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Bound Estimations on the Eigenvalues for Fan Product of M-tensors

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Abstract. In this paper, we first explore some properties of M-tensors by showing that the Fan product of two M-tensors is an M-tensor, then establish lower bound estimations and upper bound estimations on the minimal eigenvalues of for Fan product of two M-tensors. Some inclusion relations among them are also obtained.

1. Introduction

Let $\mathbb{C}$ ($\mathbb{R}$) be the set of all complex (real) numbers, $\mathbb{R}_+$ ($\mathbb{R}_{++}$) be the set of all nonnegative (positive) numbers, $\mathbb{C}^n$ ($\mathbb{R}^n$) be the set of all dimension n complex (real) vectors, and $\mathbb{R}^n_+$ ($\mathbb{R}^n_{++}$) be the set of all dimension n nonnegative (positive) vectors. An m-order n-dimensional tensor $\mathcal{A} = (a_{i_1 i_2 \ldots i_m})$ is a higher-order generalization of matrices, which consists of n^m entries:

$$a_{i_1 i_2 \ldots i_m} \in \mathbb{R}, \quad i_k \in N = \{1, 2, \ldots, n\}, \quad k = 1, 2, \ldots, m.$$

$\mathcal{A}$ is called nonnegative (positive) if $a_{i_1 i_2 \ldots i_m} \in \mathbb{R}_+$ ($a_{i_1 i_2 \ldots i_m} \in \mathbb{R}_{++}$).

Tensor has much similarities with matrix and many related results of matrices such as determinant, eigenvalue and algorithm theory can be extended to higher order tensors by exploring their multilinear algebra properties [1, 9, 13–15, 18]. Furthermore, the matrices with special structures such as nonnegative matrices and M-matrices can also be extended to higher order tensors and these are becoming the focus of tensor in recent research [2, 3, 5, 9, 13, 19–24]. In particular, M-tensor is a new developed type of tensor with a special structure [3, 23] and plays important roles in the stability study of nonlinear autonomous systems via Lyapunovs direct method in automatic control [6, 7, 14] and in the sparsest solutions to tensor complementarity problems and the numerical solution of the PDE with Dirichlet's boundary condition [8].

On the other hand, Fan product of M-matrices and Hadamard product of nonnegative matrices arise in a wide variety of ways, such as trigonometric moments of convolutions of periodic functions, products of integral equation kernels, the weak minimum principle in

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partial differential equations, characteristic functions in probability theory, the study of association schemes in combinatorial theory, and so on (see [8]). Some inequalities on the Hadamard product involving nonnegative matrices and Fan product of M-matrices can be found in [4, 10–12, 25]. Based on the above results, Sun et al. [17] investigated some inequalities for the Hadamard product of tensors and obtained some bounds on spectral radii of the Hadamard product of tensors, and used them to estimate the spectral radius of a directly weighted hypergraph. In this paper, based on the close relationship between the nonnegative tensors and M-tensors, we expect to establish some bounds for the minimal eigenvalue of M-tensor Fan product, which constitutes the motivation of this article.

The remainder of this paper is organized as follows. In Section 2, we introduce important notation and recall some preliminary results on tensor analysis. In Section 3, we explore some characterizations of M-tensors, then propose some lower bound estimations and upper bound estimations on the minimal eigenvalues for Fan product and discuss the inclusion relations among them.

2. Notations and preliminaries

In this section, we first introduce some definitions and important properties on the tensor eigenvalue needed in the subsequent analysis.

Definition 2.1. [15] Let $\mathcal{A}$ be an m-order n-dimensional tensor. Then $(\lambda, x) \in \mathcal{C} \times (\mathcal{C}^n \setminus \{0\})$ is called an eigenpair of $\mathcal{A}$ if

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

where $(\mathcal{A}x^{m-1})_i = \sum_{i_2,\dots,i_m=1}^{n} a_{ii_2\dots i_m} x_{i_2} \cdots x_{i_m}$, $x^{[m-1]} = [x_1^{m-1}, x_2^{m-1}, \dots, x_n^{m-1}]^T$, and (λ, x) is called an H-eigenpair if they are both real.

For m-order n-dimensional tensor $\mathcal{A}$, we use $\sigma(\mathcal{A})$ to denote the set of eigenvalues of $\mathcal{A}$, and the spectral radius of $\mathcal{A}$ is denoted by

$$\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}.$$

Meanwhile, we use $\tau(\mathcal{A})$ to denote the minimal value of the real part of eigenvalues of $\mathcal{A}$ [15, 23].

The following concept plays an important role in spectral analysis of nonnegative tensors [5, 9].

Definition 2.2. Let $\mathcal{A}$ be an m-order n-dimensional tensor.

(i) Nonnegative matrix $G(\mathcal{A})$ is called the representation associated to a nonnegative tensor $\mathcal{A}$, if the (i, j)-th entry of $G(\mathcal{A})$ is defined to be the summation of $a_{ii_2i_3\dots i_m}$ with indices $j \in \{i_2, i_3, \dots, i_m\}$.

(ii) Tensor $\mathcal{A}$ is called weakly reducible, if its representation $G(\mathcal{A})$ is reducible. If $\mathcal{A}$ is not weakly reducible, then it is called weakly irreducible.

The following specially structured tensors are extended from matrices [3,23].

Definition 2.3. Tensor $\mathcal{A}$ is called a Z-tensor if it can be written as $\mathcal{A} = c\mathcal{I} - \mathcal{B}$, where $c > 0$, $\mathcal{I}$ is a unit tensor with entries

$$\delta_{i_1 i_2 \ldots i_m} = \begin{cases} 1 & \text{if } i_1 = i_2 = \cdots = i_m, \\ 0 & \text{otherwise} \end{cases}$$

and $\mathcal{B}$ is a nonnegative tensor. Furthermore, if $c \geq \rho(\mathcal{B})$, then $\mathcal{A}$ is said to be an M-tensor, and if $c > \rho(\mathcal{B})$, then tensor $\mathcal{A}$ is said to be a strong M-tensor. $\mathcal{A}$ is a weakly irreducible M-tensor if $\mathcal{B}$ is weakly irreducible.

It is easy to see that all the diagonal entries of a Z-tensor are non-positive [23], and the (strong) M-tensor is closely linked with the diagonal dominance defined below.

Definition 2.4. For m-order n-dimensional tensor $\mathcal{A}$, it is called diagonally dominant if

$$|a_{i\ldots i}| \geq \sum_{\delta_{i i_2 \ldots i_m} = 0} |a_{i i_2 \ldots i_m}|, \quad \forall\, i = 1, \ldots, n.$$

Tensor $\mathcal{A}$ is called strictly diagonally dominant if the strict inequality holds for all i.

If we define positive diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$ and set

$$\mathcal{B} = (b_{i_1 i_2 \ldots i_m}) = \mathcal{A} \cdot D^{-(m-1)} \overbrace{D \ldots D}^{m-1} = (a_{i_1 \ldots i_m} d_{i_1}^{-(m-1)} d_{i_2} \ldots d_{i_m}), \tag{2.1}$$

we can obtain the following necessary and sufficient condition for identifying M-tensor [23].

Lemma 2.5. *Suppose $\mathcal{A}$ is a weakly irreducible Z-tensor and its all diagonal elements are nonnegative. Then, $\mathcal{A}$ is an (strong) M-tensor if and only if there exists a positive diagonal matrix D such that $\mathcal{B}$ defined as* (2.1) *is (strictly) diagonally dominant.*

Definition 2.6. Let $\mathcal{A}$ and $\mathcal{B}$ be two M-tensors. Fan product of $\mathcal{A}$ and $\mathcal{B}$ is defined by $\mathcal{A} \star \mathcal{B} = \mathcal{D} = (d_{i_1 i_2 \ldots i_m})$, where

$$d_{i_1 i_2 \ldots i_m} = \begin{cases} a_{i\ldots i} b_{i \ldots i} & \text{if } i_1 = i_2 = \cdots = i_m = i, \\ -|a_{i_1 i_2 \ldots i_m} b_{i_1 i_2 \ldots i_m}| & \text{otherwise.} \end{cases}$$

To end this section, we present some important properties on nonnegative tensors.

Lemma 2.7. [9] *Let $\mathcal{A}$ be an m-order n-dimensional weakly irreducible nonnegative tensor. Then, the followings hold:*

(i) *$\mathcal{A}$ has a positive eigenpair (λ, x), and x is unique up to a multiplicative constant.*

(ii) $\displaystyle\min_{x\in\mathbb{R}_{++}} \max_{1\le i\le n} \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}} = \rho(\mathcal{A}) = \max_{x\in\mathbb{R}_{++}} \min_{1\le i\le n} \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}}$.

Definition 2.8. [9,16] Let $I \subset \{1,\dots,n\}$ with $|I| = r$ and $\mathcal{A}$ be an m order n dimensional nonnegative tensor. Then $\mathcal{A}[I]$ is called a principal subtensor of the tensor $\mathcal{A}$ of order m and dimensional r with index set I with r^m elements defined by

$$\mathcal{A}[I] = (a_{i_1\dots i_m}), \quad i_1,\dots,i_m \in I.$$

For the subtensor, we have the following on the spectral radius.

Lemma 2.9. [16] *Let $\mathcal{A}$ be an m-order n-dimensional weakly irreducible nonnegative tensor, and $\mathcal{A}[I]$ be a principal subtensor of $\mathcal{A}$, $|I| = r < n$. Then $\rho(\mathcal{A}[I]) < \rho(\mathcal{A})$.*

3. Bound estimations on the minimal eigenvalues for Fan product of M-tensors

For two $n \times n$ M-matrices P and Q, Horn and Johnson [8] gave the bounds of $\tau(P \star Q)$, and the bounds are improved to the Fan product of two M-matrices [4, 10–12, 25]. In this section, we extend the results to tensors by proposing some sharp lower bounds and upper bounds on the minimal eigenvalues for Fan product of two M-tensors. To this end, we present several important lemmas.

Lemma 3.1. *Let $\mathcal{Q}$ be an M-tensor of order m dimension n, then the followings hold:*

(i) *if $\mathcal{Q}z^{m-1} \ge kz^{[m-1]}$ for $z \in \mathbb{R}^n_{++}$ and $k \in \mathbb{R}$, then $\tau(\mathcal{Q}) \ge k$;*

(ii) *if $\mathcal{Q}z^{m-1} \le kz^{[m-1]}$ for $z \in \mathbb{R}^n_{++}$ and $k \in \mathbb{R}$, then $\tau(\mathcal{Q}) \le k$.*

Proof. Since $\mathcal{Q}$ is an M-tensor, there exists a nonnegative tensor $\mathcal{A}$ such that

$$\mathcal{Q} = \lambda\mathcal{I} - \mathcal{A}, \tag{3.1}$$

where $\lambda \ge \rho(\mathcal{A})$. It is easy to see that $\tau(\mathcal{Q}) = \lambda - \rho(\mathcal{A})$, that is, $\rho(\mathcal{A}) = \lambda - \tau(\mathcal{Q})$. From the assumption and (3.1), it holds that

$$(\lambda\mathcal{I} - \mathcal{A})z^{m-1} \ge kz^{[m-1]}.$$

That is,

$$(\lambda - k)z^{[m-1]} \ge \mathcal{A}z^{m-1}.$$

It follows from Lemma 2.7 that

$$\lambda - k \ge \rho(\mathcal{A}) = \lambda - \tau(\mathcal{Q}).$$

Hence, $\tau(\mathcal{Q}) \ge k$.

(ii) is similar to the proof of (i), and we obtain the desired result. □

Lemma 3.2. *Let $\mathcal{Q}$ be a weakly irreducible strong M-tensor of order m dimension n. Then, there exists a positive vector v such that*

$$\mathcal{Q}v^{m-1} = \tau(\mathcal{Q})v^{[m-1]}.$$

Proof. Since $\mathcal{Q}$ is a strong M-tensor, there exists a nonnegative tensor $\mathcal{A}$ such that

$$\mathcal{Q} = \lambda\mathcal{I} - \mathcal{A} \quad \text{and} \quad \rho(\mathcal{A}) = \lambda - \tau(\mathcal{Q}),$$

where $\lambda > \rho(\mathcal{A})$. It follows from weak irreducibility of $\mathcal{Q}$ that $\mathcal{A}$ is weakly irreducible. By Lemma 2.7, there exists a positive vector v such that

$$\mathcal{A}v^{m-1} = \rho(\mathcal{A})v^{[m-1]} = (\lambda - \tau(\mathcal{Q}))v^{[m-1]}.$$

Hence,

$$(\lambda\mathcal{I} - \mathcal{A})v^{m-1} = \tau(\mathcal{Q})v^{[m-1]},$$

which implies

$$\mathcal{Q}v^{m-1} = \tau(\mathcal{Q})v^{[m-1]}.$$

□

Lemma 3.3. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two M-tensors of order m dimension n. Then, $\mathcal{P} \star \mathcal{Q}$ is an M-tensor. Furthermore, if $\mathcal{P}$, $\mathcal{Q}$ are strong M-tensors, then $\mathcal{P} \star \mathcal{Q}$ is a strong M-tensor.*

Proof. By the definition of $\mathcal{P} \star \mathcal{Q}$, it holds that

$$\mathcal{P} \star \mathcal{Q} = \begin{cases} p_{i\dots i}q_{i\dots i} & \text{if } i_2 = i_3 = \cdots = i_m = i, \\ -|p_{ii_2\dots i_m}q_{ii_2\dots i_m}| & \text{otherwise.} \end{cases}$$

Since $\mathcal{P}$, $\mathcal{Q}$ are M-tensors, by Lemma 2.5, there exist positive diagonal matrices C, D such that

$$\mathcal{A} = \mathcal{P} \cdot C^{-(m-1)} \overbrace{C \dots C}^{m-1}, \quad \mathcal{B} = \mathcal{Q} \cdot D^{-(m-1)} \overbrace{D \dots D}^{m-1}$$

with

$$a_{i_1\dots i_m} = p_{i_1\dots i_m} c_{i_1}^{-(m-1)} c_{i_2} \cdots c_{i_m}, \quad b_{i_1\dots i_m} = q_{i_1\dots i_m} d_{i_1}^{-(m-1)} d_{i_2} \cdots d_{i_m}.$$

Specifically,

$$a_{i\dots i} = p_{i\dots i} \quad \text{and} \quad b_{i\dots i} = q_{i\dots i}.$$

Taking into account that $\mathcal{A}$, $\mathcal{B}$ are diagonally dominant, we conclude that

$$|p_{i\dots i}| = |a_{i\dots i}| \ge \sum_{\delta_{ii_2\dots i_m}=0} |p_{ii_2\dots i_m}| c_i^{-(m-1)} c_{i_2} \cdots c_{i_m},$$
$$|q_{i\dots i}| = |b_{i\dots i}| \ge \sum_{\delta_{ii_2\dots i_m}=0} |q_{ii_2\dots i_m}| d_i^{-(m-1)} d_{i_2} \cdots d_{i_m}.$$

Furthermore, it holds that

$$
\begin{aligned}
&|p_{i\dots i}q_{i\dots i}| = |a_{i\dots i}b_{i\dots i}| \\
&\geq \sum_{\delta_{ii_2\dots i_m}=0} (|p_{ii_2\dots i_m}|c_i^{-(m-1)}c_{i_2}\cdots c_{i_m}) \sum_{\delta_{ii_2\dots i_m}=0} (|q_{ii_2\dots i_m}|d_i^{-[m-1]}d_{i_2}\cdots d_{i_m}) \\
&\geq \sum_{\delta_{ii_2\dots i_m}=0} |p_{ii_2\dots i_m}|c_i^{-(m-1)}c_{i_2}\cdots c_{i_m}|q_{ii_2\dots i_m}|d_i^{-(m-1)}d_{i_2}\cdots d_{i_m} \\
&= \sum_{\delta_{ii_2\dots i_m}=0} |p_{ii_2\dots i_m}q_{ii_2\dots i_m}|(c_id_i)^{-(m-1)}c_{i_2}d_{i_2}\cdots c_{i_m}d_{i_m}.
\end{aligned}
\tag{3.2}
$$

From (3.2), there exists a positive diagonal matrix $U = \mathrm{diag}(c_1d_1, c_2d_2, \dots, c_nd_n)$ such that

$$|p_{i\dots i}q_{i\dots i}| \geq \sum_{\delta_{ii_2\dots i_m}=0} p_{ii_2\dots i_m}q_{ii_2\dots i_m}(u_i)^{-(m-1)}u_{i_2}\cdots u_{i_m}.$$

It follows from Lemma 2.5 that $\mathcal{P} \star \mathcal{Q}$ is an M-tensor. Similar to the argument for the first conclusion, we can obtain the second conclusion. □

Lemma 3.4. *Let $\mathcal{Q}$ be a weakly irreducible strong M-tensor of order m dimension n. If $\mathcal{Q}_k$ is a principal M-subtensor of $\mathcal{Q}$, then $\tau(\mathcal{Q}_k) > \tau(\mathcal{Q})$.*

Proof. As $\mathcal{Q}$ is a strong M-tensor, there exists a nonnegative tensor $\mathcal{A}$ such that

$$\mathcal{Q} = \lambda\mathcal{I} - \mathcal{A} \quad \text{and} \quad \lambda > \rho(\mathcal{A}).$$

Then $\mathcal{Q}_k = \lambda\mathcal{I}_k - \mathcal{A}_k$, where $\mathcal{A}_k$ is a principal subtensor of $\mathcal{A}$. By Lemma 2.9, we conclude that $\tau(\mathcal{Q}_k) > \tau(\mathcal{Q})$. □

Based on characterizations of M-tensors, we can obtain a lower bound on the minimal eigenvalues in the sense of the Fan product of two M-tensors.

Theorem 3.5. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two strong M-tensors of order m dimension n. Then*

$$\tau(\mathcal{P} \star \mathcal{Q}) \geq \tau(\mathcal{P})\tau(\mathcal{Q}).$$

Proof. The argument is broken into two cases.

Case 1. $\mathcal{P}$ and $\mathcal{Q}$ are both weakly irreducible. Since $\mathcal{P}$ and $\mathcal{Q}$ are strong M-tensors, by Lemma 3.2, there exist two positive eigenvectors c and d corresponding to $\tau(\mathcal{P})$ and $\tau(\mathcal{Q})$ such that

$$p_{i\dots i}c_i^{m-1} - \sum_{\delta_{ii_2\dots i_m}=0} |p_{ii_2\dots i_m}|c_{i_2}\cdots c_{i_m} = \tau(\mathcal{P})c_i^{m-1}, \tag{3.3}$$

$$q_{i\dots i}d_i^{m-1} - \sum_{\delta_{ii_2\dots i_m}=0} |q_{ii_2\dots i_m}|q_{i_2}\cdots q_{i_m} = \tau(\mathcal{Q})d_i^{m-1}. \tag{3.4}$$

It follows from $\tau(\mathcal{P}) > 0$, $c_i > 0$, $\tau(\mathcal{Q}) > 0$ and $d_i > 0$ that

$$\left(q_{i\ldots i}d_i^{m-1} - \sum_{\delta_{ii_2\ldots i_m}=0}(|q_{ii_2\ldots i_m}|d_{i_2}\cdots d_{i_m})\right)\sum_{\delta_{ii_2\ldots i_m}=0}(|p_{ii_2\ldots i_m}|c_{i_2}\cdots c_{i_m}) \geq 0,$$

$$\left(p_{i\ldots i}c_i^{m-1} - \sum_{\delta_{ii_2\ldots i_m}=0}(|p_{ii_2\ldots i_m}|c_{i_2}\cdots c_{i_m})\right)\sum_{\delta_{ii_2\ldots i_m}=0}(|q_{ii_2\ldots i_m}|q_{i_2}\cdots q_{i_m}) \geq 0.$$

Hence,

$$\begin{aligned}&q_{i\ldots i}d_i^{m-1}\sum_{\delta_{ii_2\ldots i_m}=0}(|p_{ii_2\ldots i_m}|c_{i_2}\cdots c_{i_m}) + p_{i\ldots i}c_i^{m-1}\sum_{\delta_{ii_2\ldots i_m}=0}(|q_{ii_2\ldots i_m}|d_{i_2}\cdots d_{i_m})\\ &\geq 2\sum_{\delta_{ii_2\ldots i_m}=0}(|p_{ii_2\ldots i_m}|c_{i_2}\cdots c_{i_m})\sum_{\delta_{ii_2\ldots i_m}=0}(|q_{ii_2\ldots i_m}|d_{i_2}\cdots d_{i_m}),\end{aligned}$$

which implies

$$\begin{aligned}&p_{i\ldots i}q_{i\ldots i}c_i^{[m-1]}d_i^{[m-1]} - \sum_{\delta_{ii_2\ldots i_m}=0}(|p_{ii_2\ldots i_m}|c_{i_2}\cdots c_{i_m})\sum_{\delta_{ii_2\ldots i_m}=0}(|q_{ii_2\ldots i_m}|d_{i_2}\cdots d_{i_m})\\ &\geq \left(p_{i\ldots i}c_i^{[m-1]} - \sum_{\delta_{ii_2\ldots i_m}=0}(|p_{ii_2\ldots i_m}|c_{i_2}\cdots c_{i_m})\right)\\ &\quad\times\left(q_{i\ldots i}d_i^{[m-1]} - \sum_{\delta_{ii_2\ldots i_m}=0}(|q_{ii_2\ldots i_m}|d_{i_2}\cdots d_{i_m})\right).\end{aligned}$$

Let $z = (z_i)$ with $z_i = c_i d_i > 0$ and define $\mathcal{U} = \mathcal{P} \star \mathcal{Q}$. For $i \in N$, it holds that

$$\begin{aligned}(\mathcal{U}z^{m-1})_i &= p_{i\ldots i}q_{i\ldots i}c_i^{[m-1]}d_i^{[m-1]} - \sum_{\delta_{ii_2\ldots i_m}=0}|p_{ii_2\ldots i_m}|c_{i_2}\cdots c_{i_m}|q_{ii_2\ldots i_m}|d_{i_2}\cdots d_{i_m}\\ &\geq p_{i\ldots i}q_{i\ldots i}c_i^{[m-1]}d_i^{[m-1]}\\ &\quad - \sum_{\delta_{ii_2\ldots i_m}=0}(|p_{ii_2\ldots i_m}|c_{i_2}\cdots c_{i_m})\sum_{\delta_{ii_2\ldots i_m}=0}(|q_{ii_2\ldots i_m}|d_{i_2}\cdots d_{i_m})\\ &\geq \left[p_{i\ldots i}c_i^{[m-1]} - \sum_{\delta_{ii_2\ldots i_m}=0}(|p_{ii_2\ldots i_m}|c_{i_2}\cdots c_{i_m})\right] \qquad (3.5)\\ &\quad\times\left[q_{i\ldots i}d_i^{[m-1]} - \sum_{\delta_{ii_2\ldots i_m}=0}(|q_{ii_2\ldots i_m}|d_{i_2}\cdots d_{i_m})\right]\\ &= \tau(\mathcal{P})\tau(\mathcal{Q})z_i^{[m-1]}.\end{aligned}$$

It follows from Lemma 3.1 and (3.5) that

$$\tau(\mathcal{P} \star \mathcal{Q}) \geq \tau(\mathcal{P})\tau(\mathcal{Q}).$$

Case 2. Either $\mathcal{P}$ or $\mathcal{Q}$ is weakly reducible. Let $\mathcal{S}$ be an order m dimension n tensor with

$$s_{ii_2\dots i_m} = \begin{cases} 1 & \text{if } i_2 = i_3 = \cdots = i_m \neq i, \\ 0 & \text{otherwise.} \end{cases}$$

Then both $\mathcal{P} - \epsilon\mathcal{S}$ and $\mathcal{Q} - \epsilon\mathcal{S}$ are weakly irreducible tensors for any $\epsilon > 0$. Now, we claim that $\mathcal{P} - \epsilon\mathcal{S}$ and $\mathcal{Q} - \epsilon\mathcal{S}$ are both strong M-tensors for when $\epsilon > 0$ is sufficiently small.

In fact, since $\mathcal{P}$, $\mathcal{Q}$ are strong M-tensors, there exist positive diagonal matrices C, D such that

$$\mathcal{A} = \mathcal{P} \cdot C^{-(m-1)} \overbrace{C \dots C}^{m-1}, \quad \mathcal{B} = \mathcal{Q} \cdot D^{-(m-1)} \overbrace{D \dots D}^{m-1}$$

with

$$a_{i_1\dots i_m} = p_{i_1\dots i_m} c_{i_1}^{-(m-1)} c_{i_2} \cdots c_{i_m} \quad \text{and} \quad b_{i_1\dots i_m} = q_{i_1\dots i_m} d_{i_1}^{-(m-1)} d_{i_2} \cdots d_{i_m}.$$

In particular,

$$a_{i\dots i} = p_{i\dots i} \quad \text{and} \quad b_{i\dots i} = q_{i\dots i}.$$

By Lemma 2.5, one has

$$|p_{i\dots i}| = |a_{i\dots i}| > \sum_{\delta_{ii_2\dots i_m}=0} |p_{ii_2\dots i_m}| c_i^{-(m-1)} c_{i_2} \cdots c_{i_m},$$
$$|q_{i\dots i}| = |b_{i\dots i}| > \sum_{\delta_{ii_2\dots i_m}=0} |q_{ii_2\dots i_m}| d_i^{-(m-1)} d_{i_2} \cdots d_{i_m}.$$

Set

$$L = \max_{\substack{i,j\in N \\ i\neq j}} \left\{ \frac{c_j^{[m-1]}}{c_i^{[m-1]}}, \frac{d_j^{[m-1]}}{d_i^{[m-1]}} \right\}$$

and

$$\epsilon_0 = \min_{\substack{i,j\in N \\ i\neq j}} \left\{ \frac{|p_{i\dots i}| - \sum_{\delta_{ii_2\dots i_m}=0} |p_{i\dots i_m}| c_i^{-(m-1)} c_{i_2} \cdots c_{i_m}}{(n-1)L}, \right.$$
$$\left. \frac{|q_{i\dots i}| - \sum_{\delta_{ii_2\dots i_m}=0} |q_{i\dots i_m}| d_i^{-[m-1]} d_{i_2} \cdots d_{i_m}}{(n-1)L} \right\}.$$

Then for any $0 < \epsilon < \epsilon_0$, it can be readily verified that $\mathcal{P} - \epsilon\mathcal{S}$ and $\mathcal{Q} - \epsilon\mathcal{S}$ are strong M-tensors. Substituting $\mathcal{P} - \epsilon\mathcal{S}$ and $\mathcal{Q} - \epsilon\mathcal{S}$ for $\mathcal{P}$ and $\mathcal{Q}$ and letting $\epsilon \to 0$, we can obtain the desired results by the continuity of $\tau(\mathcal{P} - \epsilon\mathcal{S})$ and $\tau(\mathcal{Q} - \epsilon\mathcal{S})$. $\square$

The following conclusion provides a sharp lower bound on the minimal eigenvalues for Fan product of two M-tensors.

Theorem 3.6. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two strong M-tensors of order m dimension n. Then*

$$\tau(\mathcal{P}\star\mathcal{Q}) \geq \min_{i\in N}\{p_{i\dots i}\tau(\mathcal{Q}) + q_{i\dots i}\tau(\mathcal{P}) - \tau(\mathcal{P})\tau(\mathcal{Q})\}.$$

Proof. Similar to the proof of Theorem 3.5, we break the argument into two cases.

Case 1. $\mathcal{P}$ and $\mathcal{Q}$ are weakly irreducible. It follows from Lemma 3.4 that

$$p_{i\dots i} - \tau(\mathcal{P}) > 0 \quad \text{and} \quad q_{i\dots i} - \tau(\mathcal{Q}) > 0.$$

Since $\mathcal{P}$ and $\mathcal{Q}$ are strong M-tensors, from Lemma 3.2, there exist two positive eigenvectors $c = (c_i), d = (d_i) \in \mathbb{R}^n_{++}$ corresponding to $\tau(\mathcal{P})$ and $\tau(\mathcal{Q})$ satisfying (3.3) and (3.4). Let $z = (z_i)$ with $z_i = c_i d_i > 0$ and define $\mathcal{U} = \mathcal{P}\star\mathcal{Q}$. For $i \in N$, it holds that

$$\begin{aligned}
&(\mathcal{U}z^{m-1})_i \\
&= p_{i\dots i}q_{i\dots i}z_i^{[m-1]} - \sum_{\delta_{ii_2\dots i_m}=0} |p_{ii_2\dots i_m}||q_{ii_2\dots i_m}|z_{i_2}\cdots z_{i_m} \\
&\geq p_{i\dots i}q_{i\dots i}z_i^{[m-1]} - \sum_{\delta_{ii_2\dots i_m}=0} |p_{ii_2\dots i_m}|c_{i_2}\cdots c_{i_m} \sum_{\delta_{ii_2\dots i_m}=0} |q_{ii_2\dots i_m}|d_{i_2}\cdots d_{i_m} \\
&= p_{i\dots i}q_{i\dots i}z_i^{[m-1]} - (p_{i\dots i} - \tau(\mathcal{P}))(q_{i\dots i} - \tau(\mathcal{Q}))z_i^{[m-1]} \\
&= [p_{i\dots i}\tau(\mathcal{Q}) + q_{i\dots i}\tau(\mathcal{P}) - \tau(\mathcal{P})\tau(\mathcal{Q})]z_i^{[m-1]}.
\end{aligned} \tag{3.6}$$

It follows from Lemma 3.1 and (3.6) that

$$\tau(\mathcal{P}\star\mathcal{Q}) \geq \min_{i\in N}\{p_{i\dots i}\tau(\mathcal{Q}) + q_{i\dots i}\tau(\mathcal{P}) - \tau(\mathcal{P})\tau(\mathcal{Q})\}.$$

Case 2. Either $\mathcal{P}$ or $\mathcal{Q}$ is weakly reducible. Similar to the proof of Theorem 3.5, we get the conclusion. □

Theorem 3.6 improves the conclusion in Theorem 3.5 as it follows from $p_{i\dots i} - \tau(\mathcal{P}) \geq 0$ and $q_{i\dots i} - \tau(\mathcal{Q}) \geq 0$ that

$$\begin{aligned}
&\tau(\mathcal{P})\tau(\mathcal{Q}) - (p_{i\dots i}\tau(\mathcal{Q}) + q_{i\dots i}\tau(\mathcal{P}) - \tau(\mathcal{P})\tau(\mathcal{Q})) \\
&= (\tau(\mathcal{Q}) - q_{i\dots i})p_{i\dots i} + (\tau(\mathcal{P}) - p_{i\dots i})q_{i\dots i} \leq 0.
\end{aligned}$$

Hence,

$$\tau(\mathcal{P})\tau(\mathcal{Q}) \leq \min_{i\in N}\{p_{i\dots i}\tau(\mathcal{Q}) + q_{i\dots i}\tau(\mathcal{P}) - \tau(\mathcal{P})\tau(\mathcal{Q})\}.$$

The following example exhibits that Theorem 3.6 is tighter.

Example 3.7. Consider m-order n-dimensional tensors $\mathcal{P} = \mathcal{I}$ and $\mathcal{Q} = (q_{i_1i_2\dots i_m})$ defined by

$$q_{i_1i_2\dots i_m} = \begin{cases} n^{(m-1)} & \text{if } i_1 = i_2 = \cdots = i_m, \\ -1 & \text{otherwise.} \end{cases}$$

It is easy to verify that

$$\tau(\mathcal{P}) = 1, \quad \tau(\mathcal{Q}) = 1, \quad \tau(\mathcal{P} \star \mathcal{Q}) = n^{(m-1)} \gg \tau(\mathcal{P})\tau(\mathcal{Q}) = 1$$

and

$$\tau(\mathcal{P} \star \mathcal{Q}) = n^{(m-1)} = \min_{i \in N} \{p_{i \dots i}\tau(\mathcal{Q}) + q_{i \dots i}\tau(\mathcal{P}) - \tau(\mathcal{P})\tau(\mathcal{Q})\} = n^{m-1}.$$

By making use of the information of the absolute maximum in the off-diagonal elements, we are at the position to establish the following theorems.

Theorem 3.8. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two strong M-tensors of order m dimension n. Then*

$$\tau(\mathcal{P} \star \mathcal{Q}) \geq \min_{i \in N} \{p_{i \dots i} q_{i \dots i} - [\alpha_i \beta_i (p_{i \dots i} - \tau(\mathcal{P}))(q_{i \dots i} - \tau(\mathcal{Q}))]^{1/2}\},$$

where $\alpha_i = \max_{\delta_{ii_2 \dots i_m} = 0} |p_{ii_2 \dots i_m}|$ and $\beta_i = \max_{\delta_{ii_2 \dots i_m} = 0} |q_{ii_2 \dots i_m}|$.

Proof. The proof is broken into two cases.

Case 1. $\mathcal{P}$ and $\mathcal{Q}$ are weakly irreducible. Since $\mathcal{P}$ and $\mathcal{Q}$ are strong M-tensors, there exist two positive eigenvectors $c = (c_i^2), d = (d_i^2)$ corresponding to $\tau(\mathcal{P})$ and $\tau(\mathcal{Q})$ such that

$$-\sum_{\delta_{ii_2 \dots i_m} = 0} |p_{ii_2 \dots i_m}| c_{i_2}^2 \cdots c_{i_m}^2 = (\tau(\mathcal{P}) - p_{i \dots i}) c_i^{2(m-1)}, \tag{3.7}$$

$$-\sum_{\delta_{ii_2 \dots i_m} = 0} |q_{ii_2 \dots i_m}| d_{i_2}^2 \cdots d_{i_m}^2 = (\tau(\mathcal{Q}) - q_{i \dots i}) d_i^{2(m-1)}. \tag{3.8}$$

Let $z = (z_i)$ with $z_i = c_i d_i > 0$ and set $\mathcal{U} = \mathcal{P} \star \mathcal{Q}$. Then for $1 \leq i \leq n$, it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}
&(\mathcal{U} z^{m-1})_i \\
&= p_{i \dots i} q_{i \dots i} z_i^{[m-1]} - \sum_{\delta_{ii_2 \dots i_m} = 0} |p_{ii_2 \dots i_m}| |q_{ii_2 \dots i_m}| c_{i_2} d_{i_2} \cdots c_{i_m} d_{i_m} \\
&\geq p_{i \dots i} q_{i \dots i} z_i^{[m-1]} - \sum_{\delta_{ii_2 \dots i_m} = 0} |p_{ii_2 \dots i_m}| c_{i_2} \cdots c_{i_m} \sum_{\delta_{ii_2 \dots i_m} = 0} |q_{ii_2 \dots i_m}| d_{i_2} \cdots d_{i_m} \\
&\geq p_{i \dots i} q_{i \dots i} z_i^{[m-1]} \\
&\quad - \left(\sum_{\delta_{ii_2 \dots i_m} = 0} |p_{ii_2 \dots i_m}|^2 c_{i_2}^2 \cdots c_{i_m}^2 \right)^{1/2} \left(\sum_{\delta_{ii_2 \dots i_m} = 0} |q_{ii_2 \dots i_m}|^2 d_{i_2}^2 \cdots d_{i_m}^2 \right)^{1/2}.
\end{aligned} \tag{3.9}$$

On the other hand, it follows from the definitions of α_i, β_i and (3.7)–(3.9) that

$$\begin{aligned}
&(\mathcal{U} z^{m-1})_i \\
&\geq p_{i \dots i} q_{i \dots i} z_i^{[m-1]} - \alpha_i^{1/2} (p_{i \dots i} - \tau(\mathcal{P}))^{1/2} c_i^{[m-1]} \beta_i^{1/2} (q_{i \dots i} - \tau(\mathcal{Q}))^{1/2} d_i^{[m-1]} \\
&= \{p_{i \dots i} q_{i \dots i} - [\alpha_i \beta_i (p_{i \dots i} - \tau(\mathcal{P}))(q_{i \dots i} - \tau(\mathcal{Q}))]^{1/2}\} z_i^{[m-1]}.
\end{aligned} \tag{3.10}$$

Furthermore, using Lemma 3.1 and (3.10), one has

$$\tau(\mathcal{P} \star \mathcal{Q}) \geq \min_{i \in N} \{p_{i \dots i} q_{i \dots i} - [\alpha_i \beta_i (p_{i \dots i} - \tau(\mathcal{P}))(q_{i \dots i} - \tau(\mathcal{Q}))]^{1/2}\}.$$

Case 2. Either $\mathcal{P}$ or $\mathcal{Q}$ is weakly reducible. Similar to the proof of Theorem 3.5, we obtain the desired result. □

In what follows, we give the inclusion relation between Theorems 3.6 and 3.8.

Corollary 3.9. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two strong M-tensors of order m dimension n. For $i \in N$, if $(p_{i \dots i} - \tau(\mathcal{P}))(q_{i \dots i} - \tau(\mathcal{Q})) \leq \alpha_i \beta_i$, then*

$$\begin{aligned} &\min_{i \in N} \{p_{i \dots i} \tau(\mathcal{Q}) + q_{i \dots i} \tau(\mathcal{P}) - \tau(\mathcal{P}) \tau(\mathcal{Q})\} \\ &\geq \min_{i \in N} \{p_{i \dots i} q_{i \dots i} - [\alpha_i \beta_i (p_{i \dots i} - \tau(\mathcal{P}))(q_{i \dots i} - \tau(\mathcal{Q}))]^{1/2}\}; \end{aligned} \tag{3.11}$$

if $(p_{i \dots i} - \tau(\mathcal{P}))(q_{i \dots i} - \tau(\mathcal{Q})) \geq \alpha_i \beta_i$, then

$$\begin{aligned} &\min_{i \in N} \{p_{i \dots i} \tau(\mathcal{Q}) + q_{i \dots i} \tau(\mathcal{P}) - \tau(\mathcal{P}) \tau(\mathcal{Q})\} \\ &\leq \min_{i \in N} \{p_{i \dots i} q_{i \dots i} - [\alpha_i \beta_i (p_{i \dots i} - \tau(\mathcal{P}))(q_{i \dots i} - \tau(\mathcal{Q}))]^{1/2}\}. \end{aligned} \tag{3.12}$$

Proof. Observe that

$$p_{i \dots i} \tau(\mathcal{Q}) + q_{i \dots i} \tau(\mathcal{P}) - \tau(\mathcal{P}) \tau(\mathcal{Q}) = p_{i \dots i} q_{i \dots i} - (p_{i \dots i} - \tau(\mathcal{P}))(q_{i \dots i} - \tau(\mathcal{Q})).$$

For $i \in N$, when $(p_{i \dots i} - \tau(\mathcal{P}))(q_{i \dots i} - \tau(\mathcal{Q})) \leq \alpha_i \beta_i$, we see

$$\begin{aligned} &\{p_{i \dots i} q_{i \dots i} - (p_{i \dots i} - \tau(\mathcal{P}))(q_{i \dots i} - \tau(\mathcal{Q}))\} \\ &\geq \{p_{i \dots i} q_{i \dots i} - [\alpha_i \beta_i (p_{i \dots i} - \tau(\mathcal{P}))(q_{i \dots i} - \tau(\mathcal{Q}))]^{1/2}\}, \end{aligned}$$

which implies

$$\begin{aligned} &\min_{i \in N} \{p_{i \dots i} \tau(\mathcal{Q}) + q_{i \dots i} \tau(\mathcal{P}) - \tau(\mathcal{P}) \tau(\mathcal{Q})\} \\ &= \min_{i \in N} \{p_{i \dots i} q_{i \dots i} - (p_{i \dots i} - \tau(\mathcal{P}))(q_{i \dots i} - \tau(\mathcal{Q}))\} \\ &\geq \min_{i \in N} \{p_{i \dots i} q_{i \dots i} - [\alpha_i \beta_i (p_{i \dots i} - \tau(\mathcal{P}))(q_{i \dots i} - \tau(\mathcal{Q}))]^{1/2}\}. \end{aligned}$$

So, (3.11) holds.

For $i \in N$, if $(p_{i \dots i} - \tau(\mathcal{P}))(q_{i \dots i} - \tau(\mathcal{Q})) \geq \alpha_i \beta_i$, similar to the proof of (3.11), we can obtain (3.12). □

Remark 3.10. For $i \in N$, if $(p_{i \dots i} - \tau(\mathcal{P}))(q_{i \dots i} - \tau(\mathcal{Q})) \leq \alpha_i \beta_i$, from (3.11), we verify that the bound of Theorem 3.6 is sharper than that of Theorem 3.8; when $(p_{i \dots i} - \tau(\mathcal{P}))(q_{i \dots i} - \tau(\mathcal{Q})) \geq \alpha_i \beta_i$, from (3.12), we deduce that the bound of Theorem 3.8 is tighter than that of Theorem 3.6.

Theorem 3.11. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two strong M-tensors of order m dimension n. Then*

$$\tau(\mathcal{P}\star\mathcal{Q}) \geq \max\{\min_{i\in N}[p_{i\dots i}q_{i\dots i}-\alpha_i(q_{i\dots i}-\tau(\mathcal{Q}))], \min_{i\in N}[p_{i\dots i}q_{i\dots i}-\beta_i(p_{i\dots i}-\tau(\mathcal{P}))]\},$$

where $\alpha_i = \max_{\delta_{ii_2\dots i_m}=0}|p_{ii_2\dots i_m}|$ *and* $\beta_i = \max_{\delta_{ii_2\dots i_m}=0}|q_{ii_2\dots i_m}|$.

Proof. The following argument is divided into two cases.

Case 1. $\mathcal{Q}$ is weakly irreducible. Since $\mathcal{Q}$ is a strong M-tensor, there exists a positive eigenvector d corresponding to $\tau(\mathcal{Q})$ satisfying (3.4). Define $\mathcal{U}=\mathcal{P}\star\mathcal{Q}$. Since $\alpha_i = \max_{\delta_{ii_2\dots i_m}=0}|p_{ii_2\dots i_m}|$, for $1\leq i\leq n$, one has

$$\begin{aligned}(\mathcal{U}d^{m-1})_i &= p_{i\dots i}q_{i\dots i}d_i^{m-1} - \sum_{\delta_{ii_2\dots i_m}=0}|p_{ii_2\dots i_m}||q_{ii_2\dots i_m}|d_{i_2}\cdots d_{i_m}\\ &\geq p_{i\dots i}q_{i\dots i}d_i^{m-1} - \alpha_i(q_{i\dots i}-\tau(\mathcal{Q}))d_i^{m-1}\\ &= [p_{i\dots i}q_{i\dots i}-\alpha_i(q_{i\dots i}-\tau(\mathcal{Q}))]d_i^{m-1}.\end{aligned}$$

Hence, $\tau(\mathcal{P}\star\mathcal{Q})\geq \min_{i\in N}\{p_{i\dots i}q_{i\dots i}-\alpha_i(q_{i\dots i}-\tau(\mathcal{Q}))\}$ by Lemma 3.1.

Case 2. $\mathcal{Q}$ is weakly reducible. Similar to the proof of Theorem 3.5, we obtain the desired result.

Meanwhile, since $\mathcal{P}\star\mathcal{Q}=\mathcal{Q}\star\mathcal{P}$, one has $\tau(\mathcal{P}\star\mathcal{Q})\geq \min_{i\in N}\{p_{i\dots i}q_{i\dots i}-\beta_i(p_{i\dots i}-\tau(\mathcal{P}))\}$. So, the result follows. □

Next, we discuss the inclusion relations among Theorems 3.6, 3.8 and 3.11.

Corollary 3.12. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two strong M-tensors of order m dimension n. For $i\in N$, if $p_{i\dots i}-\tau(\mathcal{P})\leq\alpha_i$ and $q_{i\dots i}-\tau(\mathcal{Q})\leq\beta_i$, then*

$$\begin{aligned}&\min_{i\in N}\{p_{i\dots i}\tau(\mathcal{Q})+q_{i\dots i}\tau(\mathcal{P})-\tau(\mathcal{P})\tau(\mathcal{Q})\}\\ &\geq \max\left\{\min_{i\in N}[p_{i\dots i}q_{i\dots i}-\alpha_i(q_{i\dots i}-\tau(\mathcal{Q}))], \min_{i\in N}[p_{i\dots i}q_{i\dots i}-\beta_i(p_{i\dots i}-\tau(\mathcal{P}))]\right\};\end{aligned}\tag{3.13}$$

if $p_{i\dots i}-\tau(\mathcal{P})\geq\alpha_i$ or $q_{i\dots i}-\tau(\mathcal{Q})\geq\beta_i$, then

$$\begin{aligned}&\min_{i\in N}\{p_{i\dots i}\tau(\mathcal{Q})+q_{i\dots i}\tau(\mathcal{P})-\tau(\mathcal{P})\tau(\mathcal{Q})\}\\ &\leq \max\left\{\min_{i\in N}[p_{i\dots i}q_{i\dots i}-\alpha_i(q_{i\dots i}-\tau(\mathcal{Q}))], \min_{i\in N}[p_{i\dots i}q_{i\dots i}-\beta_i(p_{i\dots i}-\tau(\mathcal{P}))]\right\}.\end{aligned}\tag{3.14}$$

Proof. Similar to the proof of Corollary 3.9, we obtain (3.13) and (3.14). □

Remark 3.13. For $i\in N$, if $p_{i\dots i}-\tau(\mathcal{P})\leq\alpha_i$ and $q_{i\dots i}-\tau(\mathcal{Q})\leq\beta_i$, from (3.13), we deduce that the conclusion of Theorem 3.6 is sharper than that of Theorem 3.11; on the other hand, if $p_{i\dots i}-\tau(\mathcal{P})\geq\alpha_i$ or $q_{i\dots i}-\tau(\mathcal{Q})\geq\beta_i$, from (3.14), we obtain that the conclusion of Theorem 3.11 is sharper than that of Theorem 3.6.

Corollary 3.14. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two strong M-tensors of order m dimension n. Then,*

$$\begin{aligned}&\min_{i\in N}\{p_{i\dots i}q_{i\dots i}-[\alpha_i\beta_i(p_{i\dots i}-\tau(\mathcal{P}))(q_{i\dots i}-\tau(\mathcal{Q}))]^{1/2}\}\\ &\le\max\left\{\min_{i\in N}p_{i\dots i}[q_{i\dots i}-\alpha_i(q_{i\dots i}-\tau(\mathcal{Q}))],\min_{i\in N}p_{i\dots i}[q_{i\dots i}-\beta_i(p_{i\dots i}-\tau(\mathcal{P}))]\right\}.\end{aligned}\tag{3.15}$$

Proof. We divide N into two disjoint subsets I and $N\setminus I$, where $I=\{i\in N:\beta_i(p_{i\dots i}-\tau(\mathcal{P}))\le\alpha_i(q_{i\dots i}-\tau(\mathcal{Q}))\}$.

For $i\in I$, it holds that

$$p_{i\dots i}q_{i\dots i}-\alpha_i(q_{i\dots i}-\tau(\mathcal{Q}))\le p_{i\dots i}q_{i\dots i}-\beta_i(p_{i\dots i}-\tau(\mathcal{P})),\tag{3.16}$$

$$p_{i\dots i}q_{i\dots i}-\{\alpha_i\beta_i(p_{i\dots i}-\tau(\mathcal{P}))(q_{i\dots i}-\tau(\mathcal{Q}))\}^{1/2}\le p_{i\dots i}q_{i\dots i}-\beta_i(p_{i\dots i}-\tau(\mathcal{P})).\tag{3.17}$$

Combining (3.16) with (3.17) yields

$$\begin{aligned}&\min_{i\in I}\{p_{i\dots i}q_{i\dots i}-[\alpha_i\beta_i(p_{i\dots i}-\tau(\mathcal{P}))(q_{i\dots i}-\tau(\mathcal{Q}))]^{1/2}\}\\ &\le\max\left\{\min_{i\in I}[p_{i\dots i}q_{i\dots i}-\alpha_i(q_{i\dots i}-\tau(\mathcal{Q}))],\min_{i\in I}p_{i\dots i}[q_{i\dots i}-\beta_i(p_{i\dots i}-\tau(\mathcal{P}))]\right\}.\end{aligned}\tag{3.18}$$

For $i\in N\setminus I$, one has

$$p_{i\dots i}q_{i\dots i}-\alpha_i(q_{i\dots i}-\tau(\mathcal{Q}))\ge p_{i\dots i}q_{i\dots i}-\beta_i(p_{i\dots i}-\tau(\mathcal{P})),\tag{3.19}$$

$$p_{i\dots i}q_{i\dots i}-\{\alpha_i\beta_i(p_{i\dots i}-\tau(\mathcal{P}))(q_{i\dots i}-\tau(\mathcal{Q}))\}^{1/2}\le p_{i\dots i}q_{i\dots i}-\alpha_i[q_{i\dots i}-\tau(\mathcal{Q})].\tag{3.20}$$

Combining (3.19) with (3.20), we obtain

$$\begin{aligned}&\min_{i\in N\setminus I}\{p_{i\dots i}q_{i\dots i}-[\alpha_i\beta_i(p_{i\dots i}-\tau(\mathcal{P}))(q_{i\dots i}-\tau(\mathcal{Q}))]^{1/2}\}\\ &\le\max\left\{\min_{i\in N\setminus I}[p_{i\dots i}q_{i\dots i}-\alpha_i(q_{i\dots i}-\tau(\mathcal{Q}))],\min_{i\in N\setminus I}[p_{i\dots i}q_{i\dots i}-\beta_i(p_{i\dots i}-\tau(\mathcal{P}))]\right\}.\end{aligned}\tag{3.21}$$

It follows from (3.18) and (3.21) that (3.15) holds. □

Remark 3.15. From Corollary 3.14, we deduce that the result of Theorem 3.11 is always sharper than that of Theorem 3.8.

In the following, we shall give a upper bound on the minimal eigenvalues for Fan product of M-tensors.

Theorem 3.16. *If $\mathcal{P}$ and $\mathcal{Q}$ are strong M-tensors of order m dimension n, then*

$$\tau(\mathcal{P}\star\mathcal{Q})\le\min\left\{\max_{i\in N}[p_{i\dots i}q_{i\dots i}-\eta_i(q_{i\dots i}-\tau(\mathcal{Q}))],\max_{i\in N}[p_{i\dots i}q_{i\dots i}-\theta_i(p_{i\dots i}-\tau(\mathcal{P}))]\right\},$$

$\eta_i=\min_{\delta_{ii_2\dots i_m}=0}|p_{ii_2\dots i_m}|$ *and* $\theta_i=\min_{\delta_{ii_2\dots i_m}=0}|q_{ii_2\dots i_m}|$.

Proof. The proof is divided into two cases.

Case 1. $\mathcal{Q}$ is weakly irreducible. Since $\mathcal{Q}$ is a strong M-tensor, there exists a positive eigenvector $d = (d_i)$ corresponding to $\tau(\mathcal{Q})$ satisfying (3.4). Define $\mathcal{U} = \mathcal{P} \star \mathcal{Q}$. For $i \in N$, it follows from the definition of η_i that

$$
\begin{aligned}
(\mathcal{U}d^{m-1})_i &= p_{i\dots i}q_{i\dots i}d_i^{m-1} - \sum_{\delta_{ii_2\dots i_m}=0} |p_{ii_2\dots i_m}||q_{ii_2\dots i_m}|d_{i_2}\cdots d_{i_m} \\
&\le p_{i\dots i}q_{i\dots i}d_i^{m-1} - \eta_i \sum_{\delta_{ii_2\dots i_m}=0} (|q_{ii_2\dots i_m}|d_{i_2}\cdots d_{i_m}) \\
&= p_{i\dots i}q_{i\dots i}d_i^{m-1} - \eta_i(q_{i\dots i} - \tau(\mathcal{Q}))d_i^{m-1} \\
&= (p_{i\dots i}q_{i\dots i} - \eta_i(q_{i\dots i} - \tau(\mathcal{Q})))d_i^{m-1}.
\end{aligned}
\tag{3.22}
$$

It follows from Lemma 3.1 and (3.22) that $\tau(\mathcal{P}\star\mathcal{Q}) \le \max_{i\in N}\{p_{i\dots i}q_{i\dots i} - \eta_i(q_{i\dots i} - \tau(\mathcal{Q}))\}$.

Case 2. $\mathcal{Q}$ is weakly reducible. Similar to the proof of Theorem 3.5, we can obtain the desired result.

Meanwhile, since $\mathcal{P}\star\mathcal{Q} = \mathcal{Q}\star\mathcal{P}$, one has $\tau(\mathcal{P}\star\mathcal{Q}) \le \max_{i\in N}\{p_{i\dots i}q_{i\dots i} - \theta_i(p_{i\dots i} - \tau(\mathcal{P}))\}$. Hence, the desired result holds. $\square$

4. Conclusion

In this paper, we generalized important inequalities on the minimal eigenvalues for Fan product from matrices to tensors. Based on characterizations of M-tensors, we proposed lower bound estimations and upper bound estimations on the minimal eigenvalues for Fan product of two M-tensors, which all depend only on the entries to M-tensor itself. Finally, we discussed inclusion relations among different theorems.

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References

[1] K. C. Chang, K. Pearson and T. Zhang, *Perron-Frobenius theorem for nonnegative tensors*, Commun. Math. Sci. **6** (2008), no. 2, 507–520.

[2] H. Chen and Y. Wang, *On computing minimal H-eigenvalue of sign-structured tensors*, Front. Math. China. **12** (2017), no. 6, 1289–1302.

[3] W. Ding, L. Qi and Y. Wei, *$\mathcal{M}$-tensors and nonsingular $\mathcal{M}$-tensors*, Linear Algebra Appl. **439** (2013), no. 10, 3264–3278.

[4] F. Fang, *Bounds on eigenvalues of the Hadamard product and the Fan product of matrices*, Linear Algebra Appl. **425** (2007), no. 1, 7–15.

[5] S. Friedland, S. Gaubert and L. Han, *Perron-Frobenius theorem for nonnegative multilinear forms and extensions*, Linear Algebra Appl. **438** (2013), no. 2, 738–749.

[6] L. Gao and D. Wang, *Input-to-state stability and integral input-to-state stability for impulsive switched systems with time-delay under asynchronous switching*, Nonlinear Anal. Hybrid Syst. **20** (2016), 55–71.

[7] L. Gao, D. Wang and G. Wang, *Further results on exponential stability for impulsive switched nonlinear time-delay systems with delayed impulse effects*, Appl. Math. Comput. **268** (2015), 186–200.

[8] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[9] S. Hu, Z. Huang and L. Qi, *Strictly nonnegative tensors and nonnegative tensor partition*, Sci. China. Math. **57** (2014), no. 1, 181–195.

[10] R. Huang, *Some inequalities for the Hadamard product and the Fan product of matrices*, Linear Algebra Appl. **428** (2008), no. 7, 1551–1559.

[11] Y.-T. Li, F.-B. Chen and D.-F. Wang, *New lower bounds on eigenvalue of the Hadamard product of an M-matrix and its inverse*, Linear Algebra Appl. **430** (2009), no. 4, 1423–1431.

[12] Q. Liu, G. Chen and L. Zhao, *Some new bounds on the spectral radius of matrices*, Linear Algebra Appl. **432** (2010), no. 4, 936–948.

[13] M. Ng, L. Qi and G. Zhou, *Finding the largest eigenvalue of a nonnegative tensor*, SIAM J. Matrix Anal. Appl. **31** (2009), no. 3, 1090–1099.

[14] Q. Ni, L. Qi and F. Wang, *An eigenvalue method for testing positive definiteness of a multivariate form*, IEEE Trans. Automat. Control **53** (2008), no. 5, 1096–1107.

[15] L. Qi, *Eigenvalues of a real supersymmetric tensor*, J. Symbolic Comput. **40** (2005), no. 6, 1302–1324.

[16] M. Rajesh Kannan, N. Shaked-Monderer and A. Berman, *On weakly irreducible nonnegative tensors and interval hull of some classes of tensors*, Linear Multilinear Algebra **64** (2016), no. 4, 667–679.

[17] L. Sun, B. Zheng, J. Zhou and H. Yan, *Some inequalities for the Hadamard product of tensors*, Linear Multilinear Algebra **66** (2018), no. 6, 1199–1214.

[18] Y. Wang, L. Caccetta and G. Zhou, *Convergence analysis of a block improvement method for polynomial optimization over unit spheres*, Numer. Linear Algebra Appl. **22** (2015), no. 6, 1059–1076.

[19] Y. Wang, K. Zhang and H. Sun, *Criteria for strong H-tensors*, Front. Math. China. **11** (2016), no. 3, 577–592.

[20] G. Wang, G. Zhou and L. Caccetta, *Z-eigenvalue inclusion theorems for tensors*, Discrete Contin. Dyn. Syst. Ser. B **22** (2017), no. 1, 187–198.

[21] ______, *Sharp Brauer-type eigenvalue inclusion theorems for tensors*, Pac. J. Optim. **14** (2018), no. 2, 227–244.

[22] Y. Yang and Q. Yang, *Further results for Perron-Frobenius theorem for nonnegative tensors*, SIAM J. Matrix Anal. Appl. **31** (2010), no. 5, 2517–2530.

[23] L. Zhang, L. Qi and G. Zhou, *M-tensors and some applications*, SIAM J. Matrix Anal. Appl. **35** (2014), no. 2, 437–452.

[24] K. Zhang and Y. Wang, *An H-tensor based iterative scheme for identifying the positive definiteness of multivariate homogeneous forms*, J. Comput. Appl. Math. **305** (2016), 1–10.

[25] D. Zhou, G. Chen, G. Wu and X. Zhang, *On some new bounds for eigenvalues of the Hadamard product and the Fan product of matrices*, Linear Algebra Appl. **438** (2013), no. 3, 1415–1426.

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