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An Expectation Formula Based on a Maclaurin Expansion

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Abstract. In this paper, we obtain an expectation formula with respect to the q-probability distribution $W(x,y;q)$ based on a Maclaurin expansion. The formula has many applications in mathematics. Some of the applications are also given, which include a probability version of the Al-Salam and Verma q-integral.

1. Introduction

Probabilistic methods are useful tools in the study of q-series, see [5,6,9,17–19]. Recently, the present author [20] constructed the following discrete probability space $(\Omega,\mathcal{F},\mathbb{P})$, where $\Omega=\{\omega_0,\omega_1,\omega_2,\ldots\}$, $\mathcal{F}$ is the collection of all subsets of Ω and $\mathbb{P}$ is defined by

$$\mathbb{P}(\{\omega_{2k}\})=\frac{(yq^{k+1}/x,q^{k+1};q)_\infty q^k}{(q,yq/x,x/y;q)_\infty}$$

and

$$\mathbb{P}(\{\omega_{2k+1}\})=\frac{-x(q^{k+1},xq^{k+1}/y;q)_\infty q^k}{y(q,yq/x,x/y;q)_\infty}$$

for $k=0,1,2,\ldots$ and $xy<0$. We call a random variable X has a probability distribution $W(x,y;q)$, if,

$$P(X=yq^k)=\frac{(yq^{k+1}/x,q^{k+1};q)_\infty q^k}{(q,yq/x,x/y;q)_\infty},\quad k=0,2,4,\ldots$$

and

$$P(X=xq^k)=\frac{-x(q^{k+1},xq^{k+1}/y;q)_\infty q^k}{y(q,yq/x,x/y;q)_\infty},\quad k=1,3,5,\ldots.$$

In this paper, we use the Maclaurin expansion of a function $f(x)$ to obtain an expectation formula with respect to $W(x,y;q)$. We shall give some applications of our main result in later parts of the paper.

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We recall some definitions, notation and known results in [2, 7] which will be used throughout this paper. In particular, we assume $0 < q < 1$ throughout this paper. The q-shifted factorials are defined as

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

We also adopt the following compact notation for multiple q-shifted factorials:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n,$$

where n is either an integer or ∞. The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}.$$

Heine introduced the ${}_{r+1}\phi_r$ basic hypergeometric series, which is defined by

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q)_n x^n}{(q, b_1, b_2, \ldots, b_r; q)_n}.$$

We also recall the q-binomial theorem

$$\sum_{n=0}^{\infty} \frac{(a;q)_n x^n}{(q;q)_n} = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x| < 1, \tag{1.1}$$

and its special case

$$\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q;q)_n} = (-x;q)_\infty \tag{1.2}$$

as well as the q-Gauss summation formula

$${}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, \frac{c}{ab}\right) = \frac{(c/a, c/b;q)_\infty}{(c, c/ab;q)_\infty}, \quad \left|\frac{c}{ab}\right| < 1. \tag{1.3}$$

In addition to the above notation, we also need F. Jackson's q-integral defined by [8]

$$\int_0^d f(t)\, d_qt = d(1-q)\sum_{n=0}^{\infty} f(dq^n)q^n,$$

where

$$\int_c^d f(t)\, d_qt = \int_0^d f(t)\, d_qt - \int_0^c f(t)\, d_qt.$$

He also defined

$$\int_0^\infty f(t)\, d_qt = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n.$$

On the other hand, the bilateral q-integral is defined by

$$\int_{-\infty}^{\infty} f(t)\, d_q t = (1-q) \sum_{n=-\infty}^{\infty} [f(q^n) + f(-q^n)]q^n.$$

The q-integrals are important in the theory and applications of basic hypergeometric series. For example, the present author gave some applications of the q-integrals in basic hypergeometric series in [12–16].

An important class of q-hypergeometric polynomials is given by the Al-Salam-Carlitz polynomials $\varphi_n^{(a)}(x|q)$, which are defined as [11]

$$\varphi_n^{(a)}(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k (a;q)_k.$$

If $a = 0$, we get the Rogers-Szegö polynomials

$$h_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k.$$

If $X \sim W(x, y; q)$, $-1 < x < 0$, and $0 < y \le 1$, then [20]

$$\mathbb{E}\left\{ \frac{X^n}{(aX, bX; q)_\infty} \right\} = \frac{(abxy;q)_\infty}{(ax, ay, bx, by;q)_\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(ay, by;q)_k}{(abxy;q)_k} x^k y^{n-k} \tag{1.4}$$

provided that $|a| < 1$, $|b| < 1$. If $X \sim W(x, y; q)$, $-1 < x < 0$, and $0 < y \le 1$, and $f(x)$ is a measurable function, then [20]

$$\mathbb{E}\{f(X)\} = \frac{1}{y(1-q)(q, yq/x, x/y;q)_\infty} \int_x^y (qt/x, qt/y;q)_\infty f(t)\, d_q t, \tag{1.5}$$

provided that the q-integral in (1.5) converges absolutely. Here $\mathbb{E}(\cdot)$ denotes the expected value.

Finally, we recall Lebesgue's dominated convergence theorem: Suppose that $\{X_n, n \ge 1\}$ is a sequence of random variables, that $X_n \to X$ pointwise almost everywhere as $n \to \infty$, and that $|X_n| \le Y$ for all n, where the random variable Y is integrable. Then X is integrable, and

$$\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}X.$$

2. Main results

It follows from (1.5) that any expectation formula of a q-probability distribution $W(x, y; q)$ can be rewritten in terms of a q-integral formula. As a result, in order to obtain some new q-integrals, it is useful to find some new expectation formulas. In this section, we use the Maclaurin expansion of $f(x)$ to obtain a commonly encountered expectation formula.

We now state the main result of this paper:

Theorem 2.1. *Suppose $f(t)$ admits a Maclaurin expansion when $|t| \le 1$ and the series $\sum_{n=0}^{\infty} |f^{(n)}(0)|/n!$ converges absolutely. If $X \sim W(x, y; q)$, $-1 < x < 0$, and $0 < y \le 1$, then*

$$\mathbb{E}\left\{\frac{f(X)}{(aX, bX; q)_\infty}\right\} = \frac{(abxy; q)_\infty}{(ax, ay, bx, by; q)_\infty} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(ay, by; q)_k}{(abxy; q)_k} \left(\frac{x}{y}\right)^k \frac{f^{(n)}(0) y^n}{n!} \tag{2.1}$$

provided that $|a| < 1$, $|b| < 1$.

Proof. It follows from the assumption of the theorem that

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n, \quad |t| \le 1 \tag{2.2}$$

holds. First, let $t = X$ in (2.2), and then multiply both sides of (2.2) by $1/(aX, bX; q)_\infty$, we obtain

$$\frac{f(X)}{(aX, bX; q)_\infty} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0) X^n}{n!(aX, bX; q)_\infty}, \tag{2.3}$$

where the random variable $X \sim W(x, y; q)$. Applying the expectation operator $\mathbb{E}$ on both sides of (2.3) yields

$$\mathbb{E}\left\{\frac{f(X)}{(aX, bX; q)_\infty}\right\} = \mathbb{E}\left\{\sum_{n=0}^{\infty} \frac{f^{(n)}(0) X^n}{n!(aX, bX; q)_\infty}\right\}. \tag{2.4}$$

Since

$$\left|\frac{f^{(n)}(0) X^n}{n!(aX, bX; q)_\infty}\right| \le \frac{1}{(|a|, |b|; q)_\infty} \frac{|f^{(n)}(0)|}{n!}$$

and the series

$$\sum_{n=0}^{\infty} \frac{|f^{(n)}(0)|}{n!}$$

converges absolutely, Lebesgue's dominated convergence theorem and (1.4) assert that

$$\begin{aligned} \mathbb{E}\left\{\sum_{n=0}^{\infty} \frac{f^{(n)}(0) X^n}{n!(aX, bX; q)_\infty}\right\} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \mathbb{E}\left\{\frac{X^n}{(aX, bX; q)_\infty}\right\} \\ &= \frac{(abxy; q)_\infty}{(ax, ay, bx, by; q)_\infty} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(ay, by; q)_k}{(abxy; q)_k} \left(\frac{x}{y}\right)^k \frac{f^{(n)}(0) y^n}{n!} \end{aligned} \tag{2.5}$$

holds. Substituting (2.5) into (2.4) yields (2.1). □

Under the conditions of the theorem, (2.1) is equivalent to the following q-integral formula:

$$\begin{aligned}&\int_x^y \frac{(qt/x, qt/y; q)_\infty f(t)}{(at, bt; q)_\infty}\, d_qt \\ &= \frac{y(1-q)(q, yq/x, x/y, abxy; q)_\infty}{(ax, ay, bx, by; q)_\infty} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(ay, by; q)_k}{(abxy; q)_k} \left(\frac{x}{y}\right)^k \frac{f^{(n)}(0)y^n}{n!}.\end{aligned} \tag{2.6}$$

Letting $a = b = 0$ in (2.1) gives

Corollary 2.2. *Suppose $f(t)$ admits a Maclaurian expansion in $|t| \le 1$ and the series $\sum_{n=0}^{\infty} |f^{(n)}(0)|/n!$ converges absolutely. If $X \sim W(x, y; q)$, $-1 < x < 0$, and $0 < y \le 1$, then*

$$\mathbb{E}\{f(X)\} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)y^n h_n(x/y)}{n!}. \tag{2.7}$$

We observe that (2.7) is equivalent to the following q-integral formula:

$$\begin{aligned}&\int_x^y (qt/x, qt/y; q)_\infty f(t)\, d_qt \\ &= y(1-q)(q, yq/x, x/y; q)_\infty \sum_{n=0}^{\infty} \frac{f^{(n)}(0)y^n h_n(x/y)}{n!}.\end{aligned} \tag{2.8}$$

We give some applications of the above formula.

- Let $f(t) = e^t$ in (2.8). Then $f^{(n)}(0) = 1$, $n = 0, 1, 2, \ldots$. We have

$$\int_x^y (qt/x, qt/y; q)_\infty e^t\, d_qt = y(1-q)(q, yq/x, x/y; q)_\infty \sum_{n=0}^{\infty} \frac{y^n h_n(x/y)}{n!}.$$

- Let $f(t) = \ln(1 + \theta t)$, $|\theta| < 1$ in (2.8). Then

$$\frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n-1}\theta^n}{n}, \quad n = 0, 1, 2, \ldots,$$

  and the series $\sum_{n=0}^{\infty} |f^{(n)}(0)|/n!$ converges. We have

$$\begin{aligned}&\int_x^y (qt/x, qt/y; q)_\infty \ln(1 + \theta t)\, d_qt \\ &= y(1-q)(q, yq/x, x/y; q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(\theta y)^n h_n(x/y)}{n}.\end{aligned}$$

- Let $f(t) = (1 + t)^\alpha$, $\alpha > 0$, in (2.8). Then

$$\frac{f^{(n)}(0)}{n!} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, \quad n = 0, 1, 2, \ldots,$$

and the series $\sum_{n=0}^{\infty}|f^{(n)}(0)|/n!$ converges. We have

$$\int_x^y (qt/x, qt/y; q)_\infty (1+t)^\alpha \, d_q t$$
$$= y(1-q)(q, yq/x, x/y; q)_\infty \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)y^n h_n(x/y)}{n!}.$$

On the other hand, the (2.1) (and its equivalent q-integral formula (2.6)) contains some well-known results as special cases. For example, letting $f(x) = 1$ in (2.6) gives the following Andrews-Askey integral [3], was first derived from Ramanujan's ${}_1\psi_1$ summation formula:

$$\int_c^d \frac{(qt/c, qt/d; q)_\infty}{(at, bt; q)_\infty} \, d_q t = \frac{d(1-q)(q, dq/c, c/d, abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty},$$

provided that the denominator of the integral does not vanish.

3. Some applications

We can use (2.1) to derive a number of expectation formulas. Let us recall from earlier discussion that in general, for any given $f(x)$, substituting its Maclurian expansion into (2.1), we could derive an expectation formula of random variable X with distribution $W(x, y; q)$.

3.1. The Al-Salam and Verma q-integral

Al-Salam and Verma gave an extension of the Andrews-Askey integral, which is called the Al-Salam and Verma q-integral [1],

$$\int_x^y \frac{(qt/x, qt/y, dt; q)_\infty}{(at, bt, ct; q)_\infty} \, d_q t = \frac{y(1-q)(q, yq/x, x/y, d/a, d/b, d/c; q)_\infty}{(ax, ay, bx, by, cx, cy; q)_\infty}, \tag{3.1}$$

provided that the denominator of the integral does not vanish, where $d = abcxy$. The following is a probabilistic version of the Al-Salam and Verma q-integral:

Theorem 3.1. *Suppose $X \sim W(x, y; q)$, $-1 < x < 0$, and $0 < y \le 1$. Then*

$$\mathbb{E}\left\{\frac{(abcxyX; q)_\infty}{(aX, bX, cX; q)_\infty}\right\} = \frac{(abxy, acxy, bcxy; q)_\infty}{(ax, ay, bx, by, cx, cy; q)_\infty}, \tag{3.2}$$

provided that $|a| < 1$, $|b| < 1$, $|c| < 1$.

Proof. Let

$$f(t) = \frac{(abcxyt; q)_\infty}{(ct; q)_\infty}. \tag{3.3}$$

Using the q-binomial theorem (1.1), we have

$$f(t) = \frac{(abcxyt;q)_\infty}{(ct;q)_\infty} = \sum_{n=0}^{\infty} \frac{(abxy;q)_n (ct)^n}{(q;q)_n}, \quad |t| \le 1.$$

Consequently

$$\frac{f^{(n)}(0)}{n!} = \frac{(abxy;q)_n c^n}{(q;q)_n} \tag{3.4}$$

and the series $\sum_{k=n}^{\infty} |f^{(n)}(0)|/n!$ converges. Substituting (3.3) and (3.4) into (2.1) gives

$$\begin{aligned} &\mathbb{E}\left\{\frac{(abcxyX;q)_\infty}{(aX,bX,cX;q)_\infty}\right\} \\ &= \frac{(abxy;q)_\infty}{(ax,ay,bx,by;q)_\infty} \sum_{n=0}^{\infty}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(ay,by;q)_k}{(abxy;q)_k} \left(\frac{x}{y}\right)^k \frac{(abxy;q)_n (cy)^n}{(q;q)_n}. \end{aligned} \tag{3.5}$$

After some simple computations and using the q-Gauss summation formula (1.3), we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(ay,by;q)_k}{(abxy;q)_k} \left(\frac{x}{y}\right)^k \frac{(abxy;q)_n (cy)^n}{(q;q)_n} \\ &= \frac{(abcxy^2;q)_\infty}{(cy;q)_\infty} \sum_{k=0}^{\infty} \frac{(ay,by;q)_k}{(q,abcxy^2;q)_k} (cx)^k = \frac{(acxy,bcxy;q)_\infty}{(cx,cy;q)_\infty}. \end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.5) gives (3.2). □

We remark that using the expectation formula for function $f(X)$ the (1.5), (3.2) can be rewritten as (3.1).

3.2. A formula involving Al-Salam-Carlitz polynomials

The Rogers-Szegö polynomials play an important role in the theory of orthogonal polynomials, particularly in the study of the Askey-Wilson integral [4, 10].

The set of Rogers-Szegö polynomials is a special case of the Al-Salam-Carlitz polynomials $\varphi_n^{(a)}(x|q)$ when $a = 0$. Using Mehler's formula and (2.1), we obtain the following theorem.

Theorem 3.2. *Suppose $X \sim W(x,y;q)$, $-1 < x < 0$, and $0 < y \le 1$. Then*

$$\begin{aligned} &\mathbb{E}\left\{\frac{(t_1t_2c^2X^2;q)_\infty}{(aX,bX,cX,ct_1X,ct_2X,ct_1t_2X;q)_\infty}\right\} \\ &= \frac{(abxy;q)_\infty}{(ax,ay,bx,by;q)_\infty} \sum_{n=0}^{\infty} \frac{h_n(t_1|q)h_n(t_2|q)(cy)^n}{(q;q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(ay,by;q)_k}{(abxy;q)_k} \left(\frac{x}{y}\right)^k, \end{aligned} \tag{3.7}$$

provided that $|a| < 1$, $|b| < 1$, $|c| < 1$, $|t_1| < 1$, $|t_2| < 1$.

Proof. Let

$$f(x)=\frac{(t_1t_2c^2x^2;q)_\infty}{(cx,ct_1x,ct_2x,ct_1t_2x;q)_\infty}. \tag{3.8}$$

Using Mehler's formula

$$\sum_{k=0}^{\infty}h_k(t_1|q)h_k(t_2|q)\frac{x^k}{(q;q)_k}=\frac{(t_1t_2x^2;q)_\infty}{(x,t_1x,t_2x,t_1t_2x;q)_\infty},$$

we get

$$\sum_{k=0}^{\infty}h_k(t_1|q)h_k(t_2|q)\frac{(cx)^k}{(q;q)_k}=\frac{(t_1t_2c^2x^2;q)_\infty}{(cx,ct_1x,ct_2x,ct_1t_2x;q)_\infty},\quad |x|\le 1.$$

So

$$\frac{f^{(n)}(0)}{n!}=\frac{h_n(t_1|q)h_n(t_2|q)c^n}{(q;q)_n}, \tag{3.9}$$

and the series $\sum_{n=0}^{\infty}|f^{(n)}(0)|/n!$ converges. Subsisting (3.8) and (3.9) into (2.1) gives (3.7). □

Using the expectation formula for function $f(X)$ (1.5), (3.7) can be rewritten as

$$\begin{aligned}&\int_x^y\frac{(qt/x,qt/y,t_1t_2c^2t^2;q)_\infty}{(at,bt,ct,ct_1t,ct_2t,ct_1t_2t;q)_\infty}\,d_qt\\ &=\frac{y(1-q)(q,yq/x,x/y,abxy;q)_\infty}{(ax,ay,bx,by;q)_\infty}\sum_{n=0}^{\infty}\frac{h_n(t_1|q)h_n(t_2|q)(cy)^n}{(q;q)_n}\\ &\times\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(ay,by;q)_k}{(abxy;q)_k}\left(\frac{x}{y}\right)^k,\end{aligned} \tag{3.10}$$

provided that the q-integral in (3.10) converges absolutely.

If $b=0$, $y=1$ in (3.7), we obtain

$$\begin{aligned}&\mathbb{E}\left\{\frac{(t_1t_2c^2X^2;q)_\infty}{(aX,cX,ct_1X,ct_2X,ct_1t_2X;q)_\infty}\right\}\\ &=\frac{1}{(ax,a;q)_\infty}\sum_{n=0}^{\infty}\frac{h_n(t_1|q)h_n(t_2|q)\varphi_n^{(a)}(x|q)c^n}{(q;q)_n},\end{aligned}$$

which is equivalent to the formula

$$\begin{aligned}&\int_x^y\frac{(qt/x,qt,t_1t_2c^2t^2;q)_\infty}{(at,ct,ct_1t,ct_2t,ct_1t_2t;q)_\infty}\,d_qt\\ &=\frac{(1-q)(q,q/x,x;q)_\infty}{(ax,ay;q)_\infty}\sum_{n=0}^{\infty}\frac{h_n(t_1|q)h_n(t_2|q)\varphi_n^{(a)}(x|q)c^n}{(q;q)_n}.\end{aligned}$$

3.3. A formula involving Bernoulli numbers

The Bernoulli numbers B_n are a sequence of signed rational numbers that can be defined by the generating function

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}\frac{B_n x^n}{n!},\quad |x|<2\pi. \tag{3.11}$$

Theorem 3.3. *Suppose $X\sim W(x,y;q)$, $-1<x<0$, and $0<y\le 1$. Then*

$$\begin{aligned}&\mathbb{E}\left\{\frac{X}{(aX,bX;q)_\infty(e^X-1)}\right\}\\&=\frac{(abxy;q)_\infty}{(ax,ay,bx,by;q)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(ay,by;q)_k}{(abxy;q)_k}\left(\frac{x}{y}\right)^k\frac{B_n y^n}{n!},\end{aligned} \tag{3.12}$$

where B_n is the Bernoulli number and $|a|<1$, $|b|<1$.

Proof. Let

$$f(x)=\frac{x}{e^x-1}. \tag{3.13}$$

Using the formula (3.11), we know

$$\frac{f^{(n)}(0)}{n!}=\frac{B_n}{n!}, \tag{3.14}$$

and the series $\sum_{n=0}^{\infty}|f^{(n)}(0)|/n!$ converges. Substituting (3.13) and (3.14) into (2.1) gives (3.12). □

Using the expectation formula for function $f(X)$ (1.5), (3.12) can be rewritten as

$$\begin{aligned}&\int_x^y\frac{(qt/x,qt/y;q)_\infty t}{(at,bt;q)_\infty(e^t-1)}\,d_qt\\&=\frac{y(1-q)(q,yq/x,x/y,abxy;q)_\infty}{(ax,ay,bx,by;q)_\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(ay,by;q)_k}{(abxy;q)_k}\left(\frac{x}{y}\right)^k\frac{B_n y^n}{n!},\end{aligned}$$

provided that the q-integral in (3.14) converges absolutely.

3.4. New formulas from the q-binomial theorem

Using (2.1), we can also derive some new identities. Here are two examples.

Theorem 3.4. *Suppose $|a|<1$, $|b|<1$, $-1<x<0$, and $0<y\le 1$. Then*

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-by)^n}{(q;q)_n}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(ay,by;q)_k}{(abxy;q)_k}\left(\frac{x}{y}\right)^k\right)\left(\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}q^{\binom{l}{2}+\binom{n-l}{2}}\left(\frac{a}{b}\right)^l\right)\\&=\frac{(ax,ay,bx,by;q)_\infty}{(abxy;q)_\infty}.\end{aligned} \tag{3.15}$$

Proof. Let

$$f(x) = (ax, bx; q)_\infty.$$

Then

$$\mathbb{E}\left\{\frac{f(X)}{(aX, bX; q)_\infty}\right\} = 1. \tag{3.16}$$

Using the q-binomial theorem (1.2), we know

$$\frac{f^{(n)}(0)}{n!} = \frac{(-b)^n}{(q;q)_n}\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix} q^{\binom{l}{2}+\binom{n-l}{2}}\left(\frac{a}{b}\right)^l, \tag{3.17}$$

and the series $\sum_{n=0}^{\infty}|f^{(n)}(0)|/n!$ converges. Subsisting (3.16) and (3.17) into (2.1) gives (3.15). □

Theorem 3.5. *Suppose $|a| < 1$, $|b| < 1$, $-1 < x < 0$, and $0 < y \leq 1$. Then*

$$\sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{a^i b^{n-i} y^n h_n(x/y)}{(q;q)_i (q;q)_{n-i}} = \frac{(abxy;q)_\infty}{(ax, ay, bx, by; q)_\infty}. \tag{3.18}$$

Proof. Let

$$f(t) = \frac{1}{(at, bt; q)_\infty}. \tag{3.19}$$

Using the q-binomial theorem (1.1) gives

$$f(t) = \frac{1}{(at, bt; q)_\infty} = \sum_{i=0}^{\infty}\frac{a^i t^i}{(q;q)_i}\sum_{j=0}^{\infty}\frac{b^j t^j}{(q;q)_j}.$$

We know

$$\frac{f^{(n)}(0)}{n!} = \sum_{i=0}^{n}\frac{a^i b^{n-i}}{(q;q)_i (q;q)_{n-i}} \tag{3.20}$$

and the series $\sum_{n=0}^{\infty}|f^{(n)}(0)|/n!$ converges. Substituting (3.19) and (3.20) into (2.7) yields

$$\mathbb{E}\left\{\frac{1}{(aX, bX; q)_\infty}\right\} = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{a^i b^{n-i} y^n h_n(x/y)}{(q;q)_i (q;q)_{n-i}}.$$

We obtain (3.18) after applying (1.4). □

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