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Injective Chromatic Number of Outerplanar Graphs

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Abstract. An injective coloring of a graph is a vertex coloring where two vertices with common neighbor receive distinct colors. The minimum integer k such that G has a k-injective coloring is called injective chromatic number of G and denoted by $\chi_i(G)$. In this paper, the injective chromatic number of outerplanar graphs with maximum degree Δ and girth g is studied. It is shown that every outerplanar graph G has $\chi_i(G) \leq \Delta + 2$, and this bound is tight. Then, it is proved that for an outerplanar graph G with $\Delta = 3$, $\chi_i(G) \leq \Delta + 1$ and the bound is tight for outerplanar graphs of girth 3 and 4. Finally, it is proved that, the injective chromatic number of 2-connected outerplanar graphs with $\Delta = 3$, $g \geq 6$ and $\Delta \geq 4$, $g \geq 4$ is equal to Δ.

1. Introduction

All graphs we have considered here are finite, connected and simple. A plane graph is a planar drawing of a planar graph in the Euclidean plane. The vertex set, edge set, face set, minimum degree and maximum degree of a plane graph G, are denoted by $V(G)$, $E(G)$, $F(G)$, $\delta(G)$ and $\Delta(G)$, respectively. A vertex of degree k is called a k-vertex. For vertex $v \in V(G)$, $N_G(v)$ is the set of neighbors of v in G. The girth of a graph G, $g(G)$, is the length of a shortest cycle in G. If there is no confusion, we delete G in the notations. A face $f \in F(G)$ is denoted by its boundary walk $f = [v_1v_2 \ldots v_k]$, where $v_1, v_2, \ldots, v_k$ are its vertices in the clockwise order. Also, the vertices v_1 and v_k as end vertices of f are denoted by v_{L_f} and v_{R_f}, respectively. An outerplanar graph is a graph with a planar drawing for which all vertices belong to the outer face of the drawing. It is known that a graph G is an outerplanar graph if and only if G has no subdivision of complete graph K_4 and complete bipartite graph $K_{2,3}$. A path $P: v_1, v_2, \ldots, v_k$ is called a simple path in G if $v_2, \ldots, v_{k-1}$ are all 2-vertices in G. The length of a path is the number of its edges. We say that a face $f = [v_1v_2 \ldots v_k]$ is an end face of an outerplane graph G, if $P: v_1, v_2, \ldots, v_k$ is a simple path in G. An end block in graph G is a maximal 2-connected subgraph of G that contains a unique cut vertex of G.

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A proper k-coloring of a graph G is a mapping from $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that any two adjacent vertices have different colors. The chromatic number, $\chi(G)$, is the minimum integer k that G has a proper k-coloring. A coloring c of G is called an *injective coloring* if for every two vertices u and v which have common neighbor, $c(u) \neq c(v)$. That means, the restriction of c to the neighborhood of any vertex is an injective function. The *injective chromatic number*, $\chi_i(G)$, is the least integer k such that G has an injective k-coloring. Note that an injective coloring is not necessarily a proper coloring. In fact, $\chi_i(G) = \chi(G^{(2)})$, where $V(G^{(2)}) = V(G)$ and $uv \in E(G^{(2)})$ if and only if u and v have a common neighbor in G. The square of graph G, denoted by G^2, is a graph with vertex set $V(G)$, where two vertices are adjacent in G^2 if and only if they are at distance at most two in G. Since $G^{(2)}$ is a subgraph of G^2, obviously, $\chi_i(G) \leq \chi(G^2)$. The concept of injective coloring is introduced by Hahn et al. in 2002 [7]. It is clear that for every graph G, $\chi_i(G) \geq \Delta$. In general, in [7] Hahn et al. proved that $\Delta \leq \chi_i(G) \leq \Delta^2 - \Delta + 1$. In [13], Wegner raised the following conjecture for the chromatic number of the square of planar graphs.

Conjecture 1.1. [13] *If G is a planar graph with maximum degree Δ, then*

- *For $\Delta = 3$, $\chi(G^2) \leq \Delta + 2$.*
- *For $4 \leq \Delta \leq 7$, $\chi(G^2) \leq \Delta + 5$.*
- *For $\Delta \geq 8$, $\chi(G^2) \leq \lfloor 3\Delta/2 \rfloor + 1$.*

Since $\chi_i(G) \leq \chi(G^2)$, Lužar and Škrekovski in [10] proposed the following conjecture for the injective chromatic number of planar graphs.

Conjecture 1.2. [10] *If G is a planar graph with maximum degree Δ, then*

- *For $\Delta = 3$, $\chi_i(G) \leq \Delta + 2$.*
- *For $4 \leq \Delta \leq 7$, $\chi_i(G) \leq \Delta + 5$.*
- *For $\Delta \geq 8$, $\chi_i(G) \leq \lfloor 3\Delta/2 \rfloor + 1$.*

The injective coloring of planar graphs with respect to its girth and maximum degree is studied in [1–6,9,11]. In [8], Lih and Wang proved upper bound $\Delta + 2$ for the chromatic number of square of outerplanar graphs.

Theorem 1.3. [8] *If G is an outerplanar graph, then $\chi(G^2) \leq \Delta + 2$.*

Since $\chi_i(G) \leq \chi(G^2)$, Conjecture 1.2 is true for outerplanar graphs.

Corollary 1.4. *If G is an outerplanar graph, then $\chi_i(G) \leq \Delta + 2$.*

In Figure 1.1, an outerplanar graph with $\Delta = 4$, $g = 3$ and $\chi_i(G) = \Delta + 2 = 6$ is shown. Therefore, the given bound in Corollary 1.4 is tight.

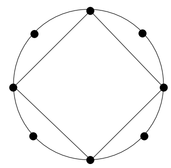


Figure 1.1: An outerplanar graph with $\Delta = 4$, $g = 3$ and $\chi_i = 6$.

In this paper, we study the injective chromatic number of outerplanar graphs. The main results of Section 2 are as follows. If G is an outerplanar graph with maximum degree Δ and girth g, then

- (Theorem 2.1) For $\Delta = 3$, $\chi_i(G) \leq \Delta + 1 = 4$.
- (Theorem 2.2) For $\Delta = 3$ and $g \geq 5$, with no face of degree k, $k \equiv 2 \pmod 4$, $\chi_i(G) = \Delta$.
- (Theorem 2.4) For $\Delta = 3$ and $g \geq 6$, $\chi_i(G) = \Delta$.
- (Theorems 2.5 and 2.8) For $\Delta \geq 4$ and $g \geq 4$, $\chi_i(G) = \Delta$.

2. Main results

First, we prove a tight bound for the injective chromatic number of outerplanar graphs with $\Delta = 3$. Note that if $\Delta = 2$, then G is an union of paths and cycles, which obviously $\chi_i(G) \leq 3 = \Delta + 1$. Moreover, if G is an arbitrary path or is a cycle of length k, where $k \equiv 0 \pmod 4$, then $\chi_i(G) = 2$. Otherwise, $\chi_i(G) = 3$ [7].

Theorem 2.1. *If G is an outerplanar graph with $\Delta = 3$, then G has a 4-injective coloring such that in every simple path of lenght three, at most three colors appear. Moreover, the bound is tight.*

Proof. We prove the theorem by the induction on $|V(G)|$. In Figure 2.1, all outerplanar graphs with $\Delta = 3$ of order 4 and 5 with an injective coloring with desired property are shown. Obviously, in the left side graph, $\chi_i(G) = 4$. Hence, bound $\Delta + 1$ is tight.

Now suppose that G is an outerplane graph with $\Delta = 3$ and the statement is true for all outerplanar graphs with $\Delta = 3$ of order less than $|V(G)|$. The following two cases can be caused.

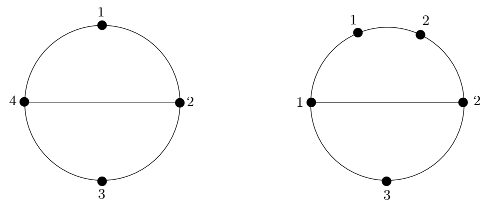


Figure 2.1: Outerplanar graphs with $\Delta = 3$ of order 4, 5.

If an end block of G is an edge, say uv, where $\deg(u) = 1$, then we consider the maximal simple path $P : (v_1 = u), (v_2 = v), v_3, \ldots, v_k$ in G. Since P is a maximal simple path and $\Delta(G) = 3$, we have $\deg(v_k) = 3$. Suppose that $N(v_k) = \{w_1, w_2, v_{k-1}\}$ and c is a 4-injective coloring of $G \setminus \{v_1, v_2, \ldots, v_{k-1}\}$ with colors $\{\alpha, \beta, \gamma, \lambda\}$ such that every simple path of length three has at most three colors. Note that w_1 and w_2 have a common neighbor v_k therefore, $c(w_1) \neq c(w_2)$. In this case, we assign to the ordered vertices $v_{k-1}, v_{k-2}, \ldots, v_2, v_1$ of path P the ordered string $(ssttsstt\ldots)$, where $s \in \{\alpha, \beta, \gamma, \lambda\} \setminus \{c(v_k), c(w_1), c(w_2)\}$ and $t = c(v_k)$.

If the minimum degree of every end block of G is at least two in G, then we consider an end face $f = [v_i v_{i+1} \ldots v_j]$ in an end block B of G in clockwise order, where v_1 is the vertex cut of G belongs to B. Note that, since $\Delta(G) = 3$, if G is a block, then G has an end face $f = [v_i v_{i+1} \ldots v_j]$. Let H be the induced subgraph of G on 2-vertices of f. If $\Delta(G \setminus H) = 2$, then we color the ordered vertices $v_j, v_{j+1}, \ldots, v_{i-1}, v_i$ of $G \setminus H$ by ordered string $(\alpha\beta\gamma\lambda\alpha\beta\gamma\lambda\ldots)$. If $|V(G \setminus H)| \equiv 2 \pmod 4$, then change the color of v_{i-1} and v_i to β and α, respectively. If $\Delta(G \setminus H) = 3$, then by the induction hypothesis $G \setminus H$ has a 4-injective coloring c with colors $\{\alpha, \beta, \gamma, \lambda\}$, such that every simple path of length three has at most three colors. Hence, in $G \setminus H$ at most three colors are used for vertices $v_{i-1}, v_i, v_j, v_{j+1}$. Now we extend c to an injective coloring of G with the desired property.

If $c(v_i) = c(v_j)$, then we assign to the ordered vertices $v_{i+1}, v_{i+2}, \ldots, v_{j-1}$ the ordered string $(ssttsstt\ldots)$, where $s \in \{\alpha, \beta, \gamma, \lambda\} \setminus \{c(v_{i-1}), c(v_i) = c(v_j), c(v_{j+1})\}$ and $t \in \{\alpha, \beta, \gamma, \lambda\} \setminus \{c(v_i) = c(v_j), c(v_{j+1}), s\}$.

If $c(v_i) \neq c(v_j)$, then we assign to the ordered vertices $v_{i+1}, v_{i+2}, \ldots, v_{j-1}$ the ordered string $(ssttsstt\ldots)$, where $s \in \{\alpha, \beta, \gamma, \lambda\} \setminus \{c(v_{i-1}), c(v_i), c(v_j), c(v_{j+1})\}$. If $j - i - 1 \equiv 1, 2 \pmod 4$, then $t \in \{\alpha, \beta, \gamma, \lambda\} \setminus \{c(v_j), s\}$. If $j - i - 1 \equiv 0, 3 \pmod 4$, then $t \in \{\alpha, \beta, \gamma, \lambda\} \setminus \{c(v_i), c(v_{j+1}), s\}$. In the case $j - i - 1 \equiv 0 \pmod 4$, if $t = c(v_j)$, then change the color of v_{j-2} to $t' \in \{\alpha, \beta, \gamma, \lambda\} \setminus \{c(v_j) = t, s\}$. Note that, since by the induction hypothesis $|\{c(v_{i-1}), c(v_i), c(v_j), c(v_{j+1})\}| \leq 3$, in each cases the colors s and t exist. It can be easily seen that the given coloring is a 4-injective coloring for G such that every simple path of length three in G has at most three colors as well. □

Graph G in Figure 2.2 is an outerplanar graph of girth 4 with maximum degree three and injective chromatic number 4. Since each pair of set $\{u, v, w\}$ have a common neighbor, in every injective coloring of G, they must have three different colors. In the similar way, we need three different colors for the vetrices $\{x, y, z\}$. Without loss of generality, color the vertices u, v, w with color α, β and γ, respectively. Now by devoting any permutation of these colors to vertices x, y and z, it can be checked that in each case we need a new color for the other vertices. Therefore, bound $\Delta+1$ in Theorem 2.1 is tight for outerplanar graphs with $\Delta = 3$, $g = 4$ and $g = 3$ (see also Figure 2.1).

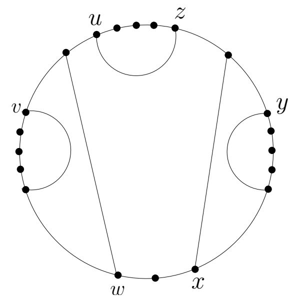


Figure 2.2: An outerplanar graph with $\Delta = 3$, $g = 4$ and $\chi_i = 4$.

In the next theorems, we improve bound $\Delta+1$ to Δ for outerplanar graphs with $\Delta = 3$ of girth greater than 4.

Theorem 2.2. *If G is a 2-connected outerplanar graph with $\Delta = 3$, $g \geq 5$ and no face of degree k, where $k \equiv 2 \pmod 4$, then G has a 3-injective coloring such that in every simple path of length three, exactly three colors appear.*

Proof. We prove it by the induction on $|V(G)|$. In Figure 2.3, the 2-connected outerplanar graphs with $\Delta = 3$ and $g \geq 5$ of order at most 10 with an injective coloring with desired property are shown.

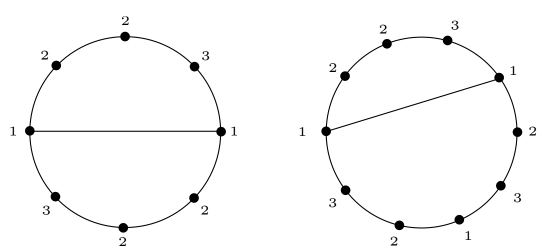


Figure 2.3: Outerplanar graphs with $\Delta = 3$ and $g \geq 5$ of order 8 and 10.

Now suppose that G is a 2-connected outerplane graph with $\Delta = 3$, $g \geq 5$ and no face of degree k, where $k \equiv 2 \pmod 4$ and the statement is true for all such 2-connected outerplanar graphs of order less than $|V(G)|$.

Let $f = [v_i v_{i+1} \dots v_j]$ be an end face of G in clockwise order and H be the induced subgraph of G on 2-vertices of f. If $\Delta(G \setminus H) = 3$, then by the induction hypothesis $G \setminus H$ has a 3-injective coloring c with colors $\{\alpha, \beta, \gamma\}$, such that every simple path of length three has exactly three colors.

If $\Delta(G \setminus H) = 2$, then we color the vertices of $G \setminus H$ as follows. If $G \setminus H = C_t$, where $t > 5$ and $t \equiv 0, 1 \pmod 3$, then color the ordered vertices $v_{i-1}, v_i, v_j, v_{j+1}, \dots, v_{i-2}$ with the ordered string $(\alpha\beta\gamma\alpha\beta\gamma\dots)$. If $t > 5$ and $t \equiv 2 \pmod 3$, then color the ordered vertices $v_{i-1}, v_i, v_j, v_{j+1}, \dots, v_{i-5}$ with the ordered string $(\alpha\beta\gamma\alpha\beta\gamma\dots)$. Then color the vertices v_{i-4}, v_{i-3} and v_{i-2} with colors β, γ and α, respectively. One can check that every simple path of length three in $G \setminus H$ has exactly three colors. If $G \setminus H = C_5$, then since $|V(G)| > 10$, $f = [v_i v_{i+1} \dots v_j]$ is a cycle of length at least 8. In this case, we consider the end face $f' = [v_j v_{j+1} \dots v_i]$ and follow the above proof when H is induced subgraph of G on 2-vertices of f'. In the following, we extend injective coloring c of $G \setminus H$ to an injective coloring of G with the desired property.

If $c(v_i) = c(v_j)$, then we assign to the ordered vertices $v_{i+1}, v_{i+2}, \dots, v_{j-1}$ the ordered string $(s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_4 \dots)$, where $s_1 = c(v_{j+1})$. Since G has no face of degree k where $k \equiv 2 \pmod 4$, we have following cases. If $j - i - 1 \equiv 1 \pmod 4$, then let $s_2 = c(v_{i-1})$, $s_3 = s_4 = c(v_i) = c(v_j)$ and change the color of vertices v_{j-2} and v_{j-1} to $c(v_{j+1})$ and $c(v_{i-1})$, respectively. If $j - i - 1 \equiv 2 \pmod 4$, then let $s_2 = s_1 = c(v_{j+1})$, $s_3 = c(v_{i-1})$ and $s_4 = c(v_i) = c(v_j)$ and change the color of v_{j-1} to $c(v_{i-1})$. If $j - i - 1 \equiv 3 \pmod 4$, then let $s_2 = s_1 = c(v_{j+1})$, $s_3 = c(v_{i-1})$ and $s_4 = c(v_i) = c(v_j)$.

If $c(v_i) \neq c(v_j)$ and $c(v_{i-1}) = c(v_{j+1})$, then we assign to the ordered vertices $v_{i+1}, v_{i+2}, \dots, v_{j-1}$ the ordered string $(s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_4 \dots)$, where $s_1 = c(v_i)$. If $j - i - 1 \equiv 1, 2 \pmod 4$, then $s_2 = c(v_j)$ and $s_3 = s_4 = c(v_{i-1}) = c(v_{j+1})$. In the case $j - i - 1 \equiv 1 \pmod 4$, we change the color of vertices v_{j-2} and v_{j-1} to $c(v_i)$ and $c(v_j)$, respectively. If $j - i - 1 \equiv 3 \pmod 4$, then let $s_2 = c(v_{i-1}) = c(v_{j+1})$ and $s_3 = s_4 = c(v_j)$.

If $c(v_i) \neq c(v_j)$ and $c(v_{i-1}) = c(v_i)$, then we assign to the ordered vertices $v_{i+1}, v_{i+2}, \dots, v_{j-1}$ the ordered string $(s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_4 \dots)$, where $s_1 = c(v_{j+1})$. If $j - i - 1 \equiv 1 \pmod 4$, then let $s_2 = c(v_j)$, $s_3 = c(v_i) = c(v_{i-1})$, $s_4 = s_1$ and change the color of vertex v_{j-1} to $c(v_j)$. If $j - i - 1 \equiv 2 \pmod 4$, then let $s_2 = c(v_j)$ and $s_3 = s_4 = c(v_{i-1}) = c(v_i)$. If $j - i - 1 \equiv 3 \pmod 4$, then we assign to the ordered vertices $v_{j-1}, v_{j-2}, \dots, v_{i+1}$ the ordered string $(s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_4 \dots)$, where $s_1 = c(v_j)$, $s_2 = c(v_{j+1})$, $s_3 = s_4 = c(v_i) = c(v_{i-1})$ and change the colors of v_{i+1} to $c(v_{j+1})$.

If $c(v_i) \neq c(v_j)$ and $c(v_j) = c(v_{j+1})$, then we assign to the ordered vertices $v_{j-1}, v_{j-2}, \dots, v_{i+1}$ the ordered string $(s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_4 \dots)$, where $s_1 = c(v_{i-1})$. If $j - i - 1 \equiv 1 \pmod 4$, then $s_2 = c(v_i)$, $s_3 = c(v_{j+1}) = c(v_j)$, $s_4 = s_1$ and change the color of v_{i+1} to $c(v_i)$. If $j - i - 1 \equiv 2 \pmod 4$, then $s_2 = c(v_i)$ and $s_3 = s_4 = c(v_{j+1})$. If $j - i - 1 \equiv 3$

(mod 4), then we assign to the ordered vertices $v_{i+1}, v_{i+2}, \ldots, v_{j-1}$ the ordered string $(s_1s_2s_3s_4s_1s_2s_3s_4\ldots)$, where $s_1 = c(v_i)$, $s_2 = c(v_{i-1})$ and $s_3 = s_4 = c(v_j) = c(v_{j+1})$ and change the color of v_{j-1} to $c(v_{i-1})$. It can be seen that the given coloring is a 3-injective coloring for G such that every simple path of length three in G has exactly three colors. □

In Theorem 2.4, we improve bound $\Delta + 1$ in Theorem 2.1 to Δ for outerplanar graph with $\Delta = 3$ and $g \geq 6$. First, we need the following theorem.

Theorem 2.3. [12] *Let G be a connected graph and L be a list-assignment to the vertices, where $|L(v)| \geq \deg(v)$ for each $v \in V(G)$. If*

(1) $|L(v)| > \deg(v)$ *for some vertex v, or*

(2) *G contains a block which is neither a complete graph nor an induced odd cycle,*

then G admits a proper coloring such that the color assign to each vertex v is in $L(v)$.

Theorem 2.4. *If G is an outerplanar graph with $\Delta = 3$ and $g \geq 6$, then $\chi_i(G) = \Delta$.*

Proof. Since $\chi_i(G) \geq \Delta$, it is enough to show that $\chi_i(G) \leq \Delta$. Let G be a minimal counterexample for this statement. That means G is an outerplane graph with $\Delta = 3$, $g \geq 6$ and $\chi_i(G) \geq \Delta+1$, such that every proper subgraph of G has a Δ-injective coloring. Obviously $\delta(G) \geq 2$. Now consider an end face $f = [v_iv_{i+1}\ldots v_j]$ in an end block B of G in clockwise order, where v_1 is the vertex cut of G belonging to B. Since $\Delta = 3$ and $g \geq 6$, the degree of face f is at least 6 and the degree of v_i and v_j are three. Let H be the induced subgraph of G on 2-vertices of f. If $\Delta(G \setminus H) = 3$, then by the minimality of G, we have $\chi_i(G \setminus H) \leq \Delta(G \setminus H) \leq \Delta(G)$. Also, if $G \setminus H$ is a cycle, then $\chi_i(G \setminus H) \leq 3 = \Delta$.

Now, we extend the Δ-injective coloring of $G \setminus H$ to a Δ-injective coloring of G, which contradicts our assumption. Each of the vertices v_i and v_j has at most $\Delta - 1 = 2$ neighbors except v_{i+1} and v_{j-1}, respectively. Hence, for each of vertices v_{i+1} and v_{j-1} there is at least one available color. Also, among the colored vertices in $G \setminus H$, the only forbbiden colors for vertices v_{i+2} and v_{j-2} are colors of the vertices v_i and v_j, respectively. The other vertices have three available colors. Now consider induced subgraph of $G^{(2)}$ on the vertices of H, denoted by $G^{(2)}[H]$, and list of available colors for each vertex of H. The components of $G^{(2)}[H]$ are some paths satisfying the assumption of Theorem 2.3. Thus, we have a proper Δ-coloring for $G^{(2)}[H]$ using the available colors which is a Δ-injective coloing of H as desired. □

Now, we are ready to determine the injective chromatic number of 2-connected outerplanar graphs with maximum degree and girth greater than three. We prove this fact by two different methods for the cases $\Delta = 4$ and $\Delta \geq 5$.

Theorem 2.5. *If G is a 2-connected outerplanar graph with $\Delta = 4$ and $g \geq 4$, then G has a 4-injective coloring c such that for every adjacent vertices v and u of degree three with $N(v) = \{u, v_1, v_2\}$ and $N(u) = \{v, u_1, u_2\}$, $\{c(u), c(v_1), c(v_2)\} \neq \{c(v), c(u_1), c(u_2)\}$.*

Proof. We prove it by the induction on $|V(G)|$. In Figure 2.4, the 2-connected outerplanar graphs with $\Delta = 4$ and $g \geq 4$ of order 8 and 9 with an injective coloring of desired property are shown.

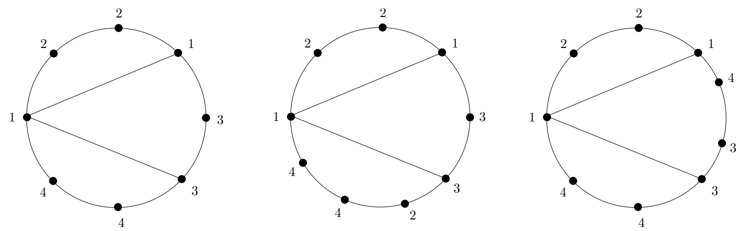


Figure 2.4: 2-connected outerplanar graphs with $\Delta = 4$ and $g \geq 4$ of order 8 and 9.

Now suppose that G is a 2-connected outerplane graph with $\Delta = 4$, $g \geq 4$ and the statement is true for all 2-connected outerplanar graphs with $\Delta = 4$ and $g \geq 4$ of order less than $|V(G)|$.

Let $f = [v_i v_{i+1} \dots v_j]$ be an end face of G in clockwise order. If $\deg(v_i) = \deg(v_j) = 3$, then consider induced subgraph H on 2-vertices of face f. Thus, $G \setminus H$ is a 2-connected outerplane graph with $\Delta(G \setminus H) = 4$ and $g(G \setminus H) \geq 4$. Hence, by the induction hypothesis, $G \setminus H$ has a 4-injective coloring such that for every adjacent vertices v and u of degree three with $N(v) = \{u, v_1, v_2\}$ and $N(u) = \{v, u_1, u_2\}$, $\{c(u), c(v_1), c(v_2)\} \neq \{c(v), c(u_1), c(u_2)\}$. If there are exactly four colors in $\{c(v_{i-1}), c(v_i), c(v_j), c(v_{j+1})\}$, then consider graph $G^{(2)}[H]$ and list of available colors for each vertex of H. Graph $G^{(2)}[H]$ satisfy the assumption of Theorem 2.3. Thus, we have a Δ-coloring for $G^{(2)}[H]$ which is a Δ-injective coloring of H. If there are at most three colors in $\{c(v_{i-1}), c(v_i), c(v_j), c(v_{j+1})\}$, then color v_{i+1} with one of its colors not in $\{c(v_{i-1}), c(v_i), c(v_j), c(v_{j+1})\}$ and color v_{j-1} with one of its available colors such that $c(v_{i+1}) \neq c(v_{j-1})$. Then color the other vertices of H with one of their available colors similar to above. It can be easily seen that for every adjacent vertices v and u of degree three with $N(v) = \{u, v_1, v_2\}$ and $N(u) = \{v, u_1, u_2\}$, $\{c(u), c(v_1), c(v_2)\} \neq \{c(v), c(u_1), c(u_2)\}$.

Now suppose that each face of G has an end vertex of degree 4. We have two following cases.

Case 1. There is an end face f with one end vertex of degree 4 and the other one of degree less than 4.

In this case, suppose that G has an end face $f = [v_i v_{i+1} \dots v_j]$, where $\deg(v_i) = 4$ and $\deg(v_j) = 3$. Consider induced subgraph H on 2-vertices of face f. If $\Delta(G \setminus H) = 4$, then by the induction hypothesis, $G \setminus H$ has a 4-injective coloring such that for every adjacent vertices v and u of degree three with $N(v) = \{u, v_1, v_2\}$ and $N(u) = \{v, u_1, u_2\}$, $\{c(u), c(v_1), c(v_2)\} \neq \{c(v), c(u_1), c(u_2)\}$. Now we extend the 4-injective coloring of $G \setminus H$ to G. If $\deg(v_{j+1}) = 3$, then suppose that v_s is the other neighbor of v_{j+1} except v_j and v_{j+2}. If there are exactly three colors in $\{c(v_i), c(v_j), c(v_{j+1}), c(v_{j+2}), c(v_s)\}$, then color vertex v_{j-1} with one of its colors not in $\{c(v_i), c(v_j), c(v_{j+1}), c(v_{j+2}), c(v_s)\}$ and color the other vertices of H with one of their available colors as explained in above. If $|\{c(v_i), c(v_j), c(v_{j+1}), c(v_{j+2}), c(v_s)\}| = 4$ or $\deg(v_{j+1}) \neq 3$, then by Theorem 2.3 color the vertices of H with one of their available colors such that obtained coloring is a 4-injective coloring of G.

If $\Delta(G \setminus H) = 3$, then $G \setminus H)$ also contains an end face. Moreover, by the assumption, each face of G has an end vertex of degree 4. Therefore, there is another end face, say f', with a common neighbor with f. Consider induced subgraph H' on 2-vertices of face f and f'. Thus, $G \setminus H'$ is a cycle and $\chi_i(G \setminus H') \leq 3$. Now each vertices of H' has at least two available colors. Hence, by applying Theorem 2.3, we obtain a 4-injective coloring of G. Note that, since $g(G) \geq 4$, in this case there is no two adjacent vertices of degree three.

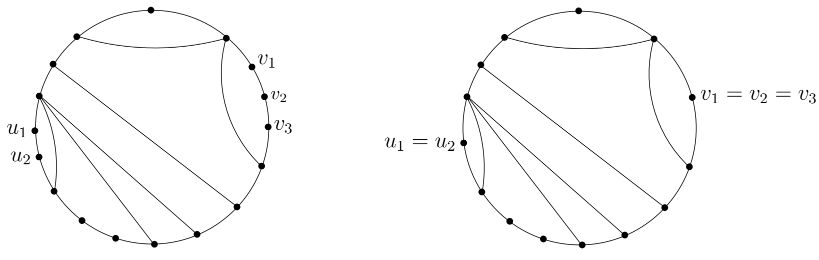
Case 2. For each end face f, its two end vertices are of degree 4.

In this case, consider the induced subgraph H on 2-vertices of $f = [v_i v_{i+1} \dots v_j]$, where $\deg(v_i) = \deg(v_j) = 4$. Since $\deg(v_i) = 4$, $G \setminus H$ has an end face f' with two ends of degree 4. Hence, $\Delta(G \setminus H) = 4$ and by the induction hypothesis, $G \setminus H$ has a 4-injective coloring such that for every adjacent vertices v and u of degree three with $N(v) = \{u, v_1, v_2\}$ and $N(u) = \{v, u_1, u_2\}$, $\{c(u), c(v_1), c(v_2)\} \neq \{c(v), c(u_1), c(u_2)\}$. Now by Theorem 2.3, color the vertices of H with their available colors such that obtained coloring is a 4-injective coloring of G. Obviously, for every adjacent vertices v and u of degree three with $N(v) = \{u, v_1, v_2\}$ and $N(u) = \{v, u_1, u_2\}$, $\{c(u), c(v_1), c(v_2)\} \neq \{c(v), c(u_1), c(u_2)\}$. □

Now we consider 2-connected outerplanar graphs with $\Delta = 5$ and $g \geq 4$. First, we need to prove the following theorem on the structure of 2-connected outerplanar graphs.

Theorem 2.6. *If G is a 2-connected outerplanar graph, then G has an end face $f = [v_i v_{i+1} \dots v_j]$, where either $\deg(v_i) < 5$ or $\deg(v_j) < 5$.*

Proof. First replace every simple path in boundary of each end face of G with a path of length two and name this graph G'. Graph G' is also a 2-connected outerplane graph that each end face of G' is of degree three (for example see Figure 2.5). If G' is a cycle, then we are done. Now, let $\Delta(G') \geq 3$ and $C : v_1 v_2 \dots v_n$ be a Hamilton cycle of G' in clockwise order. Also, let $f = [v_i v_{i+1} v_{i+2}]$ be an end face of G'. If $\deg(v_{i+2})$ is at least 5, then we present an algorithm that find an end face of G' such that the degree of at least one of its end vertices is less than 5. Since by assumption $\deg(v_{i+2}) \geq 5$, v_{i+2} has at least two other neighbors except v_i, v_{i+1} and v_{i+3}, named $v_{i'}$ and $v_{j'}$ such that the number of vertices between v_{i+3} and $v_{i'}$ in clockwise order is less than the number of vertices between v_{i+3} and $v_{j'}$ in clockwise order.



G G'

Figure 2.5: Two graphs G and G'.

Algorithm 2.7. 1. $k = 0$.

2. $f_0 = [v_i v_{i+1} v_{i+2}]$.

3. *If* $f_k = [v_t v_{t+1} v_{t+2}]$ *is an end face of* G'*, then do steps 4 to 7, respectively.*

4. *Suppose that* $v_{L_{f_k}} = v_t$ *and* $v_{R_{f_k}} = v_{t+2}$*. Let* $v_{i'_k}$ *and* $v_{j'_k}$ *be another neighbors of* v_{t+2} *except* v_t*,* v_{t+1} *and* v_{t+3} *such that the number of vertices between* $v_{R_{f_k}}$ *and* $v_{i'_k}$ *in clockwise order is less than the number of vertices between* $v_{R_{f_k}}$ *and* $v_{j'_k}$ *in clockwise order.*

5. *If* $\deg(v_{L_{f_k}}) \leq 4$ *or there is no* $v_{i'_k}$ *or* $v_{j'_k}$*, then stop the algorithm and give the face* f_k *as output of the algorithm.*

6. $k = k + 1$.

7. $f_k = [v_{R_{f_{k-1}}} v_{R_{f_{k-1}}} + 1 \dots v_{i'_{k-1}}]$ *and go to step three.*

8. *If* f_k *is not an end face of* G'*, then there exists an end face* f *in* f_k*. Do steps 9 and 10, respectively.*

9. $k = k + 1$.

10. $f_k = f$ *and go to step three.*

Note that, the neighbors of all vertices $v_{R_{f_k}}$ are between v_{i+2} and $v_{j'_0}$ in clockwise order; otherwise there is a subdivision of K_4 on G' and it is a contradiction with the assumption that G' is an outerplanar graph. Therefore, the algorithm terminates. Moreover, if $f_k = [v_k v_{k+1} v_{k+2}]$ is the output of the algorithm, then by line 5 of the algorithm, the degree of

v_{k+2} is less than 5. Finally, by returning the contracted paths to G'; we have an end face of G that one of its ends is of degree less than 5. □

Theorem 2.8. *If G is a 2-connected outerplanar graph with $\Delta \geq 5$ and $g \geq 4$, then $\chi_i(G) = \Delta$.*

Proof. Since $\chi_i(G) \geq \Delta(G)$, it is enough to show that $\chi_i(G) \leq \Delta(G)$. We prove it by the induction on $|V(G)|$. In Figure 2.6, the 2-connected outerplanar graphs with $\Delta \geq 5$ and $g \geq 4$ of order 10 and 11 with a Δ-injective coloring are shown.

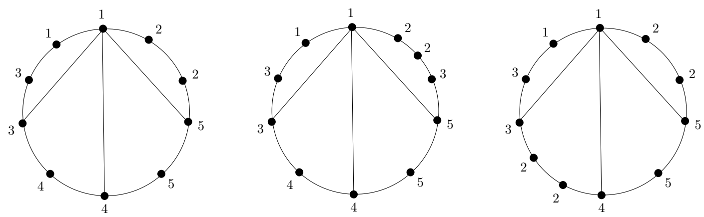


Figure 2.6: 2-connected outerplanar graphs with $\Delta \geq 5$ and $g \geq 4$ of order 10 and 11.

Now suppose that G is a 2-connected outerplane graph with $\Delta \geq 5$, $g \geq 4$ and the statement is true for all 2-connected outerplanar graphs with $\Delta \geq 5$ and $g \geq 4$ of order less that $|V(G)|$. By Theorem 2.6, G has an end face f of degree at least 4 such that at least one of its end vertices is of degree at most 4. Now consider the induced subgraph H on 2-vertices of end face f. If $\Delta(G \setminus H) \geq 5$, then by induction hypothesis, $\chi_i(G \setminus H) = \Delta(G \setminus H) \leq \Delta(G)$. If $\Delta(G \setminus H) = 4$, then by Theorem 2.5, $G \setminus H$ has a 4-injective coloring. Now consider the end face $f = [v_i v_{i+1} \ldots v_j]$ and suppose that $\deg(v_i) \leq \Delta$ and $\deg(v_j) \leq 4$. Since $\Delta \geq 5$, the vertices v_{i+1} and v_{j-1} have at least one and two available colors, respectively. The other vertices of H has at least three available colors. Now consider the graph $G^{(2)}[H]$ and list of available colors for each vertex of H. It can be easily seen that $G^{(2)}[H]$ is union of paths and isolated vertices, which satisfy the assumption of Theorem 2.3. Hence, $G^{(2)}[H]$ can be colored by at most Δ colors and the obtained coloring is a Δ-injective coloring of H. □

Remark 2.9. Applying the same idea and by laboriously proof, the results of Theorems 2.2, 2.5 and 2.8 can be generalized for the outerplanar graphs containing some cut vertices.

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