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Restricted Arc Connectivity of Unidirectional Hypercubes and Unidirectional Folded Hypercubes

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Abstract. Unidirectional hypercubes and unidirectional folded hypercubes are generalizations of hypercubes and folded hypercubes to digraphs. The super-λ property of a digraph is a index for network reliability, which can be measured by the restricted arc-connectivity quantitatively. In this paper, we first show that the restricted arc-connectivity of the n-dimensional unidirectional hypercube is $n-1$ when n is even and is $n-2$ when n is odd, and then we show that the restricted arc-connectivity of the n-dimensional unidirectional folded hypercube is $n-1$ when n is even and is n when n is odd. As a consequence, we prove that both unidirectional hypercube and unidirectional folded hypercube are super-λ.

1. Introduction

The hypercube network has been widely applied in designing massively parallel or distributed systems due to its many excellent topological properties such as simple point-to-point connection, parallel communications, short diameter, short average distance, and efficient routing [9, 14, 15, 17, 21]. Due to the lack of a bidirectional electrical/optical converter and the high cost of a full-duplex transmission, unidirectional topologies are desirable for networks. Inspired by this, Chou and Du [7] proposed the unidirectional hypercube by orienting all the edges in the hypercube. After then, many properties of unidirectional hypercubes have been researched, such as routing property [13], wide diameter [16], and Hamilton property [12].

The folded hypercube is one of the important variants of the hypercube network which has better properties than the hypercube in many measurements, such as diameter, fault diameter, and connectivity [8, 19]. Motivated by this, as one variant of the unidirectional hypercube, we will introduce the unidirectional folded hypercube in this paper.

When designing interconnection networks, one fundamental consideration is the reliability of networks, which can be measured by the edge (arc)-connectivity of graphs

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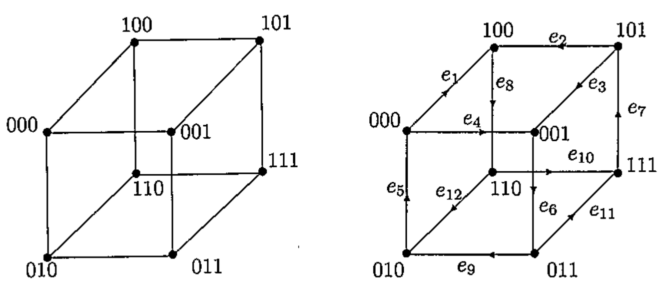
(digraphs). To maximize edge (arc)-connectivity of graphs (digraphs) and minimize the number of edge (arc)-cuts of graphs (digraphs), super-λ property was proposed by Boesch in [4]. It is a more refine index than the edge-connectivity for network reliability. As a measurement of super-λ property, the concept of restricted arc-connectivity was introduced by Volkmann in [18]. It is believed that the larger the restricted arc-connectivity of a network, the more reliable the network. Therefore, researchers tried to study upper bounds on restricted arc-connectivity. In 2008, Wang and Lin [20] introduced the concept of minimum arc-degree and proved that it is an upper bound on restricted arc-connectivity. Since then, some sufficient conditions for the restricted arc-connectivity of a digraph to reach this upper bound have been researched [1, 2, 6, 10, 20]. However, the restricted arc-connectivity of well-known networks has received little attention. In this paper, we consider the reliability of unidirectional hypercubes and unidirectional folded hypercubes, determine their restricted arc-connectivities by showing that their restricted arc-connectivities are equal to their minimum arc-degrees, and prove they are super-λ.

2. Preliminaries

In this section, we first introduce the concepts of unidirectional hypercubes and unidirectional folded hypercubes.

Definition 2.1. [15] Let $n \geq 2$ be an integer. An *n-dimensional hypercube*, denoted by Q_n, is an undirected graph with 2^n vertices. Each vertex of Q_n can be represented as an n-bit binary string. Two distinct vertices $x = a_{n-1}a_{n-2}\cdots a_0$ and $y = b_{n-1}b_{n-2}\cdots b_0$ in Q_n are adjacent if and only if there is a bit position $d \in \{0, 1, \ldots, n-1\}$ such that $b_d = 1 - a_d$ and $b_i = a_i$ for any $i \neq d$. In this case, y is denoted by x^d and the edge xy is called a d-edge.

Figure 2.1(a) shows the hypercube Q_3.



(a) Q_3 (b) UQ_3

Figure 2.1: The hypercube Q_3 and the unidirectional hypercube UQ_3.

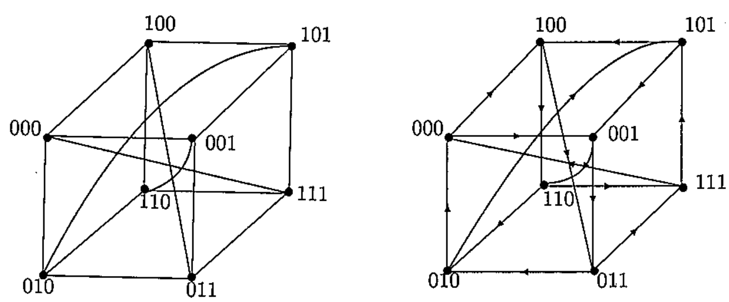
The *Hamming weight* of a vertex $x = a_{n-1}a_{n-2}\cdots a_1a_0$ in Q_n is $h(x) = a_{n-1} + a_{n-2} + \cdots + a_1 + a_0$. A vertex is *even* if its Hamming weight is even; otherwise, it is *odd.* It is easy to see that if xy is an edge of Q_n, then $h(x)$ and $h(y)$ have different parities.

Definition 2.2. [12] Let xy be a d-edge of Q_n. We orient the edge from x to y if $h(x)+d$ is even; otherwise we orient the edge from y to x. The *unidirectional hypercube* UQ_n is obtained by orienting all the edges in Q_n in this way. The arc obtained from a d-edge of Q_n by orienting is called a *d-arc of* UQ_n.

For any $d \in \{0, 1, \ldots, n-1\}$, UQ_n can be decomposed into two subgraphs, UQ_n^0 and UQ_n^1, by removing all d-arcs. In fact, UQ_n^0 and UQ_n^1 are two subgraphs of UQ_n induced by the sets $\{x = a_{n-1}\cdots a_d \cdots a_0 \in V(UQ_n) \mid a_d = 0\}$ and $\{x = a_{n-1}\cdots a_d \cdots a_0 \in V(UQ_n) \mid a_d = 1\})$, respectively. Figure 2.1(b) shows the unidirectional hypercube UQ_3.

As one of the important variants of the hypercube, the n-dimensional folded hypercube is proposed first by El-Amawy and Latifi [8].

Definition 2.3. [8] Let $n \geq 2$ be an integer. An *n-dimensional folded hypercube*, denoted by F_n, is obtained from the hypercube Q_n by adding an edge called a *complementary edge* between any pair of complementary vertices $x = a_{n-1}a_{n-2}\cdots a_1a_0$ and $\overline{x} = \overline{a}_{n-1}\overline{a}_{n-2}\cdots\overline{a}_1\overline{a}_0$, where $\overline{a}_i = 1 - a_i$ for $i = 0, 1, \ldots, n-1$.



(a) F_3 (b) UF_3

Figure 2.2: The folded hypercube F_3 and the unidirectional folded hypercube UF_3.

Figure 2.2(a) shows the folded hypercube F_3. Let $x\overline{x}$ be a complementary edge of F_n. If n is odd, then exactly one of x and $\overline{x}$ is even. We orient the edge $x\overline{x}$ from the odd vertex to the even one. If n is even, then x and $\overline{x}$ have the same parity. We orient the edge $x\overline{x}$ casually. The resulting arc is called a *complementary arc.*

Definition 2.4. Let $n \geq 2$ be an integer. An *n-dimensional unidirectional folded hypercube*, denoted by UF_n, is obtained from the unidirectional hypercube UQ_n by adding a complementary arc defined above between any pair x and $\overline{x}$ of complementary vertices.

Obviously, there exists a unique n-dimensional unidirectional folded hypercube when n is odd and $2^{2^{n-1}}$ distinct n-dimensional unidirectional folded hypercubes when n is even. The unique 3-dimensional unidirectional folded hypercube and the four 2-dimensional unidirectional folded hypercubes are shown in Figure 2.2(b) and Figure 2.3, respectively.

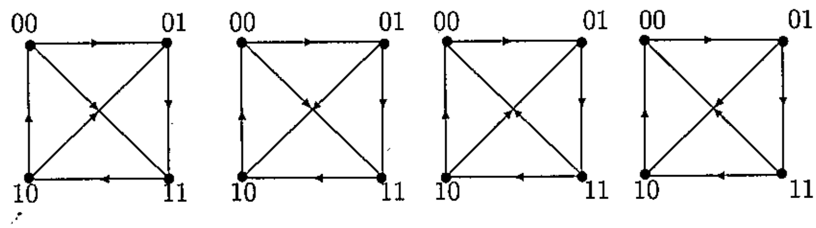


Figure 2.3: The four 2-dimensional unidirectional folded hypercubes.

Denote the set of vertices and the set of arcs of a digraph D by $V(D)$ and $A(D)$, respectively. Let $P = v_1v_2\cdots v_pv_{p+1}$ be a vertex sequence of $V(D)$ such that $v_iv_{i+1} \in A(D)$ for $i = 1, 2, \ldots, p$. If the vertices $v_1, v_2, \ldots, v_p, v_{p+1}$ are distinct, then P is a *directed path of length* p; if the vertices $v_1, v_2, \ldots, v_p$ are distinct and $v_{p+1} = v_1$, then P is a *directed cycle of length* p. An *oriented graph* is a digraph with no directed cycle of length 2. It is easy to see that both the unidirectional hypercube UQ_n and the unidirectional folded hypercube UF_n are oriented graphs. A digraph D is *strongly connected* (or simply *strong*) if there exists a directed path from x to y for every ordered pair (x, y) of vertices in D. A *strong component* of a digraph D is a maximal subdigraph of D which is strong.

Definition 2.5. [18] An arc subset S of a digraph D is an *arc-cut* if $D - S$ is not strong. The *arc-connectivity* $\lambda(D)$ of D is the minimum cardinality of all arc-cuts of D.

Definition 2.6. [18] Let D be a strong digraph. An arc subset S is a *restricted arc-cut* of D if $D - S$ has a strong component D' such that $|V(D')| \geq 2$ and $D - V(D')$ contains an arc. If such a restricted arc-cut exists, then D is called λ'*-connected.* The *restricted arc-connectivity* $\lambda'(D)$ of a λ'-connected digraph D is the minimum cardinality over all restricted arc-cuts.

The following is an example illustrating the definition above.

Example 2.7. Use the notations in Figure 2.1(b). Let $x = 010$, $y = 000$ and $S = \{e_5\}$. Then x has no out-arc, y has no in-arc and $UQ_3 - \{x, y\}$ is strong in $UQ_3 - S$, which implies that $UQ_3 - S$ has three strong components x, y and $UQ_3 - \{x, y\}$. Let $D' = UQ_3 - \{x, y\}$. It is easy to see that $UQ_3 - V(D')$ contains the arc e_5. Thus, by definition, S is a restricted arc-cut and so $\lambda'(UQ_3) \leq |S| = 1$. Combining this with the fact that UQ_3 is strong, we have $\lambda'(UQ_3) = 1$.

For a set $X \subseteq V(D)$, denote the sets of arcs of D leaving X and entering X by $\partial_D^+(X) = \partial^+(X)$ and $\partial_D^-(X) = \partial^-(X)$, respectively. Usually, abbreviate $\partial^+(\{x\})$ and $\partial^-(\{x\})$ to $\partial^+(x)$ and $\partial^-(x)$, respectively.

Definition 2.8. [20] Let D be a strong digraph. For any $xy \in A(D)$, let

$$\Omega(xy) = \{\partial^+(\{x,y\}), \partial^+(x) \cup \partial^-(y), \partial^-(x) \cup \partial^+(y), \partial^-(\{x,y\})\}.$$

The arc-degree of xy is $\xi'(xy) = \min\{|S| : S \in \Omega(xy)\}$ and the minimum arc-degree of D is $\xi'(D) = \min\{\xi'(xy) : xy \in A(D)\}$.

To illustrate the definition above, consider the arc-degree of the arc e_5 given in Figure 2.1(b). It can be seen from Figure 2.1(b) that

$$\Omega(e_5) = \{\{e_1, e_4\}, \{e_5\}, \{e_1, e_4, e_9, e_{12}\}, \{e_9, e_{12}\}\}.$$

Therefore, by Definition 2.8, $\xi'(e_5) = \min\{|S| : S \in \Omega(e_5)\} = 1$.

In a digraph D, the *out-degree* and *in-degree* of a vertex x are $d^+(x) = |\partial^+(x)|$ and $d^-(x) = |\partial^-(x)|$, respectively. The *minimum degree* of D is $\delta(D) = \min\{d^+(x), d^-(x) : x \in V(D)\}$. A digraph D is said to be *d-regular* if the out-degree and the in-degree of every vertex in D are both equal to d. In [20], it was shown that if xy is an arc of an oriented graph D, then

$$\xi'(xy) = \min\{d^+(x)+d^+(y)-1, d^-(x)+d^-(y)-1, d^+(x)+d^-(y)-1, d^-(x)+d^+(y)\}. \tag{2.1}$$

3. The minimum arc-degrees of unidirectional hypercubes and unidirectional folded hypercubes

It is clear that the degree of every vertex in Q_n is n. Let x be a vertex of UQ_n. By Definition 2.2, we obtain that

$$d^+(x) = d^-(x) = \frac{n}{2} \quad \text{if } n \text{ is even} \tag{3.1}$$

and

$$\begin{aligned} d^+(x) &= \left\lceil \frac{n}{2} \right\rceil \quad \text{and} \quad d^-(x) = \left\lfloor \frac{n}{2} \right\rfloor \quad \text{if } n \text{ is odd and } x \text{ is even},\\ d^+(x) &= \left\lfloor \frac{n}{2} \right\rfloor \quad \text{and} \quad d^-(x) = \left\lceil \frac{n}{2} \right\rceil \quad \text{if } n \text{ is odd and } x \text{ is odd}. \end{aligned} \tag{3.2}$$

Now, we determine the minimum arc-degree of the unidirectional hypercube.

Theorem 3.1. *The minimum arc-degree of UQ_n is*

$$\xi'(UQ_n) = \begin{cases} n-1 & \text{if } n \text{ is even,} \\ n-2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. First, consider the case that n is even. For an arbitrary arc xy in UQ_n, it follows from (2.1) and (3.1) that

$$\begin{aligned}\xi'(xy) &= \min\{d^+(x)+d^+(y)-1, d^-(x)+d^-(y)-1, d^+(x)+d^-(y)-1, d^-(x)+d^+(y)\} \\ &= \frac{n}{2}+\frac{n}{2}-1 = n-1.\end{aligned}$$

By Definition 2.8, we have $\xi'(UQ_n) = n-1$.

Next, consider the case that n is odd. By (2.1) and (3.2), $\xi'(uv) \geq \lfloor n/2 \rfloor + \lfloor n/2 \rfloor - 1 \geq n-2$ for each arc uv of UQ_n. Let xy be a 1-arc of UQ_n. By Definition 2.2, x is odd and y is even. It follows from (2.1) and (3.2) that $\xi'(xy) \leq d^+(x) + d^-(y) - 1 = (n-1)/2 + (n-1)/2 - 1 = n-2$. Therefore, we have $\xi'(UQ_n) = n-2$. □

Let UF_n be a unidirectional folded hypercube and x be a vertex of UF_n. By Definition 2.4, we have

$$d^+(x) = d^-(x) = \frac{n+1}{2} \quad \text{if } n \text{ is odd} \tag{3.3}$$

and

$$|d^+(x) - d^-(x)| = 1 \quad \text{and} \quad d^+(x) + d^-(x) = n+1 \quad \text{if } n \text{ is even.} \tag{3.4}$$

Next, we determine the minimum arc-degree of the unidirectional folded hypercube.

Theorem 3.2. *Let UF_n be an n-dimensional unidirectional folded hypercube. Then the minimum arc-degree of UF_n is*

$$\xi'(UF_n) = \begin{cases} n-1 & \text{if } n \text{ is even,} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

Proof. First, consider the case that n is odd. Let xy be an arc in UF_n. It follows from (2.1) and (3.3) that $\xi'(xy) = (n+1)/2 + (n+1)/2 - 1 = n$. By Definition 2.8, we obtain that $\xi'(UF_n) = n$.

Next, consider the case that n is even. It follows from (2.1) and (3.4) that $\xi'(xy) \geq n/2 + n/2 - 1 = n-1$. Let $x\overline{x}$ be a complementary arc of UF_n. Without loss of generality, assume that x is even. Then $\overline{x}$ is also even. Clearly, $xx^0, \overline{x}\overline{x}^0 \in A(UF_n)$ and there also exists a complementary arc between x^0 and $\overline{x}^0$. If $x^0\overline{x}^0 \in A(UF_n)$, then $\xi'(xx^0) \leq d^-(x) + d^-(x^0) - 1 = n/2 + n/2 - 1 = n-1$; otherwise, $\xi'(\overline{x}\overline{x}^0) \leq d^+(\overline{x}) + d^-(\overline{x}^0) - 1 = n/2 + n/2 - 1 = n-1$. By Definition 2.8, we obtain that $\xi'(UF_n) = n-1$. □

4. Restricted arc-connectivities of unidirectional hypercubes and unidirectional folded hypercubes

We now introduce the concept of restricted edge-connectivity of graphs, which is an undirected analogue of restricted arc-connectivity. A graph G is *connected* if there exists a path between any two vertices. An edge set F of G is a *restricted edge cut*, if $D - F$ is not connected and contains no isolated vertex. A connected graph G is called λ'*-connected* if it contains a restricted edge-cut. The *restricted edge-connectivity* $\lambda'(G)$ of a λ'-connected graph G is the minimum cardinality over all restricted edge-cuts. The restricted edge-connectivities of hypercubes and folded hypercubes have been determined.

Lemma 4.1. [9] *The n-dimensional hypercube Q_n with $n \geq 2$ is λ'-connected and its restricted edge-connectivity is $\lambda'(Q_n) = 2n - 2$.*

Lemma 4.2. [8] *The n-dimensional folded hypercube F_n with $n \geq 2$ is λ'-connected and its restricted edge-connectivity is $\lambda'(F_n) = 2n$.*

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For every nonempty $X \subseteq V(G)$, the subgraph of G induced by X is denoted by $G[X]$, and the set of edges in G with exactly one end in X by $[X, \overline{X}]$, where $\overline{X} = V(G) - X$. The restricted edge-connectivity of undirected graphs has the following bound.

Lemma 4.3. [5] *Let G be a λ'-connected graph. If X is a vertex set of G such that both $G[X]$ and $G[\overline{X}]$ contain an edge, then $|[X, \overline{X}]| \geq \lambda'(G)$.*

The following inequality was proved in [11]:

$$\kappa(D) \leq \lambda(D) \leq \delta(D), \tag{4.1}$$

where $\delta(D)$, $\lambda(D)$ and $\kappa(D)$ are the minimum degree, the arc-connectivity and the vertex-connectivity of a digraph D, respectively. In [13], Jow and Tuan showed that $\kappa(UQ_n) = \delta(UQ_n)$. Combining these two results, we have the following.

Lemma 4.4. *The arc-connectivity of n-dimensional unidirectional hypercube UQ_n is $\lambda(UQ_n) = \delta(UQ_n)$ when $n \geq 2$.*

Now, we turn our attention back to the restricted arc-connectivity of digraphs. In 2010, Balbuena and García-Vázquez [1] proved following results.

Lemma 4.5. [1] *Let D be a strong digraph of order at least 4. If its minimum degree $\delta(D) \geq 2$, then D is λ'-connected and $\lambda'(D) \leq \xi'(D)$.*

Definition 4.6. [1] A λ'-connected digraph D is said to be super-λ' if for every minimum restricted arc-cut S of D there exists an arc $xy \in A(D)$ such that $S \in \Omega(xy)$.

Lemma 4.7. [1] *Let D be a λ'-connected digraph and let S be a minimum restricted arc-cut of D. If D is not super-λ', then there exists a subset of vertices $X \subset V(D)$ such that $S = \partial^+(X)$ and the induced subdigraphs $D[X]$ and $D[\overline{X}]$ both contain an arc.*

In order to prove our main results, we need to prove several useful lemmas.

Lemma 4.8. *Let G be a $2k$-regular connected graph with $k \geq 2$ and $\overrightarrow{G}$ be a k-regular strong connected oriented digraph obtained by orienting all edges in G. Then $\overrightarrow{G}$ is λ'-connected and $\lambda'(\overrightarrow{G}) \geq \lambda'(G)/2$ if $\overrightarrow{G}$ is not super-λ'.*

Proof. Because G is a $2k$-regular graph, obviously, $|V(G)| \geq 4$. Combining this with $\delta(\overrightarrow{G}) = k \geq 2$, it follows from Lemma 4.5 that $\overrightarrow{G}$ is λ'-connected. Let $\overrightarrow{S}$ be a minimum restricted arc-cut of $\overrightarrow{G}$.

Suppose that $\overrightarrow{G}$ is not super-λ'. It follows from Lemma 4.7 that there exists a set of vertices X in $\overrightarrow{G}$ such that $\overrightarrow{S} = \partial^+(X)$ and both $\overrightarrow{G}[X]$ and $\overrightarrow{G}[\overline{X}]$ contain an arc. By the definition of $\overrightarrow{G}$, it is easy to see that both $G[X]$ and $G[\overline{X}]$ contain an edge. Therefore, by Lemma 4.3, $|[X, \overline{X}]| \geq \lambda'(G)$. It is well-known that $|\partial^+(Y)| = |\partial^-(Y)|$ for any subset $Y \subseteq V(D)$, if D is a regular digraph [22]. Combining this with the fact that $|[X, \overline{X}]| = |\partial^+(X)| + |\partial^-(X)|$, we have

$$2\lambda'(\overrightarrow{G}) = 2|\overrightarrow{S}| = 2|\partial^+(X)| = |\partial^+(X)| + |\partial^-(X)| = |[X, \overline{X}]| \geq \lambda'(G).$$

The proof is complete. □

Lemma 4.9. [7] *Let x be a vertex of UQ_n. Then the i-arc and j-arc incident with x can be embedded in a directed cycle of length 4 of UQ_n if and only if $i + j$ is odd.*

For any integer $i \in \{0, 1, \ldots, n-1\}$, there are exactly $n/2$ integers $j \in \{0, 1, \ldots, n-1\}$ such that $i + j$ is odd when n is even. Combining this with Lemma 4.9, we have the following observation.

Observation 4.10. If n is even, then each arc of UQ_n can be embedded in $n/2$ distinct directed cycles of length 4.

Lemma 4.11. *Let $n \geq 3$ and $d \in \{0, 1, \ldots, n-1\}$ be two integers. We decompose UQ_n into two subgraphs UQ_n^0 and UQ_n^1 by removing all d-arcs. Then*

(i) *UQ_n^0 is isomorphic to UQ_{n-1}, if both n and d are even or n and d have different parities;*

(ii) *UQ_n^0 is not isomorphic to UQ_{n-1}, if both n and d are odd.*

(iii) *UQ_n^0 is isomorphic to UQ_n^1.*

Proof. (i) First, consider the case that both n and d are even. Let θ be the mapping from $V(UQ_n^0)$ to $V(UQ_{n-1})$ defined by

$$\theta(a_{n-1}a_{n-2}\cdots a_{d+1}0a_{d-1}\cdots a_0) = a_{n-1}a_{n-2}\cdots a_{d+2}a_{d-1}\cdots a_0\overline{a_{d+1}}.$$

Obviously, θ is a bijection. In order to show that θ is an isomorphism from UQ_n^0 to UQ_{n-1}, it suffices to prove xy is an arc of UQ_n^0 if and only if $\theta(x)\theta(y)$ is an arc of UQ_{n-1}.

Suppose that xy is an arc of UQ_n^0, say an i-arc. If $i \in \{0, \ldots, d-2, d-1\}$, then $\theta(x)$ and $\theta(y)$ differ in the position $(i+1)$ and so $\theta(x)\theta(y)$ is an $(i+1)$-edge of Q_{n-1}. Obviously, $\theta(x)$ and x have different parities. Therefore, it follows that $h(\theta(x)) + i + 1$ is even from the fact that $h(x) + i$ is even, which implies that $\theta(x)\theta(y)$ is an $(i+1)$-arc of UQ_{n-1}. If $i \in \{d+2, d+3, \ldots, n-1\}$, then $\theta(x)$ and $\theta(y)$ differ in the position $(i-1)$ and so $\theta(x)\theta(y)$ is an $(i-1)$-arc of UQ_{n-1}. If $i \in \{d+1\}$, then $\theta(x)$ and $\theta(y)$ differ in the position 0 and so $\theta(x)\theta(y)$ is a 0-arc of UQ_{n-1}.

If xy is not an arc of UQ_n^0, by Definition 2.2, either x and y differ in at least two positions or yx is an arc of UQ_n^0. If x and y differ in at least two positions, then $\theta(x)$ and $\theta(y)$ differ in at least two positions, which implies that $\theta(x)\theta(y)$ is not an arc of UQ_{n-1}. If yx is an arc of UQ_n^0, according to the above argument, we have $\theta(y)\theta(x)$ is an arc of UQ_{n-1}, which implies that $\theta(x)\theta(y)$ is not an arc of UQ_{n-1}. Therefore, UQ_n^0 is isomorphic to UQ_{n-1}.

Next, consider the case that n and i have different parities. Let ϕ be the mapping from $V(UQ_n^0)$ to $V(UQ_{n-1})$ defined by

$$\phi(a_{n-1}a_{n-2}\cdots a_{d+1}0a_{d-1}\cdots a_0) = a_{d+1}a_{n-1}a_{n-2}\cdots a_{d+2}a_{d-1}\cdots a_0.$$

Obviously, ϕ is a bijection. In order to show that ϕ is an isomorphism from UQ_n^0 to UQ_{n-1}, it suffices to prove xy is an arc of UQ_n^0 if and only if $\phi(x)\phi(y)$ is an arc of UQ_{n-1}. In the same way as above, we can prove that xy is an i-arc of UQ_n^0 if and only if $\phi(x)\phi(y)$ is an i-arc of UQ_{n-1} when $i \in \{0, \ldots, d-2, d-1\}$, xy is an i-arc of UQ_n^0 if and only if $\phi(x)\phi(y)$ is an $(i-2)$-arc of UQ_{n-1} when $i \in \{d+2, d+3, \ldots, n-1\}$ and xy is an i-arc of UQ_n^0 if and only if $\phi(x)\phi(y)$ is an $i + (n-d-3)$-arc of UQ_{n-1} when $i \in \{d+1\}$. Thus, xy is an arc of UQ_n^0 if and only if $\phi(x)\phi(y)$ is an arc of UQ_{n-1}.

(ii) Let $x = 00\cdots 0$ be a vertex of UQ_n^0. We have $d^+(x) = (n+1)/2$ in UQ_n^0. By (3.1), UQ_{n-1} is $(n-1)/2$-regular, which implies that UQ_n^0 is not isomorphic to UQ_{n-1}.

(iii) Let φ be the mapping from $V(UQ_n^0)$ to $V(UQ_n^1)$ defined by

$$\varphi(a_{n-1}a_{n-2}\cdots a_{d+1}0a_{d-1}\cdots a_0) = a_{n-1}a_{n-2}\cdots a_{d+1}1a_{d-1}\cdots \overline{a_0}.$$

Obviously, φ is a bijection. Similar to the proof of (i), we can prove that xy is an arc of UQ_n^0 if and only if $\varphi(x)\varphi(y)$ is an arc of UQ_n^1. Thus, UQ_n^0 is isomorphic to UQ_n^1. $\square$

Let D be an oriented graph with 4 vertices. By Definition 2.6, if D is a λ'-connected digraph, there exists an arc subset S such that $D - S$ has a strong component D' with $|V(D')| \geq 2$ and $D - V(D')$ has an arc which implies $|V(D) - V(D')| \geq 2$. Thus, D' is a strong digraph with 2 vertices and so it is a cycle of length 2 contradicting that D is an oriented graph. Therefore, we have the following observation.

Observation 4.12. An oriented graph with 4 vertices is not λ'-connected.

By Definitions 2.2, 2.4 and Observation 4.12, we have that neither UQ_2 nor UF_2 is λ'-connected. Below, we determine the restricted arc-connectivities of UQ_n and UF_n with $n \geq 3$.

Theorem 4.13. *Let $n \geq 3$ be an integer. The n-dimensional unidirectional hypercube UQ_n is λ'-connected and its restricted arc-connectivity is*

$$\lambda'(UQ_n) = \xi'(UQ_n) = \begin{cases} n-1 & \text{if } n \text{ is even,} \\ n-2 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. It follows from Example 2.7 and Theorem 3.1 that $\lambda'(UQ_3) = 1 = n-2 = \xi'(UQ_3)$ and so the statement holds for $n = 3$.

Next, consider the case $n \geq 4$. It follows from (3.1) and (3.2) that $\delta(UQ_n) \geq 2$. By Lemma 4.5, we have that UQ_n is λ'-connected and $\lambda'(UQ_n) \leq \xi'(UQ_n)$. Combining this with Theorem 3.1, we need only to prove $\lambda'(UQ_n) \geq \xi'(UQ_n)$. Let S be a minimum restricted arc-cut of UQ_n.

Case 1. UQ_n is super-λ'.

By Definition 4.6, there exists an arc $xy \in A(UQ_n)$ such that $S \in \Omega(xy)$. Combining this with Definition 2.8, we have $\lambda'(UQ_n) = |S| \geq \xi'(xy) \geq \xi'(UQ_n)$.

Case 2. UQ_n is not super-λ' and n is even.

In this case, by (3.1), UQ_n is $n/2$-regular. It follows from Lemmas 4.1, 4.8 and Theorem 3.1, we have $\lambda'(UQ_n) \geq \lambda'(Q_n)/2 = n-1 = \xi'(UQ_n)$.

Case 3. UQ_n is not super-λ' and n is odd.

By Theorem 3.1, it is sufficient to prove that $\lambda'(UQ_n) \geq n-2$. Suppose to the contrary that $|S| = \lambda'(UQ_n) \leq n-3$.

Denote all i-arcs by F_i for $i \in \{0, 1, 2, \ldots, n-1\}$. Obviously, $|S \cap F_0| + |S \cap F_2| + |S \cap F_4| + \cdots + |S \cap F_{n-1}| \leq |S| = \lambda'(UQ_n) \leq n-3$. Thus, there exists an integer $d \in \{0, 2, 4, \ldots, n-1\}$ such that $|S \cap F_d| \leq 1$. Decompose UQ_n into two subgraphs UQ_n^0 and UQ_n^1 by removing all d-arcs. By Lemma 4.11, UQ_n^0 and UQ_n^1 are both isomorphic to UQ_{n-1} and so are both strong.

Denote $S_j = S \cap A(UQ_n^j)$ for $j = 0, 1$. Without loss of generality, we may assume $|S_1| \leq |S|/2$. By (3.1) and Lemma 4.4, we have $\lambda(UQ_n^0) = \lambda(UQ_{n-1}) = \delta(UQ_{n-1}) = (n-1)/2 > |S|/2 \geq |S_1|$. Thus, $UQ_n^1 - S_1$ is still strong. Next, we consider $UQ_n^0 - S_0$.

By Cases 1 and 2, we have $\lambda'(UQ_{n-1}) \geq \xi'(UQ_{n-1}) = n-2$. Therefore, $\lambda'(UQ_n^0) \geq n-2$. Since $|S_0| \leq |S| \leq n-3 < n-2 = \lambda'(UQ_n^0)$, S_0 is not a restricted arc-cut of UQ_n^0. By Definition 2.6, we need to consider the following two cases.

Subcase 3.1. $UQ_n^0 - S_0$ has no strong component with at least 2 vertices.

Let $P = v_1v_2\cdots v_p$ be a longest directed path of $UQ_n^0 - S_0$ and denote $O_p = \partial^+_{UQ_n^0}(v_p)$. Suppose that there exists $v_px \in O_p - S_0$. If $x \notin V(P)$, then Pv_px is a longer directed path than P in $UQ_n^0 - S_0$, which contradicts the maximality of P. If $x = v_i \in V(P) - v_p$, then $C = v_iv_{i+1}\cdots v_{p-1}v_px$ is a directed cycle. Notice that C must contain at least 4 vertices and so the strong component of $UQ_n^0 - S_0$ containing this directed cycle C has at least 4 vertices, a contradiction. Thus, $O_p \subseteq S_0$.

Denote $I_1 = \partial^-_{UQ_n^0}(v_1)$. Similarly, we can obtain $I_1 \subseteq S_0$. Obviously, $|O_p| = |I_1| = (n-1)/2$ and $|O_p \cap I_1| \leq 1$. Therefore, we can deduce a contradiction as follows:

$$n-2 = |O_p| + |I_1| - 1 \leq |O_p| + |I_1| - |O_p \cap I_1| = |O_p \cup I_1| \leq |S_0| \leq |S| \leq n-3.$$

Subcase 3.2. $UQ_n^0 - S_0$ has a strong component D_0 with at least 2 vertices, but $UQ_n^0 - V(D_0)$ contains no arc.

As UQ_n contains no directed cycle of length less than 4, by assumption, we know that D_0 contains a directed cycle of length at least 4. This implies that D_0 contains a directed path $P = x_1x_2x_3x_4$.

We first prove that the subgraph of $UQ_n - S$ induced by $V(D_0) \cup V(UQ_n^1)$, denoted D_{01}, is strong. By Definition 2.1, $x_1x_1^d, x_2x_2^d, x_3x_3^d$ and $x_4x_4^d$ are four edges of Q_n between D_0 and Q_n^1. By Definition 2.2, there are exactly two of them are oriented from D_0 to UQ_n^1. Recall that $|S \cap F_d| \leq 1$. Hence, there exists at least one d-arc from D_0 to UQ_n^1 and one d-arc from UQ_n^1 to D_0 in $UQ_n - S$ which implies that D_{01} is strong in $UQ_n - S$. Let D' be the strong component of $UQ_n - S$ containing D_{01}. It is clear that $UQ_n - V(D') \subseteq UQ_n^0 - V(D_0)$. Combining this with the fact that $UQ_n^0 - V(D_0)$ contains no arc, we know that $UQ_n - V(D')$ has no arc and so D' is the unique strong component of $UQ_n - S$ with at least 2 vertices. This contradicts our assumption that S is a restricted arc-cut of UQ_n. □

Theorem 4.14. *Let $n \geq 3$ be an integer. The n-dimensional unidirectional folded hypercube UF_n is λ'-connected and its restricted arc-connectivity is*

$$\lambda'(UF_n) = \xi'(UF_n) = \begin{cases} n-1 & \text{if } n \text{ is even,} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

Proof. It follows from (3.3) and (3.4) that $\delta(UF_n) \geq 2$. By Lemma 4.5, we have that UF_n is λ'-connected and $\lambda'(UF_n) \leq \xi'(UF_n)$. Combining this with Theorem 3.2, we need only to prove $\lambda'(UF_n) \geq \xi'(UF_n)$. Let S be a minimum restricted arc-cut of UF_n.

Case 1. UF_n is super-λ'.

By Definition 4.6, there exists an arc $xy \in A(UF_n)$ such that $S \in \Omega(xy)$. Combining this with Definition 2.8, we have $\lambda'(UF_n) = |S| \geq \xi'(xy) \geq \xi'(UF_n)$.

Case 2. UF_n is not super-λ' and n is odd.

In this case, by (3.4), UF_n is $(n+1)/2$-regular. It follows from Lemmas 4.2, 4.8 and Theorem 3.2, we have $\lambda'(UF_n) \geq \lambda'(F_n)/2 = n = \xi'(UF_n)$.

Case 3. UF_n is not super-λ' and n is even.

By Theorem 3.2, it is sufficient to prove that $|S| = \lambda'(UF_n) \geq n-1$. We know that UF_n is obtained from the unidirectional hypercube UQ_n by adding complementary arcs. Denote $S' = S \cap A(UQ_n)$. We claim $|S'| \geq n-1$ which implies $|S| \geq n-1$ and so the theorem follows. Suppose to the contrary that $|S'| \leq n-2$. By Theorem 4.13, S' is not a restricted arc-cut of UQ_n. Thus, by Definition 2.6, we need to consider the following two cases.

Subcase 3.1. $UQ_n - S'$ has no strong component with at least 2 vertices.

By Observation 4.10, there are $\frac{n2^{n-1}(n/2)}{4} = 2^{n-4}n^2$ directed cycles of length 4 in UQ_n and the remove of S' will destroy at most $|S'|\frac{n}{2}$ directed cycles of length 4 in UQ_n. It is clear that $2^{n-4}n^2 > (n-2)\frac{n}{2} \geq |S'|\frac{n}{2}$ when $n \geq 4$. Thus, $UQ_n - S'$ contains a directed cycle of length 4 and so contains one strong component with at least 4 vertices, a contradiction.

Subcase 3.2. $UQ_n - S'$ has a strong component D' with at least 2 vertices, but $UQ_n - V(D')$ contains no arc.

Let D'' be the strong component of $UF_n - S$ containing D'. If $UF_n - V(D'')$ contains at most 1 vertex, it is easy to see that S is not a restricted arc-cut of UF_n, a contradiction. Therefore, $UF_n - V(D'')$ contains at least 2 vertices which implies $UQ_n - V(D')$ also contains at least 2 vertices.

Let u and v be 2 vertices of $UQ_n - V(D')$. Then each arc incident with u has exactly one end in D' and so u and v are not adjacent. If $\partial^-_{UQ_n}(u) \nsubseteq S'$ and $\partial^+_{UQ_n}(u) \nsubseteq S'$, then the vertex u and D' can reach each other in $UQ_n - S'$, which contradicts the fact that $V(D')$ is a strong component of $UQ_n - S'$. So we have either $\partial^-_{UQ_n}(u) \subseteq S'$ or $\partial^+_{UQ_n}(u) \subseteq S'$. Combining this with (3.1), we can know that S' contains at least $n/2$ arcs incident with u. Similarly, S' also contains at least $n/2$ arcs incident with v. Hence, $|S'| \geq n$, a contradiction to the assumption that $|S'| \leq n-2$. □

5. Super-λ property of unidirectional hypercubes and unidirectional folded hypercubes

The concept of super-λ was originally introduced by Boesch in [4].

Definition 5.1. [4] A digraph D is super arc-connected or super-λ if every minimum arc-cut of D is either the set of in-arcs of some vertex or the set of out-arcs of some vertex.

In the following we will show that the super-λ property of a digraph can be justified by the restricted arc-connectivity. First, we introduce a helpful concept. If a digraph D has p strong components, then these can be labeled $D_1, \ldots, D_p$ such that there is no arc from D_j to D_i unless $j < i$ [3]. We call such an ordering *an acyclic ordering of the strong components* of D. It is clear that D_1 has no in-arc and D_p has no out-arc.

Theorem 5.2. *Let D be a λ'-connected digraph. If $\lambda'(D) > \lambda(D)$, then D is super-λ.*

Proof. Let S be a minimum arc-cut of D and let $D_1, \ldots, D_p$ be an acyclic ordering of the strong components of $D - S$. If $|V(D_1)| \geq 2$ and $|V(D_p)| \geq 2$, then S is a restricted arc-cut of D. Therefore, $\lambda(D) = |S| \geq \lambda'(D)$, a contradiction. So we have $|V(D_1)| = 1$ or $|V(D_p)| = 1$. If $|V(D_1)| = 1$, say $V(D_1) = \{x\}$, then by the definition of acyclic ordering of the strong components, we have $\partial^-(x) \subseteq S$. Combining this with the minimality of S, we have $S = \partial^-(x)$. Similarly, if $|V(D_p)| = 1$, say $V(D_p) = \{y\}$, then we can obtain $S = \partial^+(y)$. Therefore, by Definition 5.1, D is super-λ. $\square$

Corollary 5.3. *For an integer $n \geq 2$, let D be an n-dimensional unidirectional hypercube or an n-dimensional unidirectional folded hypercube. Then D is super-λ.*

Proof. It is easy to verify that D is super-λ if $D = UQ_2$, $D = UQ_3$ or $D = UF_2$. Now let us consider the remaining cases. By Theorems 4.13, 4.14 and formulas (3.2), (3.3), (3.4) and (4.1), we have $\lambda'(D) > \delta(D) \geq \lambda(D)$. Therefore, the proof follows from Theorem 5.2. $\square$

It easy to see that if D is super-λ, then $\lambda(D) = \delta(D)$. So we have the following corollary.

Corollary 5.4. *For an integer $n \geq 2$, let D be an n-dimensional unidirectional hypercube or n-dimensional unidirectional folded hypercube. Then $\lambda(D) = \delta(D)$.*

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