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Arbitrary High-order EQUIP Methods for Stochastic Canonical Hamiltonian Systems

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Abstract. This paper is concerned with arbitrary high-order energy-preserving numerical methods for stochastic canonical Hamiltonian systems. Energy and quadratic invariants-preserving (EQUIP) methods for deterministic Hamiltonian systems are applied to stochastic canonical Hamiltonian systems and analyzed accordingly. A class of stochastic parametric Runge-Kutta methods with a truncation technique of random variables are obtained. Increments of Wiener processes are replaced by some truncated random variables. We prove the replacement doesn't change the convergence order under some conditions. The methods turn out to be symplectic for any given parameter. It is shown that there exists a parameter α_n^* at each step such that the energy-preserving property holds, and the energy-preserving methods retain the order of the underlying stochastic Gauss Runge-Kutta methods. Numerical results illustrate the effectiveness of EQUIP methods when applied to stochastic canonical Hamiltonian systems.

1. Introduction

Stochastic differential equations (SDEs) have been widely used in modelling many physical and social systems [24]. Since most SDEs cannot be solved explicitly, it is important to design numerical methods for solving them. Many efforts have been made to develop effective numerical methods for SDEs (see [9,20,21,25,33] and so on). Generally, it is natural to look forward to numerical methods which preserve as much as possible the intrinsic properties of the original system. Those numerical methods are called structure-preserving methods. Many numerical experiments have demonstrated the significant superiority of structure-preserving numerical methods in comparison with general-purpose ones in long-time numerical simulations.

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Symplecticity is an intrinsic property of Hamiltonian systems. Numerical methods that can preserve the symplectic structure are called symplectic methods. Symplectic methods for deterministic Hamiltonian systems have been studied by many scholars (see [15,17,30,32] and references therein), and symplectic methods for stochastic Hamiltonian systems also have sprung up in recent years [2, 10, 22, 23, 26, 27, 29, 31]. Milstein proves the flow of stochastic Hamiltonian systems can preserve the symplectic structure, and proposes symplectic methods for stochastic Hamiltonian systems with multiplicative noise and stochastic Hamiltonian systems with additive noise in [26,27]. Symplectic conditions of stochastic Runge-Kutta methods and stochastic partitioned Runge-Kutta methods for stochastic Hamiltonian systems are studied in [22,23,36]. Symplectic methods based on stochastic generating function are constructed in [1,31]. Stochastic variational integrators for preserving symplectic structure are proposed in [2]. Symplectic schemes for stochastic Hamiltonian dynamical systems through composition methods are proposed in [29].

Energy conservation is also an important intrinsic property of some systems. Great efforts have been made to look for numerical methods that can inherit the property. Many energy-preserving numerical methods for deterministic problems have been studied, for example, [3–8,12,16] and references therein. However, less research has been done for stochastic cases. Examples of energy-preserving methods for stochastic problems are found in literatures such as the conservative stochastic difference method [28], discrete gradient method [19], projection method [35] and stochastic average vector field method [13].

In this paper, we consider the stochastic canonical Hamiltonian systems in the sense of Stratonovich of the form [28]

$$dy = J^{-1}\nabla H(y)(dt + \lambda \circ dW_t), \quad y(0) = y_0, \tag{1.1}$$

where y is an m-dimensional ($m = 2l$) column vector, $\lambda \in R$ is a constant factor and W_t is a one-dimensional standard Wiener process, defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_t\}_{t\geq 0})$ which fulfills the usual conditions. y_0 is an $\mathcal{F}_0$-measurable random variable with $E\|y_0\|^2 < \infty$, where $\|\cdot\|$ denotes the Euclidean norm. The sufficiently smooth real-valued function $H(y)$ is the Hamiltonian which is usually called energy function. $J = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}$ is an m-dimensional skew-symmetric matrix with I_l denoting an identity matrix. The system (1.1) arises frequently in applications that usually describes a Hamiltonian motion perturbed by a multiplicative white noise.

All results in the present work can be extended to the following stochastic canonical Hamiltonian systems with multiple Wiener processes

$$dy = J^{-1}\nabla H(y)\left(dt + \sum_{i=1}^{r} \lambda_i \circ dW_t^i\right), \quad y(0) = y_0,$$

because this case can be reduced to (1.1) by introducing $\lambda := \sqrt{\sum_{i=1}^{r} \lambda_i^2}$ and $W_t := \frac{1}{\lambda} \sum_{i=1}^{r} \lambda_i W_t^i$ according to [14].

System (1.1) has properties of both symplecticity and energy conservation. It is natural for us to search for numerical methods that inherit the properties. Unfortunately, the symplectic schemes do not preserve the energy function $H(y)$ generally [34]. Brugnano et al. [7] propose a new family of symplectic methods depending on a parameter α for deterministic Hamiltonian systems, and show the parameter α may be properly tuned at each step so as to guarantee energy conservation. The methods are called energy and quadratic invariants-preserving (EQUIP) methods in [7]. The present paper mainly applies EQUIP methods along with a truncation technique of Wiener increments to the class of stochastic problems (1.1) and analyzes the symplecticity, energy conservation and mean-square convergence order.

The paper is organized as follows. In Section 2, we review a family of parametric Gauss Runge-Kutta (RK) methods for ordinary differential equations (ODEs). In Section 3, the stochastic parametric Gauss RK methods with truncated random variables are applied to (1.1). We prove the methods are symplectic and discuss the mean-square convergence order. Then we prove there exists a parameter at each step such that the energy conservation is satisfied. In Section 4, some numerical experiments are presented to verify the theoretical results and show the effectiveness of EQUIP methods when applied to stochastic canonical Hamiltonian systems.

2. The parametric Gauss RK methods

Brugnano et al. [7] introduce a family of parametric Gauss RK methods for solving deterministic Hamiltonian systems, and based on the parametric Gauss RK methods they propose EQUIP methods. This section is mainly a review of the parametric Gauss RK methods. Consider the deterministic autonomous Hamiltonian system

$$\dot{z} = J^{-1} \nabla H(z), \quad z(0) = z_0 \in \mathbb{R}^m. \tag{2.1}$$

Recall the shifted and normalized Legendre polynomials [18]

$$P_k(x) = \frac{\sqrt{2k+1}}{k!} \frac{d^k}{dx^k} (x^k (x-1)^k), \quad k = 1, 2, \ldots.$$

These polynomials satisfy the following formulas

$$\int_0^1 P_k(x) P_l(x)\, dx = \delta_{kl}, \quad \int_0^x P_0(t)\, dt = \xi_1 P_1(x) + \frac{1}{2} P_0(x),$$
$$\int_0^x P_k(t)\, dt = \xi_{k+1} P_{k+1}(x) - \xi_k P_{k-1}(x), \quad k = 1, 2, \ldots,$$

where δ_{kl} is the Kronecker symbol and $\xi_k = 1/(2\sqrt{4k^2-1})$.

Let $W = (w_{ij})_{s\times s}$ be the matrix defined by

$$w_{ij} = P_{j-1}(c_i), \quad i = 1,\ldots,s,\ j = 1,\ldots,s,$$

and (c, A, b) be the coefficients of Gauss RK method of order $2s$, with c_i the distinct abscissae and b_i the weights. Then the W-transformation for the Gauss RK method with coefficients (c, A, b) is

$$X_G = W^{-1}AW = \begin{pmatrix} \frac{1}{2} & -\xi_1 & 0 & & & \\ \xi_1 & 0 & -\xi_2 & & & \\ & \xi_2 & 0 & -\xi_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \xi_{s-2} & 0 & -\xi_{s-1} \\ & & & & \xi_{s-1} & 0 \end{pmatrix}.$$

Based on X_G, we consider the following perturbed matrix

$$\begin{aligned} \widetilde{X}_G &= \begin{pmatrix} \frac{1}{2} & -\xi_1-\alpha_1 & 0 & & & \\ \xi_1+\alpha_1 & 0 & -\xi_2-\alpha_2 & & & \\ & \xi_2+\alpha_2 & 0 & -\xi_3-\alpha_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \xi_{s-2}+\alpha_{s-2} & 0 & -\xi_{s-1}-\alpha_{s-1} \\ & & & & \xi_{s-1}+\alpha_{s-1} & 0 \end{pmatrix} \\ &= X_G + \begin{pmatrix} 0 & -\alpha_1 & 0 & & & \\ \alpha_1 & 0 & -\alpha_2 & & & \\ & \alpha_2 & 0 & -\alpha_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \alpha_{s-2} & 0 & -\alpha_{s-1} \\ & & & & \alpha_{s-1} & 0 \end{pmatrix} \\ &= X_G + X_{\mathrm{per}}. \end{aligned}$$

Here $\alpha_1, \alpha_2, \ldots, \alpha_{s-1}$ are $s-1$ real parameters. Applying the W-transformation and $\widetilde{X}_G$, we define a family of parametric Gauss RK methods for solving (2.1), denoted by the Butcher tableau [11]

$$\begin{array}{c|c} & \widetilde{A} \\ \hline & b^T \end{array} \tag{2.2}$$

where

$$\widetilde{A} = W\widetilde{X}_G W^{-1} = (\widetilde{a}_{ij})_{s\times s}. \tag{2.3}$$

Lemma 2.1. *The family of parametric Gauss RK methods* (2.2) *for solving Hamiltonian system* (2.1) *are symplectic.*

Proof. Let

$$b = (b_1, b_2, \ldots, b_s)^T, \quad B = \mathrm{diag}(b_1, b_2, \ldots, b_s).$$

According to [17], the family of RK methods denoted by (2.2) are symplectic if the following condition

$$B\widetilde{A} + \widetilde{A}^T B = bb^T \tag{2.4}$$

holds. By a straightforward computation we have

$$\begin{aligned}\widetilde{A} &= W\widetilde{X}_G W^{-1} = W(X_G + X_{\mathrm{per}})W^{-1}\\ &= WX_G W^{-1} + WX_{\mathrm{per}}W^{-1} = A + WX_{\mathrm{per}}W^{-1}.\end{aligned}$$

Then we get

$$\begin{aligned}B\widetilde{A} + \widetilde{A}^T B &= B(A + WX_{\mathrm{per}}W^{-1}) + (A + WX_{\mathrm{per}}W^{-1})^T B\\ &= BA + A^T B + BWX_{\mathrm{per}}W^{-1} + (W^{-1})^T X_{\mathrm{per}}^T W^T B.\end{aligned} \tag{2.5}$$

Since the method with coefficients (c, A, b) is Gauss RK method, so it satisfies the symplectic condition, i.e., $BA + A^T B = bb^T$, meanwhile, it satisfies $W^T BW = I$ [18]. Therefore, (2.5) is equal to

$$bb^T + BW(X_{\mathrm{per}} + X_{\mathrm{per}}^T)W^{-1} = bb^T,$$

where $X_{\mathrm{per}} + X_{\mathrm{per}}^T$ is equal to 0 because X_{per} is skew-symmetric. Hence (2.4) holds and the proof is completed. □

3. The stochastic parametric Gauss RK methods with truncated random variables

In this section, we apply EQUIP methods with a truncation technique of random variables to the stochastic canonical Hamiltonian system (1.1). Considering the relation between the drift term and diffusion term in (1.1), a family of stochastic parametric Gauss RK methods is quite naturally denoted by a tableau as

$$\begin{array}{c|cc} & \widetilde{A} & \widetilde{A} \\ \hline & b^T & b^T \end{array} \tag{3.1}$$

where $\widetilde{A}$ is defined in (2.3) and contains $s-1$ parameters $\alpha_1, \ldots, \alpha_{s-1}$. Now we present the following iterated scheme with truncated random variables of (3.1) by

$$
\begin{aligned}
Y_i &= y_n + (h + \lambda \Delta\widehat{W}(h)) \sum_{j=1}^{s} \widetilde{a}_{ij} J^{-1} \nabla H(Y_j), \quad i = 1, \ldots, s, \\
y_{n+1} &= y_n + (h + \lambda \Delta\widehat{W}(h)) \sum_{i=1}^{s} b_i J^{-1} \nabla H(Y_i),
\end{aligned} \tag{3.2}
$$

where $h > 0$ is the step size. Usually the Wiener increment $\Delta W(h) = W_{t_{n+1}} - W_{t_n} = \xi\sqrt{h}$ is used in common numerical methods, where ξ is an $\mathcal{N}(0,1)$-distributed random variable, but here we employ a truncated random variable $\Delta\widehat{W}(h) = \zeta\sqrt{h}$, with

$$
\zeta = \zeta_h = \begin{cases} \xi & \text{if } |\xi| \le A_h, \\ A_h & \text{if } \xi > A_h, \\ -A_h & \text{if } \xi < -A_h, \end{cases}
$$

where $A_h := \sqrt{2k|\ln h|}$, and k is a positive integer. The truncation guarantees $\Delta\widehat{W}(h) = \zeta\sqrt{h}$ is bounded, for details see [27]. Now we proceed to the symplecticity results.

Theorem 3.1. *The family of stochastic parametric Gauss RK methods defined by* (3.2) *for solving* (1.1) *are symplectic.*

Proof. According to [23], a general stochastic RK method with a standard Wiener increment $\Delta W(h)$ denoted by

$$
\begin{array}{c|cc} & A & \overline{A} \\ \hline & b^T & \overline{b}^T \end{array}
$$

is symplectic if the following four conditions hold

$$
BA + A^T B = bb^T, \quad \overline{B}\overline{A} + \overline{A}^T\overline{B} = \overline{b}\overline{b}^T, \quad \overline{B}A + \overline{A}^T B = \overline{b}b^T, \quad B\overline{A} + A^T\overline{B} = b\overline{b}^T,
$$

where $B = \mathrm{diag}(b_1, b_2, \ldots, b_s)$, $\overline{B} = \mathrm{diag}(\overline{b}_1, \overline{b}_2, \ldots, \overline{b}_s)$. Obviously, the truncation $\Delta\widehat{W}(h)$ doesn't affect the symplectic conditions. Since $A = \overline{A}$, $B = \overline{B}$ and $b = \overline{b}$ in (3.2), the four symplectic conditions of (3.2) reduce to only one condition, i.e., $B\widetilde{A} + \widetilde{A}^T B = bb^T$, which we have proved in Lemma 2.1. This completes the proof. □

Remark 3.2. Since the sufficient conditions of preserving symplectic structure and preserving quadratic invariants coincide for stochastic RK methods, the family of stochastic parametric Gauss RK methods (3.2) can preserve all quadratic invariants of (1.1).

The family of methods (3.2) contain $s-1$ parameters $\alpha_1, \alpha_2, \ldots, \alpha_{s-1}$, and for arbitrary α_i $(i = 1, 2, \ldots, s-1)$, (3.2) are symplectic. For the discussion of order conditions and construction of energy-preserving scheme in the remaining part of the work, we make some simplification by setting $\alpha_1 = \alpha_2 = \cdots = \alpha_{s-2} = 0$ and leaving α_{s-1} the only free parameter, then denoting the parameter α_{s-1} by α for brief. Now we derive a family of methods with one parameter α denoted by

$$\begin{array}{c|cc} & \widetilde{A}(\alpha) & \widetilde{A}(\alpha) \\ \hline & b^T & b^T \end{array}$$

or by the corresponding iterated scheme

$$\begin{aligned} Y_i &= y_n + (h + \lambda\Delta\widehat{W}(h)) \sum_{j=1}^{s} \widetilde{a}_{ij}(\alpha) J^{-1} \nabla H(Y_j), \quad i = 1, \ldots, s, \\ y_{n+1} &= y_n + (h + \lambda\Delta\widehat{W}(h)) \sum_{i=1}^{s} b_i J^{-1} \nabla H(Y_i), \end{aligned} \tag{3.3}$$

where $\widetilde{A}(\alpha) = (\widetilde{a}_{ij}(\alpha))_{s\times s}$ is $\widetilde{A}$ in (2.3) with $\alpha_1 = \alpha_2 = \cdots = \alpha_{s-2} = 0$ and $\alpha_{s-1} = \alpha$. For example [7], when $s = 2$,

$$\widetilde{A}(\alpha) = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} - \frac{\sqrt{3}}{6} - \alpha \\ \frac{1}{4} + \frac{\sqrt{3}}{6} + \alpha & \frac{1}{4} \end{pmatrix}, \quad b^T = \left(\frac{1}{2}, \frac{1}{2}\right), \tag{3.4}$$

when $s = 3$,

(3.5)

$$\widetilde{A}(\alpha) = \begin{pmatrix} \frac{5}{36} & \frac{2}{9} - \frac{\sqrt{15}}{15} - \frac{2}{3}\alpha & \frac{5}{36} - \frac{\sqrt{15}}{30} + \frac{2}{3}\alpha \\ \frac{5}{36} + \frac{\sqrt{15}}{24} + \frac{5}{12}\alpha & \frac{2}{9} & \frac{5}{36} - \frac{\sqrt{15}}{24} - \frac{5}{12}\alpha \\ \frac{5}{36} + \frac{\sqrt{15}}{30} - \frac{2}{3}\alpha & \frac{2}{9} + \frac{\sqrt{15}}{15} + \frac{2}{3}\alpha & \frac{5}{36} \end{pmatrix}, \quad b^T = \left(\frac{5}{18}, \frac{4}{9}, \frac{5}{18}\right),$$

which will be used in Section 4.

Before discussing the convergence order of (3.3) for solving (1.1), we first prove the method with truncated random variables (3.3) has the same mean-square order as that of the corresponding method with standard Wiener increments

$$\begin{aligned} \widetilde{Y}_i &= \widetilde{y}_n + (h + \lambda\Delta W(h)) \sum_{j=1}^{s} \widetilde{a}_{ij}(\alpha) J^{-1} \nabla H(\widetilde{Y}_j), \quad i = 1, \ldots, s, \\ \widetilde{y}_{n+1} &= \widetilde{y}_n + (h + \lambda\Delta W(h)) \sum_{i=1}^{s} b_i J^{-1} \nabla H(\widetilde{Y}_i). \end{aligned} \tag{3.6}$$

Theorem 3.3. *Assume the necessary coefficients of Stratonovich-Taylor expansion of* (1.1) *are bounded. Suppose the one-step approximation* $\widetilde{y}_{n+1}$ *of* (3.6) *for solving* (1.1) *satisfies*

$$\|E(y(t_{n+1}) - \widetilde{y}_{n+1})\| = O(h^{p_1}), \tag{3.7}$$

$$(E\|y(t_{n+1}) - \widetilde{y}_{n+1}\|^2)^{1/2} = O(h^{p_2}), \tag{3.8}$$

$$p_2 > \frac{1}{2}, \quad p_1 \geq p_2 + \frac{1}{2},$$

where $y(t_{n+1})$ *represents the exact solution at* t_{n+1}. *Then the method with truncated random variables* (3.3) *and the corresponding method with standard Wiener increments* (3.6) *have the same mean-square convergence order* $p_2 - 1/2$ *for solving* (1.1) *if* $k \geq \max\{2p_1 - 1, 4p_2 - 2\}$.

Proof. According to [25], the method (3.6) is of mean-square order $p_2 - 1/2$. Now we prove the method with truncated random variables (3.3) is also of mean-square order $p_2 - 1/2$ if $k \geq \max\{2p_1 - 1, 4p_2 - 2\}$.

Assume that the numerical solution of the method (3.6) for solving (1.1) has the form

$$\widetilde{y}_{n+1} = y_n + \sum_{m=1}^{M} d_m(y_n)\widetilde{\mu}^m(h) + R_1, \quad \widetilde{\mu}(h) = h + \lambda\sqrt{h}\xi,$$

and the numerical solution of the method (3.3) for solving (1.1) has the form

$$y_{n+1} = y_n + \sum_{m=1}^{M} d_m(y_n)\mu^m(h) + R_2, \quad \mu(h) = h + \lambda\sqrt{h}\zeta,$$

where $d_m(y_n)$ $(m = 1, \ldots, M)$ are coefficients of Taylor-expansion, satisfying $\|Ed_m(y_n)\| \leq M_0$ and $E\|d_m(y_n)\|^2 \leq M_0$ for some positive constant M_0, R_1 and R_2 are remainder terms with higher order about $\widetilde{\mu}(h)$ and $\mu(h)$, respectively. So there is

$$y_{n+1} - \widetilde{y}_{n+1} = \sum_{m=1}^{M} d_m(y_n)(\mu^m(h) - \widetilde{\mu}^m(h)) + R,$$

where $R = R_2 - R_1$.

Set $M = 2p_1$, then

$$\begin{aligned}
\|E(y_{n+1} - \widetilde{y}_{n+1})\| &= \left\| E\left(\sum_{m=1}^{2p_1} d_m(y_n)(\mu^m(h) - \widetilde{\mu}^m(h)) + R \right) \right\| \\
&\leq \sum_{m=1}^{2p_1} \|E(d_m(y_n))\| \sum_{l=1}^{m} C_m^l h^{m-l} |\lambda|^l h^{l/2} |E(\zeta^l - \xi^l)| + R_3 \\
&\leq M_0 \sum_{m=1}^{2p_1} \sum_{l=1}^{m} C_m^l h^{m-l} |\lambda|^l h^{l/2} |E(\zeta^l - \xi^l)| + R_3,
\end{aligned} \tag{3.9}$$

where $R_3 = \|ER\| = O(h^{p_1+1})$.

Set $M = 2p_2$, then

$$
\begin{aligned}
E\|y_{n+1} - \widetilde{y}_{n+1}\|^2 &= E\left\|\sum_{m=1}^{2p_2} d_m(y_n)(\mu^m(h) - \widetilde{\mu}^m(h)) + R\right\|^2 \\
&\le 2E\left\|\sum_{m=1}^{2p_2} d_m(y_n)(\mu^m(h) - \widetilde{\mu}^m(h))\right\|^2 + R_4 \\
&\le 4p_2 \sum_{m=1}^{2p_2} E\|d_m(y_n)\|^2 E|\mu^m(h) - \widetilde{\mu}^m(h)|^2 + R_4 \\
&\le 4p_2 M_0 \sum_{m=1}^{2p_2} E|\mu^m(h) - \widetilde{\mu}^m(h)|^2 + R_4 \\
&= 4p_2 M_0 \sum_{m=1}^{2p_2} E\left|\sum_{l=1}^{m} C_m^l h^{m-l} \lambda^l h^{l/2} (\zeta^l - \xi^l)\right|^2 + R_4 \\
&\le 4p_2 M_0 \sum_{m=1}^{2p_2} m \sum_{l=1}^{m} (C_m^l)^2 h^{2m-2l} \lambda^{2l} h^l E|\zeta^l - \xi^l|^2 + R_4,
\end{aligned}
\tag{3.10}
$$

where $R_4 = 2E\|R\|^2 = O(h^{2p_2+1})$.

Now it remains to check $|E(\zeta^l - \xi^l)|$ and $E|\zeta^l - \xi^l|^2$ in (3.9) and (3.10).

For $l = 1$, there is $|E(\zeta - \xi)| = 0$. For $l \ge 2$, it holds that

$$
\begin{aligned}
|E(\zeta^l - \xi^l)| &= |E[(\zeta - \xi)(\zeta^{l-1} + \zeta^{l-2}\xi + \cdots + \xi^{l-1})]| \\
&\le E[|\zeta - \xi||\zeta^{l-1} + \zeta^{l-2}\xi + \cdots + \xi^{l-1}|] \\
&\le E[|\zeta - \xi| l |\xi|^{l-1}] \\
&\le l(E|\zeta - \xi|^2)^{1/2} (E|\xi|^{2l-2})^{1/2},
\end{aligned}
$$

where we use $|\zeta| \le |\xi|$ and Hölder's inequality. According to [27], $E|\zeta - \xi|^2 \le h^k$, in addition, $E|\xi|^{2l-2}$ is bounded, so

$$
|E(\zeta^l - \xi^l)| \le Ch^{k/2}, \tag{3.11}
$$

where C is a positive constant. For simplicity, unless otherwise specified, we always denote a positive constant by C throughout this paper, which may be different from line to line.

Similarly, we check $E|\zeta^l - \xi^l|^2$. For $l = 1$, due to [27], there is $E|\zeta - \xi|^2 \le h^k$, for $l \ge 2$, it holds that

$$
\begin{aligned}
E|\zeta^l - \xi^l|^2 &= E[|\zeta - \xi|^2 |\zeta^{l-1} + \zeta^{l-2}\xi + \cdots + \xi^{l-1}|^2] \\
&= E[|\zeta - \xi||\zeta - \xi||\zeta^{l-1} + \zeta^{l-2}\xi + \cdots + \xi^{l-1}|^2]
\end{aligned}
$$

$$
\begin{aligned}
&\leq E[|\zeta-\xi|2l^2|\xi|^{2l-1}] \\
&\leq 2l^2(E|\zeta-\xi|^2)^{1/2}(E|\xi|^{4l-2})^{1/2} \\
&\leq Ch^{k/2}.
\end{aligned} \tag{3.12}
$$

Substituting (3.11) into (3.9), we get if $k \geq 2p_1 - 1$, then

$$
\|E(y_{n+1} - \widetilde{y}_{n+1})\| = O(h^{p_1}). \tag{3.13}
$$

Similarly, inserting (3.12) into (3.10), we see for $k \geq 4p_2 - 2$, it holds

$$
(E\|y_{n+1} - \widetilde{y}_{n+1}\|^2)^{1/2} = O(h^{p_2}). \tag{3.14}
$$

Then if $k \geq \max\{2p_1 - 1, 4p_2 - 2\}$, by (3.7), (3.8), (3.13) and (3.14), we derive

$$
\|E(y_{n+1} - y(t_{n+1}))\| = \|E(y_{n+1} - \widetilde{y}_{n+1} + \widetilde{y}_{n+1} - y(t_{n+1}))\| = O(h^{p_1}),
$$

and

$$
(E\|y_{n+1} - y(t_{n+1})\|^2)^{1/2} = (E\|y_{n+1} - \widetilde{y}_{n+1} + \widetilde{y}_{n+1} - y(t_{n+1})\|^2)^{1/2} = O(h^{p_2}).
$$

Consequently, due to [25], the method (3.3) for solving (1.1) has mean-square order $p_2 - 1/2$. The proof is completed. □

Theorem 3.3 shows the truncation $\Delta\widehat{W}(h)$ doesn't change the convergence order if k satisfies some conditions. In the rest of the paper, we always assume k is suitably chosen. Next, we want to prove that the method (3.3) for solving (1.1) can achieve arbitrary high order. Since (1.1) is a single integrand SDE, we may use the results in [14] to prove the following theorem.

Theorem 3.4. *The family of stochastic one-parameter Gauss RK methods with truncated random variables defined by* (3.3) *for solving* (1.1) *are of mean-square order* $s-1$ *for any fixed* $\alpha \neq 0$.

Proof. On the basis of [14], if the order of a deterministic RK method for ODEs is p, then the mean-square order of the according stochastic RK method for single integrand SDEs is $\lfloor p/2 \rfloor$. Since the family of deterministic RK methods denoted by

$$
\begin{array}{c|c}
 & \widetilde{A}(\alpha) \\
\hline
 & b^T
\end{array}
$$

are of order $2s-2$ for any fixed $\alpha \neq 0$ (see [7] for details), the family of stochastic RK methods (3.6) are of mean-square order $s-1$ for any fixed $\alpha \neq 0$. Therefore by Theorem 3.3, the stochastic RK methods (3.3) have the same mean-square order $s-1$ for any fixed $\alpha \neq 0$. The proof is completed. □

Theorem 3.4 shows the method (3.3) for solving (1.1) can achieve arbitrary high order as long as the stage s is chosen properly. Note that for $\alpha = 0$, the stochastic RK method (3.3) is of mean-square order s because the according RK method with coefficients $(c, \widetilde{A}(0), b)$ is Gauss method of order $2s$.

The family of stochastic symplectic one-parameter Gauss RK methods (3.3) defined a map $y_1 = \Phi_{h,\Delta\widehat{W}(h)}(y_0, \alpha)$. We are now in the position to discuss whether there exists some suitable parameter $\alpha^*(y_0, \mu(h))$, where the notation $\mu(h) = h + \lambda\Delta\widehat{W}(h)$ is similar as that used in [14], such that energy preservation holds, i.e., $H(y_1(\alpha^*, \mu(h))) = H(y_0)$. To this end, we define the energy error function as

$$g(\alpha, \mu(h)) = H(y_1(\alpha, \mu(h))) - H(y_0). \tag{3.15}$$

Similar to the way in [7], we employ the implicit function theorem with respect to the energy error function (3.15) to prove the existence of α^*. First, we make the following mild and reasonable assumptions:

$(\mathcal{A}_1)$ The function $g(\alpha, \mu(h))$ is analytic near the origin.

$(\mathcal{A}_2)$ When the underlying method ($\alpha = 0$) is applied, $g(0, \mu(h))$ satisfies

$$g(0, \mu(h)) = H(y_1(0, \mu(h))) - H(y_0) = c_0\mu^d(h) + R_5,$$

where y_0 is the initial value, d is some positive integer representing the power of $\mu(h)$, $c_0 \neq 0$, and R_5 is the higher order remainder term. Further, we assume that for any $\alpha \in R$ ($\alpha \neq 0$),

$$g(\alpha, \mu(h)) = c(\alpha)\mu^{d-2}(h) + R_6,$$

where $c'(0) \neq 0$ and R_6 is the higher order remainder term.

Theorem 3.5. *Suppose the assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ hold, then there exist a function $\alpha^* = \alpha^*(\mu(h))$ and a positive h_0, such that*

(i) $g(\alpha^*(\mu(h)), \mu(h)) = 0$ *for all* $h \in (0, h_0)$,

(ii) $\alpha^*(\mu(h)) = C\mu^2(h) + R_7$, *where C is some constant and R_7 is the higher order remainder term.*

Proof. From $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, $g(\alpha, \mu(h))$ can be expanded as

$$g(\alpha, \mu(h)) = \sum_{j=d}^{\infty} \frac{1}{j!} \frac{\partial^j g}{\partial \mu(h)^j}(0,0)\mu^j(h) + \sum_{i=1}^{\infty}\sum_{j=d-2}^{\infty} \frac{1}{i!j!} \frac{\partial^{i+j} g}{\partial \alpha^i \partial \mu(h)^j}(0,0)\mu^j(h)\alpha^i. \tag{3.16}$$

We expect to apply the implicit function theorem to search for a solution to $g(\alpha,\mu(h))=0$ in the form $\alpha^*(\mu(h))=\eta(\mu(h))\mu^2(h)$, where $\eta(\mu(h))$ is a real-valued function of $\mu(h)$. Following the way in [7], we let $\alpha=\eta\mu^2(h)$ and insert it into (3.16) to obtain

$$\begin{aligned} g(\alpha,\mu(h)) &= \frac{1}{d!}\frac{\partial^d g}{\partial\mu(h)^d}(0,0)\mu^d(h)+\frac{1}{(d-2)!}\frac{\partial^{d-1}g}{\partial\alpha\partial\mu(h)^{d-2}}(0,0)\mu^d(h)\eta \\ &\quad+\frac{1}{(d-1)!}\frac{\partial^d g}{\partial\alpha\partial\mu(h)^{d-1}}(0,0)\mu^{d+1}(h)\eta+R_8, \end{aligned} \tag{3.17}$$

where R_8 is the higher order term. Dividing by $\mu^d(h)/(d-2)!$ on the right-hand side of (3.17) yields

$$\begin{aligned} \widetilde{g}(\eta,\mu(h)) &= \frac{1}{d(d-1)}\frac{\partial^d g}{\partial\mu(h)^d}(0,0)+\frac{\partial^{d-1}g}{\partial\alpha\partial\mu(h)^{d-2}}(0,0)\eta \\ &\quad+\frac{1}{d-1}\frac{\partial^d g}{\partial\alpha\partial\mu(h)^{d-1}}(0,0)\mu(h)\eta+R_9, \end{aligned} \tag{3.18}$$

where R_9 is the higher order term. Now we turn to $\widetilde{g}(\eta,\mu(h))$ for an answer. From assumption $(\mathcal{A}_2)$, $\frac{\partial^{d-1}g}{\partial\alpha\partial\mu(h)^{d-2}}(0,0)=c'(0)(d-2)!\neq 0$, hence we can employ the implicit function theorem to ensure that there exists a function $\eta=\eta(\mu(h))$ such that $\widetilde{g}(\eta,\mu(h))=0$. For sufficiently small h, $\eta(\mu(h))$ can be calculated from (3.18) as

$$\begin{aligned} \eta(\mu(h)) &= -\frac{\frac{1}{d(d-1)}\frac{\partial^d g}{\partial\mu(h)^d}(0,0)+R_9}{\frac{\partial^{d-1}g}{\partial\alpha\partial\mu(h)^{d-2}}(0,0)+\frac{1}{d-1}\frac{\partial^d g}{\partial\alpha\partial\mu(h)^{d-1}}(0,0)\mu(h)} \\ &= -\frac{1}{d(d-1)}\frac{\frac{\partial^d g}{\partial\mu(h)^d}(0,0)}{\frac{\partial^{d-1}g}{\partial\alpha\partial\mu(h)^{d-2}}(0,0)}+R_{10}, \end{aligned}$$

where R_{10} is the higher order remainder term. By assumption $(\mathcal{A}_2)$, $\frac{\partial^d g}{\partial\mu(h)^d}(0,0)$ is different from zero, so the solution $\alpha^*(\mu(h))$ to $g(\alpha,\mu(h))=0$ takes the form

$$\begin{aligned} \alpha^*(\mu(h)) &= \eta(\mu(h))\mu^2(h) \\ &= -\frac{1}{d(d-1)}\frac{\frac{\partial^d g}{\partial\mu(h)^d}(0,0)}{\frac{\partial^{d-1}g}{\partial\alpha\partial\mu(h)^{d-2}}(0,0)}\mu^2(h)+R_7=C\mu^2(h)+R_7, \end{aligned}$$

where R_7 is the higher order remainder term. This completes the proof. □

It should be pointed out the truncation $\Delta\widehat{W}(h)$ ensures $\mu(h)\to 0$ as $h\to 0$, so that the implicit function theorem can be applied here. The reason is the standard Wiener increment $\Delta W(h)$ is unbounded for any arbitrarily small h. Theorem 3.5 shows there exists a suitable parameter at each step such that the method (3.3) preserves the energy function. We denote the variable-parameter energy-preserving method by a mapping form $y_{n+1}=\Phi_{h,\Delta\widehat{W}(h)}(y_n,\alpha_n^*)$. Next we will discuss the convergence order of this method.

Theorem 3.6. *The energy-preserving method* $y_{n+1} = \Phi_{h,\Delta\widehat{W}(h)}(y_n, \alpha_n^*)$ *is of mean-square order* s *for solving* (1.1) *under the assumptions* $(\mathcal{A}_1)$ *and* $(\mathcal{A}_2)$.

Proof. Let $y_1(\alpha, \mu(h))$ be the solution computed by (3.3) at time $t_0 + h$ with $y_0 = y(t_0)$. Applying the mean value theorem we get

$$y_1(\alpha, \mu(h)) = y_1(0, \mu(h)) + \alpha \int_0^1 y_1'(t\alpha, \mu(h))\, dt,$$

where $y_1'(t\alpha, \mu(h)) = \frac{\partial y_1(v, \mu(h))}{\partial v}\big|_{v=t\alpha}$. Recall that $y_1(0, \mu(h))$ is of mean-square order s while $y_1(\alpha, \mu(h))$ is of mean-square order $s-1$ when $\alpha \neq 0$, which implies that

$$\int_0^1 y_1'(t\alpha, \mu(h))\, dt = \widetilde{C}_1\mu^{2s-1}(h) + \widetilde{C}_2\mu^{2s}(h) + \widetilde{C}_3\mu^{2s+1}(h) + \cdots,$$

where $\widetilde{C}_i$ $(i = 1, 2, \ldots)$ are vector-valued coefficients. Denote $R_{11} = \widetilde{C}_2\mu^{2s}(h) + \widetilde{C}_3\mu^{2s+1}(h) + \cdots$, then $\|ER_{11}\| = O(h^s)$ and $E\|R_{11}\|^2 = O(h^{2s})$. Thus,

$$y_1(\alpha, \mu(h)) = y_1(0, \mu(h)) + \alpha(\widetilde{C}_1\mu^{2s-1}(h) + R_{11}).$$

Therefore,

$$\begin{aligned} &\|E(y_1(\alpha, \mu(h)) - y(t_0 + h))\| \\ &= \|E(y_1(0, \mu(h)) + \alpha(\widetilde{C}_1\mu^{2s-1}(h) + R_{11}) - y(t_0 + h))\| \\ &\leq \|E(y_1(0, \mu(h)) - y(t_0 + h))\| + \|E(\alpha(\widetilde{C}_1\mu^{2s-1}(h) + R_{11}))\| \\ &\leq Ch^{s+1} + \|E(\alpha(\widetilde{C}_1\mu^{2s-1}(h) + R_{11}))\|. \end{aligned} \tag{3.19}$$

Substituting $\alpha = \alpha^* = C\mu^2(h) + R_7$ into (3.19) yields

$$\|E(y_1(\alpha^*, \mu(h)) - y(t_0 + h))\| \leq Ch^{s+1} + Ch^{s+1} = Ch^{s+1}.$$

Hence,

$$\|E(y_1(\alpha^*, \mu(h)) - y(t_0 + h))\| = O(h^{s+1}). \tag{3.20}$$

In addition, it holds

$$\begin{aligned} &E\|y_1(\alpha, \mu(h)) - y(t_0 + h)\|^2 \\ &= E\|y_1(0, \mu(h)) + \alpha(\widetilde{C}_1\mu^{2s-1}(h) + R_{11}) - y(t_0 + h)\|^2 \\ &\leq 2E\|y_1(0, \mu(h)) - y(t_0 + h)\|^2 + 2E\|\alpha(\widetilde{C}_1\mu^{2s-1}(h) + R_{11})\|^2 \\ &\leq Ch^{2s+1} + 2E\|\alpha(\widetilde{C}_1\mu^{2s-1}(h) + R_{11})\|^2. \end{aligned} \tag{3.21}$$

Inserting $\alpha = \alpha^* = C\mu^2(h) + R_7$ into (3.21), similar to the argument of (3.20), we obtain

$$E\|y_1(\alpha^*, \mu(h)) - y(t_0 + h)\|^2 = O(h^{2s+1}). \tag{3.22}$$

By an application of Theorem 1.1 in [25], together with (3.20) and (3.22), we can easily derive the conclusion and the proof is completed. □

Theorem 3.6 implies the energy-preserving method $y_{n+1} = \Phi_{h,\Delta\widehat{W}(h)}(y_n, \alpha_n^*)$ retains the convergence order of the underlying stochastic Gauss RK method ($\alpha = 0$).

4. Numerical experiments

In this section, we will illustrate the superiority of the parametric numerical method $y_{n+1} = \Phi_{h,\Delta\widehat{W}(h)}(y_n, \alpha_n^*)$ when preserving the energy function of the original system for a long-term simulation by comparing with the corresponding non-parametric ones. With respect to $A_h = \sqrt{2k|\ln h|}$, $k = 12$ is chosen in the three examples of this section. For convenience, we introduce the shorthand notations in the following table.

Methods	Shorthand notations
2-stage stochastic Gauss RK method (3.3) with (3.4) and $\alpha = 0$	SGRK2
3-stage stochastic Gauss RK method (3.3) with (3.5) and $\alpha = 0$	SGRK3
Energy-preserving 2-stage stochastic parametric RK method (3.3) with (3.4)	EPSPRK2
Energy-preserving 3-stage stochastic parametric RK method (3.3) with (3.5)	EPSPRK3

Example 4.1 (The stochastic mathematical pendulum)**.** Consider the following stochastic mathematical pendulum

$$d\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} -\sin(q(t)) \\ p(t) \end{pmatrix}(dt + \beta \circ dW_t), \quad p(0) = p_0, \quad q(0) = q_0, \tag{4.1}$$

where β denotes the noise intensity and $H(p(t), q(t)) = p^2/2 - \cos q$ is the energy function.

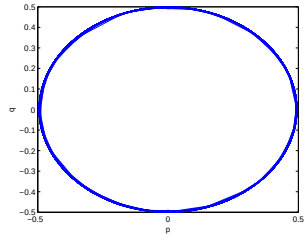


Figure 4.1: Phase space plot of the numerical solution to (4.1) computed by EPSPRK3 with $h = 0.1$.

We employ the methods SGRK3 and EPSPRK3 to solve the system (4.1), respectively. Note that the parameter α_n^* is determined at each step by bisection such that the energy error function $g(\alpha_n^*, \mu(h)) = H(y_{n+1}(\alpha_n^*, \mu(h))) - H(y_n)$ is equal to zero. We choose the step size $h = 0.1$, the initial values $p_0 = 0$, $q_0 = 0.5$, and the noise intensity $\beta = 0.5$.

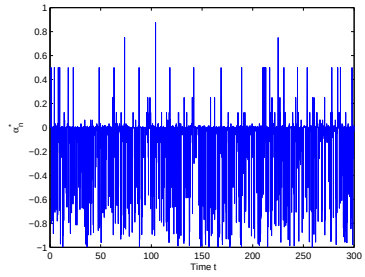
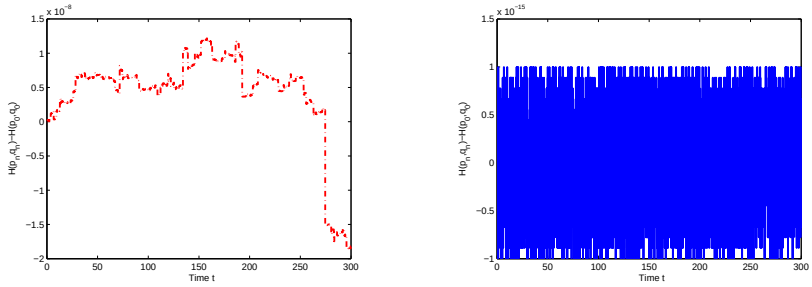


Figure 4.2: Values of the sequence $\{\alpha_n^*\}$ in EPSPRK3 such that the energy conservation holds for (4.1) with $h = 0.1$.



(a) SGRK3 (b) EPSPRK3

Figure 4.3: Energy errors of numerical solutions for solving the system (4.1) with $h = 0.1$.

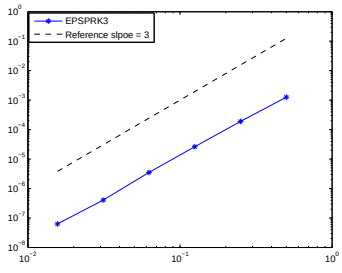


Figure 4.4: Convergence rate of the method EPSPRK3 for solving (4.1).

Figure 4.1 reports the numerical solutions of EPSPRK3 for simulating a sample phase trajectory of (4.1) on the interval $[0, 300]$. Figure 4.2 shows the values of the sequence $\{\alpha_n^*\}$,

such that the energy function is invariant at each step. Figure 4.3 exhibits the energy errors of SGRK3 and EPSPRK3 on the interval $[0, 300]$, respectively, where we can see EPSPRK3 has greater advantage than SGRK3 in preserving energy function. Figure 4.4 demonstrates the convergence rate of EPSPRK3 for solving (4.1), where we use 1000 independent sample paths, and for each path, the method EPSPRK3 is implemented with six different step sizes: $h = 2^{-1}$, 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}. Note here the numerical solution computed by SGRK3 with smaller step size $h = 2^{-10}$ is used as a reference solution because the exact solution of (4.1) cannot be expressed explicitly. We calculate the mean-square errors at the terminal $T = 1$ by $\sqrt{\big(\sum_{i=1}^{1000}(|p(1,\omega_i) - p_N(\omega_i)|^2 + |q(1,\omega_i) - q_N(\omega_i)|^2)\big)/1000}$, and show the results in a log-log plot in Figure 4.4.

Example 4.2 (The stochastic Kepler problem)**.** Consider the stochastic Kepler problem

$$d\begin{pmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} -\frac{q_1}{(q_1^2+q_2^2)^{3/2}} \\ -\frac{q_2}{(q_1^2+q_2^2)^{3/2}} \\ p_1 \\ p_2 \end{pmatrix}(dt + \beta \circ dW_t), \tag{4.2}$$

$$p_1(0) = p_{10}, \quad p_2(0) = p_{20}, \quad q_1(0) = q_{10}, \quad q_2(0) = q_{20},$$

where β denotes the noise intensity and $H(p_1, p_2, q_1, q_2) = \frac{1}{2}(p_1^2+p_2^2) - 1/(\sqrt{q_1^2+q_2^2})$ is the energy function representing the total energy of system (4.2). In addition, system (4.2) possesses a quadratic invariant $I(p_1, p_2, q_1, q_2) = q_1p_2 - q_2p_1$ which represents the angular momentum.

Apply the methods SGRK2 and EPSPRK2 to solving the system (4.2), respectively. The parameter α_n^* is determined at each step by bisection such that the energy error function $g(\alpha_n^*, \mu(h)) = H(y_{n+1}(\alpha_n^*, \mu(h))) - H(y_n)$ is equal to zero. We choose the step size $h = 0.1$, the initial values $p_{10} = 0$, $p_{20} = \sqrt{(1+e)/(1-e)}$, $q_{10} = 1-e$, $q_{20} = 0$ with $e = 0.5$, and the noise intensity $\beta = 0.1$.

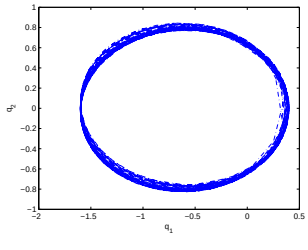


Figure 4.5: Phase space plot of the numerical solution to (4.2) computed by EPSPRK2 with $h = 0.1$.

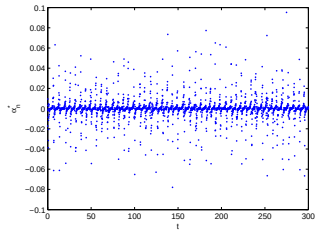
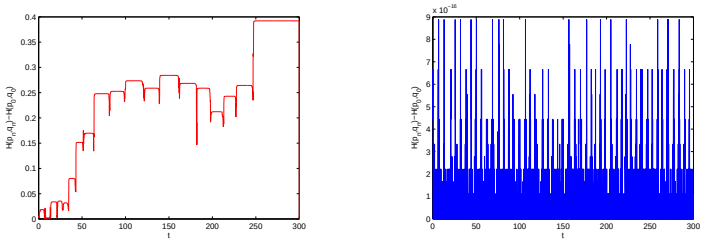


Figure 4.6: Values of the sequence $\{\alpha_n^*\}$ in EPSPRK2 such that the energy conservation holds for (4.2) with $h = 0.1$.



(a) SGRK2 (b) EPSPRK2

Figure 4.7: Energy errors computed by the two numerical methods for the system (4.2) with $h = 0.1$.

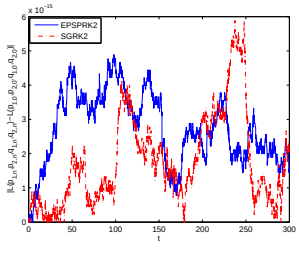


Figure 4.8: Errors in the angular momentum computed by the two numerical methods for the system (4.2) with $h = 0.1$. Dashdotted line: SGRK2; Solid line: EPSPRK2.

Figure 4.5 reports the phase portrait by using EPSPRK2 to simulate a sample path on the interval $[0, 300]$. Figure 4.6 shows the values of the sequence $\{\alpha_n^*\}$ at each step such that the energy conservation holds. Figure 4.7 reports the energy errors computed by SGRK2 and EPSPRK2 on the interval $[0, 300]$, respectively, where we can see EPSPRK2 can preserve the energy function while SGRK2 can not. Figure 4.8 reports the errors in angular momentum $I(p_1, p_2, q_1, q_2) = q_1p_2 - q_2p_1$ computed by SGRK2 and EPSPRK2 on the interval $[0, 300]$, respectively, where we can see the two methods both can preserve

the angular momentum because the angular momentum is a quadratic invariant of (4.2), which coincides with Remark 3.2.

Example 4.3 (The stochastic cyclic Lotka-Volterra system)**.** Consider the stochastic cyclic Lotka-Volterra system

$$d\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} x(t)(z(t)-y(t)) \\ y(t)(x(t)-z(t)) \\ z(t)(y(t)-x(t)) \end{pmatrix}(dt+\sigma\circ dW_t), \tag{4.3}$$
$$x(0)=x_0, \quad y(0)=y_0, \quad z(0)=z_0,$$

where σ is a real-valued constant. Note this system is not a stochastic canonical Hamiltonian system, but it possesses an energy function $H(x,y,z)=xyz$. We try to test how the energy-preserving method behaves in this case.

Apply the methods SGRK3 and EPSPRK3 to solving the system (4.3), respectively. The parameter α_n^* is determined at each step by bisection such that the energy error function $g(\alpha_n^*,\mu(h))=H(y_{n+1}(\alpha_n^*,\mu(h)))-H(y_n)$ is equal to zero. We choose the step size $h=0.1$, the initial values $x_0=1$, $y_0=2$, $z_0=3$, and the constant $\sigma=0.5$.

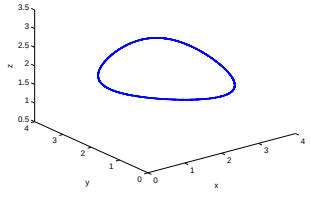


Figure 4.9: Phase space plot of the numerical solution to (4.3) computed by EPSPRK3 with $h=0.1$.

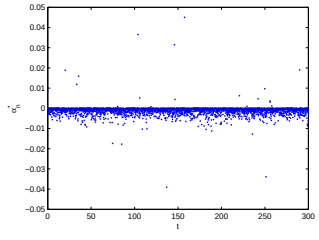
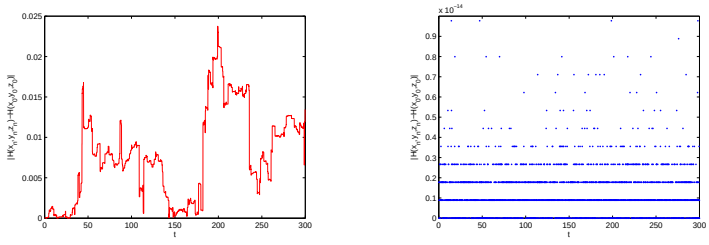


Figure 4.10: Values of the sequence $\{\alpha_n^*\}$ in EPSPRK3 such that the energy function H is invariant for (4.3) with $h=0.1$.



(a) SGRK3 (b) EPSPRK3

Figure 4.11: Errors in the energy function H computed by the two numerical methods for the system (4.3) with $h = 0.1$.

Figure 4.9 reports the phase portrait by using EPSPRK3 to simulate a sample path on the interval $[0, 300]$. Figure 4.10 shows the values of the sequence $\{\alpha_n^*\}$ at each step such that the energy function $H(x, y, z)$ is invariant. Figure 4.11 reports the errors in the energy function $H(x, y, z)$ computed by SGRK3 and EPSPRK3 on the interval $[0, 300]$, respectively, where we can see EPSPRK3 can preserve the energy function while SGRK3 can not. It shows that although the system (4.3) is not a stochastic canonical Hamiltonian system but a stochastic Poisson system [13], the constructed stochastic parametric method still behaves well in preserving energy.

At the end of this section, we should point out the numerical solutions of the energy-preserving methods as well as the parameters $\{\alpha_n^*\}$ in the above tests are derived by solving nonlinear systems at each step. So how to improve the efficiency of the numerical methods is still a problem to be settled in our future work.

5. Conclusions

In this paper, EQUIP methods are applied to the stochastic canonical Hamiltonian systems and analyzed. Standard Wiener increments are replaced by some truncated random variables. We prove the truncation doesn't change the convergence order under some conditions. It is shown that EQUIP methods can preserve the energy and the quadratic invariants, and retain the mean-square convergence order of the underlying methods when applied to the stochastic canonical Hamiltonian systems. Numerical results illustrate the effectiveness of the methods in preserving energy and quadratic invariants, and show the convergence order results. Furthermore, experimental results indicate the methods are appropriate for not only stochastic canonical Hamiltonian systems but also stochastic Poisson systems when preserving energy.

References

[1] C. A. Anton, Y. S. Wong and J. Deng, *Symplectic schemes for stochastic Hamiltonian systems preserving Hamiltonian functions*, Int. J. Numer. Anal. Model. **11** (2014), no. 3, 427–451.

[2] N. Bou-Rabee and H. Owhadi, *Stochastic variational integrators*, IMA J. Numer. Anal. **29** (2009), no. 2, 421–443.

[3] L. Brugnano, G. F. Caccia and F. Iavernaro, *Line integral formulation of energy and quadratic invariants preserving (EQUIP) methods for Hamiltonian systems*, AIP Conference Proceedings **1738** (2016), 100002.

[4] L. Brugnano, G. Gurioli and F. Iavernaro, *Analysis of energy and quadratic invariant preserving (EQUIP) methods*, J. Comput. Appl. Math. **335** (2018), 51–73.

[5] L. Brugnano and F. Iavernaro, *Line Integral Methods for Conservative Problems*, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2016.

[6] L. Brugnano, F. Iavernaro and D. Trigiante, *Energy and quadratic invariants preserving integrators of Gaussian type*, AIP Conference Proceedings **1281** (2010), 227–230.

[7] ———, *Energy- and quadratic invariants–preserving integrators based upon Gauss collocation formulae*, SIAM J. Numer. Anal. **50** (2012), no. 6, 2897–2916.

[8] L. Brugnano and Y. Sun, *Multiple invariants conserving Runge-Kutta type methods for Hamiltonian problems*, Numer. Algorithms **65** (2014), no. 3, 611–632.

[9] K. Burrage and P. M. Burrage, *High strong order explicit Runge-Kutta methods for stochastic ordinary differential equations*, Appl. Numer. Math. **22** (1996), no. 1-3, 81–101.

[10] ———, *Low rank Runge-Kutta methods, symplecticity and stochastic Hamiltonian problems with additive noise*, J. Comput. Appl. Math. **236** (2012), no. 16, 3920–3930.

[11] J. C. Butcher, *Numerical Methods for Ordinary Differential Equations*, John Wiley & Sons, Chichester, 2003.

[12] E. Celledoni, R. I. McLachlan, D. I. McLaren, B. Owren, G. R. W. Quispel and W. M. Wright, *Energy-preserving Runge-Kutta methods*, M2AN Math. Model. Numer. Anal. **43** (2009), no. 4, 645–649.

[13] D. Cohen and G. Dujardin, *Energy-preserving integrators for stochastic Poisson systems*, Commun. Math. Sci. **12** (2014), no. 8, 1523–1539.

[14] K. Debrabant and A. Kværnø, *Cheap arbitrary high order methods for single integrand SDEs*, BIT **57** (2017), no. 1, 153–168.

[15] K. Feng and M. Qin, *Symplectic Geometric Algorithms for Hamiltonian Systems*, Zhejiang Science and Technology, Hangzhou, Springer, Heidelberg, 2010.

[16] O. Gonzalez, *Time integration and discrete Hamiltonian systems*, J. Nonlinear Sci. **6** (1996), no. 5, 449–467.

[17] E. Hairer, C. Lubich and G. Wanner, *Geometric Numerical Integration: Structure-preserving algorithms for ordinary differential equations*, Second edition, Springer Series in Computational Mathematics **31**, Springer-Verlag, Berlin, 2006.

[18] E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II: Stiff and differential-algebraic problems*, Second edition, Springer Series in Computational Mathematics **14**, Springer-Verlag, Berlin, 1996.

[19] J. Hong, S. Zhai and J. Zhang, *Discrete gradient approach to stochastic differential equations with a conserved quantity*, SIAM J. Numer. Anal. **49** (2011), no. 5, 2017–2038.

[20] C. Huang, *Exponential mean square stability of numerical methods for systems of stochastic differential equations*, J. Comput. Appl. Math. **236** (2012), no. 16, 4016–4026.

[21] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Applications of Mathematics (New York) **23**, Springer-Verlag, Berlin, 1992.

[22] Q. Ma and X. Ding, *Stochastic symplectic partitioned Runge-Kutta methods for stochastic Hamiltonian systems with multiplicative noise*, Appl. Math. Comput. **252** (2015), 520–534.

[23] Q. Ma, D. Ding and X. Ding, *Symplectic conditions and stochastic generating functions of stochastic Runge-Kutta methods for stochastic Hamiltonian systems with multiplicative noise*, Appl. Math. Comput. **219** (2012), no. 2, 635–643.

[24] X. Mao, *Stochastic Differential Equations and Their Applications*, Horwood Publishing Series in Mathematics & Applications, Horwood, Chichester, 1997.

[25] G. N. Milstein, *Numerical Integration of Stochastic Differential Equations*, Mathematics and its Applications **313**, Kluwer Academic Publishers Group, Dordrecht, 1995.

[26] G. N. Milstein, Y. M. Repin and M. V. Tretyakov, *Symplectic integration of Hamiltonian systems with additive noise*, SIAM J. Numer. Anal. **39** (2002), no. 6, 2066–2088.

[27] ———, *Numerical methods for stochastic systems preserving symplectic structure*, SIAM J. Numer. Anal. **40** (2002), no. 4, 1583–1604.

[28] T. Misawa, *Energy conservative stochastic difference scheme for stochastic Hamilton dynamical systems*, Japan. J. Indust. Appl. Math. **17** (2000), no. 1, 119–128.

[29] ———, *Symplectic integrators to stochastic Hamiltonian dynamical systems derived from composition methods*, Math. Probl. Eng. **2010** (2010), Art. ID 384937, 12 pp.

[30] G. Sun, *Construction of high order symplectic PRK methods*, J. Comput. Math. **13** (1995), no. 1, 40–50.

[31] L. Wang and J. Hong, *Generating functions for stochastic symplectic methods*, Discrete Contin. Dyn. Syst. **34** (2014), no. 3, 1211–1228.

[32] D. Wang, A. Xiao and X. Li, *Parametric symplectic partitioned Runge-Kutta methods with energy-preserving properties for Hamiltonian systems*, Comput. Phys. Commun. **184** (2013), no. 2, 303–310.

[33] A. Xiao and X. Tang, *High strong order stochastic Runge-Kutta methods for Stratonovich stochastic differential equations with scalar noise*, Numer. Algorithms **72** (2016), no. 2, 259–296.

[34] G. Zhong and J. E. Marsden, *Lie-Poisson Hamilton-Jacobi theory and Lie-Poisson integrators*, Phys. Lett. A **133** (1988), no. 3, 134–139.

[35] W. Zhou, L. Zhang, J. Hong and S. Song, *Projection methods for stochastic differential equations with conserved quantities*, BIT **56** (2016), no. 4, 1497–1518.

[36] W. Zhou, J. Zhang, J. Hong and S. Song, *Stochastic symplectic Runge-Kutta methods for the strong approximation of Hamiltonian systems with additive noise*, J. Comput. Appl. Math. **325** (2017), 134–148.

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