|y\|_X)(e^\lambda - 1)$$

and

$$f(x) \ge f(z) - \frac{\varepsilon}{\lambda(1 + \|z\|_X)} \|x - z\|_X \quad \text{for all } x \in X.$$

Proof. To employ Lemma 2.1, we set $d(x, y) := \|x - y\|_X$ for any $x, y \in X$, where $d(x, y)$ is given in Lemma 2.1. Let $\phi\colon [0, \infty) \to [0, \infty)$ be defined by $\phi(r) = r + \|x_0\|_X$. Thus ϕ is continuous and nondecreasing. Then we see that $\phi(\|x_0 - z\|_X) = \|x_0 - z\|_X + \|x_0\|_X \ge \|z\|_X$, and hence for every $\varepsilon > 0$ and $y \in X$, we have

$$-\frac{\varepsilon}{\lambda(1 + \phi(\|x_0 - z\|_X))} \ge -\frac{\varepsilon}{\lambda(1 + \|z\|_X)}.$$

If we take $x_0 := y$, where y is given in Lemma 2.1, then $r_0 = 0$ and (2.1) imply that

$$\lambda \le \int_0^{\overline{r}} \frac{1}{1 + \phi(r)}\, dr = \int_0^{\overline{r}} \frac{1}{1 + r + \|y\|_X}\, dr = [\ln|1 + r + \|y\|_X|]_0^{\overline{r}} = \ln\left|1 + \frac{\overline{r}}{1 + \|y\|_X}\right|,$$

which is equivalent to

$$(1 + \|y\|_X)(e^\lambda - 1) \le \overline{r}.$$

Therefore, by applying Lemma 2.1, our conclusion holds. □

First, for a differentiable functional with (C)-condition based on Zhong's Ekeland variational principle stated in Corollary 2.2, we establish the relationship between the mountain pass geometry and the existence of at least one local minima in X which is not strictly global. For the case where the functional is continuously Gâteaux differentiable and satisfies (PS)-condition, we refer to [10, Theorem 2.1]; see also [8, Theorem 2.1].

Theorem 2.3. *Assume that a function $I\colon X \to \mathbb{R}$ is continuously Gâteaux differentiable and bounded from below on every bounded set of X. Assume that I satisfies (C)-condition. Then the following statements are equivalent:*

(i) *There exist* x_1, x_2 *and* $r \in \mathbb{R}$ *with* $0 < r < \|x_2 - x_1\|_X$ *such that*

$$\inf_{u \in X} \{I(u) : \|u - x_1\|_X = r\} \geq \max\{I(x_1), I(x_2)\}.$$

(ii) I *admits at least one local minimum in* X *which is not strictly global.*

Proof. (ii) $\Rightarrow$ (i). Suppose that x_1 is a local minimal point for I which is not strictly global. Then there exists a point $x_2 \in X$ such that $x_1 \neq x_2$ and $I(x_2) \leq I(x_1)$. We can choose positive numbers ρ_1 and ρ_2 such that $0 < \rho_1 < \|x_2 - x_1\|_X$ and $I(x_1) \leq I(u)$ for all $u \in X$ with $\|u - x_1\|_X < \rho_2$. If we set $r := \min\{\rho_1, \rho_2/2\}$, then $I(u) \geq I(x_1) \geq I(x_2)$ for all $u \in X$ such that $\|u - x_1\|_X = r$. Hence the statement (i) is achieved.

(i) $\Rightarrow$ (ii). Consider a set $E := \{u \in X : \|u - x_1\|_X \leq r\}$. It is obvious that E is a complete metric space, I is lower semicontinuous, from the fact that I is bounded from below on every bounded set of X, bounded from below on E. Then there exists a sequence $\{u_n\}$ in E such that $I(u_n) \to \alpha$ as $n \to \infty$, where $\alpha := \inf_E I$. Let $\{\varepsilon_n\}$ be a positive sequence defined by

$$\varepsilon_n := \begin{cases} \left\{\ln\left(1 + \frac{1}{n(1+\|u_n\|_X)}\right)\right\}^2 & \text{if } I(u_n) = \alpha, \\ 2(I(u_n) - \alpha) & \text{if } I(u_n) > \alpha. \end{cases}$$

Then we have

$$I(u_n) < \alpha + \varepsilon_n.$$

Denote a positive sequence $\{\lambda_n\}$ by $\lambda_n := \sqrt{\varepsilon_n}$. By applying Corollary 2.2 with $x_0 := u_n$ we obtain a sequence $\{z_n\}$ in E such that for all $n \in \mathbb{N}$,

$$I(z_n) \leq I(u_n), \quad \|u_n - z_n\|_X \leq (1 + \|u_n\|_X)(e^{\sqrt{\varepsilon_n}} - 1) \tag{2.2}$$

and for every $w \in E$ with $w \neq z_n$,

$$I(w) \geq I(z_n) - \frac{\varepsilon_n}{\lambda_n(1 + \|z_n\|_X)}\|w - z_n\|_X. \tag{2.3}$$

Then there are two possibilities to be considered: either there is a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \in S := \{u \in X : \|u - x_1\|_X = r\}$ for all $k \in \mathbb{N}$ or there exists an integer n_0 such that $z_n \in E \setminus S$ for all $n > n_0$.

For the first case, by the statement (i), we assert $I(x_1) \leq I(z_{n_k}) \leq I(u_{n_k}) \leq \alpha + \varepsilon_{n_k}$ for all $k \in \mathbb{N}$ and so, $I(x_1) = \alpha$. This yields that $I(x_1)$ is a global minimum in E, that is, it is a local minimum for I in X.

In the second case, if we consider $h \in X$ with $\|h\| \leq 1$, we may suppose that $w := z_n + th \in E \setminus S$ for sufficiently small $t > 0$ and for all $n > n_0$. By taking the limits as $t \to 0$, we have from (2.3) that

$$\langle I'(z_n), h\rangle \geq -\frac{\sqrt{\varepsilon_n}\|h\|_X}{1 + \|z_n\|_X}$$

and thus by replacing h with $-h$,

$$|\langle I'(z_n), h\rangle| \leq \frac{\sqrt{\varepsilon_n}\|h\|_X}{1+\|z_n\|_X}.$$

Hence we obtain that $(1+\|z_n\|_X)\|I'(z_n)\|_{X^*} \to 0$ as $n \to \infty$. Since I satisfies (C)-condition, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \to z^*$ for some $z^* \in E$ as $k \to \infty$. Now, we prove that $I(z^*) = \alpha$. Since (2.2) implies that

$$\frac{\|u_n - z_n\|_X}{1+\|u_n\|_X} \leq e^{\sqrt{\varepsilon_n}} - 1 \to 0 \quad \text{as } n \to \infty,$$

we have by the boundedness of $\{u_n\}$ that $\|u_n - z_n\|_X \to 0$ as $n \to \infty$. It follows from $u_{n_k} \to z^*$ that $I(u_{n_k}) \to \alpha = I(z^*)$ as $k \to \infty$ as claimed. If $z^* \in E \setminus S$, then $I(z^*)$ is a local minimum. Otherwise, if $z^* \in S$, then $I(z^*) \geq I(x_1)$ due to the statement (i) and thus the local minimum is $I(x_1)$. Consequently, we have shown that there exists one local minimum belonging to $E \setminus S$.

Now, let z_0 denote the local minimum previously obtained which belongs to $E \setminus S$. There are two cases to consider. If I is unbounded from below, there is $z_1 \in X$ such that $I(z_1) \leq I(z_0)$ for which z_0 is not a strictly global minimum. Otherwise, if I is bounded from below, then we know that I satisfies (PS)-condition due to the coercivity (see [30, Corollary 3]). As before (see also [8, Theorem 2.1]), we can prove that there is a local minimum z_1 belonging to $X \setminus E$. Therefore, one of them, between z_0 and z_1, is not a strict global minimum. □

To get main results in this section, we use the following lemma which is derived from Zhong's variational principle. The proof is analogous to that of [9, Lemma 3.1]; see also [42].

Lemma 2.4. *Let X be a real Banach space and let $I\colon X \to \mathbb{R}$ be a locally Lipschitz function bounded from below. Then, for all minimizing sequence of I, $\{u_n\} \subseteq X$, there exists a minimizing sequence of I, $\{v_n\} \subseteq X$, such that for any $n \in \mathbb{N}$*

$$I(v_n) \leq I(u_n) \quad \text{and} \quad \langle I^\circ(v_n), h\rangle \geq \frac{-\varepsilon_n\|h\|_X}{1+\|v_n\|_X}$$

for all $h \in X$ and $n \in \mathbb{N}$, where $\varepsilon_n \to 0^+$ as $n \to \infty$.

Next, we give abstract critical point theorems for a functional $I_\lambda := \Phi - \lambda\Psi$ satisfying (C)-condition cut off upper at μ for a fixed $\mu \in \mathbb{R}$ ($(\mathrm{C})^{[\mu]}$-condition for short), that is, if any Cerami sequence $\{u_n\} \subset X$ for I_λ with $\Phi(u_n) < \mu$ has a convergent subsequence of $\{u_n\}$. For this, let us introduce a function

$$\varphi_1(\mu) = \inf_{v\in\Phi^{-1}((-\infty,\mu))} \frac{\sup_{u\in\Phi^{-1}((-\infty,\mu))} \Psi(u) - \Psi(v)}{\mu - \Phi(v)}$$

for all $\mu \in \mathbb{R}$.

Now, we give a variant of the work of Bonanno [9, Theorem 5.2].

Theorem 2.5. *Let $\Phi, \Psi \colon X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below. Moreover, fix $\mu > \inf_{u \in X} \Phi(u)$ and assume that for each $\lambda \in \Lambda := (0, 1/\varphi_1(\mu))$ the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies $(C)^{[\mu]}$-condition. Then, for each $\lambda \in \Lambda$ there is an element u_0 in $\Phi^{-1}((-\infty, \mu))$ such that $I_\lambda(u_0) \le I_\lambda(u)$ for all $u \in \Phi^{-1}((-\infty, \mu))$ and $I'_\lambda(u_0) = 0$.*

Proof. Let $\lambda \in \Lambda$ be arbitrary but fixed, that is, $\lambda < 1/\varphi_1(\mu)$. Now, we will show that there exists a local minimal point u_1 in $\Phi^{-1}((-\infty, \mu))$ for I_λ such that $\Phi(u_1) < \mu$. In view of the definition of $\varphi_1(\mu)$, we can find a point u_1 in $\Phi^{-1}((-\infty, \mu))$ such that

$$\frac{\sup_{u \in \Phi^{-1}((-\infty,\mu))} \Psi(u) - \Psi(u_1)}{\mu - \Phi(u_1)} < \frac{1}{\lambda}.$$

This implies that

$$\sup_{u \in \Phi^{-1}((-\infty,\mu))} \lambda\Psi(u) < \mu - \Phi(u_1) + \lambda\Psi(u_1) =: M. \tag{2.4}$$

Define a functional Ψ_M by

$$\Psi_M(u) = \begin{cases} \lambda\Psi(u) & \text{if } \lambda\Psi(u) < M, \\ M & \text{if } \lambda\Psi(u) \ge M. \end{cases}$$

It is clear that $I := \Phi - \Psi_M$ is locally Lipschitz and bounded from below on X because Φ is bounded from below and $-\Psi_M \ge -M$. Then we find a sequence $\{u_n\}$ in X such that $I(u_n) \to \inf_X I$ as $n \to \infty$. From Lemma 2.4, it follows that there exists a sequence $\{v_n\}$ in X such that

$$I(v_n) \to \inf_X I \text{ as } n \to \infty \quad \text{and} \quad \langle I'(v_n), h\rangle \ge -\frac{\varepsilon_n \|h\|_X}{1 + \|v_n\|_X} \tag{2.5}$$

for all $h \in X$ and $n \in \mathbb{N}$, where $\varepsilon_n \to 0^+$ as $n \to \infty$. Then there are two possibilities to consider:

If $I(u_1) = \inf_X I$, then u_1 is a local minimal point for I_λ. Indeed, if $u \in \Phi^{-1}((-\infty, \mu))$, the relation (2.4) implies that $\lambda\Psi(u) \le M$ and $I(u) = I_\lambda(u)$ for all $u \in \Phi^{-1}((-\infty, \mu))$. Hence $I_\lambda(u_1) \le I_\lambda(u)$ for all $u \in \Phi^{-1}((-\infty, \mu))$.

Otherwise, if $\inf_X I < I(u_1)$, then there exists an integer N such that $I(v_n) < I(u_1)$ for all $n > N$. Since $\Phi(v_n) - \Psi_M(v_n) < \Phi(u_1) - \Psi_M(u_1)$, we have $\Phi(v_n) < \Psi_M(v_n) + \Phi(u_1) - \Psi(u_1) \le M + \Phi(u_1) - \Psi(u_1) = \mu$ and so $\Phi(v_n) < \mu$ for all $n > N$. Therefore, by Lemma 2.4 we obtain that

$$I(v_n) = I_\lambda(v_n) \to \inf_X I \quad \text{and} \quad \langle I'(v_n), h\rangle = \langle I'_\lambda(v_n), h\rangle \ge -\frac{\varepsilon_n \|h\|_X}{(1 + \|v_n\|_X)},$$

which implies $\|I'_\lambda(v_n)\|_{X^*}(1+\|v_n\|_X) \to 0$ as $n\to\infty$. By $(C)^{[\mu]}$-condition of I_λ, we may suppose that $v_n \to v^*$ as $n\to\infty$ for some $v^*\in X$ and so

$$I_\lambda(v^*) \le I_\lambda(u) \tag{2.6}$$

for all $u\in\Phi^{-1}((-\infty,\mu))$. Combining $\Phi(v_n)<\mu$ for all $n>N$ with the continuity of Φ, we infer $v^*\in\Phi^{-1}((-\infty,\mu])$. If $v^*\in\Phi^{-1}((-\infty,\mu))$, (2.6) immediately ensures the conclusion. Otherwise, if $\Phi(v^*)=\mu$, we first observe that $\Psi_M(v^*)\le M$. Note by $I_\lambda(v^*)=I(v^*)$ that $\mu-\lambda\Psi(v^*)=\mu-\Psi_M(v^*)$ and thus $\lambda\Psi(v^*)=\Psi_M(v^*)\le M$. Thus we obtain that $I_\lambda(v^*)=\mu-\lambda\Psi(v^*)\ge\mu-M=\Phi(u_1)-\lambda\Psi(u_1)=I_\lambda(u_1)$. Therefore, by (2.6) we conclude that $I_\lambda(u_1)\le I_\lambda(u)$ for all $u\in\Phi^{-1}((-\infty,\mu))$ and also the conclusion holds. This completes the proof. □

From Theorem 2.5, we give the next result. As an application of this theorem, we obtain the solvability of the problem (1.1) in Subsection 3.2 (see Theorem 3.14). For the case of the functional I_λ satisfying $(PS)^{[\mu]}$-condition, we refer to [10, Theorem 2.4].

Corollary 2.6. *Let X be a real Banach space, $\Phi,\Psi\colon X\to\mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0)=\Psi(0)=0$. Fix $\mu>0$ and assume that, for each*

$$\lambda\in\Lambda_0 := \left(0, \frac{\mu}{\sup_{u\in\Phi^{-1}((-\infty,\mu))}\Psi(u)}\right),$$

the functional I_λ satisfies $(C)^{[\mu]}$-condition for all $\lambda\in\Lambda_0$. Then, for each $\lambda\in\Lambda_0$, there is an element u_0 in $\Phi^{-1}((-\infty,\mu))$ such that $I_\lambda(u_0)\le I_\lambda(u)$ for all $u\in\Phi^{-1}((-\infty,\mu))$ and $I'_\lambda(u_0)=0$.

Proof. Let $\lambda\in\Lambda_0$ be arbitrary but fixed. Then we have $\frac{\sup_{u\in\Phi^{-1}((-\infty,\mu))}\Psi(u)}{\mu}<\frac{1}{\lambda}$. According to the definition of $\varphi_1(\mu)$, there is an element x_0 in $\Phi^{-1}((-\infty,\mu))$ such that

$$\frac{\sup_{u\in\Phi^{-1}((-\infty,\mu))}\Psi(u)-\Psi(x_0)}{\mu-\Phi(x_0)} \le \frac{\sup_{u\in\Phi^{-1}((-\infty,\mu))}\Psi(u)}{\mu} < \frac{1}{\lambda}.$$

This says

$$\sup_{u\in\Phi^{-1}((-\infty,\mu))}\lambda\Psi(u) < \mu-\Phi(x_0)+\lambda\Psi(x_0).$$

Arguing the same argument as in the proof of Theorem 2.5, I_λ admits a local minimum. This completes the proof. □

As a consequence of Theorem 2.5, we get another critical point theorem with (C)-condition. This is crucial to ensure the existence of at least two distinct weak solutions for the problem (1.1) in Subsection 3.3 (see Theorem 3.18). See Theorem 3.2 of [10] for functionals satisfying (PS)-condition.

Corollary 2.7. *Let X be a real Banach space, $\Phi, \Psi \colon X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Fix $\mu > 0$ and assume that, for each*

$$\lambda \in \Lambda_0 = \left(0, \frac{\mu}{\sup_{u \in \Phi^{-1}((-\infty,\mu))} \Psi(u)}\right),$$

the functional I_λ satisfies (C)-condition for all $\lambda \in \Lambda_0$ and it is unbounded from below. Then, for each $\lambda \in \Lambda_0$, the functional I_λ admits two distinct critical points.

Proof. Taking into account that I_λ satisfies $(\mathrm{C})^{[\mu]}$-condition, Theorem 2.5 ensures that I_λ admits one local minimum. Since I_λ is unbounded from below, it is not strictly global and the mountain pass theorem ensures the conclusion. This completes the proof. □

Finally, we study another abstract critical point theorem as a particular case of [9, Theorem 5.1] when I_λ satisfies $(\mathrm{PS})^{[\mu]}$-condition. To do this, we introduce two functions

$$\chi_1(\mu) = \inf_{v \in \Phi^{-1}((0,\mu))} \frac{\sup_{u \in \Phi^{-1}((0,\mu))} \Psi(u) - \Psi(v)}{\mu - \Phi(v)}$$

and

$$\chi_2(\mu) = \sup_{v \in \Phi^{-1}((0,\mu))} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}((-\infty,0])} \Psi(u)}{\Phi(v)}$$

for all $\mu \in \mathbb{R}$ with $\mu > 0$.

Theorem 2.8. *Let $\Phi, \Psi \colon X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is bounded from below. Suppose that*

$$\text{there exists } \mu \in \mathbb{R} \text{ such that } \chi_1(\mu) < \chi_2(\mu).$$

Moreover, assume that for each $\lambda \in \Lambda := (1/\chi_2(\mu), 1/\chi_1(\mu))$ the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies $(C)^{[\mu]}$-condition. Then, for each $\lambda \in \Lambda$, the functional I_λ has a nontrivial point $u_{0,\lambda}$ in $\Phi^{-1}((0,\mu))$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all u in $\Phi^{-1}((0,\mu))$ with $u_{0,\lambda}$ being a critical point of I_λ.

Proof. For each $\lambda \in (1/\chi_2(\mu), 1/\chi_1(\mu))$, we have $\chi_1(\mu) < 1/\lambda < \chi_2(\mu)$ which implies the existence of $u_1, u_2 \in \Phi^{-1}((0,\mu))$ such that

$$(2.7) \qquad \frac{1}{\lambda} > \frac{\sup_{u \in \Phi^{-1}((0,\mu))} \Psi(u) - \Psi(u_1)}{\mu - \Phi(u_1)} \quad \text{and} \quad \frac{1}{\lambda} < \frac{\Psi(u_2) - \sup_{u \in \Phi^{-1}((-\infty,0])} \Psi(u)}{\Phi(u_2)}.$$

Now, let $x_0 \in \Phi^{-1}((0,\mu))$ be such that

$$\Phi(x_0) - \lambda\Psi(x_0) = \min\{\Phi(u_1) - \lambda\Psi(u_1), \Phi(u_2) - \lambda\Psi(u_2)\}.$$

From (2.7), we have

$$\sup_{u\in\Phi^{-1}((0,\mu))} \lambda\Psi(u) \le \mu - \Phi(x_0) + \lambda\Psi(x_0)$$

and

$$\sup_{u\in\Phi^{-1}((-\infty,0])} \lambda\Psi(u) \le -\Phi(x_0) + \lambda\Psi(x_0). \tag{2.8}$$

Set

$$\mathcal{M} = \mu - \Phi(x_0) + \lambda\Psi(x_0). \tag{2.9}$$

Define

$$\Phi_0(u) = \max\{\Phi(u), 0\}, \quad \lambda\Psi_{\mathcal{M}}(u) = \min\{\lambda\Psi(u), \mathcal{M}\}$$

and

$$I = \Phi_0 - \lambda\Psi_{\mathcal{M}}.$$

Clearly, I is locally Lipschitz continuous and bounded from below. Given a sequence $\{u_n\}$ in X such that $\lim_{n\to\infty} I(u_n) = \inf_X I$, it follows from Lemma 2.4 that we choose a sequence $\{v_n\}$ in X such that (2.5) holds for all $h \in X$ and for all $n \in \mathbb{N}$, where $\varepsilon_n \to 0^+$. We first prove that $\Phi(v_n) > 0$ for all $n > n_0$. Suppose to the contrary that $\Phi(v_n) \le 0$ for all $n > n_0$. This yields

$$-\lambda\Psi(v_n) = \Phi_0(v_n) - \lambda\Psi(v_n) \le \Phi(x_0) - \lambda\Psi(x_0)$$

or equivalently

$$-\Phi(x_0) + \lambda\Psi(x_0) < \lambda\Psi(v_n).$$

Due to (2.8), we deduce a contradiction.

Next it follows from the analogous arguments as in the proof of Theorem 2.5 that $\Phi(v_n) < \mu$ for all $n > n_0$ and thus

$$I_\lambda(v^*) = \inf_X I \le I(u) = I_\lambda(u) \tag{2.10}$$

for all $u \in \Phi^{-1}((0,\mu))$ and for some v^* in X satisfying $v_n \to v^*$ as $n \to \infty$. From the continuity of Φ and the fact that $0 < \Phi(v_n) < \mu$ for all $n > n_0$, we have $v^* \in \Phi^{-1}([0,\mu])$. If $v^* \in \Phi^{-1}((0,\mu))$, by (2.10) the conclusion holds. If $\Phi(v^*)$ is either 0 or μ, taking into account (2.8), (2.9) and (2.10) it is easy to show that $I_\lambda(x_0) \le I_\lambda(u)$ for all $u \in \Phi^{-1}((0,\mu))$. This completes the proof. □

As a consequence of Theorem 2.8, we have the following result which is a main tool to guarantee the existence of a nontrivial weak solution for the problem (1.1) in Subsection 3.2 (see Theorem 3.14). It can be found in [10, Theorem 2.3] when the functional satisfies $(\mathrm{PS})^{[\mu]}$-condition.

Corollary 2.9. *Let $\Phi\colon X \to \mathbb{R}$ be a continuously Gâteaux differentiable and $\Psi\colon X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\inf_{u\in X} \Phi(u) = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist a positive constant μ and an element $\widetilde{u} \in X$, with $0 < \Phi(\widetilde{u}) < \mu$, such that

$$\frac{\sup_{u\in\Phi^{-1}((-\infty,\mu])} \Psi(u)}{\mu} < \frac{\Psi(\widetilde{u})}{\Phi(\widetilde{u})} \tag{2.11}$$

holds and the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies (C)$^{[\mu]}$-condition. Then, for each

$$\lambda \in \Lambda_\mu := \left(\frac{\Phi(\widetilde{u})}{\Psi(\widetilde{u})}, \frac{\mu}{\sup_{u\in\Phi^{-1}((-\infty,\mu])} \Psi(u)}\right),$$

the functional I_λ has a nontrivial point $u_{0,\lambda}$ in $\Phi^{-1}((0,\mu))$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all u in $\Phi^{-1}((0,\mu))$ with $u_{0,\lambda}$ being a critical point of I_λ.

Proof. In view of definitions of $\chi_1(\mu)$ and $\chi_2(\mu)$, we assert that

$$\chi_1(\mu) \le \min\left\{\frac{\sup_{u\in\Phi^{-1}((-\infty,\mu))} \Psi(u) - \Psi(\widetilde{u})}{\mu - \Phi(\widetilde{u})}, \frac{\sup_{u\in\Phi^{-1}((-\infty,\mu))} \Psi(u)}{\mu}\right\}$$

and

$$\begin{aligned}\chi_2(\mu) &= \sup_{v\in\Phi^{-1}((0,\mu))} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}((-\infty,0])} \Psi(u)}{\Phi(v)} \ge \frac{\Psi(\widetilde{u}) - \sup_{u\in\Phi^{-1}((-\infty,0])} \Psi(u)}{\Phi(\widetilde{u}) - \mu} \\ &\ge \frac{\Psi(\widetilde{u}) - \mu\frac{\Psi(\widetilde{u})}{\Phi(\widetilde{u})}}{\Phi(\widetilde{u}) - \mu} = \frac{\Psi(\widetilde{u})}{\Phi(\widetilde{u})}.\end{aligned}$$

Combining this together with the assumption, we infer that

$$\chi_1(\mu) \le \frac{\sup_{u\in\Phi^{-1}((-\infty,\mu))} \Psi(u)}{\mu} < \frac{\Psi(\widetilde{u})}{\Phi(\widetilde{u})} \le \chi_2(\mu).$$

Hence we obtain that

$$\Lambda_\mu \subseteq \left(\frac{1}{\chi_2(\mu)}, \frac{1}{\chi_1(\mu)}\right).$$

Since Φ is bounded from below, Theorem 2.8 ensures our conclusion. Hence I_λ admits a local minimum. Since I_λ is unbounded from below, it is not strictly global and the mountain pass theorem ensures the conclusion. This completes the proof. □

The following theorem for (C)-condition is an analogous result to Corollary 2.9, as a variant of [10, Theorem 2.3]. This is crucial to guarantee the existence of two distinct weak solutions for the problem (1.1) in Subsection 3.3 (see Theorem 3.19).

Corollary 2.10. *Let $\Phi\colon X \to \mathbb{R}$ be a continuously Gâteaux differentiable and $\Psi\colon X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\inf_{u \in X} \Phi(u) = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist a positive constant μ and an element $\widetilde{u} \in X$, with $0 < \Phi(\widetilde{u}) < \mu$, such that (2.11) holds and for each $\lambda \in \Lambda_\mu := \left(\frac{\Phi(\widetilde{u})}{\Psi(\widetilde{u})}, \frac{\mu}{\sup_{u \in \Phi^{-1}((-\infty,\mu])} \Psi(u)} \right)$, the functional $I_\lambda := \Phi - \lambda\Psi$ satisfies (C)-condition and it is unbounded from below. Then, for each $\lambda \in \Lambda_\mu$ the functional I_λ admits two distinct critical points.

Proof. Since I_λ satisfies $(\mathrm{C})^{[\mu]}$-condition and it is unbounded from below, Theorem 2.8 and the mountain pass theorem ensures the conclusion. This completes the proof. □

3. Applications to equations of $p(x)$-Laplace type in $\mathbb{R}^N$

In this section, we deal with the elliptic equations with variable exponents as applications of our abstract critical points theorems introduced in Section 2.

3.1. Some properties of the variable exponent spaces

We recall some definitions of the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^N)$ and the variable exponent Lebesgue-Sobolev space $W^{1,p(\cdot)}(\mathbb{R}^N)$ which will be treated in the next subsections. For a deeper treatment on these spaces, we refer to [21, 22, 25].

Set

$$C_+(\mathbb{R}^N) = \{h \in C(\mathbb{R}^N) : \inf_{x \in \mathbb{R}^N} h(x) > 1\}.$$

For any $h \in C_+(\mathbb{R}^N)$, we define

$$h_+ = \sup_{x \in \mathbb{R}^N} h(x) \quad \text{and} \quad h_- = \inf_{x \in \mathbb{R}^N} h(x).$$

For any $p \in C_+(\mathbb{R}^N)$, we introduce the variable exponent Lebesgue space

$$L^{p(\cdot)}(\mathbb{R}^N) := \left\{ u : u \text{ is a measurable real-valued function, } \int_{\mathbb{R}^N} |u|^{p(x)}\, dx < +\infty \right\}$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} \left| \frac{u}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

The dual space of $L^{p(\cdot)}(\mathbb{R}^N)$ is $L^{p'(\cdot)}(\mathbb{R}^N)$, where $1/p(x) + 1/p'(x) = 1$.

The variable exponent Lebesgue-Sobolev space $X := W^{1,p(\cdot)}(\mathbb{R}^N)$ is defined by

$$W^{1,p(\cdot)}(\mathbb{R}^N) = \{u \in L^{p(\cdot)}(\mathbb{R}^N) : |\nabla u| \in L^{p(\cdot)}(\mathbb{R}^N)\},$$

where the norm is

$$\|u\|_{W^{1,p(\cdot)}(\mathbb{R}^N)} = \|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} + \|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^N)}. \tag{3.1}$$

Let the potential $V \in C(\mathbb{R}^N)$ be continuous. Suppose that

(V) $V \in C(\mathbb{R}^N)$, $V_- := \inf_{x\in\mathbb{R}^N} V(x) > 0$, $\operatorname{meas}\{x \in \mathbb{R}^N : -\infty < V(x) \le M\} < +\infty$ for all $M \in \mathbb{R}$.

Define the linear subspace

$$X = \left\{ u \in W^{1,p(\cdot)}(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^{p(x)} + V(x)|u|^{p(x)})\, dx < +\infty \right\}$$

with the norm

$$\|u\|_X = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^N} \left(\left|\frac{\nabla u}{\lambda}\right|^{p(x)} + V(x) \left|\frac{u}{\lambda}\right|^{p(x)} \right) dx \le 1 \right\},$$

which is equivalent to the norm (3.1).

In this subsection, we first collect some preliminary properties that will be used later.

Lemma 3.1. [25] *The space $L^{p(\cdot)}(\mathbb{R}^N)$ is a separable, uniformly convex Banach space, and its conjugate space is $L^{p'(\cdot)}(\mathbb{R}^N)$ where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(\cdot)}(\mathbb{R}^N)$ and $v \in L^{p'(\cdot)}(\mathbb{R}^N)$, we have*

$$\left| \int_{\mathbb{R}^N} uv\, dx \right| \le \left(\frac{1}{p_-} + \frac{1}{(p')_-} \right) \|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} \|v\|_{L^{p'(\cdot)}(\mathbb{R}^N)} \le 2\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} \|v\|_{L^{p'(\cdot)}(\mathbb{R}^N)}.$$

Lemma 3.2. [25] *Denote*

$$\rho(u) = \int_{\mathbb{R}^N} |u|^{p(x)}\, dx \quad \text{for all } u \in L^{p(\cdot)}(\mathbb{R}^N).$$

Then

(1) *$\rho(u) > 1$ $(= 1; < 1)$ if and only if $\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} > 1$ $(= 1; < 1)$, respectively;*

(2) *if $\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} > 1$, then $\|u\|^{p_-}_{L^{p(\cdot)}(\mathbb{R}^N)} \le \rho(u) \le \|u\|^{p_+}_{L^{p(\cdot)}(\mathbb{R}^N)}$;*

(3) *if $\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} < 1$, then $\|u\|^{p_+}_{L^{p(\cdot)}(\mathbb{R}^N)} \le \rho(u) \le \|u\|^{p_-}_{L^{p(\cdot)}(\mathbb{R}^N)}$.*

Remark 3.3. [25] Denote

$$\rho(u) = \int_{\mathbb{R}^N} \left(|\nabla u|^{p(x)} + |u|^{p(x)} \right) dx \quad \text{for all } u \in X.$$

Then

(1) $\rho(u) > 1$ ($= 1$; < 1) if and only if $\|u\|_X > 1$ ($= 1$; < 1), respectively;

(2) if $\|u\|_X > 1$, then $\|u\|_X^{p_-} \le \rho(u) \le \|u\|_X^{p_+}$;

(3) if $\|u\|_X < 1$, then $\|u\|_X^{p_+} \le \rho(u) \le \|u\|_X^{p_-}$.

Lemma 3.4. [22] *Let $q \in L^\infty(\mathbb{R}^N)$ be such that $1 \le p(x)q(x) \le \infty$ for almost all $x \in \mathbb{R}^N$. If $u \in L^{q(\cdot)}(\mathbb{R}^N)$ with $u \neq 0$, then*

(1) *if $\|u\|_{L^{p(\cdot)q(\cdot)}(\mathbb{R}^N)} > 1$, then $\|u\|^{q_-}_{L^{p(\cdot)q(\cdot)}(\mathbb{R}^N)} \le \||u|^{q(x)}\|_{L^{p(\cdot)}(\mathbb{R}^N)} \le \|u\|^{q_+}_{L^{p(\cdot)q(\cdot)}(\mathbb{R}^N)}$;*

(2) *if $\|u\|_{L^{p(\cdot)q(\cdot)}(\mathbb{R}^N)} < 1$, then $\|u\|^{q_+}_{L^{p(\cdot)q(\cdot)}(\mathbb{R}^N)} \le \||u|^{q(x)}\|_{L^{p(\cdot)}(\mathbb{R}^N)} \le \|u\|^{q_-}_{L^{p(\cdot)q(\cdot)}(\mathbb{R}^N)}$.*

Lemma 3.5. [22, 24] *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with Lipschitz boundary and let $p \in C_+(\overline{\Omega})$ with $1 < p_- \le p_+ < \infty$. If $q \in L^\infty(\Omega)$ with $q_- > 1$ satisfies*

$$q(x) \le p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } N > p(x), \\ +\infty & \text{if } N \le p(x), \end{cases}$$

then there is a continuous embedding

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

and the embedding is compact if $\inf_{x\in\Omega}(p^*(x) - q(x)) > 0$.

Lemma 3.6. [24] *Suppose that $p\colon \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous with $1 < p_- \le p_+ < N$. Let $q \in L^\infty(\mathbb{R}^N)$ and $p(x) \le q(x) \le p^*(x)$ for almost all $x \in \mathbb{R}^N$. Then there is a continuous embedding $W^{1,p(\cdot)}(\mathbb{R}^N) \hookrightarrow L^{q(\cdot)}(\mathbb{R}^N)$.*

Lemma 3.7. [1,24] *Suppose that $p\colon \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous with $1 < p_- \le p_+ < N$ and the condition* (V) *is fulfilled. Let $q \in L^\infty(\mathbb{R}^N)$ with $p(x) < q(x)$ for all $x \in \mathbb{R}^N$. Then there is a compact embedding $X \hookrightarrow L^{q(\cdot)}(\mathbb{R}^N)$ if $\inf_{x\in\mathbb{R}^N}(p^*(x) - q(x)) > 0$.*

In the sequel we will denote by $\mathcal{S}_q$ the embedding's constant for which one has

$$\mathcal{S}_q = \inf_{u\in X\setminus\{0\}} \frac{\|u\|_X}{\|u\|_{L^{q(\cdot)}(\mathbb{R}^N)}}, \tag{3.2}$$

where $p, q \in C_+(\mathbb{R}^N)$ and $p(x) \le q(x) < p^*(x)$ for all $x \in \mathbb{R}^N$.

In what follows, let $p \in C_+(\mathbb{R}^N)$ be Lipschitz continuous with $1 < p_- \le p_+ < N$ and the potential V satisfies the condition (V). We denote by the space X^* be a dual space of X. Furthermore, $\langle \cdot, \cdot \rangle$ denotes the pairing of X and its dual X^* and Euclidean scalar product on $\mathbb{R}^N$, respectively.

Definition 3.8. We say that $u \in X$ is a weak solution of the problem (1.1) if

$$\int_{\mathbb{R}^N} \varphi(x, \nabla u) \cdot \nabla v \, dx + \int_{\mathbb{R}^N} V(x)|u|^{p(x)-2} uv \, dx = \lambda \int_{\mathbb{R}^N} f(x, u)v \, dx$$

for all $v \in X$.

We assume that $\varphi \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous function with the continuous derivative with respect to v of the mapping $\Phi_0 \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $\Phi_0 = \Phi_0(x, v)$, that is, $\varphi(x, v) = \frac{d}{dv}\Phi_0(x, v)$. Suppose that φ and Φ_0 satisfy the following assumptions:

(J1) The identity

$$\Phi_0(x, \mathbf{0}) = 0$$

holds for almost all $x \in \mathbb{R}^N$.

(J2) Assume that there is a nonnegative constant c^* such that

$$|\varphi(x, v)| \le c^* |v|^{p(x)-1}$$

holds for almost all $x \in \mathbb{R}^N$ and for all $v \in \mathbb{R}^N$.

(J3) $\Phi_0(x, \cdot\,)$ is strictly convex in $\mathbb{R}^N$ for all $x \in \mathbb{R}^N$.

(J4) The relations

$$c_* |v|^{p(x)} \le p_+ \Phi_0(x, v) \quad \text{and} \quad c_* |v|^{p(x)} \le \varphi(x, v) \cdot v$$

hold for all $x \in \mathbb{R}^N$ and $v \in \mathbb{R}^N$, where c_* is a positive constant with $c_* < c^*$.

(J5) There exists a constant $\theta \ge p_+$ such that

$$\mathcal{H}(x, sv) \le \mathcal{H}(x, v),$$

holds for $v \in \mathbb{R}^N$ and $s \in [0, 1]$, where $\mathcal{H}(x, v) = \theta \Phi_0(x, v) - \varphi(x, v) \cdot v$ for all $x \in \mathbb{R}^N$.

Let us define the functional $\Phi \colon X \to \mathbb{R}$ by

$$\Phi(u) = \int_{\mathbb{R}^N} \Phi_0(x, \nabla u) \, dx + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)} |u|^{p(x)} \, dx.$$

Under the assumptions (V), (J1), (J2) and (J4), it follows from [32, Lemma 3.2] that the functional Φ is well-defined on X, $\Phi \in C^1(X, \mathbb{R})$ and its Fréchet derivative is given by

$$\langle \Phi'(u), v \rangle = \int_{\mathbb{R}^N} \varphi(x, \nabla u) \cdot \nabla v \, dx + \int_{\mathbb{R}^N} V(x)|u|^{p(x)-2} uv \, dx.$$

Example 3.9. Let us consider

$$\varphi(x,v)=\left(1+\frac{|v|^{p(x)}}{\sqrt{1+|v|^{2p(x)}}}\right)|v|^{p(x)-2}v$$

for all $v\in\mathbb{R}^N$. In this case, put

$$\Phi_0(x,v)=\frac{1}{p(x)}\left(|v|^{p(x)}+\sqrt{1+|v|^{2p(x)}}-1\right)$$

for all $v\in\mathbb{R}^N$. It is immediate that the conditions (J1)–(J4) hold. The same argument in [29] implies that the assumption (J5) holds when $\theta=2p_+$.

Next, taking inspiration from the argument given in [19], we give that the operator Φ' is a mapping of type (S_+). On a bounded domain Ω in $\mathbb{R}^N$, Le [31] proved this fact for the case of variable exponent $p(x)$; see also [19] for the case of a constant exponent.

Lemma 3.10. *Assume that* (V) *and* (J1)–(J4) *hold. Then the functional* $\Phi\colon X\to\mathbb{R}$ *is convex and weakly lower semicontinuous on* X. *Moreover, the operator* Φ' *is a mapping of type* (S_+), *i.e., if* $u_n\rightharpoonup u$ *in* X *as* $n\to\infty$ *and* $\limsup_{n\to\infty}\langle\Phi'(u_n)-\Phi'(u),u_n-u\rangle\le 0$, *then* $u_n\to u$ *in* X *as* $n\to\infty$.

Proof. It can be found in [32, Lemma 3.4]. □

The following consequence follows from Lemma 3.10 immediately.

Corollary 3.11. *Assume that* (V) *and* (J1)–(J4) *hold. Then the operator* $\Phi'\colon X\to X^*$ *is strictly monotone, coercive and hemicontinuous on* X. *Furthermore, the operator* Φ' *is homeomorphism onto* X^*.

Proof. It is obvious that the operator Φ' is strictly monotone, coercive and hemicontinuous on X. By the Browder-Minty theorem, the inverse operator $(\Phi')^{-1}$ exists. If we apply Lemma 3.10, then the proof of continuity of the inverse operator $(\Phi')^{-1}$ is analogous to that in the case of a constant exponent and is omitted. □

3.2. Existence of nontrivial solution

In this subsection, we investigate the existence of at least one nontrivial solution for the problem (1.1) by applying Corollary 2.9; see also [7,11]. It is worth noticing that we prove the existence result when the nonlinear term f has a subcritical growth condition.

Until now, we considered some properties for the integral operator corresponding to the divergence part in the problem (1.1). To deal with our main results in this subsection, we need the following assumptions on f. Let us put $F(x,t)=\int_0^t f(x,s)\,ds$. For $p(x)<N$, we assume that

(H1) $p, q \in C_+(\mathbb{R}^N)$ and $1 < p_- \le p_+ < q_- \le q_+ < p^*(x)$ for all $x \in \mathbb{R}^N$;

(F1) $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition in the sense that $f(\,\cdot\,, s)$ is measurable for all $s \in \mathbb{R}$ and $f(x, \cdot\,)$ is continuous for almost all $x \in \mathbb{R}^N$;

(F2) there exist nonnegative functions $\rho_1 \in L^{\gamma_1'(\cdot)}(\mathbb{R}^N) \cap L^{p'(\cdot)}(\mathbb{R}^N)$ and $\sigma_1 \in L^\infty(\mathbb{R}^N) \cap L^{\frac{q(\cdot)}{q(\cdot)-\gamma_1(\cdot)}}(\mathbb{R}^N)$ such that

$$|f(x,t)| \le \rho_1(x) + \sigma_1(x)|t|^{\gamma_1(x)-1}$$

holds for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, where $1 < \gamma_1(x) < q_-$ for all $x \in \mathbb{R}^N$.

Define the functional $\Psi\colon X \to \mathbb{R}$ by

$$\Psi(u) = \int_{\mathbb{R}^N} F(x,u)\,dx.$$

Then it is easy to check that $\Psi \in C^1(X, \mathbb{R})$ and its Fréchet derivative is

$$\langle \Psi'(u), v\rangle := \int_{\mathbb{R}^N} f(x,u)v\,dx$$

for any $u, v \in X$. Next we define the functional $I_\lambda\colon X \to \mathbb{R}$ by

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u).$$

Then it follows that the functional $I_\lambda \in C^1(X, \mathbb{R})$ and its Fréchet derivative is

$$\langle I_\lambda'(u), v\rangle = \int_{\mathbb{R}^N} \varphi(x, \nabla u)\cdot \nabla v\,dx + \int_{\mathbb{R}^N} V(x)|u|^{p(x)-2}uv\,dx - \lambda\int_{\mathbb{R}^N} f(x,u)v\,dx$$

for any $u, v \in X$.

Lemma 3.12. *Assume that* (V), (H1) *and* (F1)–(F2) *hold. Then Ψ' is weakly strongly continuous on X.*

Proof. Let $\{u_n\}$ be a sequence in X such that $u_n \rightharpoonup u$ in X as $n \to \infty$. Since $\{u_n\}$ is bounded in X, Lemma 3.7 guarantees that there exists a subsequence such that

$$u_{n_k}(x) \to u(x) \text{ a.e. in } \mathbb{R}^N \text{ and } u_{n_k} \to u \text{ in } L^{q(\cdot)}(\mathbb{R}^N) \text{ as } k \to \infty.$$

By the convergence principle, there exists a function $g \in L^{q(\cdot)}(\mathbb{R}^N)$ such that $|u_{n_k}(x)| \le g(x)$ for all $k \in \mathbb{N}$ and for almost all $x \in \mathbb{R}^N$. Taking into account (F2) and the Young inequality, we derive that

$$\begin{aligned} &|f(x,u_{n_k})|^{\gamma_1'(x)}\,dx \\ &\le C_1\left(|\rho_1(x)|^{\gamma_1'(x)} + |\sigma_1|^{\gamma_1'(x)}|u_{n_k}|^{\gamma_1(x)}\right) \\ &\le C_1\left(|\rho_1(x)|^{\gamma_1'(x)} + \|\sigma_1\|_{L^\infty(\mathbb{R}^N)}^{\gamma_1'(x)-1}\left(\frac{q(x)-\gamma_1(x)}{q(x)}|\sigma_1|^{\frac{q(x)}{q(x)-\gamma_1(x)}} + \frac{\gamma_1(x)}{q(x)}|u_{n_k}|^{q(x)}\right)\right) \end{aligned} \tag{3.3}$$

and analogously,

$$\begin{aligned}
&|f(x,u)|^{\gamma_1'(x)}\,dx \\
&\le C_2\left(|\rho_1(x)|^{\gamma_1'(x)} + |\sigma_1|^{\gamma_1'(x)}|u|^{\gamma_1(x)}\right) \\
&\le C_2\left(|\rho_1(x)|^{\gamma_1'(x)} + \|\sigma_1\|_{L^\infty(\mathbb{R}^N)}^{\gamma_1'(x)-1}\left(\frac{q(x)-\gamma_1(x)}{q(x)}|\sigma_1|^{\frac{q(x)}{q(x)-\gamma_1(x)}} + \frac{\gamma_1(x)}{q(x)}|u|^{q(x)}\right)\right)
\end{aligned} \tag{3.4}$$

for some positive constants C_1, C_2. By (3.3) and (3.4), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} |f(x,u_{n_k}) - f(x,u)|^{\gamma_1'(x)}\,dx \\
&\le C_3\int_{\mathbb{R}^N} |f(x,u_{n_k})|^{\gamma_1'(x)} + |f(x,u)|^{\gamma_1'(x)}\,dx \\
&\le C_4\int_{\mathbb{R}^N} |\rho_1(x)|^{\gamma_1'(x)} \\
&\qquad + \|\sigma_1\|_{L^\infty(\mathbb{R}^N)}^{\gamma_1'(x)-1}\left(\frac{q(x)-\gamma_1(x)}{q(x)}|\sigma_1|^{\frac{q(x)}{q(x)-\gamma_1(x)}} + \frac{\gamma_1(x)}{q(x)}(|g|^{q(x)}+|u|^{q(x)})\right)dx
\end{aligned} \tag{3.5}$$

for some positive constants C_3, C_4 and thus the integral at the the left-hand side is dominated by an integrable function. Since $u_{n_k} \to u$ in $L^{q(\cdot)}(\mathbb{R}^N)$ and f satisfies the Carathéodory condition, we have $f(x,u_{n_k}) \to f(x,u)$ as $k \to \infty$ for almost all $x \in \mathbb{R}^N$. Combining this with (3.5), it follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned}
\|\Psi'(u_{n_k}) - \Psi'(u)\|_{X^*} &= \sup_{\|v\|_X\le 1} |\langle \Psi'(u_{n_k}) - \Psi'(u), v\rangle| \\
&= \sup_{\|v\|_X\le 1} \left|\int_{\mathbb{R}^N} (f(x,u_{n_k}) - f(x,u))v\,dx\right| \\
&\le 2\|f(x,u_{n_k}) - f(x,u)\|_{L^{\gamma_1'(\cdot)}(\mathbb{R}^N)} \to 0 \quad \text{as } k\to\infty.
\end{aligned}$$

Consequently, we derive that $\Psi'(u_{n_k}) \to \Psi'(u)$ in X^* as $k \to \infty$. This completes the proof. □

First, we prove the existence of at least one weak solution for the problem (1.1) provided that λ is suitable, by using Corollary 2.6 given in Section 2.

Theorem 3.13. *Assume that* (V), (J1)–(J5), (H1) *and* (F1)–(F2) *hold. Then there exists a positive constant* $\widetilde{\lambda}$ *such that the problem* (1.1) *admits at least one weak solution in* X *for each* $\lambda \in (0, \widetilde{\lambda})$.

Proof. It is obvious that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. The assumption (F2) and the relation (3.2) imply that

$$
\begin{aligned}
\Psi(u) &= \int_{\mathbb{R}^N} F(x,u)\,dx \\
&\le \int_{\mathbb{R}^N} \rho_1(x)|u(x)| + \frac{\sigma_1(x)}{\gamma_1(x)}|u(x)|^{\gamma_1(x)}\,dx \\
&\le 2\|\rho_1\|_{L^{p'(\cdot)}(\mathbb{R}^N)}\|u\|_{L^{p(\cdot)}(\mathbb{R}^N)} \\
&\quad + \frac{1}{(\gamma_1)_-}\|\sigma_1\|_{L^{\frac{q(\cdot)}{q(\cdot)-\gamma_1(\cdot)}}(\mathbb{R}^N)} \max\{\|u\|^{(\gamma_1)_+}_{L^{q(\cdot)}(\mathbb{R}^N)}, \|u\|^{(\gamma_1)_-}_{L^{q(\cdot)}(\mathbb{R}^N)}\} \\
&\le 2\mathcal{S}_p^{-1}\|\rho_1\|_{L^{p'(\cdot)}(\mathbb{R}^N)}\|u\|_X \\
&\quad + \max\{\mathcal{S}_q^{-(\gamma_1)_-}, \mathcal{S}_q^{-(\gamma_1)_+}\}\|\sigma_1\|_{L^{\frac{q(\cdot)}{q(\cdot)-\gamma_1(\cdot)}}(\mathbb{R}^N)} \max\{\|u\|_X^{(\gamma_1)_+}, \|u\|_X^{(\gamma_1)_-}\}.
\end{aligned}
\tag{3.6}
$$

To apply Corollary 2.6, by choosing $\mu = 1$, we observe that for each $u \in \Phi^{-1}((-\infty, 1))$,

$$
\begin{aligned}
\|u\|_X &\le \max\left\{(p_m\Phi(u))^{1/p_-}, (p_m\Phi(u))^{1/p_+}\right\} \\
&\le \max\left\{(p_m)^{1/p_-}, (p_m)^{1/p_+}\right\} = (p_m)^{1/p_+},
\end{aligned}
\tag{3.7}
$$

where $p_m := \min\{c_*, 1\}/p_+$. Denote

$$
\widetilde{\lambda} = \left(2\mathcal{S}_p^{-1}(p_m)^{\frac{1}{p_+}}\|\rho\|_{L^{p'(\cdot)}(\mathbb{R}^N)} + \max\{\mathcal{S}_q^{-(\gamma_1)_-}, \mathcal{S}_q^{-(\gamma_1)_+}\}(p_m)^{\frac{(\gamma_1)_-}{p_+}}\|\sigma_1\|_{L^{\frac{q(\cdot)}{q(\cdot)-\gamma_1(\cdot)}}(\mathbb{R}^N)}\right)^{-1},
$$

where $\mathcal{S}_p$ and $\mathcal{S}_q$ come from (3.2). Combining (3.6) with (3.7) we assert that

$$
\begin{aligned}
\sup_{u\in\Phi^{-1}((-\infty,1))} \Psi(u) &\le 2\mathcal{S}_p^{-1}(p_m)^{1/p_+}\|\rho\|_{L^{p'(\cdot)}(\mathbb{R}^N)} \\
&\quad + \max\{\mathcal{S}_q^{-(\gamma_1)_-}, \mathcal{S}_q^{-(\gamma_1)_+}\}(p_m)^{(\gamma_1)_-/p_+}\|\sigma_1\|_{L^{\frac{q(\cdot)}{q(\cdot)-\gamma_1(\cdot)}}(\mathbb{R}^N)} \\
&= \frac{1}{\widetilde{\lambda}}
\end{aligned}
$$

and so $(0, \widetilde{\lambda}) \subset \Lambda_0$.

Now, we claim that I_λ satisfies $(\mathrm{C})^{[\mu]}$-condition for all $\lambda \in \mathbb{R}$. Let μ be a fixed positive number and let $\{u_n\}$ be a Cerami sequence in X with $\Phi(u_n) < \mu$. It follows from Lemma 3.6, Remark 3.3, and (J4) that we have

$$
\begin{aligned}
\mu > \Phi(u_n) &= \int_{\mathbb{R}^N} \Phi_0(x, \nabla u_n)\,dx + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)}|u_n|^{p(x)}\,dx \\
&\ge \int_{\mathbb{R}^N} \frac{c_*}{p(x)}|\nabla u_n|^{p(x)}\,dx + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)}|u_n|^{p(x)}\,dx \ge \frac{\min\{c_*, 1\}}{\alpha}\|u_n\|_X^\alpha,
\end{aligned}
$$

where α is either p_+ or p_-. Thus, $\{u_n\}$ is bounded in X and so there is a subsequence of $\{u_n\}$ denoted again by $\{u_n\}$, such that $u_n \rightharpoonup u_0$ as $n \to \infty$ for some $u_0 \in X$. By the compactness of Ψ' stated in Lemma 3.12, we know that $\lambda\Psi'(u_n) \to \lambda\Psi'(u_0)$ as $n \to \infty$, which implies that

$$\lim_{n\to\infty} \langle \lambda\Psi'(u_n), u_n - u_0\rangle = 0. \tag{3.8}$$

From definition of the Cerami sequence and the boundedness of $\{u_n\}$, it follows that $\langle I'_\lambda(u_n), v\rangle \to 0$ as $n \to \infty$ for any $v \in X$. This implies together with $v = u_n - u_0$ that

$$\lim_{n\to\infty} \langle \Phi'(u_n) - \lambda\Psi'(u_n), u_n - u_0\rangle = 0.$$

Combining this with (3.8), we have

$$\limsup_{n\to\infty} \langle \Phi'(u_n), u_n - u_0\rangle \le \lim_{n\to\infty} \langle \lambda\Psi'(u_n), u_n - u_0\rangle = 0.$$

Since Φ is of type (S_+), we conclude that $u_n \to u_0$ in X as $n \to \infty$. Therefore, all assumptions of Corollary 2.6 are satisfied, so that, for each $\lambda \in (0, \widetilde{\lambda}) \subset \Lambda_0$ the problem (1.1) admits at least one weak solution in X. This completes the proof. □

Now, we localize the precise interval of λ for which the problem (1.1) has at least one nontrivial weak solution with the aid of Corollary 2.9.

Theorem 3.14. *Assume that* (V), (J1)–(J4), (H1) *and* (F1)–(F2) *hold. If furthermore f satisfies the following assumption:*

(F3) *There exist a real number s_0 and a positive constant r_0 with $(2\max\{1, \sup_{x\in B_N(x_0,r_0)} V(x)\}(c^*+1)\omega_N r_0^N |2s_0|^{p_+})/p_- < 1$ such that*

$$\int_{B_N(x_0,r_0)} F(x,|s_0|)\,dx > 0 \quad \textit{and} \quad F(x,t) \ge 0 \tag{3.9}$$

for almost all $x \in B_N(x_0,r_0) \setminus B_N(x_0,r_0/2)$ and for all $0 \le t \le |s_0|$, where

$$B_N(x_0,r_0) = \{x \in \mathbb{R}^N : |x - x_0| \le r_0\}$$

and

$$\begin{aligned} \frac{1}{\varrho^*} &:= 2\mathcal{S}_p^{-1} p_m^{\frac{1}{p_+}} \|\rho_1\|_{L^{p'(\cdot)}(\mathbb{R}^N)} + \max\{\mathcal{S}_q^{-(\gamma_1)_-}, \mathcal{S}_q^{-(\gamma_1)_+}\} p_m^{\frac{(\gamma_1)_-}{p_+}} \|\sigma_1\|_{L^{\frac{q(\cdot)}{q(\cdot)-\gamma_1(\cdot)}}(\mathbb{R}^N)} \\ &< \frac{p_- \operatorname{ess\,inf}_{B_N(x_0,r_0/2)} F(x,|s_0|)}{2^{N+p_++1}(c^*+1)\max\{1, \sup_{x\in B_N(x_0,r_0)} V(x)\}|s_0|^{p_+}} =: \frac{1}{\varrho_*}, \end{aligned}$$

where ω_N is the volume of $B_N(0,1)$, p_m was given in the proof of Theorem 3.13*, and c^* and c_* are given in* (J2), (J4)*, respectively. Then, for every $\lambda \in \Lambda^* := (\varrho_*, \varrho^*]$ the problem* (1.1) *has at least one nontrivial weak solution.*

Proof. Thanks to Lemmas 3.10, 3.12 and Corollary 3.11, it suffices to show the condition (2.11) in Corollary 2.9 hold. To verify the assumption (2.11), let $s_0 \neq 0$. Fix $\sigma \in (0,1)$ and define

$$\widetilde{u}_1(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}^N \setminus B_N(x_0, r_0), \\ |s_0| & \text{if } x \in B_N(x_0, r_0/2), \\ \frac{2|s_0|}{r_0}(r_0 - |x - x_0|) & \text{if } x \in B_N(x_0, r_0) \setminus B_N(x_0, r_0/2), \end{cases} \tag{3.10}$$

then it is obvious that $0 \leq \widetilde{u}_1(x) \leq |s_0|$ for all $x \in \mathbb{R}^N$ and $\widetilde{u}_1 \in X$. Then it follows from Lemma 3.6, Remark 3.3 and due to the condition (J4), we have

$$\begin{aligned} \Phi(\widetilde{u}_1) &= \int_{\mathbb{R}^N} \Phi_0(x, \nabla \widetilde{u}_1)\, dx + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)} |\widetilde{u}_1|^{p(x)}\, dx \\ &\geq \frac{\min\{c_*, V_-, 1\}}{p_+} \left(\int_{\mathbb{R}^N} |\nabla \widetilde{u}_1|^{p(x)}\, dx + \int_{\mathbb{R}^N} |\widetilde{u}_1|^{p(x)}\, dx \right) \\ &\geq \frac{\min\{c_*, V_-, 1\}}{p_+} |2s_0|^{p_-} \left(1 - \left(\frac{1}{2}\right)^N \right) r_0^{N-p_-} \omega_N > 0, \end{aligned}$$

where ω_N is the volume of $B_N(0,1)$ and

$$\begin{aligned} \Phi(\widetilde{u}_1) &\leq \frac{(c^*+1)}{p_-} \left(\int_{\mathbb{R}^N} |\nabla \widetilde{u}_1|^{p(x)}\, dx + \int_{\mathbb{R}^N} V(x) |\widetilde{u}_1|^{p(x)}\, dx \right) \\ &\leq \frac{(c^*+1) \max\{1, \sup_{x \in B_N(x_0,r_0)} V(x)\}}{p_-} \\ &\quad \times \left(\int_{B_N(x_0,r_0)} |\nabla \widetilde{u}_1|^{p(x)}\, dx + \int_{B_N(x_0,r_0)} |\widetilde{u}_1|^{p(x)}\, dx \right) \\ &\leq \frac{(c^*+1) \max\{1, \sup_{x \in B_N(x_0,r_0)} V(x)\} |s_0|^{p_+} \omega_N r_0^N}{p_-} \left(2^{p_+}(1 - 2^{-N}) + 1 \right) \\ &< \frac{2(c^*+1) \max\{1, \sup_{x \in B_N(x_0,r_0)} V(x)\} |2s_0|^{p_+} \omega_N r_0^N}{p_-} < 1. \end{aligned}$$

Owing to the assumption (F3), we deduce that

$$\begin{aligned} \Psi(\widetilde{u}_1) &\geq \int_{B_N(x_0, r_0/2)} F(x, \widetilde{u}_1)\, dx \\ &\geq \operatorname*{ess\,inf}_{B_N(x_0, r_0/2)} F(x, |s_0|) \left(\frac{\omega_N r_0^N}{2^N} \right) \end{aligned}$$

and thus

$$\frac{\Psi(\widetilde{u}_1)}{\Phi(\widetilde{u}_1)} \geq \frac{p_- \operatorname{ess\,inf}_{B_N(x_0,r_0/2)} F(x, |s_0|)}{2^{N+p_++1}(c^*+1) \max\{1, \sup_{x \in B_N(x_0,r_0)} V(x)\} |s_0|^{p_+}}. \tag{3.11}$$

Taking into account the relations (3.6) and (3.7), we derive that

$$\sup_{u\in\Phi^{-1}((-\infty,1])} \Psi(u)$$
$$\leq \left\{ 2\mathcal{S}_p^{-1} p_m^{\frac{1}{p_+}} \|\rho_1\|_{L^{p'(\cdot)}(\mathbb{R}^N)} + \max\{\mathcal{S}_q^{-(\gamma_1)_-}, \mathcal{S}_q^{-(\gamma_1)_+}\} p_m^{\frac{(\gamma_1)_-}{p_+}} \|\sigma_1\|_{L^{\frac{q(\cdot)}{q(\cdot)-\gamma_1(\cdot)}}(\mathbb{R}^N)} \right\}.$$

From (3.11) and due to the assumption (F3), we infer

$$\sup_{u\in\Phi^{-1}((-\infty,1])} \Psi(u) < \frac{\Psi(\widetilde{u}_1)}{\Phi(\widetilde{u}_1)}.$$

Therefore, we conclude that $(\varrho_*, \varrho^*) \subseteq \left(\frac{\Phi(\widetilde{u}_1)}{\Psi(\widetilde{u}_1)}, \frac{1}{\sup_{u\in\Phi^{-1}((-\infty,1])}\Psi(u)} \right)$. Since the functional I_λ satisfies $(\mathrm{C})^{[\mu]}$-condition, it follows from Corollary 2.9 with $\mu = 1$ that for each $\lambda \in (\varrho_*, \varrho^*)$, the problem (1.1) admits at least one nontrivial weak solution u_λ such that

$$0 < \Phi(u_\lambda) < 1. \tag{3.12}$$

To complete the proof, from the similar argument as in the proof of [7, Theorem 3.1], we will show that (1.1) with $\lambda = \varrho^*$ has at least one nontrivial solution. By the relation (3.12), for each $\lambda \in (\varrho_*, \varrho^*)$, we obtain a nontrivial solution u_λ for the problem (1.1) in $\Phi^{-1}((0,1))$ such that $I_\lambda(u_\lambda) \leq I_\lambda(v)$ for all $v \in \Phi^{-1}((0,1))$. Taking into account this, we have a sequence $\{v_n\} \subset \Phi^{-1}((0,1))$ such that $\|v_n\|_X \to 0$ as $n \to \infty$ and

$$I_\lambda(u_\lambda) \leq I_\lambda(v_n).$$

By the continuity of I_λ, we assert

$$I_\lambda(u_\lambda) \leq 0 \quad \text{for all } \lambda \in (\varrho_*, \varrho^*). \tag{3.13}$$

Fix $\lambda_0 \in (\varrho_*, \varrho^*)$. Then we can choose a sequence $\{\lambda_n\}$ such that $\varrho_* < \lambda_0 < \lambda_n < \varrho^*$ and $\lambda_n \to \varrho^*$ as $n \to \infty$. Then there exists a corresponding sequence $\{u_{\lambda_n}\}$ in $\Phi^{-1}((0,1))$ such that u_{λ_n} is a minimal point for the problem (1.1). In view of Definition 3.8, we have

$$\langle \Phi'(u_{\lambda_n}), v\rangle = \langle \lambda_n \Psi'(u_{\lambda_n}), v\rangle \tag{3.14}$$

for any $v \in X$. Since $\{u_{\lambda_n}\}$ is bounded, we may suppose that $u_{\lambda_n} \rightharpoonup u^*$ in X and hence $\lambda_n\Psi'(u_{\lambda_n}) \to \varrho^*\Psi'(u^*)$ as $n \to \infty$ due to the compactness of Ψ'. It follows from (3.14) with $v = u_{\lambda_n} - u^*$ that we obtain

$$\limsup_{n\to\infty} \langle \Phi'(u_{\lambda_n}), u_{\lambda_n} - u^*\rangle = \lim_{n\to\infty} \langle \lambda_n \Psi'(u_{\lambda_n}), u_{\lambda_n} - u^*\rangle = 0.$$

Since Φ is of type (S_+), we conclude that $u_{\lambda_n} \to u^*$ in X as $n \to \infty$. Taking the limit of both sides in (3.14) gives

$$\langle \Phi'(u^*), v \rangle = \langle \varrho^* \Psi'(u^*), v \rangle$$

for all $v \in X$. Therefore, u^* is a weak solution for (1.1) with $\lambda = \varrho^*$. Now, we claim that $u^* \neq 0$. Note that every u_λ is a minimal point for I_λ and $I_\lambda(u_\lambda) \leq I_\lambda(v)$ for all $v \in \Phi^{-1}((0,1))$ and all $\lambda \in (\varrho_*, \varrho^*)$. Since $u_\lambda \in \Phi^{-1}((0,1))$ for every $\lambda \in (\varrho_*, \varrho^*)$, we see that

$$I_{\lambda_n}(u_{\lambda_n}) \leq I_{\lambda_n}(u_{\lambda_0}) \quad \text{and} \quad I_{\lambda_0}(u_{\lambda_0}) \leq I_{\lambda_0}(u_{\lambda_n}) \quad \text{for all } n \in \mathbb{N}.$$

According to $\lambda_0 < \lambda_n$ for all $n \in \mathbb{N}$, these inequalities imply that

$$\Psi(u_{\lambda_0}) \leq \Psi(u_{\lambda_n}) \quad \text{for all } n \in \mathbb{N},$$

and therefore

$$\Psi(u_{\lambda_0}) \leq \Psi(u^*). \tag{3.15}$$

In fact, if we let $u^* = 0$ in (3.15), then we deduce $\Psi(u_{\lambda_0}) \leq 0$ and thus $I_{\lambda_0}(u_{\lambda_0}) > 0$ which contradicts (3.13). This completes the proof. □

3.3. Existence of multiple solutions

In this subsection we deal with the existence of at least two weak solutions for the problem (1.1) as applications of critical points theorems in the previous section. Applying our critical point theorem, we determine precisely the interval of λ's for which (1.1) has at least two nontrivial weak solutions.

Let us put $F(x,t) = \int_0^t f(x,s)\,ds$. For $p(x) < N$, we assume that

(H2) $\gamma_2 \in C_+(\mathbb{R}^N)$ and $1 < p_- \leq p_+ < (\gamma_2)_- \leq (\gamma_2)_+ < p^*(x)$ for all $x \in \mathbb{R}^N$;

(F4) there exist nonnegative functions $\rho_2 \in L^{\gamma_2'(\cdot)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\sigma_2 \in L^\infty(\mathbb{R}^N)$ such that

$$|f(x,t)| \leq \rho_2(x) + \sigma_2(x)|t|^{\gamma_2(x)-1}$$

holds for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$;

(F5) $\lim_{|t|\to\infty} F(x,t)/|t|^{p_+} = \infty$ uniformly for almost all $x \in \mathbb{R}^N$;

(F6) there exists a constant $\chi \geq 1$ such that

$$\chi \mathcal{F}(x,t) \geq \mathcal{F}(x,st)$$

holds for $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and $s \in [0,1]$, where $\mathcal{F}(x,t) = f(x,t)t - \theta F(x,t)$ and θ was given in (J5).

We give the following example satisfying the assumption (F6) which is weaker than (AR)-condition.

Example 3.15. Let us consider

$$f(x,t) = |t|^{\gamma_2(x)-2}t\log(1+|t|)$$

for all $t \in \mathbb{R}$. It is clear that the function f satisfies conditions (F1), (F3)–(F5) hold. Since the following relation

$$\frac{f(x,t)}{|t|^{p_+-2}t} = \frac{|t|^{\gamma_2(x)-2}t\log(1+|t|)}{|t|^{p_+-2}t} = |t|^{\gamma_2(x)-p_+}\log(1+|t|)$$

is increasing in $t > 0$ and decreasing in $t < 0$ if $\gamma_2(x) > p_+ = \theta$ for all $x \in \mathbb{R}^N$, it follows that assumption (F6) holds.

Remark 3.16. One of the key assumptions for proving that the functional I_λ satisfies (C)-condition for $\lambda > 0$ is

$$f(x,t) = o(|t|^{p_+-1}) \quad \text{as } |t| \to 0 \text{ uniformly for } x \in \mathbb{R}^N. \tag{3.16}$$

However, we prove the following result without the assumption (3.16).

With the aid of Lemmas 3.10 and 3.12, we prove that the energy functional I_λ satisfies (C)-condition.

Lemma 3.17. *Assume that the conditions* (V), (H2), (J1)–(J5), (F1) *and* (F4)–(F6) *hold. Then I_λ satisfies the (C)-condition for any $\lambda > 0$.*

Proof. Proceeding an argument analogous to Lemma 3.12, we know that the functionals Ψ' is weakly strongly continuous on X. Combining this and Lemma 3.10, it follows that I'_λ is a mapping of type (S_+). Let $\{u_n\}$ be a (C)-sequence for I_λ in X, that is, $\{I_\lambda(u_n)\}$ is bounded and $\|I'_\lambda(u_n)\|_{X^*}(1+\|u_n\|_X) \to 0$ as $n \to \infty$. Then there exists a positive constant $\mathcal{C}$ such that $\sup|I_\lambda(u_n)| \le \mathcal{C}$ and $\langle I'_\lambda(u_n), u_n\rangle = o(1)$. Since X is reflexive and I'_λ is of type (S_+) according to Lemmas 3.1 and 3.10, it suffices to verify that the sequence $\{u_n\}$ is bounded in X. In fact, if the sequence $\{u_n\}$ is unbounded in X, then we may assume that $\|u_n\|_X > 1$ and $\|u_n\|_X \to \infty$ as $n \to \infty$. Define a sequence $\{w_n\}$ by $w_n = u_n/\|u_n\|_X$. Then it is obvious that $\{w_n\} \subset X$ and $\|w_n\|_X = 1$. Hence, up to a subsequence, still denoted by $\{w_n\}$, we obtain $w_n \rightharpoonup w$ in X as $n \to \infty$ and by Lemma 3.7

$$w_n(x) \to w(x) \text{ a.e. in } \mathbb{R}^N \text{ and } w_n \to w \text{ in } L^{\gamma_2(\cdot)}(\mathbb{R}^N) \text{ as } n \to \infty. \tag{3.17}$$

Set $\Omega_1 = \{x \in \mathbb{R}^N : w(x) \neq 0\}$. Using the assumption (J4) and Remark 3.3, we assert that

$$\begin{aligned} I_\lambda(u_n) &= \int_{\mathbb{R}^N} \Phi_0(x, \nabla u_n)\, dx + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)} |u_n|^{p(x)}\, dx - \lambda \int_{\mathbb{R}^N} F(x, u_n)\, dx \\ &\geq \frac{\min\{c_*, 1\}}{p_+} \|u_n\|_X^{p_-} - \lambda \int_{\mathbb{R}^N} F(x, u_n)\, dx. \end{aligned}$$

Since the sequence $\{u_n\}$ is the (C)-sequence, we have

$$\int_{\mathbb{R}^N} F(x, u_n)\, dx \geq \frac{\min\{c_*, 1\}}{\lambda p_+} \|u_n\|_X^{p_-} - \frac{1}{\lambda} I_\lambda(u_n) \to \infty \quad \text{as } n \to \infty. \tag{3.18}$$

From Lemma 3.1 and the assumption (J2), we deduce that

$$\begin{aligned} I_\lambda(u_n) &\leq \frac{c^*}{p_-} \int_{\mathbb{R}^N} |\nabla u_n|^{p(x)}\, dx + \frac{1}{p_-} \int_{\mathbb{R}^N} V(x) |u_n|^{p(x)}\, dx - \lambda \int_{\mathbb{R}^N} F(x, u_n)\, dx \\ &\leq (c^* + 1) \|u_n\|_X^{p_+} - \lambda \int_{\mathbb{R}^N} F(x, u_n)\, dx. \end{aligned}$$

Then we obtain

$$c^* + 1 \geq \frac{1}{\|u_n\|_X^{p_+}} \left(I_\lambda(u_n) + \lambda \int_{\mathbb{R}^N} F(x, u_n)\, dx \right) \tag{3.19}$$

for sufficiently large n. The condition (F5) implies that there exists $t_0 > 1$ such that $F(x,t) > |t|^{p_+}$ for all $x \in \mathbb{R}^N$ and $|t| > t_0$. According to the assumptions (F1) and (F4), there exists a positive constant M such that $|F(x,t)| \leq M$ for almost all $x \in \mathbb{R}^N$ and for all $t \in [-t_0, t_0]$. Therefore we can choose a real number M_0 such that $F(x,t) \geq M_0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, and thus

$$\frac{F(x, u_n(x)) - M_0}{\|u_n\|_X^{p_+}} \geq 0 \tag{3.20}$$

for all $x \in \mathbb{R}^N$ and for all $n \in \mathbb{N}$. Taking (3.17) into account, we know that $|u_n(x)| = |w_n(x)| \|u_n\|_X \to \infty$ as $n \to \infty$ for all $x \in \Omega_1$. Furthermore, it follows from the condition (F5) that

$$\lim_{n\to\infty} \frac{F(x, u_n(x))}{\|u_n\|_X^{p_+}} = \lim_{n\to\infty} \frac{F(x, u_n(x))}{|u_n(x)|^{p_+}} |w_n(x)|^{p_+} = \infty \tag{3.21}$$

for all $x \in \Omega_1$. Hence we get that $\operatorname{meas}(\Omega_1) = 0$. In fact, if $\operatorname{meas}(\Omega_1) \neq 0$, then according to (3.18), (3.19), (3.20), (3.21) and the Fatou lemma, we assert

$$\begin{aligned} \frac{1}{\lambda}(c^* + 1) &\geq \int_{\Omega_1} \liminf_{n\to\infty} \frac{F(x, u_n(x))}{|u_n(x)|^{p_+}} |w_n(x)|^{p_+}\, dx - \int_{\Omega_1} \limsup_{n\to\infty} \frac{M_0}{\|u_n\|_X^{p_+}}\, dx \\ &= \infty, \end{aligned}$$

which is a contradiction. Thus $w(x) = 0$ for almost all $x \in \mathbb{R}^N$. Since $I_\lambda(tu_n)$ is continuous in $t \in [0,1]$, for each $n \in \mathbb{N}$ there is $t_n \in [0,1]$ such that

$$I_\lambda(t_n u_n) := \max_{t\in[0,1]} I_\lambda(tu_n).$$

Let $\{d_k\}$ be a positive sequence of real numbers such that $\lim_{k\to\infty} d_k = \infty$ and $d_k > 1$ for any k. Then it is clear that $\|d_k w_n\|_X = d_k > 1$ for any k and n. Fix k, since $w_n \to 0$ strongly in the spaces $L^{\gamma_2(\cdot)}(\mathbb{R}^N)$ as $n \to \infty$, it follows from the continuity of the Nemytskii operator that $F(x, d_k w_n) \to 0$ in $L^1(\mathbb{R}^N)$ as $n \to \infty$. Hence we assert

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(x, d_k w_n)\, dx = 0. \tag{3.22}$$

Since $\|u_n\|_X \to \infty$ as $n \to \infty$, we have $\|u_n\|_X > d_k$ for sufficiently large n. Thus we know by (J4), Remark 3.3 and (3.22) that

$$\begin{aligned} I_\lambda(t_n u_n) &\geq I_\lambda\left(\frac{d_k}{\|u_n\|_X} u_n\right) = I_\lambda(d_k w_n) \\ &= \int_{\mathbb{R}^N} \Phi_0(x, \nabla d_k w_n)\, dx + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)} |d_k w_n|^{p(x)}\, dx - \lambda \int_{\mathbb{R}^N} F(x, d_k w_n)\, dx \\ &\geq C_5 \|d_k w_n\|_X^{p_-} - \lambda \int_{\mathbb{R}^N} F(x, d_k w_n)\, dx \\ &\geq \frac{C_5}{2} d_k^{p_-} \end{aligned}$$

for a positive constant C_5 and for sufficiently large n. Then letting n and k tend to infinity, it follows that

$$\lim_{n\to\infty} I_\lambda(t_n u_n) = \infty. \tag{3.23}$$

Since $I_\lambda(0) = 0$ and $|I_\lambda(u_n)| \leq \mathcal{C}$, it is trivial that $t_n \in (0,1)$, and $\langle I'_\lambda(t_n u_n), t_n u_n\rangle = o(1)$. Therefore, due to the assumptions (J5) and (F6), for n large enough we deduce that

$$\begin{aligned} &\frac{1}{\chi} I_\lambda(t_n u_n) \\ &= \frac{1}{\chi} I_\lambda(t_n u_n) - \frac{1}{\theta\chi} \langle I'_\lambda(t_n u_n), t_n u_n\rangle \\ &= \frac{1}{\chi} \int_{\mathbb{R}^N} \Phi_0(x, t_n \nabla u_n)\, dx + \frac{1}{\chi} \int_{\mathbb{R}^N} \frac{V(x)}{p(x)} |t_n u_n|^{p(x)}\, dx - \frac{\lambda}{\chi} \int_{\mathbb{R}^N} F(x, t_n u_n)\, dx \\ &\quad - \frac{1}{\theta\chi} \int_{\mathbb{R}^N} \varphi(x, t_n \nabla u_n) \cdot (t_n \nabla u_n)\, dx - \frac{1}{\theta\chi} \int_{\mathbb{R}^N} V(x) |t_n u_n|^{p(x)}\, dx \\ &\quad + \frac{\lambda}{\theta\chi} \int_{\mathbb{R}^N} f(x, t_n u_n) t_n u_n\, dx \\ &= \frac{1}{\theta\chi} \int_{\mathbb{R}^N} \mathcal{H}(x, t_n \nabla u_n)\, dx + \frac{1}{\chi} \int_{\mathbb{R}^N} \frac{V(x)}{p(x)} |t_n u_n|^{p(x)}\, dx \end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\theta\chi}\int_{\mathbb{R}^N} |t_n u_n|^{p(x)}\,dx + \frac{\lambda}{\theta\chi}\int_{\mathbb{R}^N} \mathcal{F}(x, t_n u_n)\,dx \\
&\le \frac{1}{\theta}\int_{\mathbb{R}^N} \mathcal{H}(x, \nabla u_n)\,dx + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)}|u_n|^{p(x)}\,dx - \frac{1}{\theta}\int_{\mathbb{R}^N} V(x)|u_n|^{p(x)}\,dx \\
&\quad + \frac{\lambda}{\theta}\int_{\mathbb{R}^N} \mathcal{F}(x, u_n)\,dx \\
&= \int_{\mathbb{R}^N} \Phi_0(x, \nabla u_n)\,dx + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)}|u_n|^{p(x)}\,dx - \lambda\int_{\mathbb{R}^N} F(x, u_n)\,dx \\
&\quad - \frac{1}{\theta}\left(\int_{\mathbb{R}^N} \varphi(x, \nabla u_n)\cdot\nabla u_n\,dx + \int_{\mathbb{R}^N} V(x)|u_n|^{p(x)}\,dx - \lambda\int_{\mathbb{R}^N} f(x, u_n)u_n\,dx\right) \\
&= I_\lambda(u_n) - \frac{1}{\theta}\langle I_\lambda'(u_n), u_n\rangle \\
&\le \mathcal{C} \quad \text{as } n \to \infty,
\end{aligned}
$$

which contradicts (3.23). This completes the proof. □

First we apply Corollary 2.7 to establish the existence of two distinct weak solutions for the problem (1.1).

Theorem 3.18. *Assume that* (V), (J1)–(J5), (H2), (F1) *and* (F4)–(F6) *hold. Then there exists a positive constant* $\widetilde{\lambda}$ *such that the problem* (1.1) *admits at least two distinct weak solutions in* X *for each* $\lambda \in (0, \widetilde{\lambda})$.

Proof. It is obvious that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. By the condition (F5), for any $M > 0$, there exists a constant $\delta > 0$ such that

$$
F(x, \eta) \ge M|\eta|^{p_+} \tag{3.24}
$$

for $|\eta| > \delta$ and for almost all $x \in \mathbb{R}^N$. Take $v \in X \setminus \{0\}$. Then the relation (3.24) implies that

$$
\begin{aligned}
I_\lambda(tv) &= \int_{\mathbb{R}^N} \Phi_0(x, t\nabla v)\,dx + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)}|tv|^{p(x)}\,dx - \lambda\int_{\mathbb{R}^N} F(x, tv)\,dx \\
&\le \int_{\mathbb{R}^N} \frac{c^*}{p(x)}|t\nabla v|^{p(x)}\,dx + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)}|tv|^{p(x)}\,dx - \lambda\int_{\mathbb{R}^N} F(x, tv)\,dx \\
&\le t^{p_+}\left(\int_{\mathbb{R}^N} \frac{c^*}{p(x)}|\nabla v|^{p(x)}\,dx + \int_{\mathbb{R}^N} \frac{V(x)}{p(x)}|v|^{p(x)}\,dx - \lambda M\int_{\mathbb{R}^N} |v|^{p_+}\,dx\right)
\end{aligned}
$$

for sufficiently large $t > 1$. If M is large enough, then we assert that $I_\lambda(tv) \to -\infty$ as $t \to \infty$. Hence the functional I_λ is unbounded from below.

The assumption (F4) and the relation (3.2) imply that

$$
\begin{aligned}
\Psi(u) &= \int_{\mathbb{R}^N} F(x,u)\,dx \\
&\le \int_{\mathbb{R}^N} \rho_2(x)|u(x)| + \frac{\sigma_2(x)}{\gamma_2(x)}|u(x)|^{\gamma_2(x)}\,dx \\
&\le 2\|\rho_2\|_{L^{\gamma_2'(\cdot)}(\mathbb{R}^N)}\|u\|_{L^{\gamma_2(\cdot)}(\mathbb{R}^N)} \\
&\quad + \frac{\|\sigma_2\|_{L^\infty(\mathbb{R}^N)}}{(\gamma_2)_-}\max\{\|u\|^{(\gamma_2)_+}_{L^{\gamma_2(\cdot)}(\mathbb{R}^N)}, \|u\|^{(\gamma_2)_-}_{L^{\gamma_2(\cdot)}(\mathbb{R}^N)}\} \\
&\le 2\mathcal{S}_{\gamma_2}^{-1}\|\rho_2\|_{L^{\gamma_2'(\cdot)}(\mathbb{R}^N)}\|u\|_X \\
&\quad + \max\{\mathcal{S}_{\gamma_2}^{-(\gamma_2)_-}, \mathcal{S}_{\gamma_2}^{-(\gamma_2)_+}\}\|\sigma_2\|_{L^\infty(\mathbb{R}^N)}\max\{\|u\|_X^{(\gamma_2)_+}, \|u\|_X^{(\gamma_2)_-}\}.
\end{aligned}
\tag{3.25}
$$

For each $u \in \Phi^{-1}((-\infty,1))$, it follows from (3.7) that $\|u\|_X \le (p_m)^{1/p_+} < 1$. Let us denote

$$
\widetilde{\lambda} := \left(2\mathcal{S}_{\gamma_2}^{-1}(p_m)^{\frac{1}{p_+}}\|\rho_2\|_{L^{\gamma_2'(\cdot)}(\mathbb{R}^N)} + \max\{\mathcal{S}_{\gamma_2}^{-(\gamma_2)_-}, \mathcal{S}_{\gamma_2}^{-(\gamma_2)_+}\}(p_m)^{\frac{(\gamma_2)_-}{p_+}}\|\sigma_2\|_{L^\infty(\mathbb{R}^N)}\right)^{-1}.
$$

Taking into account (3.25) we assert that

$$
\begin{aligned}
\sup_{u\in\Phi^{-1}((-\infty,1))}\Psi(u) &\le 2\mathcal{S}_{\gamma_2}^{-1}(p_m)^{1/p_+}\|\rho_2\|_{L^{\gamma_2'(\cdot)}(\mathbb{R}^N)} \\
&\quad + \max\{\mathcal{S}_{\gamma_2}^{-(\gamma_2)_-}, \mathcal{S}_{\gamma_2}^{-(\gamma_2)_+}\}(p_m)^{(\gamma_2)_-/p_+}\|\sigma_2\|_{L^\infty(\mathbb{R}^N)} \\
&= \frac{1}{\widetilde{\lambda}}.
\end{aligned}
$$

According to Lemma 3.17, we have that the functional I_λ satisfies (C)-condition for each $\lambda > 0$.

To apply Corollary 2.7, by choosing $\mu = 1$, we see that $(0,\widetilde{\lambda}) \subset \Lambda_0$. Therefore, all assumptions of Corollary 2.7 are satisfied, so that, for each $\lambda \in (0,\widetilde{\lambda}) \subset \Lambda_0$ the problem (1.1) admits at least two distinct weak solutions in X. This completes the proof. □

Next, we show that the problem (1.1) has at least two nontrivial weak solutions provided that λ is suitable, by applying Corollary 2.10. The conclusion of the following theorem is more precise that of Theorem 3.18.

Theorem 3.19. *Assume that* (V), (J1)–(J4), (H2), (F1) *and* (F4)–(F6) *hold. If furthermore f satisfies the following assumption:*

(F7) *There exist positive constants r_0 and $s_0 \in \mathbb{R}$ with* $\frac{\min\{c_*, V_-, 1\}}{p_+}|2s_0|^{p_-}(1-2^{-N})r_0^{N-p_-}\omega_N < 1$ *such that* (3.9) *for almost all $x \in B_N(x_0,r_0) \setminus B_N(x_0,r_0/2)$ and for all*

$0 \le t \le |s_0|$ *and*

$$\frac{1}{\varrho^{**}} := 2\mathcal{S}_{\gamma_2}^{-1} p_m^{\frac{1}{p_+}} \|\rho_2\|_{L^{\gamma_2'(\cdot)}(\mathbb{R}^N)} + \max\{\mathcal{S}_{\gamma_2}^{-(\gamma_2)_-}, \mathcal{S}_{\gamma_2}^{-(\gamma_2)_+}\} p_m^{\frac{(\gamma_2)_-}{p_+}} \|\sigma_2\|_{L^\infty(\mathbb{R}^N)}$$
$$< \frac{p_- \operatorname{ess\,inf}_{B_N(x_0,r_0/2)} F(x,|s_0|)}{2^{N+p_++1}(c^*+1)\max\{1, \sup_{x\in B_N(x_0,r_0)} V(x)\}|s_0|^{p_+}} =: \frac{1}{\varrho_{**}},$$

where ω_N *is the volume of* $B_N(0,1)$ *and* c^*, c_* *are given in* (J2), (J4), *respectively. Then, for every* $\lambda \in \widetilde{\Lambda} := (\varrho_{**}, \varrho^{**}]$, *the problem* (1.1) *has at least two distinct nontrivial weak solutions.*

Proof. Let us define $\widetilde{u}_2$ as in (3.10). Then it follows from (F7) and the same arguments as in Theorem 3.14 that $\widetilde{u}_2 \in X$,

$$0 < \Phi(\widetilde{u}_2) < \frac{2(c^*+1)\max\{1, \sup_{x\in B_N(x_0,r_0)} V(x)\}|2s_0|^{p_+}\omega_N r_0^N}{p_-} < 1,$$

and

$$\frac{\Psi(\widetilde{u}_2)}{\Phi(\widetilde{u}_2)} \ge \frac{p_- \operatorname{ess\,inf}_{B_N(x_0,r_0/2)} F(x,|s_0|)}{2^{N+p_++1}(c^*+1)\max\{1, \sup_{x\in B_N(x_0,r_0)} V(x)\}|s_0|^{p_+}}. \tag{3.26}$$

As in the proof of Theorem 3.18, for each $u \in \Phi^{-1}((-\infty,1])$, we deduce that

$$\sup_{u\in\Phi^{-1}((-\infty,1])} \Psi(u) \le 2\mathcal{S}_{\gamma_2}^{-1}(p_m)^{1/p_+}\|\rho_2\|_{L^{\gamma_2'(\cdot)}(\mathbb{R}^N)} + \max\{\mathcal{S}_{\gamma_2}^{-(\gamma_2)_-}, \mathcal{S}_{\gamma_2}^{-(\gamma_2)_+}\}(p_m)^{(\gamma_2)_-/p_+}\|\sigma_2\|_{L^\infty(\mathbb{R}^N)}.$$

By (3.26) and the assumption (F7), we infer

$$\sup_{u\in\Phi^{-1}((-\infty,1])} \Psi(u) < \frac{\Psi(\widetilde{u}_2)}{\Phi(\widetilde{u}_2)}$$

and thus

$$(\varrho_{**}, \varrho^{**}) \subseteq \left(\frac{\Phi(\widetilde{u}_2)}{\Psi(\widetilde{u}_2)}, \frac{1}{\sup_{u\in\Phi^{-1}((-\infty,1])}\Psi(u)}\right).$$

Combining Lemma 3.17 and Corollary 2.10 with $\mu = 1$, we conclude that for each $\lambda \in (\varrho_{**}, \varrho^{**})$, the problem (1.1) has at least two nontrivial weak solutions. The analogous argument as in the proof of Theorem 3.14 implies that the problem (1.1) with $\lambda = \varrho^{**}$ has at least one nontrivial solutions u^{**} satisfying $I'_{\varrho^{**}}(u^{**}) = 0$ and $I_{\varrho^{**}}(u^{**}) \le I_{\varrho^{**}}(v)$ for all $v \in \Phi^{-1}((0,1))$. Since I_{ϱ^*} is unbounded from below, it is not strictly global and the mountain pass theorem ensures that there is a nontrivial weak solution u_1^{**} which is different from u^{**}. This completes the proof. □

A. Appendix

In this section, we investigate the Pucci-Serrin theorem for a Gâteaux differentiable functional satisfying (C)-condition in place of (PS)-condition via Corollary 2.2 due to Zhong's Ekeland variational principle [43]. For our aim, we give the Mountain pass theorem for a functional satisfying (C)-condition which may be found in [42]. For the case of (PS)-condition, we refer to [2].

Lemma A.1. [20] *Assume that a function $I\colon X \to \mathbb{R}$ is continuously Gâteaux differentiable and bounded from below. Assume that I satisfies (C)-condition. If, in addition, there exist $x_1, x_2 \in X$ with $x_1 \neq x_2$ and $r \in (0, \|x_2 - x_1\|_X)$ such that*

$$\inf_{u\in X}\{I(u) : \|u - x_1\|_X = r\} > \max\{I(x_1), I(x_2)\},$$

then

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) > \max\{I(x_1), I(x_2)\}$$

is a critical value for I, where Γ is the family of paths $\gamma\colon [0,1] \to (X, \|\cdot\|_X)$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$.

Theorem A.2. *Assume that a function $I\colon X \to \mathbb{R}$ is continuously Gâteaux differentiable. Assume that I satisfies (C)-condition. If I has two different local minima in X with respect to the norm $\|\cdot\|_X$, then it has at least one more critical point in X.*

Proof. Suppose that I has two different local minima. Then there exist $x_1, x_2 \in X$ with $x_1 \neq x_2$ and $r_0 > 0$ such that

$$I(x_i) \le I(u) \text{ if } u \in X \quad \text{and} \quad \|u - x_i\|_X \le r_0 \text{ for } i = 1, 2.$$

We may suppose without loss of generality that $I(x_2) \le I(x_1)$. Then, it follows that there are two cases to consider. If there exists $r \in (0, r_0)$ such that

$$I(x_1) < \inf_{u\in X}\{I(u) : \|u - x_1\|_X = r\} =: \alpha,$$

then Lemma A.1 ensures that I admits a third critical point.

Otherwise, we have $\alpha = I(x_1)$ for every r. Fix $r \in (0, r_0)$. Then we can find a sequence $\{u_n\}$ in X such that $I(u_n) \to I(x_1)$ as $n \to \infty$. Let $\{\varepsilon_n\}$ be a positive sequence defined by

$$\varepsilon_n := \begin{cases} \left\{\ln\left(1 + \frac{1}{n(1+\|u_n\|_X)}\right)\right\}^2 & \text{if } I(u_n) = \alpha, \\ 2(I(u_n) - \alpha) & \text{if } I(u_n) > \alpha. \end{cases}$$

Then we have

$$\|u_n - x_1\|_X = r \quad \text{and} \quad I(u_n) < I(x_1) + \varepsilon_n \quad \text{for all } n \in \mathbb{N}$$

and we can choose a positive number δ such that $0 < r - \delta < r + \delta < r_0$. We point out that

$$\inf_{u \in X} \{I(u) : r - \delta \leq \|u - x_1\|_X \leq r + \delta\} = I(x_1).$$

Denote a positive sequence $\{\lambda_n\}$ by $\lambda_n := \sqrt{\varepsilon_n}$. It follows from Corollary 2.2 with $x_0 := u_n$ that there exists a sequence $\{z_n\}$ in X such that for all $n \in \mathbb{N}$,

$$\begin{aligned} r - \delta \leq \|z_n - x_1\|_X &\leq r + \delta, \\ I(z_n) \leq I(u_n) &\leq I(x_1) + \varepsilon_n, \\ \|u_n - z_n\|_X &\leq (1 + \|u_n\|_X)(e^{\sqrt{\varepsilon_n}} - 1), \end{aligned} \tag{A.1}$$

and for every $w \in X$ such that $r - \delta \leq \|w - x_1\|_X \leq r + \delta$,

$$I(w) \geq I(z_n) - \frac{\varepsilon_n}{\lambda_n(1 + \|z_n\|_X)} \|w - z_n\|_X. \tag{A.2}$$

According to the inequalities in (A.1), we see that

$$\frac{\|u_n - z_n\|_X}{1 + \|u_n\|_X} \leq e^{\sqrt{\varepsilon_n}} - 1 \to 0 \quad \text{as } n \to \infty.$$

By the boundedness of $\{u_n\}$, we have $\|u_n - z_n\|_X \to 0$ as $n \to \infty$ and so $r - \delta < \|z_n - x_1\|_X < r + \delta$ for large n. Hence, if we consider $h \in X$ with $\|h\|_X \leq 1$, we may suppose that $w := z_n + th$ satisfies $r - \delta \leq \|w - x_1\|_X \leq r + \delta$ for sufficiently small $t > 0$. By taking the limits as $t \to 0$ in (A.2), we have

$$|\langle I'(z_n), h\rangle| \leq \frac{\sqrt{\varepsilon_n}}{1 + \|z_n\|_X} \|h\|_X \tag{A.3}$$

for n large enough. Since I satisfies (C)-condition, we find a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \to z$ for some $z \in X$ as $k \to \infty$. From the continuity of I' and (A.3) it follows that $I'(z_{n_k}) \to I'(z) = 0$ as $k \to \infty$, which asserts that z is a critical point for I. Therefore, we have shown that z is different from x_1 because $\|z - x_1\|_X = r$. As a similar argument from (A.1), we also infer that z is different from x_2. This completes the proof. □

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