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Semi-classical Limit for the Quantum Zakharov System

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Abstract. In this paper, we prove the semi-classical limit for the quantum Zakharov system, that is, the quantum Zakharov system converges to the classical Zakharov system as the quantum parameter goes to zero, including a convergence rate. We improve the results of Guo-Zhang-Guo [11].

1. Introduction

In this paper, we consider the semi-classical limit for the quantum Zakharov system, which describes the propagation of Langmuir waves in an ionized plasma. Langmuir waves are rapid oscillations of the electron density in conducting media, such as plasmas or metals, see [6, 13, 14]. The system reads as

$$\begin{aligned} & iE_t + \Delta E - \varepsilon^2 \Delta^2 E = nE, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}; \\ & n_{tt} - \Delta n + \varepsilon^2 \Delta^2 n = \Delta |E|^2; \\ & E(0) = E_0, \quad n(0) = n_0, \quad \partial_t n(0) = n_1, \end{aligned} \tag{1.1}$$

where E is slowly varying envelope of the rapidly oscillating electric field and n is the deviation of the ion density from its mean value. E is complex valued and n is real valued. The quantum parameter ε typically goes from the values of order 10^{-5} to the values of order unity, see [6, 13, 14] for more physical background.

When the quantum effect is absent, the system is reduced to the classical Zakharov system which is as follows:

$$\begin{aligned} & iE_t + \Delta E = nE, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R}; \\ & n_{tt} - \Delta n = \Delta |E|^2; \\ & E(0) = E_0, \quad n(0) = n_0, \quad \partial_t n(0) = n_1. \end{aligned} \tag{1.2}$$

The regular solutions of (1.2) satisfy the conservation of mass

$$\int |E(t)|^2 \, dx = \int |E(0)|^2 \, dx = \text{constant}$$

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and the conservation of the Hamiltonian

$$\|\nabla E\|_{L^2}^2 + \frac{1}{2}\left(\|n_t\|_{\dot{H}^{-1}}^2 + \|n\|_{L^2}^2\right) + \int n|E|^2\,dx = \text{constant}.$$

The Zakharov system (1.2) has been extensively studied for the local and global well-posedness [1, 4, 7, 16, 18], for blow-up [8, 9], for scattering results [10], and for adiabatic limit and subsonic limit of the solution [17, 21]. In 1992, Ozawa and Tsutsumi proved the Schrödinger limit for the Zakharov system with the optimal convergence rate, see [19].

Analogous to (1.2), (1.1) has the conservation of mass

$$\int |E(t)|^2\,dx = \int |E(0)|^2\,dx \tag{1.3}$$

and the conservation of the Hamiltonian

$$\begin{aligned}\mathcal{H}(E,n,n_t)(t) &:= \||\nabla|\langle\varepsilon\nabla\rangle E\|_{L^2}^2 + \frac{1}{2}\left(\|n_t\|_{\dot{H}^{-1}}^2 + \|\langle\varepsilon\nabla\rangle n\|_{L^2}^2\right) + \int n|E|^2\,dx \\ &= \text{constant},\end{aligned} \tag{1.4}$$

where $|\nabla|$ and $\langle\varepsilon\nabla\rangle$ are defined in (2.1). We list some known results for (1.1). In 2013, Guo, Zhang, and Guo proved the global well-posedness and the classical limit of (1.1) for dimensions $d = 1, 2, 3$, see [11]. In 2016, Jiang, Lin, and Shao proved the local well-posedness for (1.1) in 1D with initial data $(E_0, n_0, n_1) \in H^k \oplus H^\ell \oplus H^{\ell-2}$ provided that $|k| - 3/2 < \ell < \min\{k + 3/2, 2k + 3/2\}$ and $k > -3/4$, see [15]. In 2016, Fang, Lin, and Segata showes that the Schrödinder limit for the quantum Zakharov system with optimal convergence rate as the wave speed goes to infinity. In 2017, Chen, Fang, and Wang obtained the global well-posedness for (1.1) in 1D with $(E_0, n_0, n_1) \in L^2 \oplus H^\ell \oplus H^{\ell-2}$ provided that $-3/2 \le \ell \le 3/2$, see [3]. In 2017, Fang, Shih, and Wang obtained a local well-posednes for (1.1) in 1D for a wider range of initial data, see [5].

Following an idea used in a work of Masmoudi and Nakanishi, see [17], we can rewrite the wave equation in (1.1) as follows. We set

$$\mathcal{N} = n + \frac{i\partial_t n}{|\nabla|\langle\varepsilon\nabla\rangle}, \tag{1.5}$$

where $|\nabla|$ and $\langle\varepsilon\nabla\rangle$ are defined in (2.1). Thus the quantum Zakharov system (1.1) becomes

$$\begin{aligned}&iE_t + \Delta\langle\varepsilon\nabla\rangle^2 E = \mathrm{Re}(\mathcal{N})E, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R};\\ &i\partial_t\mathcal{N} - |\nabla|\langle\varepsilon\nabla\rangle\mathcal{N} = |\nabla|\langle\varepsilon\nabla\rangle^{-1}|E|^2;\\ &E(0,x) = E_0(x), \quad \mathcal{N}(0,x) = \mathcal{N}_0(x),\end{aligned} \tag{1.6}$$

where $\mathcal{N}_0 = n_0 + \frac{in_1}{|\nabla|\langle\varepsilon\nabla\rangle}$ and $\mathrm{Re}(\mathcal{N})$ is the real part of $\mathcal{N}$. Invoking (1.5) with $\varepsilon = 0$, analogously the Zakharov system (1.2) becomes

$$
\begin{aligned}
&iE_t + \Delta E = \mathrm{Re}(\mathcal{N})E, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R};\\
&i\partial_t \mathcal{N} - |\nabla|\mathcal{N} = |\nabla||E|^2;\\
&E(0,x) = E_0(x), \quad \mathcal{N}(0,x) = \mathcal{N}_0(x),
\end{aligned}
\tag{1.7}
$$

where $\mathcal{N}_0 = n_0 + i\, n_1/|\nabla|$. Thus the Hamiltonian (1.4) can be rewritten as

$$
\mathcal{H}(E,\mathcal{N})(t) := \||\nabla|\langle\varepsilon\nabla\rangle E\|_{L^2}^2 + \frac{1}{2}\|\langle\varepsilon\nabla\rangle\mathcal{N}\|_{L^2}^2 + \int \mathrm{Re}(\mathcal{N})|E|^2\, dx = \text{constant}. \tag{1.8}
$$

Let $0 < \varepsilon \leq 1$ throughout the paper, unless it is specified. For the system (1.6), we define the function spaces

$$
V_k := H^k(\mathbb{R}^d) \times H^{k-1}(\mathbb{R}^d) \quad \text{and} \quad W_k := H^k(\mathbb{R}^d) \times H^k(\mathbb{R}^d),
$$

where $k \geq 0$. The function spaces V_k and W_k are endorsed with the natural norms

$$
\|(f,g)\|_{V_k} := \|f\|_{H^k} + \|g\|_{H^{k-1}} \quad \text{and} \quad \|(f,g)\|_{W_k} := \|f\|_{H^k} + \|g\|_{H^k}
$$

respectively. Our main results are as follows. The results of Theorems 1.1 and 1.3 improve those of [11] with lower regularities. The results of Theorems 1.2 and 1.4 improve those of [11] with regularities in a wider range.

Theorem 1.1. (Guo-Zhang-Guo [11] *Let $k \geq 2$ be an integer and $d = 1, 2, 3$. Assume $(E_0, \mathcal{N}_0) \in V_k$. Then the system* (1.6) *with initial data $(E_0, \mathcal{N}_0)$ admits a unique solution $(E, \mathcal{N}) \in C(\mathbb{R}; V_k)$, and the solution depends continuously on the initial data, that is,*

$$
\lim_{j\to\infty} \sup_{t\in[-T,T]} \|(E^j - E, \mathcal{N}^j - \mathcal{N})(t)\|_{V_k} = 0
$$

for all $T > 0$, where $(E^j, \mathcal{N}^j)$ are the solutions to (1.6) *with initial datum $(E_0^j, \mathcal{N}_0^j)$ such that $(E_0^j, \mathcal{N}_0^j) \to (E_0, \mathcal{N}_0)$ in V_k as $j \to \infty$. Moreover, the solution $(E, \mathcal{N})$ satisfies $\|E(t)\|_{L^2} = \|E_0\|_{L^2}$ and $\mathcal{H}(t) = \mathcal{H}(0)$, where $\mathcal{H}$ is defined in* (1.8).

Theorem 1.2. *Under the same assumptions in Theorem* 1.1 *with the space V_k replaced by the space W_k, we have the same conclusion as those of Theorem* 1.1 *in the corresponding space W_k for $k \geq 0$.*

Theorem 1.3. *Let $k \geq 6$ be an integer, $(E_0^\varepsilon, \mathcal{N}_0^\varepsilon)$ and $(E_0^0, \mathcal{N}_0^0) \in V_k$, and satisfy*

$$
(E_0^\varepsilon, \mathcal{N}_0^\varepsilon) \quad \textit{uniformly bounded in } V_k \tag{1.9}
$$

and

$$
(E_0^\varepsilon, \mathcal{N}_0^\varepsilon) \to (E_0^0, \mathcal{N}_0^0) \quad \textit{in } V_{k-2} \textit{ as } \varepsilon \to 0.
$$

Assume that $(E^\varepsilon, \mathcal{N}^\varepsilon)$ *and* $(E^0, \mathcal{N}^0)$ *are solutions to* (1.6) *and* (1.7) *with initial datum* $(E_0^\varepsilon, \mathcal{N}_0^\varepsilon)$ *and* $(E_0^0, \mathcal{N}_0^0)$ *respectively. Then, for any* $T > 0$ *when* $d = 1$ *and for some* $T > 0$ *when* $d = 2, 3$, *there holds that*

$$\|(E^\varepsilon - E^0, \mathcal{N}^\varepsilon - \mathcal{N}^0)\|_{V_{k-4}} \lesssim \|(E_0^\varepsilon - E_0^0, \mathcal{N}_0^\varepsilon - \mathcal{N}_0^0)\|_{V_{k-4}} + \varepsilon \|\Delta(\mathcal{N}_0^\varepsilon - \mathcal{N}_0^0)\|_{H^{k-6}} + \varepsilon^2 \tag{1.10}$$

and

$$\|(E^\varepsilon - E^0, \mathcal{N}^\varepsilon - \mathcal{N}^0)\|_{V_s} \lesssim \|(E_0^\varepsilon - E_0^0, \mathcal{N}_0^\varepsilon - \mathcal{N}_0^0)\|^\alpha_{V_{k-4}} + \varepsilon^\alpha \|\Delta(\mathcal{N}_0^\varepsilon - \mathcal{N}_0^0)\|^\alpha_{H^{k-6}} + \varepsilon^{2\alpha}, \tag{1.11}$$

where $\alpha = (k - 2 - s)/2$, $s \in [k - 4, k - 2)$ *for all* $t \in [-T, T]$, *and the constants in the inequalities are independent of* ε. *Moreover, we have*

$$(E^\varepsilon, \mathcal{N}^\varepsilon) \to (E^0, \mathcal{N}^0) \quad \text{in } C([-T, T]; V_{k-2}) \text{ as } \varepsilon \to 0. \tag{1.12}$$

Theorem 1.4. *Let* $k \geq 6$ *be an integer,* $(E_0^\varepsilon, \mathcal{N}_0^\varepsilon)$ *and* $(E_0^0, \mathcal{N}_0^0) \in W_k$, *and satisfy*

$$(E_0^\varepsilon, \mathcal{N}_0^\varepsilon) \quad \text{uniformly bounded in } W_k$$

and

$$(E_0^\varepsilon, \mathcal{N}_0^\varepsilon) \to (E_0^0, \mathcal{N}_0^0) \quad \text{in } W_{k-3} \text{ as } \varepsilon \to 0.$$

Assume that $(E^\varepsilon, \mathcal{N}^\varepsilon)$ *and* $(E^0, \mathcal{N}^0)$ *are solutions to* (1.6) *and* (1.7) *with initial datum* $(E_0^\varepsilon, \mathcal{N}_0^\varepsilon)$ *and* $(E_0^0, \mathcal{N}_0^0)$ *respectively. Then, for any* $T > 0$ *when* $d = 1$ *and for some* $T > 0$ *when* $d = 2, 3$, *there holds that*

$$\|(E^\varepsilon - E^0, \mathcal{N}^\varepsilon - \mathcal{N}^0)\|_{W_{k-5}} \lesssim \|(E_0^\varepsilon - E_0^0, \mathcal{N}_0^\varepsilon - \mathcal{N}_0^0)\|_{W_{k-4}} + \varepsilon \|\Delta(\mathcal{N}_0^\varepsilon - \mathcal{N}_0^0)\|_{H^{k-6}} + \varepsilon^2 \tag{1.13}$$

and

$$\|(E^\varepsilon - E^0, \mathcal{N}^\varepsilon - \mathcal{N}^0)\|_{W_s} \lesssim \|(E_0^\varepsilon - E_0^0, \mathcal{N}_0^\varepsilon - \mathcal{N}_0^0)\|^\beta_{W_{k-4}} + \varepsilon^\beta \|\Delta(\mathcal{N}_0^\varepsilon - \mathcal{N}_0^0)\|^\beta_{H^{k-6}} + \varepsilon^{2\beta}, \tag{1.14}$$

where $\beta = (k - 3 - s)/2$, $s \in [k - 5, k - 3)$ *for all* $t \in [-T, T]$, *and the constants in the inequalities are independent of* ε. *Moreover, we have*

$$(E^\varepsilon, \mathcal{N}^\varepsilon) \to (E^0, \mathcal{N}^0) \quad \text{in } C([-T, T]; W_{k-3}) \text{ as } \varepsilon \to 0. \tag{1.15}$$

The outline of the paper is as follows. In Section 2, we introduce some notations and solution formulae. In Section 3, we apply energy method to derive some uniform bounds of solutions in time for global well-posedness and some uniform bounds for the semi-classical limit. We state some Strichartz estimates of the 4th order Schrödinger equation and that of the 4th order wave equation for the global well-posedness of (1.6). In Section 4, we prove the global well-posedness of the quantum Zakharov system. In Section 5, we prove that solution to the quantum Zakharov system converges to the solution to Zakharov system in an appropriate norm.

2. Notations and solution formulae

Let us denote

$$|\nabla| := \sqrt{-\Delta} \quad \text{and} \quad \langle \varepsilon\nabla\rangle := (1+\varepsilon^2|\nabla|^2)^{1/2} = (1-\varepsilon^2\Delta)^{1/2}. \tag{2.1}$$

We also denote the projection on the low frequency part by $\mathcal{P}_L$ and the projection on the high frequency part by $\mathcal{P}_H$,

$$\widehat{\mathcal{P}_L u}(\xi) = \psi_\varepsilon(|\xi|)\widehat{u}(\xi) \quad \text{and} \quad \widehat{\mathcal{P}_H u}(\xi) = (1-\psi_\varepsilon(|\xi|))\widehat{u}(\xi),$$

where the cut-off function $\psi_\varepsilon(|\xi|)$ is 1 for $\varepsilon|\xi| \le 1$, 0 for $\varepsilon|\xi| \ge 2$.

For $s \in \mathbb{R}$, we denote $H^s(\mathbb{R}^d)$ and $\dot{H}^s(\mathbb{R}^d)$ the standard inhomogeneous and homogeneous Sobolev spaces equipped with the norms

$$\|u\|_{H^s} := \|\langle\nabla\rangle^s u\|_{L^2} = \|(1+|\xi|^2)^{s/2}\widehat{u}\|_{L^2} \quad \text{and} \quad \|u\|_{\dot{H}^s} := \||\nabla|^s u\|_{L^2} = \||\xi|^s\widehat{u}\|_{L^2}$$

respectively, where $\widehat{u}$ is the Fourier transform of u over space variables. Also we denote the Sobolev space $H_r^k = W^{k,r}$.

Consider the 4th order Schrödinger equation

$$\begin{aligned} &iE_t + \Delta E - \varepsilon^2\Delta^2 E = F, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R};\\ &E(0) = E_0. \end{aligned} \tag{2.2}$$

We have the solution formula

$$E(t,x) = U_\varepsilon(t)E_0(x) - i\int_0^t U_\varepsilon(t-s)F(s,x)\,ds, \tag{2.3}$$

where $U_\varepsilon(t) := e^{it\Delta\langle\varepsilon\nabla\rangle^2}$ is the 4th order Schrödinger propagator. In general, for the equation of $\mathcal{N}$,

$$\begin{aligned} &i\partial_t\mathcal{N} - |\nabla|\langle\varepsilon\nabla\rangle\mathcal{N} = h, \quad (x,t) \in \mathbb{R}^d \times \mathbb{R};\\ &\mathcal{N}(0,x) = \mathcal{N}_0(x), \end{aligned} \tag{2.4}$$

we can obtain the solution formula

$$\mathcal{N}(t,x) = \Phi_\varepsilon(t)\mathcal{N}_0(x) - i\int_0^t \Phi_\varepsilon(t-s)h(s,x)\,ds, \tag{2.5}$$

where $\Phi_\varepsilon(t) := e^{-it|\nabla|\langle\varepsilon\nabla\rangle}$ is the propagator of (2.4). The Duhamel operators for (2.3) and (2.5) are respectively

$$U_\varepsilon *_R F(t,x) = -i\int_0^t U_\varepsilon(t-s)F(s,x)\,ds \quad \text{and} \quad \Phi_\varepsilon *_R F(t,x) = -i\int_0^t \Phi_\varepsilon(t-s)F(s,x)\,ds.$$

3. Energy estimates and Strichartz estimates

A pair (q,r) is called Schrödinger admissible, for short S-admissible, if $2 \le q, r \le \infty$, $(q,r,d) \neq (2,\infty,2)$, and
$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}.$$
A pair (q,r) is called biharmonic admissible, for short B-admissible, if $2 \le q, r \le \infty$, $(q,r,d) \neq (2,\infty,4)$, and
$$\frac{4}{q} + \frac{d}{r} = \frac{d}{2}.$$
A pair (q,r) is called wave admissible, for short W-admissible, if $2 \le q, r \le \infty$, $(q,r,d) \neq (2,\infty,3)$, and
$$\frac{2}{q} + \frac{d-1}{r} \le \frac{d-1}{2}.$$

We recall some known results.

Lemma 3.1. (Pausader [20]) *Let $E \in C([0,T], H^{-4}(\mathbb{R}^d))$ be a solution of (2.2) with $\varepsilon = 1$. For any B-admissible pairs (q,r) and $(\widetilde{q},\widetilde{r})$, it satisfies*

$$\|E\|_{L_t^q([0,T];L_x^r(\mathbb{R}^d))} \le C\left(\|E_0\|_{L_x^2(\mathbb{R}^d)} + \|F\|_{L_t^{\widetilde{q}'}([0,T];L_x^{\widetilde{r}'}(\mathbb{R}^d))}\right), \tag{3.1}$$

where C depends only on $\widetilde{q}$ and $\widetilde{r}$. Besides, for any S-admissible pairs (q,r) and (a,b), and any $s \ge 0$, we have

$$\||\nabla|^s E\|_{L_t^q([0,T];L_x^r(\mathbb{R}^d))} \le C\left(\||\nabla|^{s-2/q} E_0\|_{L_x^2(\mathbb{R}^d)} + \||\nabla|^{s-2/q-2/a} F\|_{L_t^{a'}([0,T];L_x^{b'}(\mathbb{R}^d))}\right), \tag{3.2}$$

where C depends only on a' and b'.

Consider the quantum effect, the Strichartz estimates (3.1) and (3.2) can be derived as follows.

Corollary 3.2. *Let $E \in C([0,T], H^{-4}(\mathbb{R}^d))$ be a solution of (2.2). For any B-admissible pairs (q,r) and $(\widetilde{q},\widetilde{r})$, it satisfies*

$$\|E\|_{L_t^q([0,T];L_x^r(\mathbb{R}^d))} \le C\left(\varepsilon^{-2/q}\|E_0\|_{L_x^2(\mathbb{R}^d)} + \varepsilon^{-2/q-2/\widetilde{q}}\|F\|_{L_t^{\widetilde{q}'}([0,T];L_x^{\widetilde{r}'}(\mathbb{R}^d))}\right), \tag{3.3}$$

where C depends only on $\widetilde{q}'$ and $\widetilde{r}'$. Besides, for any S-admissible pairs (q,r) and (a,b), and any $s \ge 0$, we have

$$\begin{aligned} &\||\nabla|^s E\|_{L_t^q([0,T];L_x^r(\mathbb{R}^d))} \\ &\le C\left(\varepsilon^{-2/q}\||\nabla|^{s-2/q} E_0\|_{L_x^2(\mathbb{R}^d)} + \varepsilon^{-2/q-2/a}\||\nabla|^{s-2/q-2/a} F\|_{L_t^{a'}([0,T];L_x^{b'}(\mathbb{R}^d))}\right), \end{aligned} \tag{3.4}$$

where C depends only on a' and b'. For $\varepsilon = 1$, we also have

$$\|\langle\nabla\rangle^{s-2/q} E\|_{L_t^q([0,T];L_x^r(\mathbb{R}^d))} \le C\left(\|\langle\nabla\rangle^s E_0\|_{L_x^2(\mathbb{R}^d)} + \|\langle\nabla\rangle^{s-2/a} F\|_{L_t^{a'}([0,T];L_x^{b'}(\mathbb{R}^d))}\right). \tag{3.5}$$

Proof. To derive the homogeneous terms of (3.3) and (3.4), we use the L^2-norm invariant scaling, $E_{0,\lambda}(x) = \lambda^{d/2} E_0(\lambda x)$ and the homogeneous estimates of (3.1) and (3.2).

To derive the inhomogeneous terms of (3.3) and (3.4), we use the $L_t^{a'}(L_x^{b'})$-norm invariant scaling, $F_\lambda(t,x) = \lambda^{2/a'+d/b'} F(\lambda^2 t, \lambda x)$ and the inhomogeneous estimates of (3.1) and (3.2). (3.5) is an inhomogeneous version of (3.2). $\square$

Lemma 3.3. (Gustafson-Nakanishi-Tsai [12, Theorem 2.1]) *Let $\mathcal{N}$ be a solution of* (2.4). *If (q_i, r_i) are S-admissible with $r_i < \infty$ for $i = 1, 2$, we have*

$$\left\| \left(\frac{|\nabla|}{\langle \nabla \rangle} \right)^{-\gamma_1} \mathcal{N} \right\|_{L_t^{q_1}([0,T];L_x^{r_1}(\mathbb{R}^d))} \lesssim \|\mathcal{N}_0\|_{L^2(\mathbb{R}^d)} + \left\| \left(\frac{|\nabla|}{\langle \nabla \rangle} \right)^{\gamma_2} h \right\|_{L_t^{q_2'}([0,T];L_x^{r_2'}(\mathbb{R}^d))},$$

where the constant depends on d and

$$\gamma_i = \left(1 - \frac{2}{d}\right) \frac{1}{q_i}.$$

For the proof, the readers are referred to the work of Gustafson-Nakanishi-Tsai [12].

Lemma 3.4. (Gagliardo-Nirenberg inequality [2]) *We have the following estimates*

$$\|u\|_{L^4(\mathbb{R}^d)}^4 \lesssim \|\nabla u\|_{L^2(\mathbb{R}^d)}^d \|u\|_{L^2(\mathbb{R}^d)}^{4-d}, \tag{3.6}$$

$$\|u\|_{L^4(\mathbb{R}^d)}^4 \lesssim \|\Delta u\|_{L^2(\mathbb{R}^d)}^d \|u\|_{L^2(\mathbb{R}^d)}^{8-d} \tag{3.7}$$

and

$$\|u\|_{H^{k-6}(\mathbb{R}^d)} \lesssim \|u\|_{H^{k-4}(\mathbb{R}^d)}^{(k-6)/(k-4)} \|u\|_{L^2(\mathbb{R}^d)}^{2/(k-4)}. \tag{3.8}$$

Proposition 3.5. *For smooth solutions to* (1.6)*, there hold two conserved quantities:*

$$\|E(t)\|_{L^2}^2 = \|E_0\|_{L^2}^2, \quad \mathcal{H}(t) = \mathcal{H}(0) \quad \text{for all } t \in \mathbb{R}, \tag{3.9}$$

where $\mathcal{H}(t) = \mathcal{H}(E(t), \mathcal{N}(t))$ is given in (1.8).

Moreover, if $(E_0, \mathcal{N}_0) \in V_2$, then

$$\|(E(t), \mathcal{N}(t))\|_{V_2} \le C \quad \text{for all } t \in \mathbb{R}, \tag{3.10}$$

where C depends only on $\|(E_0, \mathcal{N}_0)\|_{V_2}$ and ε. If $(E_0, \mathcal{N}_0) \in V_k$ for $k \ge 3$, then

$$\|(E(t), \mathcal{N}(t))\|_{V_k} \le C \quad \text{for } t \in [-T, T], \tag{3.11}$$

where C depends only on $\|(E_0, \mathcal{N}_0)\|_{V_k}$, T and ε.

Proof. The derivations of conservation of mass and the conservation of the Hamiltonians (1.4) for (1.1) and (1.8) for (1.6) are standard.

To prove (3.10), we define $\mathcal{H}_N := \int \mathrm{Re}(\mathcal{N})|E|^2\,dx$. To estimate the inhomogeneous part of the Hamiltonian, we apply (3.7),

$$\|E(t)\|_{L_x^4} \lesssim \|E(t)\|_{L_x^2}^{1-d/8} \cdot \|\Delta E(t)\|_{L_x^2}^{d/8}.$$

Hence we have

$$\begin{aligned}|\mathcal{H}_N| &\le \frac{1}{4}\|\mathcal{N}(t)\|_{L_x^2}^2 + C\|E_0\|_{L_x^2}^{4(1-d/8)} \cdot \|\Delta E(t)\|_{L_x^2}^{d/2} \\ &\le \frac{1}{4}\|\mathcal{N}(t)\|_{L_x^2}^2 + \frac{1}{2}\varepsilon^2\|\Delta E(t)\|_{L_x^2}^2 + C.\end{aligned}$$

Now we can estimate the homogeneous part of the Hamiltonian:

$$\begin{aligned}\mathcal{H}_L(t) &= \mathcal{H}(t) - \mathcal{H}_N(t) \\ &\le |\mathcal{H}(0)| + \frac{1}{4}\|\mathcal{N}(t)\|_{L_x^2}^2 + \frac{1}{2}\varepsilon^2\|\Delta E(t)\|_{L_x^2}^2 + C.\end{aligned}$$

Thus we obtain

$$\mathcal{H}_L(t) \lesssim |\mathcal{H}(0)| + C.$$

Hence we get

$$\|(E(t),\mathcal{N}(t))\|_{V_2} \sim \mathcal{H}_L(t) \le C.$$

To show (3.11) for $k \ge 3$, we proceed the proof by induction on k. First we show (3.11) for $k = 3$. We apply (3.10) to obtain $\|E(t)\|_{H^2} \le C$ for all $t \in \mathbb{R}$. Since $H^2(\mathbb{R}^d)$ is a Banach algebra for $d \le 3$, we have $\|\Delta|E|^2\|_{L^2} \le \|E\|_{H^2}^2 \le C$. Taking the inner product of the second equation of (1.6) with $\langle\nabla\rangle^4\overline{\mathcal{N}}$ and then taking the imaginary part of the resulting equation gives

$$\mathrm{Re}\int \langle\nabla\rangle^2\mathcal{N}_t\langle\nabla\rangle^2\overline{\mathcal{N}}\,dx = -\,\mathrm{Im}\int \langle\nabla\rangle^2(|\nabla|\langle\varepsilon\nabla\rangle^{-1}|E|^2)\langle\nabla\rangle^2\overline{\mathcal{N}}\,dx.$$

Hence we obtain

$$\frac{d}{dt}\|\langle\nabla\rangle^2\mathcal{N}\|_{L^2}^2 \lesssim \|\langle\nabla\rangle^2\mathcal{N}\|_{L^2}^2 + C.$$

Invoking Gronwall's inequality, we get

$$\|\langle\nabla\rangle^2\mathcal{N}\|_{L^2} \lesssim C$$

for $t \in [-T,T]$.

To estimate E, we differentiate the first equation of (1.6) to get

$$iE_{tt} + \Delta\langle\varepsilon\nabla\rangle^2 E_t = \mathrm{Re}(\mathcal{N}_t)E + \mathrm{Re}(\mathcal{N})E_t. \tag{3.12}$$

Analogously we take the inner product of the above equation with $\langle\nabla\rangle^{-2}\overline{E}_t$ and then take the imaginary part of the resulting equation, thus we have

$$\mathrm{Re}\int\langle\nabla\rangle^{-1}E_{tt}\langle\nabla\rangle^{-1}\overline{E}_t\,dx=\mathrm{Im}\int\langle\nabla\rangle^{-1}(\mathrm{Re}(\mathcal{N}_t)E+\mathrm{Re}(\mathcal{N})E_t)\langle\nabla\rangle^{-1}\overline{E}_t\,dx.$$

Hence we obtain

$$\frac{d}{dt}\|E_t\|^2_{H^{-1}}\lesssim\|\mathcal{N}_tE\|^2_{H^{-1}}+\|\mathcal{N}E_t\|^2_{H^{-1}}+\|E_t\|^2_{H^{-1}}\lesssim\|E_t\|^2_{H^{-1}}+C.$$

Since

$$\|\mathcal{N}_tE\|_{H^{-1}}=\sup_{\|\varphi\|_{H^1}\le1}|\langle\mathcal{N}_tE,\varphi\rangle|\le\sup_{\|\varphi\|_{H^1}\le1}\|\mathcal{N}_t\|_{L^2}\|E\|_{H^1}\|\varphi\|_{H^1}\le C$$

and

$$\|\mathcal{N}E_t\|_{H^{-1}}=\sup_{\|\varphi\|_{H^1}\le1}|\langle\mathcal{N}E_t,\varphi\rangle|\le\sup_{\|\varphi\|_{H^1}\le1}\|E_t\|_{H^{-1}}\|\mathcal{N}\|_{H^2}\|\varphi\|_{H^1}\le\|E_t\|^2_{H^{-1}}+C.$$

Notice that using the second equation of (1.6), we have $\|\mathcal{N}_t\|_{L^2}\le C$. Invoking Gronwall's inequality, we get

$$\|E_t\|_{H^{-1}}\lesssim C\quad\text{for } t\in[-T,T].$$

Finally we use the first equation of (1.6),

$$\Delta E=\langle\varepsilon\nabla\rangle^{-2}(-iE_t+\mathrm{Re}(\mathcal{N})E)\tag{3.13}$$

to obtain

$$\|E\|_{H^3}\lesssim C\quad\text{for } t\in[-T,T].$$

Hence we have

$$\|(E(t),\mathcal{N}(t))\|_{V_3}\le C\quad\text{for all } t\in[-T,T].$$

Now we assume that the estimate (3.10) holds for $k=3,4,\ldots,m$. Now we want to show that (3.10) holds for $k=m+1$. Notice that $\|E(t)\|_{H^m}\le C$ implies $\|\Delta E(t)\|_{H^{m-2}}\le C$. We take the inner product of the second equation of (1.6) with $\langle\nabla\rangle^{2m}\overline{\mathcal{N}}$ and then we take the imaginary part of the resulting equation to get

$$\mathrm{Re}\int\langle\nabla\rangle^m\mathcal{N}_t\langle\nabla\rangle^m\overline{\mathcal{N}}\,dx=-\,\mathrm{Im}\int\langle\nabla\rangle^m(|\nabla|\langle\varepsilon\nabla\rangle^{-1}|E|^2)\langle\nabla\rangle^m\overline{\mathcal{N}}\,dx.$$

Hence we obtain

$$\frac{d}{dt}\|\langle\nabla\rangle^m\mathcal{N}\|^2_{L^2}\lesssim\|\langle\nabla\rangle^m\mathcal{N}\|^2_{L^2}+C.$$

Invoking Gronwall's inequality, we get

$$\|\langle\nabla\rangle^m\mathcal{N}\|_{L^2}\lesssim C\quad\text{for } t\in[-T,T].$$

To estimate E, analogously we take the inner product of (3.12) with $\langle\nabla\rangle^{2m-6}\overline{E}_t$ and then take the imaginary part of the resulting equation, thus we have

$$\operatorname{Re}\int\langle\nabla\rangle^{m-3}E_{tt}\langle\nabla\rangle^{m-3}\overline{E}_t\,dx=\operatorname{Im}\int\langle\nabla\rangle^{m-3}(\operatorname{Re}(\mathcal{N}_t)E+\operatorname{Re}(\mathcal{N})E_t)\langle\nabla\rangle^{m-3}\overline{E}_t\,dx.$$

Hence we obtain

$$\begin{aligned}\frac{d}{dt}\|E_t\|^2_{H^{m-3}}&\lesssim\|\mathcal{N}_tE\|^2_{H^{m-3}}+\|\mathcal{N}E_t\|^2_{H^{m-3}}+\|E_t\|^2_{H^{m-3}}\\&\lesssim\|\mathcal{N}_t\|^2_{H^{m-3}}\|E\|^2_{H^{m-1}}+\|\mathcal{N}\|^2_{H^{m-1}}\|E_t\|^2_{H^{m-3}}+\|E_t\|^2_{H^{m-3}}\\&\lesssim\|E_t\|^2_{H^{m-3}}+C.\end{aligned}$$

Notice that using the second equation of (1.6) gives $\|\mathcal{N}_t\|_{H^{m-3}}\le C$. Invoking Gronwall's inequality, we get

$$\|E_t\|_{H^{m-3}}\lesssim C\quad\text{for } t\in[-T,T].$$

Finally we use (3.13) to obtain

$$\|E\|_{H^{m+1}}\lesssim C\quad\text{for } t\in[-T,T].$$

This completes the proof. □

Proposition 3.6. *Let $T>0$, $k\ge 2$ and $(E_0,\mathcal{N}_0)\in W_k$. Then*

$$\|(E(t),\mathcal{N}(t))\|_{W_k}\le C\quad\textit{for all } t\in[-T,T],$$

where the constant C depends on $\|(E_0,\mathcal{N}_0)\|_{W_k}$, T and ε.

Proof. Since the initial data $(E_0,\mathcal{N}_0)\in H^k\times H^k\subset H^k\times H^{k-1}$, we can then apply Proposition 3.5 to get

$$\|(E(t),\mathcal{N}(t))\|_{H^k\times H^{k-1}}\le C\quad\text{for all } t\in[-T,T].$$

As in the proof of Proposition 3.5, we take the inner product of the second equation of (1.6) with $\langle\nabla\rangle^{2k}\overline{\mathcal{N}}$ and then we take the imaginary part of the resulting equation to get

$$\operatorname{Re}\int\langle\nabla\rangle^{k}\mathcal{N}_t\langle\nabla\rangle^{k}\overline{\mathcal{N}}\,dx=-\operatorname{Im}\int\langle\nabla\rangle^{k}(|\nabla|\langle\varepsilon\nabla\rangle^{-1}|E|^2)\langle\nabla\rangle^{k}\overline{\mathcal{N}}\,dx.$$

Hence we obtain

$$\frac{d}{dt}\|\langle\nabla\rangle^k\mathcal{N}\|^2_{L^2}\lesssim\|\langle\nabla\rangle^k\mathcal{N}\|^2_{L^2}+C.$$

Invoking Gronwall's inequality, we get

$$\|\langle\nabla\rangle^k\mathcal{N}\|_{L^2}\lesssim C\quad\text{for } t\in[-T,T].$$

Hence we have the desired result. □

Proposition 3.7. *Let $d = 1$ and $k \geq 4$. Assume that $(E_0^\varepsilon, \mathcal{N}_0^\varepsilon) \in V_k$ and*

$$\|(E_0^\varepsilon, \mathcal{N}_0^\varepsilon)\|_{V_k} \leq C_0, \tag{3.14}$$

where C_0 is independent of ε. If $(E^\varepsilon, \mathcal{N}^\varepsilon) \in C(\mathbb{R}; V_k)$ is the solution to (1.6) with the initial data $(E_0^\varepsilon, \mathcal{N}_0^\varepsilon)$, then for all $T > 0$, we have

$$\|(E^\varepsilon(t), \mathcal{N}^\varepsilon(t))\|_{V_{k-2}} + \|E_t^\varepsilon(t)\|_{H^{k-4}} \leq C \quad \text{for all } t \in [-T, T], \tag{3.15}$$

where C depends on C_0 and T, but not depends on ε.

Proof. We first prove the case for $k = 4$. From Proposition 3.5, we have the conservation of mass and conservation of Hamiltonian in (3.9), that is

$$\|E(t)\|_{L^2}^2 = \|E_0\|_{L^2}^2, \quad \mathcal{H}(t) = \mathcal{H}(0) \quad \text{for all } t \in \mathbb{R}.$$

Invoking (3.6),

$$\|u\|_{L^4}^4 \lesssim \|u\|_{L^2}^3 \|u_x\|_{L^2}, \tag{3.16}$$

we get

$$\left| \int \mathrm{Re}(\mathcal{N}^\varepsilon) |E^\varepsilon|^2 \, dx \right| \leq \frac{1}{4} \|\mathcal{N}^\varepsilon\|_{L^2} + \|E^\varepsilon\|_{L^4}^4 \leq \frac{1}{4} \|\mathcal{N}^\varepsilon\|_{L^2}^2 + \frac{1}{2} \|E_x^\varepsilon\|_{L^2}^2 + \frac{1}{2} \|E_0^\varepsilon\|_{L^2}^6.$$

Thus we have

$$\left| \int \mathrm{Re}(\mathcal{N}^\varepsilon) |E^\varepsilon|^2 \, dx \right| - \frac{1}{4} \|\mathcal{N}^\varepsilon\|_{L^2}^2 - \frac{1}{2} \|E_x^\varepsilon\|_{L^2}^2 \leq C$$

and then

$$\begin{aligned} \||\nabla| \langle \varepsilon \nabla \rangle E^\varepsilon\|_{L^2}^2 + \frac{1}{2} \|\langle \varepsilon \nabla \rangle \mathcal{N}^\varepsilon\|_{L^2}^2 &\sim \mathcal{H}(E^\varepsilon, \mathcal{N}^\varepsilon)(t) - \frac{1}{4} \|\mathcal{N}^\varepsilon\|_{L^2}^2 \\ &\quad - \frac{1}{2} \|E_x^\varepsilon\|_{L^2}^2 - \int \mathrm{Re}(\mathcal{N}^\varepsilon) |E^\varepsilon|^2 \, dx \\ &\lesssim \mathcal{H}(E^\varepsilon, \mathcal{N}^\varepsilon)(0) + C. \end{aligned} \tag{3.17}$$

We also have that

$$\|E^\varepsilon\|_{L^\infty} \leq \|E^\varepsilon\|_{H^1} \leq C. \tag{3.18}$$

Now we want to estimate $\|E^\varepsilon\|_{\dot{H}^2}$ and $\|\mathcal{N}^\varepsilon\|_{\dot{H}^1}$. Taking inner product of (3.12) and the second equation of (1.6) with E_t^ε and $|\nabla|^2 \langle \varepsilon \nabla \rangle^2 \mathcal{N}^\varepsilon$ respectively, then we apply (3.16) and (3.18) to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|E_t^\varepsilon\|_{L^2}^2 &= \mathrm{Im} \int \mathrm{Im}(|\nabla| \langle \varepsilon \nabla \rangle \mathcal{N}^\varepsilon) E^\varepsilon \overline{E}_t^\varepsilon \, dx \\ &\lesssim \||\nabla| \langle \varepsilon \nabla \rangle \mathcal{N}^\varepsilon\|_{L^2}^2 + \|E_t^\varepsilon\|_{L^2}^2 \end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\||\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}^\varepsilon\|_{L^2}^2 &= \operatorname{Im}\int |\nabla|^2|E^\varepsilon|^2|\nabla|\langle\varepsilon\nabla\rangle\overline{\mathcal{N}}^\varepsilon\,dx \\ &\lesssim \left(\|\Delta E^\varepsilon\|_{L^2}^2+1\right)\||\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}^\varepsilon\|_{L^2}.\end{aligned} \tag{3.20}$$

Since

$$i\langle\varepsilon\nabla\rangle^{-2}E_t^\varepsilon+\Delta E^\varepsilon=\langle\varepsilon\nabla\rangle^{-2}(\operatorname{Re}(\mathcal{N}^\varepsilon)E^\varepsilon), \tag{3.21}$$

we have

$$\begin{aligned}\|\Delta E^\varepsilon\|_{L^2} &\le \|\langle\varepsilon\nabla\rangle^{-2}E_t^\varepsilon\|_{L^2}+\|\langle\varepsilon\nabla\rangle^{-2}(\operatorname{Re}(\mathcal{N}^\varepsilon)E^\varepsilon)\|_{L^2} \\ &\le \|E_t^\varepsilon\|_{L^2}+C.\end{aligned} \tag{3.22}$$

Combining (3.19)–(3.22), we have

$$\frac{d}{dt}\left(\|E_t^\varepsilon\|_{L^2}^2+\||\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}^\varepsilon\|_{L^2}^2\right)\lesssim\|E_t^\varepsilon\|_{L^2}^2+\||\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}^\varepsilon\|_{L^2}^2+C.$$

Invoking Gronwall's inequality, (3.14) and (3.17), one can derive

$$\sup_{t\in[-T,T]}\left(\|E^\varepsilon\|_{H^2}^2+\|E_t^\varepsilon\|_{L^2}^2+\||\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}^\varepsilon\|_{L^2}^2+\|\langle\varepsilon\nabla\rangle\mathcal{N}^\varepsilon\|_{L^2}^2\right)\le C,$$

where C is independent of ε. Hence we have proved (3.15) for $k=4$.

Repeating the same arguments as above, we can prove (3.15) for $k>4$. □

Proposition 3.8. *Let $d=2,3$ and $k\ge 4$. Assume that $(E_0^\varepsilon,\mathcal{N}_0^\varepsilon)\in V_k$ and $\|(E_0^\varepsilon,\mathcal{N}_0^\varepsilon)\|_{V_k}\le C_0$, where C_0 is independent of ε. If $(E^\varepsilon,\mathcal{N}^\varepsilon)\in C(\mathbb{R};V_k)$ is the solution to (1.6) with the initial data $(E_0^\varepsilon,\mathcal{N}_0^\varepsilon)$, then there exist constants T and C such that*

$$\|(E^\varepsilon(t),\mathcal{N}^\varepsilon(t))\|_{V_{k-2}}+\|E_t^\varepsilon(t)\|_{H^{k-4}}\le C\quad\text{for all } t\in[-T,T], \tag{3.23}$$

where C and T are independent of ε.

Proof. In the case of $d=2,3$, the Hamiltonian possesses a nonlinear part which can not be absorbed by the linear part. Thus, unlike the case for $d=1$ in Proposition 3.7, we use a different approach at the cost of that the size of the time interval T is not arbitrary.

Let us consider the case of $k=4$. First we compute

$$\begin{aligned}\frac{\partial}{\partial t}|E^\varepsilon|^2 &= 2\operatorname{Im}\left[\left(-\Delta E^\varepsilon+\varepsilon^2\Delta^2E^\varepsilon+\operatorname{Re}(\mathcal{N}^\varepsilon)E^\varepsilon\right)\overline{E^\varepsilon}\right] \\ &\lesssim 2\operatorname{Im}\left[\left(-\Delta E^\varepsilon+\varepsilon^2\Delta^2E^\varepsilon\right)\overline{E^\varepsilon}\right].\end{aligned}$$

So we have that

$$\|E^\varepsilon(t)\|_{L^4}^4-\|E^\varepsilon(0)\|_{L^4}^4=\int_0^t\int 2|E^\varepsilon|^2\frac{\partial}{\partial s}|E^\varepsilon|^2\,dxds\lesssim\int_0^t\|E^\varepsilon(s)\|_{H^2}^4\,ds. \tag{3.24}$$

Analogously we have

$$\|\operatorname{Re}\mathcal{N}^\varepsilon(t)\|_{L^4}^4 \lesssim \|\operatorname{Re}\mathcal{N}^\varepsilon(0)\|_{L^4}^4 + \int_0^t \|\operatorname{Re}\mathcal{N}^\varepsilon(s)\|_{H^1}^6 + \|\operatorname{Re}\mathcal{N}_t^\varepsilon(s)\|_{L^2}^2\,ds. \tag{3.25}$$

Invoking (3.21), (3.24) and (3.25), we then obtain

$$\begin{aligned}\|\Delta E^\varepsilon(t)\|_{L^2}^2 &\lesssim \|E_t^\varepsilon(t)\|_{L^2}^2 + \|\operatorname{Re}(\mathcal{N}^\varepsilon)(t)\|_{L^4}^4 + \|E^\varepsilon(t)\|_{L^4}^4 \\ &\lesssim C + C\int_0^t \|E^\varepsilon(s)\|_{H^2}^4 + \|\operatorname{Re}\mathcal{N}^\varepsilon(s)\|_{H^1}^6 + \|\operatorname{Re}\mathcal{N}_t^\varepsilon(s)\|_{L^2}^2\,ds.\end{aligned} \tag{3.26}$$

Next we apply (3.24) to bound the integral

$$\left|\int \operatorname{Re}(\mathcal{N}^\varepsilon)|E^\varepsilon|^2\,dx\right| \le \frac{1}{4}\|\operatorname{Re}(\mathcal{N}^\varepsilon)\|_{L^2}^2 + C + C\int_0^t \|E^\varepsilon(s)\|_{H^2}^4\,ds.$$

Thus we can apply (3.19) and (3.20) to estimate the following terms

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|E_t^\varepsilon(t)\|_{L^2}^2 + \||\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}^\varepsilon(t)\|_{L^2}^2\right) \\ &= \operatorname{Im}\int \operatorname{Im}(|\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}^\varepsilon)E^\varepsilon\overline{E}_t^\varepsilon\,dx + \operatorname{Im}\int |\nabla|^2|E^\varepsilon|^2|\nabla|\langle\varepsilon\nabla\rangle\overline{\mathcal{N}}^\varepsilon\,dx \\ &\lesssim \||\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}_t^\varepsilon\|_{L^2}^2 + \|E^\varepsilon(s)\|_{H^2}^4 + \|E_t^\varepsilon(s)\|_{L^2}^4.\end{aligned}$$

Thus we integrate the above estimate over the time interval $[0,T]$ and then we get

$$\begin{aligned}&\|E_t^\varepsilon(t)\|_{L^2}^2 + \||\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}^\varepsilon(t)\|_{L^2}^2 \\ &\le C + C\int_0^t \||\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}_t^\varepsilon\|_{L^2}^2 + \|E^\varepsilon(s)\|_{H^2}^4 + \|E_t^\varepsilon(s)\|_{L^2}^4\,ds.\end{aligned}$$

Now we apply the conservation of mass, the conservation of Hamiltonian and (3.24) to compute

$$\begin{aligned}&\|E^\varepsilon(t)\|_{H^1}^2 + \|\langle\varepsilon\nabla\rangle\mathcal{N}^\varepsilon(t)\|_{L^2}^2 \\ &\sim \|E^\varepsilon\|_{L^2}^2 + \|\nabla E^\varepsilon\|_{L^2}^2 + \frac{1}{4}\|\operatorname{Re}\mathcal{N}^\varepsilon\|_{L^2}^2 + \frac{1}{2}\|\operatorname{Im}\mathcal{N}^\varepsilon\|_{L^2}^2 + \frac{1}{2}\|\varepsilon\nabla\mathcal{N}^\varepsilon\|_{L^2}^2 \\ &\le \|E^\varepsilon(t)\|_{L^2}^2 + \mathcal{H}^\varepsilon(t) - \frac{1}{4}\|\operatorname{Re}\mathcal{N}^\varepsilon\|_{L^2}^2 - \int \operatorname{Re}(\mathcal{N}^\varepsilon)|E^\varepsilon|^2\,dx \\ &\le \|E^\varepsilon(0)\|_{L^2}^2 + \mathcal{H}^\varepsilon(0) + \|E^\varepsilon(t)\|_{L^4}^4 \\ &\le C + C\int_0^t \|E^\varepsilon(s)\|_{H^2}^4\,ds.\end{aligned} \tag{3.27}$$

Now we set

$$\Phi(t) = \|E^\varepsilon(t)\|_{H^2}^2 + \|E_t^\varepsilon(t)\|_{L^2}^2 + \|\langle\varepsilon\nabla\rangle\mathcal{N}^\varepsilon(t)\|_{L^2}^2 + \||\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}^\varepsilon(t)\|_{L^2}^2.$$

Invoking (3.26), (3.27) and (3.19), we can show that

$$\Phi(t) \lesssim C + C \int_0^t (1 + \Phi(s))^3 \, ds.$$

By Gronwall's inequality, there exist $T > 0$ and $C > 0$ such that (3.23) holds.

Finally, for $k > 4$, we set

$$\Phi_k(t) = \|E^\varepsilon(t)\|_{H^{k-2}}^2 + \|E_t^\varepsilon(t)\|_{H^{k-4}}^2 + \|\langle \varepsilon\nabla\rangle \mathcal{N}^\varepsilon(t)\|_{H^{k-4}}^2 + \||\nabla|\langle \varepsilon\nabla\rangle \mathcal{N}^\varepsilon(t)\|_{H^{k-4}}^2.$$

Analogously we can prove that

$$\Phi(t) \lesssim C + C \int_0^t (1 + \Phi(s))^3 \, ds.$$

Again we apply Gronwall's inequality to obtain that there exist $T > 0$ and $C > 0$ such that (3.23) holds. □

Proposition 3.9. *Let $d = 1$ and $k \geq 4$. Assume that $(E_0^\varepsilon, \mathcal{N}_0^\varepsilon) \in W_k$ and*

$$\|(E_0^\varepsilon, \mathcal{N}_0^\varepsilon)\|_{W_k} \leq C_0,$$

where C_0 is independent of ε. If $(E^\varepsilon, \mathcal{N}^\varepsilon) \in C(\mathbb{R}; W_k)$ is the solution to (1.6) with the initial data $(E_0^\varepsilon, \mathcal{N}_0^\varepsilon)$, then for all $T > 0$, we have

$$\|(E^\varepsilon(t), \mathcal{N}^\varepsilon(t))\|_{W_{k-2}} + \|E_t^\varepsilon(t)\|_{H^{k-4}} \leq C \quad \text{for all } t \in [-T, T],$$

where C depends on C_0 and T, but not depends on ε.

Proof. The proof is analogous to that of Proposition 3.6 so that we skip it. □

Proposition 3.10. *Let $d = 2, 3$ and $k \geq 4$. Assume that $(E_0^\varepsilon, \mathcal{N}_0^\varepsilon) \in W_k$ and $\|(E_0^\varepsilon, \mathcal{N}_0^\varepsilon)\|_{W_k} \leq C_0$, where C_0 is independent of ε. If $(E^\varepsilon, \mathcal{N}^\varepsilon) \in C(\mathbb{R}; W_k)$ is the solution to (1.6) with the initial data $(E_0^\varepsilon, \mathcal{N}_0^\varepsilon)$, then there exist constants T and C such that*

$$\|(E^\varepsilon(t), \mathcal{N}^\varepsilon(t))\|_{W_{k-2}} + \|E_t^\varepsilon(t)\|_{H^{k-4}} \leq C \quad \text{for all } t \in [-T, T],$$

where C and T are independent of ε.

The proof is analogous to that of Proposition 3.9 so that we skip it.

4. Global well-posedness for QZ system

Now we prove Theorem 1.1.

Proof of Theorem 1.1. For the existence and the uniqueness of the solution, the readers are referred to Theorem 1.1 in [11]. Thus (1.6) admits a unique local solution. The estimates in Proposition 3.5 imply that the solution $(E, \mathcal{N})$ obtained by the iteration argument can be extended globally in time.

Secondly, we prove the continuous dependence of the solution to (1.6) on the initial data. Let $(E^j, \mathcal{N}^j)$ and $(E, \mathcal{N})$ be the solutions to (1.6) with initial datum $(E_0^j, \mathcal{N}_0^j) \in V_k$ and $(E_0, \mathcal{N}_0) \in V_k$ respectively, such that

$$\|(E_0^j, \mathcal{N}_0^j) - (E_0, \mathcal{N}_0)\|_{V_k} \to 0 \quad \text{as } j \to \infty. \tag{4.1}$$

Let us consider the difference of $(E^j, \mathcal{N}^j)$ and $(E, \mathcal{N})$. Denote

$$F^j := E^j - E \quad \text{and} \quad M^j := \mathcal{N}^j - \mathcal{N},$$

and then (F^j, M^j) solves

$$\begin{aligned} iF_t^j + \Delta\langle\varepsilon\nabla\rangle^2 F^j &= \mathrm{Re}(M^j)E^j + \mathrm{Re}(\mathcal{N})F^j, \\ iM_t^j + |\nabla|\langle\varepsilon\nabla\rangle M^j &= |\nabla|\langle\varepsilon\nabla\rangle^{-1}(F^j\overline{E^j} + E\overline{F^j}). \end{aligned} \tag{4.2}$$

Invoking (3.10), we have

$$\sup_{t\in[-T,T]} \|(E^j, \mathcal{N}^j)\|_{V_k} + \|E_t^j\|_{H^{k-4}} \le C, \tag{4.3}$$

where C is independent of j.

As in the statement of Proposition 3.5, the solution $(E^j, \mathcal{N}^j)$ satisfies

$$\|E^j(t)\|_{L^2}^2 = \|E_0^j\|_{L^2}^2, \quad \mathcal{H}^j(t) = \mathcal{H}^j(0) \quad \text{for all } t \in \mathbb{R},$$

where $\mathcal{H}^j(t) = \mathcal{H}(E^j(t), \mathcal{N}^j(t))$ is given in (1.8). Invoking (4.1) and (4.3), one can get

$$\sup_{t\in[-T,T]} \|(E^j - E, \mathcal{N}^j - \mathcal{N})\|_{L^2\times L^2} \to 0.$$

Thus we can use the interpolation in Sobolev spaces to derive

$$\sup_{t\in[-T,T]} \|(E^j - E, \mathcal{N}^j - \mathcal{N})\|_{H^{k_1}\times H^{k_1-1}} \to 0 \quad \text{for } k_1 < k. \tag{4.4}$$

Hence we can also get

$$(E^j, \mathcal{N}^j) \to (E, \mathcal{N}) \quad \text{weakly} * \text{ in } L^\infty(-T, T; V_k). \tag{4.5}$$

From (4.3) and the second equation of (4.2), by the energy method, we have

$$\frac{d}{dt}\|M^j\|_{H^{k-1}} \le C\left(\|F^j\|_{H^k} + \|M^j\|_{H^{k-1}}\right). \tag{4.6}$$

Since M^j has one regularity less than F^j, we differentiate the equation of F^j to get

$$iF^j_{tt} + \Delta\langle\varepsilon\nabla\rangle^2 F^j_t = M^j_t E^j + M^j E^j_t + \mathcal{N}^j_t F^j + \mathcal{N}^j F^j_t.$$

Invoking the energy method, we obtain

$$\frac{d}{dt}\|F^j_t\|_{H^{k-4}} \le C\left(\|M^j_t\|^2_{H^{k-3}} + \|M^j\|^2_{H^{k-1}} + \|F^j\|^2_{H^k} + \|F^j_t\|^2_{H^{k-4}}\right) \tag{4.7}$$

for $d = 2, 3$ and $k \ge 3$. Since $\mathcal{N}$ is given in (1.5), we can see that $\mathcal{N}^j$ and $\mathcal{N}$ do not have L^∞-estimates when $k = 2$ and $d = 2, 3$. We apply the first equation of (4.2),

$$\Delta F^j = \langle\varepsilon\nabla\rangle^{-2}(-iF^j_t + \mathrm{Re}(M^j)E^j + \mathrm{Re}(\mathcal{N})F^j),$$

together with (4.3) and (4.4),

$$\|\Delta F^j\|_{H^{k-2}} \lesssim \|F^j_t\|_{H^{k-4}} + \|M^j\|_{H^{k-2}}\|E^j\|_{H^{k-1}} + \|\mathcal{N}\|_{H^{k-2}}\|F^j\|_{H^{k-1}}.$$

Thus we derive

$$\|F^j\|_{H^k} \lesssim \|F^j_t\|_{H^{k-4}} + o(1).$$

Combining the the above estimate with (4.1), (4.6) and (4.7), we get

$$\sup_{t\in[-T,T]} \|(E^j - E, \mathcal{N}^j - \mathcal{N})\|_{V_k} \to 0 \quad \text{as } j \to \infty \text{ for } k \ge 3. \tag{4.8}$$

For $k = 2$, (4.7) does not hold, so that we apply the conserved quantity $\mathcal{H}$. Let us denote

$$A^j(t) := \|\varepsilon\Delta E^j\|^2_{L^2} + \frac{1}{2}\|\varepsilon|\nabla|\mathcal{N}^j\|^2_{L^2} \quad \text{and} \quad A(t) := \|\varepsilon\Delta E\|^2_{L^2} + \frac{1}{2}\|\varepsilon|\nabla|\mathcal{N}\|^2_{L^2}.$$

Compute

$$\begin{aligned} A^j(t) - A(t) = \mathcal{H}(E^j, \mathcal{N}^j) - \mathcal{H}(E, \mathcal{N}) - \|\nabla E^j\|^2_{L^2} + \|\nabla E\|^2_{L^2} - \frac{1}{2}\|\mathcal{N}^j\|^2_{L^2} + \frac{1}{2}\|\mathcal{N}\|^2_{L^2} \\ - \int \mathrm{Re}(\mathcal{N}^j)|E^j|^2\,dx + \int \mathrm{Re}(\mathcal{N})|E|^2\,dx. \end{aligned}$$

Invoking (4.3) and (4.4), we can estimate the inhomogeneous part,

(4.9)

$$\begin{aligned} &\sup_{t\in[-T,T]} \int \mathrm{Re}(\mathcal{N}^j)|E^j|^2 - \mathrm{Re}(\mathcal{N})|E|^2\,dx \\ &\lesssim \sup_{t\in[-T,T]} \left(\|M^j\|_{L^2}\|E^j\|^2_{L^4} + \|\mathrm{Re}(\mathcal{N})\|_{L^2}\|F^j\|_{L^2}\|E^j\|_{L^4} + \|\mathrm{Re}(\mathcal{N})\|_{L^4}\|E\|_{L^4}\|F^j\|_{L^2}\right) \end{aligned}$$

which tends to zero as $j \to \infty$. Thus we can combine (1.8), (4.1), (4.4) and (4.9) to obtain

$$\lim_{j\to\infty} \sup_{t\in[-T,T]} |A^j(t) - A(t)| = 0. \tag{4.10}$$

Invoking the equality

$$\|f-g\|_{L^2}^2 = \|f\|_{L^2}^2 - \|g\|_{L^2}^2 - 2\,\mathrm{Re}\int (f-g)\overline{g}\,dx,$$

we can rewrite the following quantity

$$\begin{aligned}
&\|\varepsilon\Delta E^j - \varepsilon\Delta E\|_{L^2}^2 + \frac{1}{2}\|\varepsilon|\nabla|\mathcal{N}^j - \varepsilon|\nabla|\mathcal{N}\|_{L^2}^2 \\
&= A^j(t) - A(t) - 2\varepsilon^2\,\mathrm{Re}\int \Delta(E^j - E)\Delta\overline{E}\,dx - \varepsilon^2\,\mathrm{Re}\int |\nabla|(\mathcal{N}^j - \mathcal{N})|\nabla|\overline{\mathcal{N}}\,dx.
\end{aligned}$$

Combining (4.5) and (4.10), we can get

$$\lim_{j\to\infty} \sup_{t\in[-T,T]} \left(\|\varepsilon\Delta(E^j - E)\|_{L^2}^2 + \frac{1}{2}\|\varepsilon|\nabla|(\mathcal{N}^j - \mathcal{N})\|_{L^2}^2 \right) = 0.$$

Therefore we have

$$\sup_{t\in[-T,T]} \|(E^j - E, \mathcal{N}^j - \mathcal{N})\|_{V_2} \to 0 \quad \text{as } j \to \infty. \tag{4.11}$$

Finally (4.8) and (4.11) imply that the solution to (1.1) depends continuously on the initial data. For $k > 2$, the argument is analogous to the previous case. This completes the proof. $\square$

Proof of Theorem 1.2. Denote the notations

$$q = \frac{8}{d}, \quad \widetilde{q} = \frac{16}{d}, \quad 1 = \frac{1}{q'} + \frac{1}{q}, \quad \alpha = \frac{1}{q'} - \frac{1}{q}, \quad \beta = \frac{1}{q'} - \frac{1}{\widetilde{q}} \quad \text{and} \quad I = [0,T].$$

Invoking the Strichartz estimates in Lemmas 3.1 and 3.3, for $k \geq 0$, we estimate the Schrödinger part E and the wave part $\mathcal{N}$ as follows. For $d = 1$, we have

$$\begin{aligned}
&\|\mathcal{P}_L\langle\nabla\rangle^s\mathcal{N}\|_{L^{q_1}(I;L^4(\mathbb{R}^d))} + \|\mathcal{P}_H\langle\nabla\rangle^s\mathcal{N}\|_{L^q(I;L^4(\mathbb{R}^d))} \\
&\lesssim \left\| \left\{\frac{|\nabla|}{\langle\nabla\rangle}\right\}^{1/q_1} \mathcal{P}_L\langle\nabla\rangle^s\mathcal{N} \right\|_{L^{q_1}(I;L^{\widetilde{r}}(\mathbb{R}^d))} + \left\| \left\{\frac{|\nabla|}{\langle\nabla\rangle}\right\}^{1/q} \mathcal{P}_H\langle\nabla\rangle^s\mathcal{N} \right\|_{L^q(I;L^4(\mathbb{R}^d))} \\
&\lesssim \|\mathcal{P}_L\langle\nabla\rangle^s\mathcal{N}_0\|_{L^2(\mathbb{R}^d)} + \left\| \left\{\frac{|\nabla|}{\langle\nabla\rangle}\right\}^{-1/q} \frac{|\nabla|}{\langle\nabla\rangle}\mathcal{P}_L\langle\nabla\rangle^s|E|^2 \right\|_{L^{q'}(I;L^{4'}(\mathbb{R}^d))} \\
&\quad + \|\mathcal{P}_H\langle\nabla\rangle^s\mathcal{N}_0\|_{L^2(\mathbb{R}^d)} + \left\| \left\{\frac{|\nabla|}{\langle\nabla\rangle}\right\}^{-1/q} \frac{|\nabla|}{\langle\nabla\rangle}\mathcal{P}_H\langle\nabla\rangle^s|E|^2 \right\|_{L^{q'}(I;L^{4'}(\mathbb{R}^d))} \\
&\lesssim \|\mathcal{N}_0\|_{H^s(\mathbb{R}^d)} + \|\langle\nabla\rangle^s E\|_{L^{q'}(I;L^4(\mathbb{R}^d))}\|E\|_{L^\infty(I;L^2(\mathbb{R}^d))} \\
&\lesssim \|\mathcal{N}_0\|_{H^s(\mathbb{R}^d)} + T^\beta\|\langle\nabla\rangle^s E\|_{L^{\widetilde{q}}(I;L^4(\mathbb{R}^d))}\|E\|_{L^\infty(I;L^2(\mathbb{R}^d))},
\end{aligned} \tag{4.12}$$

where $(q,4)$ and $(q_1,\widetilde{r})$ are S-admissible, and $1/4 = 1/\widetilde{r} - 1/q_1$. For $d = 2,3$, we have

$$\begin{aligned}
&\|\langle\nabla\rangle^s \mathcal{N}\|_{L^q(I;L^4(\mathbb{R}^d))} \\
&\lesssim \left\| \left\{ \frac{|\nabla|}{\langle\nabla\rangle} \right\}^{-(1-\frac{2}{d})\frac{1}{q}} \langle\nabla\rangle^s \mathcal{N} \right\|_{L^q(I;L^4(\mathbb{R}^d))} \\
&\lesssim \|\mathcal{P}_L \langle\nabla\rangle^s \mathcal{N}_0\|_{L^2(\mathbb{R}^d)} + \left\| \left\{ \frac{|\nabla|}{\langle\nabla\rangle} \right\}^{(1-\frac{2}{d})\frac{1}{q}} \frac{|\nabla|}{\langle\nabla\rangle} \langle\nabla\rangle^s |E|^2 \right\|_{L^{q'}(I;L^{4'}(\mathbb{R}^d))} \\
&\lesssim \|\mathcal{N}_0\|_{H^s(\mathbb{R}^d)} + \|\langle\nabla\rangle^s E\|_{L^{q'}(I;L^4(\mathbb{R}^d))} \|E\|_{L^\infty(I;L^2(\mathbb{R}^d))} \\
&\lesssim \|\mathcal{N}_0\|_{H^s(\mathbb{R}^d)} + T^\beta \|\langle\nabla\rangle^s E\|_{L^{\widetilde{q}}(I;L^4(\mathbb{R}^d))} \|E\|_{L^\infty(I;L^2(\mathbb{R}^d))}.
\end{aligned} \tag{4.13}$$

For $d = 1,2,3$, we can obtain

$$\|\langle\nabla\rangle^s \mathcal{N}\|_{L^\infty(I;L^2(\mathbb{R}^d))} \lesssim \|\mathcal{N}_0\|_{H^s(\mathbb{R}^d)} + T^\beta \|\langle\nabla\rangle^s E\|_{L^{\widetilde{q}}(I;L^4(\mathbb{R}^d))} \|E\|_{L^\infty(I;L^2(\mathbb{R}^d))}. \tag{4.14}$$

For $d = 1,2,3$ and $0 \le s \le 2/q$, we can obtain estimates for E:

$$\begin{aligned}
&\|\langle\nabla\rangle^s E\|_{L^\infty(I;L^2(\mathbb{R}^d))} + \|\langle\nabla\rangle^s E\|_{L^{\widetilde{q}}(I;L^4(\mathbb{R}^d))} \\
&\lesssim \|\langle\nabla\rangle^s E\|_{L^\infty(I;L^2(\mathbb{R}^d))} + \|\langle\nabla\rangle^{s+2/\widetilde{q}} E\|_{L^{\widetilde{q}}(I;L^{r_1}(\mathbb{R}^d))} \\
&\lesssim \|\langle\nabla\rangle^s E_0\|_{L^2(\mathbb{R}^d)} + \|\langle\nabla\rangle^{s-2/q} (\mathcal{N}E)\|_{L^{q'}(I;L^{4'}(\mathbb{R}^d))} \\
&\lesssim \|E_0\|_{H^s} + \|\mathcal{N}E\|_{L^{q'}(I;L^{4'}(\mathbb{R}^d))} \\
&\lesssim \|E_0\|_{H^s} + \|\mathcal{N}\|_{L^{q'}(I;L^4(\mathbb{R}^d))} \|E\|_{L^\infty(I;L^2(\mathbb{R}^d))} \\
&\lesssim \|E_0\|_{H^s} + T^\alpha \|\mathcal{N}\|_{L^q(I;L^4(\mathbb{R}^d))} \|E\|_{L^\infty(I;L^2(\mathbb{R}^d))},
\end{aligned} \tag{4.15}$$

where $(\widetilde{q},4)$ is B-admissible and $(\widetilde{q},r_1)$ is S-admissible. For $2/q \le s \le 4/q$, we can obtain estimates

$$\begin{aligned}
&\|\langle\nabla\rangle^s E\|_{L^\infty(I;L^2(\mathbb{R}^d))} + \|\langle\nabla\rangle^s E\|_{L^{\widetilde{q}}(I;L^4(\mathbb{R}^d))} \\
&\lesssim \|\langle\nabla\rangle^s E\|_{L^\infty(I;L^2(\mathbb{R}^d))} + \|\langle\nabla\rangle^{s+2/\widetilde{q}} E\|_{L^{\widetilde{q}}(I;L^{r_1}(\mathbb{R}^d))} \\
&\lesssim \|\langle\nabla\rangle^s E_0\|_{L^2(\mathbb{R}^d)} + \|\langle\nabla\rangle^{s-2/q} (\mathcal{N}E)\|_{L^{q'}(I;L^{4'}(\mathbb{R}^d))} \\
&\lesssim \|E_0\|_{H^s} + \|\langle\nabla\rangle^{s-2/q} \mathcal{N} E\|_{L^{q'}(I;L^{4'})} + \|\mathcal{N} \langle\nabla\rangle^{s-2/q} E\|_{L^{q'}(I;L^{4'})} \\
&\lesssim \|E_0\|_{H^s} + \|\langle\nabla\rangle^{s-2/q} \mathcal{N}\|_{L^{q'}(I;L^4)} \|E\|_{L^\infty(I;L^2)} + \|\mathcal{N}\|_{L^{q'}(I;L^4)} \|\langle\nabla\rangle^{s-2/q} E\|_{L^\infty(I;L^2)} \\
&\lesssim \|E_0\|_{H^s} + T^\alpha \left(\|E\|_{L^\infty(I;L^2)} + \|\langle\nabla\rangle^{s-2/q} E\|_{L^\infty(I;L^2)} \right) \|\langle\nabla\rangle^{s-2/q} \mathcal{N}\|_{L^q(I;L^4)}.
\end{aligned}$$

For $2(k-1)/q \le s \le 2k/q$, we can inductively obtain estimates

$$\begin{aligned}
&\|\langle\nabla\rangle^s E\|_{L^\infty(I;L^2(\mathbb{R}^d))} + \|\langle\nabla\rangle^s E\|_{L^{\widetilde{q}}(I;L^4(\mathbb{R}^d))} \\
&\lesssim \|E_0\|_{H^s} + T^\alpha \left(\|E\|_{L^\infty(I;L^2)} + \|\langle\nabla\rangle^{s-2/q} E\|_{L^\infty(I;L^2)} \right) \|\langle\nabla\rangle^{s-2/q} \mathcal{N}\|_{L^q(I;L^4)}.
\end{aligned} \tag{4.16}$$

To make the iteration arguments work, we denote the following norms

$$\|\mathcal{N}\|_{W(\mathbb{R}^d)} := \begin{cases} \|\langle\nabla\rangle^s\mathcal{N}\|_{L^\infty(I;L^2)} + \|\mathcal{P}_L\langle\nabla\rangle^s\mathcal{N}\|_{L^{q_1}(I;L^4)} + \|\mathcal{P}_H\langle\nabla\rangle^s\mathcal{N}\|_{L^q(I;L^4)} & \text{if } d=1, \\ \|\langle\nabla\rangle^s\mathcal{N}\|_{L^\infty(I;L^2)} + \|\langle\nabla\rangle^s\mathcal{N}\|_{L^q(I;L^4)} & \text{if } d=2,3, \end{cases}$$

$$\|E\|_{S(\mathbb{R}^d)} := \|\langle\nabla\rangle^s E\|_{L^\infty(I;L^2(\mathbb{R}^d))} + \|\langle\nabla\rangle^s E\|_{L^{\tilde{q}}(I;L^4(\mathbb{R}^d))}.$$

For $0 \le s \le 2/q$, we combine (4.12), (4.13), (4.14) and (4.15) to obtain

$$\|\mathcal{N}\|_{W(\mathbb{R}^d)} \le 2\|\mathcal{N}_0\|_{H^s} + CT^\beta\|E\|_{L^\infty(I;L^2)}\|E\|_{S(\mathbb{R}^d)}$$

and

$$\|E\|_{S(\mathbb{R}^d)} \le 2\|E_0\|_{H^s} + CT^\alpha\|E\|_{L^\infty(I;L^2)}\|\mathcal{N}\|_{W(\mathbb{R}^d)}.$$

Thus we can show that the iteration argument works for T sufficiently small. Due to the conservation law (1.3), we can choose the same size of time interval T for each step of iteration which ensures that the solution exists globally in time.

We proceed the proof by induction on s. Assuming that $2(j-1)/q \le s \le 2j/q$ for $j = 1, 2, \ldots, k$, there is a unique global solution $(E, \mathcal{N})$ to (1.6) with initial data in W_s. For $2k/q \le s \le 2(k+1)/q$, we combine (4.12), (4.13), (4.14) and (4.16) to obtain

$$\|\mathcal{N}\|_{W(\mathbb{R}^d)} \le 2\|\mathcal{N}_0\|_{H^s} + CT^\beta\|E\|_{L^\infty(I;L^2)}\|E\|_{S(\mathbb{R}^d)}$$

and

$$\|E\|_{S(\mathbb{R}^d)} \le 2\|E_0\|_{H^s} + CT^\alpha\left(\|E\|_{L^\infty(I;L^2)} + \|\langle\nabla\rangle^{s-2/q}E\|_{L^\infty(I;L^2)}\right)\|\mathcal{N}\|_{W(\mathbb{R}^d)}.$$

Since $2(k-1)/q \le s-2/q \le 2k/q$ such that $\|\langle\nabla\rangle^{s-2/q}E\|_{L^\infty([0,T];L^2)}$ is finite for any finite T, we can then perform an iteration argument at any T which is finite. Thus we have solution $(E, \mathcal{N}) \in C([0,T]; W_s(\mathbb{R}^d))$ for any finite T with initial data $(E_0, \mathcal{N}_0) \in W_s(\mathbb{R}^d)$. □

5. The limit behavior of the QZ system

Now we prove Theorem 1.3.

Proof of Theorem 1.3. From the works of [7, 21], we know that $(E^0, \mathcal{N}^0) \in C(\mathbb{R}, V_k)$. Denote the difference between the quantum solution and the classical solution by

$$E = E^\varepsilon - E^0 \quad \text{and} \quad \mathcal{N} = \mathcal{N}^\varepsilon - \mathcal{N}^0.$$

$(E, \mathcal{N})$ satisfies the system

$$
\begin{aligned}
& iE_t + \Delta\langle\varepsilon\nabla\rangle^2 E = \varepsilon^2\Delta^2 E^0 + \mathrm{Re}(\mathcal{N})E^\varepsilon + \mathrm{Re}(\mathcal{N}^0)E, \\
& i\mathcal{N}_t - |\nabla|\langle\varepsilon\nabla\rangle\mathcal{N} = |\nabla|(\langle\varepsilon\nabla\rangle - I)\mathcal{N}^0 + |\nabla|\langle\varepsilon\nabla\rangle^{-1}[E\overline{E}^\varepsilon + E^0\overline{E}] \\
& \qquad\qquad + |\nabla|(\langle\varepsilon\nabla\rangle^{-1} - I)|E^0|^2.
\end{aligned}
\tag{5.1}
$$

First we take the inner product of the first equation of (5.1) with E, and then take the imaginary part of it. Thus we can obtain

$$
\frac{1}{2}\frac{d}{dt}\|E\|_{L^2}^2 = \mathrm{Im}\int \varepsilon^2\Delta^2 E^0\overline{E} + \mathrm{Re}(\mathcal{N})E^\varepsilon\,dx \lesssim \varepsilon^4 + \|E\|_{L^2}^2 + \|\mathcal{N}\|_{L^2}^2.
$$

Next we take the inner product of the second equation of (5.1) with $\langle\varepsilon\nabla\rangle^2\mathcal{N}$, and then take the imaginary part of it. Thus we can derive

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|\langle\varepsilon\nabla\rangle\mathcal{N}\|_{L^2}^2 \\
& = \mathrm{Im}\int \left(|\nabla|(\langle\varepsilon\nabla\rangle - I)\mathcal{N}^0 + |\nabla|\langle\varepsilon\nabla\rangle^{-1}[E\overline{E}^\varepsilon + E^0\overline{E}] + |\nabla|(\langle\varepsilon\nabla\rangle^{-1} - I)|E^0|^2\right)\langle\varepsilon\nabla\rangle^2\overline{\mathcal{N}}\,dx \\
& \lesssim \varepsilon^4 + \|E\|_{H^1}^2 + \|\langle\varepsilon\nabla\rangle\mathcal{N}\|_{L^2}^2.
\end{aligned}
$$

Combining the above, we have

$$
\frac{d}{dt}(\|E\|_{L^2}^2 + \|\langle\varepsilon\nabla\rangle\mathcal{N}\|_{L^2}^2) \lesssim \varepsilon^4 + \|E\|_{H^1}^2 + \|\langle\varepsilon\nabla\rangle\mathcal{N}\|_{L^2}^2.
\tag{5.2}
$$

To derive the bounds of higher order derivative of E and $\mathcal{N}$, we invoke (3.8) and Young's inequality to get

$$
\begin{aligned}
\|E\|_{H^{k-6}} & \le C\|E\|_{L^2}^{2/(k-4)}\|E\|_{H^{k-4}}^{(k-6)/(k-4)} \\
& \le C\|E\|_{L^2} + \frac{1}{2}\|E\|_{H^{k-4}} \le C\|E\|_{L^2} + \frac{1}{2}\|\Delta E\|_{H^{k-6}}.
\end{aligned}
\tag{5.3}
$$

Now we take the H^{k-6}-norm of the first equation of (5.1) and then we invoke (5.3) and Propositions 3.7 and 3.8 to get

$$
\begin{aligned}
\|\Delta E\|_{H^{k-6}} & = \|-iE_t + \varepsilon^2\Delta^2 E^\varepsilon + \mathrm{Re}(\mathcal{N})E^\varepsilon + \mathrm{Re}(\mathcal{N}^0)E\|_{H^{k-6}} \\
& \le C\varepsilon^2 + C\|E_t\|_{H^{k-6}} + C\|\mathrm{Re}(\mathcal{N})\|_{H^{k-6}} + C\|E\|_{H^{k-6}} \\
& \le C\varepsilon^2 + C\|E_t\|_{H^{k-6}} + C\|\mathrm{Re}(\mathcal{N})\|_{H^{k-6}} + C\|E\|_{L^2} + \frac{1}{2}\|\Delta E\|_{H^{k-6}}.
\end{aligned}
$$

Thus we can obtain

$$
\|\Delta E\|_{H^{k-6}} \lesssim \varepsilon^2 + \|E_t\|_{H^{k-6}} + \|\mathrm{Re}(\mathcal{N})\|_{H^{k-6}} + \|E\|_{L^2}.
$$

Also we have

$$\begin{aligned}\|\mathcal{N}\|_{H^{k-5}} &\sim \|\mathcal{N}\|_{L^2} + \||\nabla|\mathcal{N}\|_{H^{k-6}} \\ &\lesssim \|\mathcal{N}\|_{L^2} + \|\varepsilon|\nabla|\mathcal{N}\|_{L^2} + \||\nabla|\mathcal{N}\|_{H^{k-6}} + \|\varepsilon\Delta\mathcal{N}\|_{H^{k-6}}.\end{aligned}$$

Hence we have

$$\begin{aligned}\|E\|_{H^{k-4}} &\lesssim \varepsilon^2 + \|E_t\|_{H^{k-6}} + \|\operatorname{Re}(\mathcal{N})\|_{H^{k-6}} + \|E\|_{L^2}, \\ \|\mathcal{N}\|_{H^{k-5}} &\lesssim \|\langle\varepsilon\nabla\rangle\mathcal{N}\|_{L^2} + \||\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}\|_{H^{k-6}}.\end{aligned} \tag{5.4}$$

Here we have an alternative for deriving (5.4). We can take the H^{k-6}-norm of the first equation of (5.1) in a slightly different way and then we invoke (5.3) to obtain

$$\begin{aligned}\|\Delta E\|_{H^{k-6}} &= \|\langle\varepsilon\nabla\rangle^{-2}(-iE_t - \varepsilon^2\Delta^2E^0 + \operatorname{Re}(\mathcal{N})E^\varepsilon + \operatorname{Re}(\mathcal{N}^0)E)\|_{H^{k-6}} \\ &\leq C\varepsilon^2 + C\|E_t\|_{H^{k-6}} + C\|\operatorname{Re}(\mathcal{N})\|_{H^{k-6}} + C\|E\|_{H^{k-6}} \\ &\leq C\varepsilon^2 + C\|E_t\|_{H^{k-6}} + C\|\operatorname{Re}(\mathcal{N})\|_{H^{k-6}} + C\|E\|_{L^2} + \frac{1}{2}\|\Delta E\|_{H^{k-6}}.\end{aligned}$$

The computation only requires the uniform boundedness of E^ε in a norm with lower regularity.

Now we take the time derivative of the first equation of (5.1) and we get

$$iE_{tt} + \Delta\langle\varepsilon\nabla\rangle^2E_t = \varepsilon^2\Delta^2E_t^0 + \operatorname{Re}(\mathcal{N}_t)E^\varepsilon + \operatorname{Re}(\mathcal{N})E_t^\varepsilon + \operatorname{Re}(\mathcal{N}_t^0)E + \operatorname{Re}(\mathcal{N}^0)E_t. \tag{5.5}$$

Invoking (1.5), we have

$$\operatorname{Re}(\mathcal{N}_t^0) = \operatorname{Im}(|\nabla|\mathcal{N}^0) \quad \text{and} \quad \operatorname{Re}(\mathcal{N}_t) = \operatorname{Im}(|\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}^\varepsilon - |\nabla|\mathcal{N}^0)$$

which enables us to transfer the time derivative to space derivative. Now we take the inner product of (5.5) with $\langle\nabla\rangle^{2k-12}E_t$ and then we take the imaginary part of it. Invoking (5.4), we can get

$$\frac{d}{dt}\|E_t\|^2_{H^{k-6}} \lesssim \varepsilon^4 + \|E\|^2_{L^2} + \|E_t\|^2_{H^{k-6}} + \|\operatorname{Im}(|\nabla|\langle\varepsilon\nabla\rangle\mathcal{N})\|^2_{H^{k-6}} + \|\operatorname{Re}(\mathcal{N})\|^2_{H^{k-6}}. \tag{5.6}$$

To derive the bound of higher order derivative of $\mathcal{N}$, analogously we take the inner product of the second equation of (5.1) with $\langle\nabla\rangle^{2k-12}|\nabla|^2\langle\varepsilon\nabla\rangle^2\mathcal{N}$ and then we take the imaginary part of it. Thus we can get

$$\begin{aligned}\frac{d}{dt}\||\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}\|^2_{H^{k-6}} &\lesssim \varepsilon^4 + \|E\|^2_{L^2} + \|E_t\|^2_{H^{k-6}} \\ &\quad + \|\operatorname{Im}(|\nabla|\langle\varepsilon\nabla\rangle\mathcal{N})\|^2_{H^{k-6}} + \|\operatorname{Re}(\mathcal{N})\|^2_{H^{k-6}}.\end{aligned} \tag{5.7}$$

Combining (5.2), (5.6) and (5.7), we get

$$\begin{aligned}&\frac{d}{dt}\left(\|E\|^2_{L^2} + \|E_t\|^2_{H^{k-6}} + \|\langle\varepsilon\nabla\rangle\mathcal{N}\|^2_{L^2} + \||\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}\|^2_{H^{k-6}}\right) \\ &\lesssim \varepsilon^4 + \|E\|^2_{L^2} + \|E_t\|^2_{H^{k-6}} + \|\langle\varepsilon\nabla\rangle\mathcal{N}\|^2_{L^2} + \||\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}\|^2_{H^{k-6}}.\end{aligned} \tag{5.8}$$

Notice that

$$\|\mathcal{N}\|_{H^{k-5}} \lesssim \|\langle\varepsilon\nabla\rangle\mathcal{N}\|_{L^2} + \||\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}\|_{H^{k-6}}.$$

Let us denote

$$\Psi(t) = \|E(t)\|_{L^2}^2 + \|E_t(t)\|_{H^{k-6}}^2 + \|\langle\varepsilon\nabla\rangle\mathcal{N}(t)\|_{L^2}^2 + \||\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}(t)\|_{H^{k-6}}^2. \tag{5.9}$$

Integrate (5.8) over the time interval $[0,t]$ and invoke the Gronwall's inequality, we have

$$\Psi(t) \lesssim \varepsilon^4 + \Psi(0) \quad \text{for } t \in [0,T],$$

where T is given in Propositions 3.7 and 3.8.

Now we analyze the quantity $\Psi(0)$. Invoking (5.9), we get

$$\Psi(0) = \|E_0\|_{L^2}^2 + \|E_t(0)\|_{H^{k-6}}^2 + \|\langle\varepsilon\nabla\rangle\mathcal{N}_0\|_{L^2}^2 + \||\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}_0\|_{H^{k-6}}^2.$$

First we estimate the term $\|E_t(0)\|_{H^{k-6}}$. Invoking the first equation of (5.1), we can derive

$$\begin{aligned}\|E_t(0)\|_{H^{k-6}} &\lesssim \|\Delta E_0\|_{H^{k-6}} + \|\varepsilon^2\Delta^2 E_0^\varepsilon\|_{H^{k-6}} + \|\operatorname{Re}(\mathcal{N})E^\varepsilon\|_{H^{k-6}} + \|\operatorname{Re}(\mathcal{N}^0)E\|_{H^{k-6}}\\ &\lesssim \|E_0\|_{H^{k-4}} + \varepsilon^2\|E_0^\varepsilon\|_{H^{k-2}} + \|\operatorname{Re}(\mathcal{N}_0)E_0^\varepsilon\|_{H^{k-6}} + \|\operatorname{Re}(\mathcal{N}_0^0)E_0\|_{H^{k-6}}.\end{aligned}$$

Invoking (1.9), we have $\|E_0^\varepsilon\|_{H^{k-2}} \le C$. Next we have

$$\|\operatorname{Re}(\mathcal{N}_0)E_0^\varepsilon\|_{H^{k-6}} \lesssim \begin{cases}\|\mathcal{N}_0\|_{L^2}\|E_0^\varepsilon\|_{H^2} & \text{for } k=6,\\ \|\mathcal{N}_0\|_{H^1}\|E_0^\varepsilon\|_{H^2} & \text{for } k=7,\\ \|\mathcal{N}_0\|_{H^{k-6}}\|E_0^\varepsilon\|_{H^{k-6}} & \text{for } k\ge 8.\end{cases}$$

Analogously we can get

$$\|\operatorname{Re}(\mathcal{N}_0^0)E_0\|_{H^{k-6}} \lesssim \begin{cases}\|\mathcal{N}_0^0\|_{L^2}\|E_0\|_{H^2} & \text{for } k=6,\\ \|\mathcal{N}_0^0\|_{H^1}\|E_0\|_{H^2} & \text{for } k=7,\\ \|\mathcal{N}_0^0\|_{H^{k-6}}\|E_0\|_{H^{k-6}} & \text{for } k\ge 8.\end{cases}$$

For the last two terms in the expression of $\Psi(0)$, we can get

$$\|\langle\varepsilon\nabla\rangle\mathcal{N}_0\|_{L^2} \lesssim \|\varepsilon|\nabla|\mathcal{N}_0\|_{L^2} + \|\mathcal{N}_0\|_{L^2}$$

and

$$\||\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}_0\|_{H^{k-6}} \lesssim \|\varepsilon\Delta\mathcal{N}_0\|_{H^{k-6}} + \||\nabla|\mathcal{N}_0\|_{H^{k-6}}.$$

Now we can estimate $\Psi(0)$ as follows:

$$\begin{aligned}\Psi(0) &\lesssim \|E_0\|_{H^{k-4}}^2 + \varepsilon^4 + \|\mathcal{N}_0\|_{H^{k-6}}^2 + \|\varepsilon\Delta\mathcal{N}_0\|_{H^{k-6}}^2 + \||\nabla|\mathcal{N}_0\|_{H^{k-6}}^2 + \|\varepsilon|\nabla|\mathcal{N}_0\|_{L^2}^2\\ &\lesssim \|E_0\|_{H^{k-4}}^2 + \varepsilon^4 + \|\mathcal{N}_0\|_{H^{k-5}}^2 + \|\varepsilon\Delta\mathcal{N}_0\|_{H^{k-6}}^2 + \|\varepsilon|\nabla|\mathcal{N}_0\|_{L^2}^2.\end{aligned} \tag{5.10}$$

Therefore we can compute

$$\begin{aligned}\|(E,\mathcal{N})(t)\|_{V_{k-4}} &\sim \|E(t)\|_{H^{k-4}} + \|\mathcal{N}(t)\|_{H^{k-5}}\\ &\lesssim \varepsilon^2 + \|E_t\|_{H^{k-6}} + \|\operatorname{Re}(\mathcal{N})\|_{H^{k-6}} + \|E\|_{L^2}\\ &\quad + \|\langle\varepsilon\nabla\rangle\mathcal{N}\|_{L^2} + \||\nabla|\langle\varepsilon\nabla\rangle\mathcal{N}\|_{H^{k-6}}\\ &\lesssim \varepsilon^2 + \Psi^{1/2}(t)\\ &\lesssim \varepsilon^2 + \Psi^{1/2}(0)\\ &\lesssim \|E_0\|_{H^{k-4}} + \varepsilon^2 + \|\mathcal{N}_0\|_{H^{k-5}} + \|\varepsilon\Delta\mathcal{N}_0\|_{H^{k-6}} + \|\varepsilon|\nabla|\mathcal{N}_0\|_{L^2}\\ &\lesssim \|(E_0,\mathcal{N}_0)\|_{V_{k-4}} + \|\varepsilon|\nabla|\mathcal{N}_0\|_{L^2} + \varepsilon^2\end{aligned}$$

which implies the result of (1.10). Interpolate between (1.10) with (3.15), we can obtain (1.11). For the derivation of (1.12), the readers are referred to the proof of Theorem 1.3 in [11]. □

Remark 5.1. Invoking (1.5), the formula (5.10) can be rewritten as follows:

$$\Psi(0) \lesssim \|E_0\|^2_{H^{k-4}} + \varepsilon^4 + \|n_0\|^2_{H^{k-5}} + \|n_1\|^2_{H^{k-6}\cap\dot{H}^{-1}} + \|\varepsilon\Delta n_0\|^2_{H^{k-6}}.$$

Now we prove Theorem 1.4.

Proof of Theorem 1.4. Since

$$\|(E^\varepsilon - E^0, \mathcal{N}^\varepsilon - \mathcal{N}^0)\|_{W_{k-5}} \le \|(E^\varepsilon - E^0, \mathcal{N}^\varepsilon - \mathcal{N}^0)\|_{V_{k-4}},$$

we can apply (1.10) to obtain (1.13)–(1.15). The argument is analogous to that of Theorem 1.3 so that we skip the details. □

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