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# Well-posedness and Stability of Two Classes of Plate Equations with Memory and Strong Time-dependent Delay

Baowei Feng* and Gongwei Liu

Abstract. Two classes of plate equations with past history and strong time-dependent delay in the internal feedback are considered. Our results contain the global well-posedness and exponential stability of the two systems. We prove the global well-posedness of a system with rotational inertia without any restrictions on $\mu_1$, $\mu_2$, and the system without rotational inertia under the assumption $|\mu_2| \leq \mu_1$. For the system with rotational inertia, we establish exponential stability to the plate equation with the memory term only to control the delay term if the amplitude of the time delay term is small, and the stability result also holds for the plate equation with strong anti-damping. For the system without rotational inertia, we obtain the exponential stability under the assumption $|\mu_2| < \sqrt{1-d}\mu_1$.

## 1. Introduction

### 1.1. The model

In this paper, we consider the following plate equation:

$$u_{tt}+\alpha\Delta^2u-\int_{-\infty}^{t} g(t-s)\Delta^2u(s)\,ds-\nu\Delta u_{tt}-\mu_1\Delta u_t-\mu_2\Delta u_t(t-\tau(t))+f(u)=h(x), \tag{1.1}$$

defined in a bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) with a sufficiently smooth boundary $\partial\Omega$. The function $u(x,t)$ represents the transverse displacement of a plate filament with prescribed history $u_0(x,t)$, $t \leq 0$. $\alpha$ is a positive constant and $\mu_1$ and $\mu_2$ are constants satisfying some assumptions. Here $\nu \geq 0$. The function $\tau(t)$ represents the time delay. The functions $f(u)$ and $h(x)$ are source term and nonhomogeneous term, respectively. The knowledge of the value of $u$ for all past time is assumed, i.e., $u(-s)|_{s>0} = \phi_0(s)$, where the function $\phi_0$ is a given datum.

To (1.1), we consider the following initial conditions

$$u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad x\in\Omega, \tag{1.2}$$

$$u_t(x,t)=f_0(x,t), \quad x\in\Omega,\ t\in[-\tau(0),0), \tag{1.3}$$

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*Corresponding author.

and simply supported boundary conditions:

$$u(x,t)=0, \quad \Delta u(x,t)=0 \quad \text{on } \partial\Omega\times\mathbb{R}^+. \tag{1.4}$$

Equation (1.1) is a plate equation with strong damping, memory with history and strong time-dependent delay in the internal feedback. The strong damping term $-\mu_1\Delta u_t$ can be regarded as Kelvin-Voigt damping, which occurs in the study of the motion of viscoelastic materials, for instance, the string is made up of the viscoelastic material of rate-type [5], and it indicates that the stress is proportional not only to the stain, as with the Hooke law, but also to the stain rate as in a linearized Kelvin-Voigt material. The term $-\nu\Delta u_{tt}$ is rotational inertia. The viscoelastic term can be regarded as a natural weak damping which is related to their special property of retaining a long-time range memory of their past histories.

Since the memory with past history, following the same arguments in Dafermos [2], we define a new variable $\eta=\eta^t(x,s)$ as

$$\eta^t(x,s)=u(x,t)-u(x,t-s), \quad (t,s)\in\mathbb{R}^+\times\mathbb{R}^+,$$

then

$$\eta_t+\eta_s=u_t, \quad (x,t,s)\in\mathbb{R}^n\times\mathbb{R}^+\times\mathbb{R}^+$$

with

$$\eta^t(0)=0 \quad \text{in } \mathbb{R}^n,\ t\geq 0.$$

Thus the original history can be rewritten as

$$\begin{aligned}\int_{-\infty}^{t} g(t-s)\Delta^2u(s)\,ds &= \int_0^\infty g(s)\Delta^2u(t-s)\,ds\\ &= \left(\int_0^\infty g(s)\,ds\right)\Delta^2u(t)-\int_0^\infty g(s)\Delta^2\eta^t(s)\,ds.\end{aligned} \tag{1.5}$$

Then combining (1.1)–(1.4) with (1.5) and assuming for simplicity that $\alpha-\int_0^\infty g(s)\,ds=1$, we can get the following problem which is equivalent to problem (1.1)–(1.4):

$$u_{tt}+\Delta^2u+\int_0^\infty g(s)\Delta^2\eta^t(s)\,ds-\nu\Delta u_{tt}-\mu_1\Delta u_t-\mu_2\Delta u_t(t-\tau(t))+f(u)=h(x), \tag{1.6}$$

$$\eta_t^t+\eta_s^t=u_t, \tag{1.7}$$

$$u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), \quad \eta^t(x,0)=0, \quad x\in\mathbb{R}^n, \tag{1.8}$$

$$u_t(x,t)=f_0(x,t), \quad x\in\mathbb{R}^n,\ t\in[-\tau(0),0), \tag{1.9}$$

$$\eta^0(x,s)=\eta_0(x,s), \quad (x,s)\in\mathbb{R}^n\times\mathbb{R}^+, \tag{1.10}$$

$$u(x,t)=0, \quad \Delta u(x,t)=0 \quad \text{on } \partial\Omega\times\mathbb{R}^+, \tag{1.11}$$

where

$$\eta_0(x,s)=u_0(x)-\phi_0(s), \quad s\in\mathbb{R}^+.$$

### 1.2. Literature overview

In recent years, many mathematical researchers have been studying partial differential equations with time delay effects, and established so many results concerning the global well-posedness of these systems. The delay effects often arise in many practical problems, for instance, chemical, physical, thermal and economic phenomena and so on, and may turn a well-behaved system into a wild one. The presence of delay can be a source of instability and an arbitrarily small delay may destabilize a system that is uniformly asymptotically stable in the absence of delay unless additional control terms are added. For some results on wave equation with time delay, one can refer Datko, Lagnese and Polis [4], Feng [9], Kafini, Messaoudi and Nicaise [13], Liu [16,17], Nicaise and Pignotti [21], Nicaise, Valein and Fridman [25], Nicaise and Valein [24], Xu, Yung and Li [30], and the references therein. Here we mention the work of Nicaise and Pignotti [20]. In this work, the authors studied a wave equation with time delay

$$u_{tt} - \Delta u + a(x)(\mu_1 u_t + \mu_2 u_t(t-\tau)) = 0,$$

and established stability results under the assumption $0 < \mu_2 < \mu_1$. They also studied the instability of the system. On the other hand, they considered a wave equation with a delay term in the boundary and proved the energy decay if $0 < \mu_2 < \mu_1$. In [22], the same authors extended the results to a problem with time-dependent delay and studied a wave equation with boundary or internal time-varying delay feedback in a bounded and smooth domain. They considered the following wave equation with strong damping of the form

$$u_{tt} - \Delta u - a\Delta u_t = 0,$$

and with a boundary time-varying delay feedback

$$\mu u_{tt} = -\frac{\partial(u + a u_t)}{\partial \nu} - k u_t(t - \tau(t)) \quad \text{on } \Gamma_1 \times (0, \infty).$$

They proved the well-posedness and decay of energy to the problem under the assumption $|k| \le \frac{a}{C_P}\sqrt{1-d}$, where $C_P$ is the Poincaré constant. In addition, they investigated the case of a internal time-varying delay feedback

$$u_{tt} - \Delta u + a_0 u_t + a_1 u_t(t - \tau(t)) = 0,$$

and established the global well-posedness and stability with $|a_1| < \sqrt{1-d}a_0$. For viscoelastic wave equation with delay, Kirane and Said-Houari [14] studied the following equation

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)\, ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0,$$

and obtained the global well-posedness and energy decay under the condition $0 < \mu_2 \leq \mu_1$. Dai and Yang [3] considered the same equation as in [14], and improved the results in [14]. In this paper, the authors proved the global existence of solutions without restrictions of $\mu_1, \mu_2 > 0$ and $\mu_2 < \mu_1$, and obtained an exponential decay result of energy in the case $\mu_1 = 0$. In Liu and Zhang [18], the authors considered a wave equation with past history and time delay of the form

$$u_{tt} - \alpha\Delta u + \int_{-\infty}^{t} \mu(t-s)\Delta u(s)\,ds + \mu_1 u_t + \mu_2 u_t(t-\tau) + f(u) = h.$$

They proved the global well-posedness of the problem and established the exponential decay of energy under the assumption $0 < |\mu_2| < \mu_1$. Recently, Alabau-Boussouira, Nicaise and Pignotti [1] investigated a wave equation of the form

$$u_{tt} - \Delta u + \int_{0}^{\infty} \mu(s)\Delta u(t-s)\,ds + k u_t(t-\tau) = 0,$$

and proved that the system is exponentially stable if the constant $k$ is small enough. They also established the stability of an anti-damping system, i.e., the case $\tau = 0$ and $k < 0$. Messaoudi, Fareh and Doudi [19] considered a wave equation of the form

$$u_{tt} - \Delta u - \mu_1 \Delta u_t - \mu_2 \Delta u_t(t-\tau) = 0,$$

and proved the well-posedness with $|\mu_2| \leq \mu_1$ and energy decay with $|\mu_2| < \mu_1$. In addition, they also considered a wave equation with distributed delay. Feng considered a viscoelastic wave equation with strong delay and established a general decay result of energy, see [8].

For plate equation with time delay effects, Park [26] considered a weak viscoelastic plate equation with a time-varying delay

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u + \sigma(t)\int_{0}^{t} g(t-s)\Delta u(s)\,ds + a_0 u_t + a_1 u_t(t-\tau(t)) = 0,$$

and proved a general decay result of energy under the assumption $|a_1| < \sqrt{1-d}\,a_0$. Yang [31] studied a viscoelastic plate equation with a time delay

$$u_{tt} + \Delta^2 u - \int_{0}^{t} g(t-s)\Delta^2 u(s)\,ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0.$$

The author proved the global existence of solutions for any real numbers $\mu_1, \mu_2 > 0$, and established the exponential stability with $0 < |\mu_2| < \mu_1$ and $\mu_1 = 0$. Very recently, the first author of the present paper considered a plate equation with past history, source term and time-dependent delay

$$u_{tt} + \alpha\Delta^2 u - \int_{-\infty}^{t} g(t-s)\Delta^2 u(s)\,ds + \mu_1 u_t + \mu_2 u_t(t-\tau(t)) + f(u) = h(x).$$

The author obtained the global well-posedness without any restrictions on $\mu_1$ and $\mu_2$. In addition, the exponential stability was achieved when $f(u) \neq 0$ under the assumption $|\mu_2| < \sqrt{1-d}\mu_1$ and when $f(u) = 0$, $\mu_1 = 0$ if the coefficient of delay $|\mu_2|$ is small, and then the author obtained the existence of a global attractor and exponential attractor, see [7] and [10], respectively. Feng [6] considered the following equation

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u - \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = 0,$$

and obtained the global well-posedness with $|\mu_2| \leq \mu_1$ and decay rate of energy under the assumption $|\mu_2| < \mu_1$. Recently, Feng and co-authors [11] considered an extensible viscoelastic plate equation with a nonlinear time-varying delay feedback and nonlinear source term, and established a general decay of energy. Nicaise and Pignotti [23] considered an abstract evolution equations with constant time delay of the form

$$\begin{aligned} &U_t(t) = \mathcal{A}U(t) + F(U(t)) + k\mathcal{B}U(t-\tau), \\ &U(0) = U_0, \quad \mathcal{B}U(t-\tau) = f(t), \end{aligned}$$

where $\mathcal{B}$ is a bounded operator. They proved that the system is exponentially stable without delay, and also obtained that the model with delay remains exponentially stable if the coefficient of time delay feedback is sufficiently small. Recently, Pignotti [29] studied a second-order evolution equations with memory and intermittent delay feedback

$$u_{tt} + Au - \int_0^\infty \mu(s)Au(t-s)\,ds + b(t)u_t(t-\tau) = 0,$$

and showed that the stability if $b \in L^1(0,\infty)$ and the length of time intervals is very large. Furthermore, the author established the stability results for a problem with anti-damping. On the other hand, there are so many results on plate equation in absence of delay. It has been stabilized by means of different controls, for example, internal damping, boundary controls, dynamic boundary conditions, distributed damping and heat damping, and so on.

### 1.3. Goals and features

Our goals in the present work are to study the global well-posedness and exponential stability of two classes of plate equations with strong time-dependent delay. The main features of this work are as follows:

- Since the delay is time-dependent, so our results are more general than those in [1], where the authors considered a wave equation with a constant time delay.

- For global well-posedness, we first consider equation (1.6) with $\nu > 0$, which the system is new, and prove the global well-posedness of this system without any restrictions of $\mu_1$ and $\mu_2$. The main result is presented in Theorem 3.2. We also obtain the global well-posedness of system without rotational inertia, i.e., $\nu = 0$ in (1.6), under the assumption $|\mu_2| \le \mu_1$. The result is different from the result in [22], where the existence of solution holds for $|\mu_2| < \frac{\mu_1}{C_P}\sqrt{1-d}$. The global well-posedness is presented in Theorem 3.4. From the proof of Theorems 3.2 and 3.4, one will see that the rotational inertia $-\nu\Delta u_{tt}$ plays a crucial role in restrictions between $\mu_1$ and $\mu_2$ for the global well-posedness. Since the delay is strong, we need much more regularity of $u_t$.

- For stability, we consider the following two classes of plate equations:

$$u_{tt} + \Delta^2 u + \int_0^\infty g(s)\Delta^2\eta^t(s)\,ds - \Delta u_{tt} - \mu_2\Delta u_t(t-\tau(t)) = 0, \tag{1.12}$$

and

$$u_{tt} + \Delta^2 u + \int_0^\infty g(s)\Delta^2\eta^t(s)\,ds - \mu_1\Delta u_t - \mu_2\Delta u_t(t-\tau(t)) + f(u) = 0. \tag{1.13}$$

For the system with rotational inertia (1.12), we consider the system with the memory term only to control the strong time-dependent delay term. To establish the stability, we consider an auxiliary problem. We add the strong damping to the equation, see (4.3), and obtain the exponential stability to the two systems by using a perturbative approach. Our results are presented in Theorems 4.1 and 4.7. The stability result also holds for the plate equation with a strong anti-damping, i.e., the case $\tau(t) = 0$ and $\mu_2 < 0$, namely a strong damping with an opposite sign with respect to the standard strong dissipation, and thus it induces instability, see, for example, Freitas and Zuazua [12]. The result is presented in Remark 4.9. For the system without rotational inertia (1.13), we establish some multipliers to get exponential stability under the assumption $|\mu_2| < \sqrt{1-d}\mu_1$. The result is presented in Theorem 4.10.

The plan of the paper is as follows. In Section 2, we give some preliminaries. In Section 3, we shall state and prove the global well-posedness of the problem. The stability result and proofs will be given in Section 4.

## 2. Preliminaries

In the following, $L^q(\Omega)$, $(1 \le q \le \infty)$ and $H^i(\Omega)$, $(i = 1, 2, 3)$, are the standard notations of Lebesgue integral and Sobolev spaces. The $L^2$-inner product is denoted by $(\,\cdot\,,\cdot\,)$ and

$\|\cdot\|_B$ denotes the norm in the space $B$. For simplicity, we use $\|u\|$ instead of $\|u\|_2$ when $q = 2$. The constants $\lambda_1$, $\lambda_2$, $\lambda_3$ are the embedding constants

$$\lambda_1\|u\|^2 \le \|\Delta u\|^2, \quad \lambda_2\|\nabla u\|^2 \le \|\Delta u\|^2, \quad \lambda_3\|u\|^2 \le \|\nabla u\|^2$$

for $u \in H^2 \cap H_0^1$.

Now we give some assumptions used in this paper. The source term $f(u)$ is a nonlinear functional that $f(0) = 0$, and

$$|f(u) - f(v)| \le c_f(1 + |u|^p + |v|^p)|u - v|, \quad \forall\, u, v \in \mathbb{R}, \tag{2.1}$$

where $c_f$ is a positive constant and

$$0 < p \le \frac{4}{n-4} \text{ if } n \ge 5 \quad \text{and} \quad p > 0 \text{ if } 1 \le n \le 4. \tag{2.2}$$

We assume further that

$$0 \le \widehat{f}(u) \le f(u)u, \quad \forall\, u \in \mathbb{R}, \tag{2.3}$$

where $\widehat{f}(u) = \int_0^u f(z)\,dz$. Assumptions (2.1) and (2.3) include the following nonlinear type

$$f(u) \approx |u|^p u + |u|^\alpha u, \quad 0 < \alpha < p.$$

Concerning the relaxation function $g$, we assume that $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function satisfying

$$g(0) > 0, \quad \int_0^\infty g(s)\,ds = l_0 > 0, \tag{2.4}$$

and there exists a positive constant $k$ such that

$$g'(t) \le -kg(t) \quad \text{for } t \ge 0. \tag{2.5}$$

With respect to the delay $\tau(t)$, we first assume

$$0 < \tau_0 \le \tau(t) \le \tau_1, \quad \forall\, t > 0, \tag{2.6}$$

where the constants $\tau_0$ and $\tau_1$ are two positive constants. We assume further that

$$\tau(t) \in W^{2,\infty}(0,T), \quad \text{and} \quad \tau'(t) \le d < 1, \quad \forall\, T, t > 0. \tag{2.7}$$

In the following, we consider the Hilbert spaces that will be used in the present work. Let

$$V_0 = L^2(\Omega), \quad V_1 = H_0^1(\Omega), \quad V_2 = H^2(\Omega) \cap H_0^1(\Omega)$$

and

$$V_3 = \{u \in H^3(\Omega) \cap H^1(\Omega) : \Delta u \in H_0^1(\Omega)\}.$$

To consider the new variable $\eta$, we define the weighted $L^2$-spaces

$$\mathcal{M}_i = L_g^2(\mathbb{R}^+, V_i) = \left\{\eta\colon \mathbb{R}^+ \to V_i : \int_0^\infty g(s)\|\eta(s)\|_{V_i}^2 \, ds < \infty\right\}, \quad i = 0, 1, 2, 3,$$

which are Hilbert spaces endowed with inner products and norms

$$(\eta, \zeta)_{\mathcal{M}_i} = \int_0^\infty g(s)(\eta(s), \zeta(s))_{V_i} \, ds \quad \text{and} \quad \|\eta\|_{\mathcal{M}_i}^2 = \int_0^\infty g(s)\|\eta(s)\|_{V_i}^2 \, ds, \quad i = 0, 1, 2, 3.$$

Finally, we introduce the phase spaces

$$\mathcal{H} = V_2 \times V_1 \times \mathcal{M}_2, \quad \mathcal{H}_1 = V_3 \times V_2 \times \mathcal{M}_3$$

and

$$\mathcal{H}_2 = V_2 \times V_0 \times \mathcal{M}_2, \quad \mathcal{H}_3 = V_3 \times V_1 \times \mathcal{M}_3$$

equipped with the norms

$$\|(u, v, \eta)\|_{\mathcal{H}}^2 = \|\Delta u\|^2 + \|\nabla v\|^2 + \|\eta\|_{\mathcal{M}_2}^2,$$
$$\|(u, v, \eta)\|_{\mathcal{H}_1}^2 = \|\nabla \Delta u\|^2 + \|\Delta v\|^2 + \|\eta\|_{\mathcal{M}_3}^2$$

and

$$\|(u, v, \eta)\|_{\mathcal{H}_2}^2 = \|\Delta u\|^2 + \|v\|^2 + \|\eta\|_{\mathcal{M}_2}^2,$$

respectively.

The following lemma plays a very important role to get the exponential stability of the problem. One can find the proof in Komornik [15, Theorem 8.1].

**Lemma 2.1.** *Let $V(\cdot)$ be a non-negative decreasing function defined on $[0, \infty)$. If there exists some constant $C > 0$ such that for any $S > 0$,*

$$\int_S^\infty V(t) \, dt \le CV(S),$$

*then we have for any $t \ge 0$,*

$$V(t) \le V(0) \exp\left(1 - \frac{t}{C}\right).$$

We will use the following theorem to prove exponential stability of the problem with rotational inertia and $\mu_1 = 0$, which can be found in Pazy [27, Theorem 1.1 in Chapter 3].

**Theorem 2.2.** *Let $X$ be a Banach space and $A$ is the infinitesimal generator of a $C_0$-semigroup $T(t)$ on $X$ satisfying $\|T(t)\| \le Me^{\omega t}$. If $B$ is a bounded linear operator on $X$, then $A + B$ is the infinitesimal generator of a $C_0$-semigroup $S(t)$ on $X$ satisfying $\|S(t)\| \le Me^{(\omega + M\|B\|)t}$.*

## 3. Well-posedness

In this section, we shall establish the global well-posedness of problem (1.6)–(1.11), which will be divided into the following two subsections.

### 3.1. The system with rotational inertia

In this subsection, we consider (1.6) with $\nu > 0$ and take $\nu = 1$ without loss of generality. Let us first give the definition of weak solutions.

**Definition 3.1.** We call a function $U(t) = (u, u_t, \eta^t) \in C([0,T], \mathcal{H})$ a weak solution of the problem (1.6)–(1.11) for given $T > 0$ if $U(0) = (u_0, u_1, \eta_0)$, and

$$\begin{gathered}\frac{d}{dt}[(u_t(t),\omega) + (\nabla u_t(t), \nabla\omega)] + (\Delta u(t), \Delta\omega) + (\eta^t, \omega)_{\mathcal{M}_2}\\ + \mu_1(\nabla u_t(t), \nabla\omega) + \mu_2(\nabla u_t(t-\tau(t)), \nabla\omega) + (f(u) - h, \omega) = 0,\\ (\partial_t \eta^t, \xi)_{\mathcal{M}_2} = -(\partial_s \eta^t, \xi)_{\mathcal{M}_2} + (u_t(t), \xi)_{\mathcal{M}_2},\end{gathered}$$

a.e. in $[0,T]$, for all $\omega \in V_2$, $\xi \in \mathcal{M}_2$.

The following theorem is concerned with the global well-posedness of problem (1.6)–(1.11).

**Theorem 3.2.** *Under the assumptions* (2.1)–(2.7), *we have*

(i) *If initial data* $U(0) = (u_0, u_1, \eta_0) \in \mathcal{H}$, $f_0(x,t) \in H^1(\Omega \times (-\tau(0), 0))$ *and* $h(x) \in L^2(\Omega)$, *then problem* (1.6)–(1.11) *has a weak solution* $(u, u_t, \eta^t) \in C(0,T;\mathcal{H})$, $\forall\, T > 0$, *satisfying*

$$u \in L^\infty(0,T;V_2), \quad u_t \in L^\infty(0,T;V_1) \quad \text{and} \quad \eta^t \in L^\infty(0,T;\mathcal{M}_2).$$

(ii) *If initial data* $U(0) = (u_0, u_1, \eta_0) \in \mathcal{H}_1$, $f_0(x,t) \in H^2(\Omega \times (-\tau(0), 0))$ *and* $h(x) \in L^2(\Omega)$, *then problem* (1.6)–(1.11) *has a stronger weak solution such that for any* $T > 0$,

$$u \in L^\infty(0,T;V_3), \quad u_t \in L^\infty(0,T;V_2) \quad \text{and} \quad \eta^t \in L^\infty(0,T;\mathcal{M}_3).$$

(iii) *In both cases the weak solution depends continuously on the initial data in* $\mathcal{H} \times H^1(\Omega \times (-\tau(0), 0))$, *i.e., given any two weak solutions* $U_1$ *and* $U_2$ *of problem* (1.6)–(1.11) *corresponding the initial data* $U_1(0), U_2(0) \in \mathcal{H}$ *and* $f_0(x,t), \widetilde{f}_0(x,t) \in H^1(\Omega \times (-\tau(0), 0))$, *then for some constant* $C_T > 0$ *depending the initial data in the phase space* $\mathcal{H}$ *and any time* $T > 0$, *for all* $t \in [0,T]$,

$$\|U_1(t) - U_2(t)\|_{\mathcal{H}}^2 \le C_T(\|U_1(0) - U_2(0)\|_{\mathcal{H}}^2 + \|f_0(x,t) - \widetilde{f}_0(x,t)\|_{H^1(\Omega\times(-\tau(0),0))}^2).$$

*In particular, the weak solution of problem* (1.6)–(1.11) *is unique.*

*Proof. Step* 1*: Approximate problem.* By using the theory of ODEs, we can find a local approximate solution $(u^m(t), u^m_t(t), \eta^{t,m})$ in the following form

$$u^m(t) = \sum_{j=1}^{m} a_{mj}(t)\omega_j(x) \in \mathrm{Span}\{\omega_1, \ldots, \omega_n\}$$

and

$$\eta^{t,m}(s) = \sum_{j=1}^{m} b_{mj}(t)\xi_j(x,s) \in \mathrm{Span}\{\xi_1, \ldots, \xi_m\},$$

satisfying the following approximate problem

$$\begin{aligned}(u^m_{tt}(t), \omega_j) + (\nabla u^m_{tt}, \nabla\omega) + (\Delta u^m(t), \Delta\omega_j) + (\eta^{t,m}, \omega_j)_{\mathcal{M}_2}\\ + \mu_1(\nabla u^m_t(t), \nabla\omega_j) + \mu_2(\nabla u^m_t(t-\tau(t)), \nabla\omega_j) + (f(u^m) - h, \omega_j) = 0,\end{aligned} \tag{3.1}$$

$$(\partial_t\eta^{t,m}, \xi_j)_{\mathcal{M}_2} = -(\partial_s\eta^{t,m}, \xi_j)_{\mathcal{M}_2} + (u^m_t(t), \xi_j)_{\mathcal{M}_2} \tag{3.2}$$

with initial conditions

$$(u^m(0), u^m_t(0), \eta^{0,m}, u^m_t(t)) = (u^m_0, u^m_1, \eta^m_0, f^m_0),$$

on some small time interval $[0, T_m)$ with $0 < T_m \le T$ for every $m \in \mathbb{N}$, where $\{\omega_j\}_{j=1}^{\infty}$ is the Galerkin basis given by the eigenfunctions of $\Delta^2$ in $\Omega$ with boundary condition (1.11) and a smooth orthonormal basis $\{\xi_j(x,s)\}_{j=1}^{\infty}$ for $\mathcal{M}_2$.

Next we consider the initial data $(u_0, u_1, \eta_0) \in \mathcal{H}_1$ and $f_0(x,t) \in H^2_0(\Omega \times (-\tau(0), 0))$ in the approximate problem (3.1)–(3.2) satisfying

$$u^m_0 \to u_0 \text{ in } V_3, \quad u^m_1 \to u_1 \text{ in } V_2,$$

and

$$\eta^m_0 \to \eta_0 \text{ in } \mathcal{M}_3, \quad f^m_0 \to f_0 \text{ in } H^2_0(\Omega \times (-\tau(0), 0)).$$

*Step* 2*: A prior estimate I.* Multiplying the equation (1.6) by $u^m_t$, and using the equation (1.7), we can get for any $t > 0$,

$$\begin{aligned}\frac{d}{dt}E^m(t) = -\mu_1\|\nabla u^m_t\|^2 - \mu_2\int_\Omega \nabla u^m_t(t)\nabla u^m_t(t-\tau(t))\,dx\\ + \frac{1}{2}\int_0^\infty g'(s)\|\Delta\eta^{t,m}(s)\|^2\,ds + \int_\Omega hu^m_t(t)\,dx\end{aligned} \tag{3.3}$$

with

$$\begin{aligned}E^m(t) = \frac{1}{2}\|u^m_t(t)\|^2 + \frac{1}{2}\|\nabla u^m_t(t)\|^2 + \frac{1}{2}\|\Delta u^m(t)\|^2\\ + \frac{1}{2}\|\eta^{t,m}\|^2_{\mathcal{M}_2} + \int_\Omega \widehat{f}(u^m(t))\,dx,\end{aligned} \tag{3.4}$$

where we used the following fact

$$(\partial_s \eta^{t,m}, \eta^{t,m})_{\mathcal{M}_2} = \frac{1}{2}\int_\Omega \int_0^\infty g(s)\frac{d}{ds}|\Delta \eta^{t,m}|^2\, dsdx = -\frac{1}{2}\int_0^\infty g'(s)\|\Delta \eta^{t,m}(s)\|^2\, ds.$$

By using Young's inequality, we can easily obtain

$$-\mu_2 \int_\Omega \nabla u_t^m(t)\nabla u_t^m(t-\tau(t))\, dx \le \frac{|\mu_2|}{2}\|\nabla u_t^m(t)\|^2 + \frac{|\mu_2|}{2}\|\nabla u_t^m(t-\tau(t))\|^2. \tag{3.5}$$

It follows from (3.4) that

$$\int_\Omega h u_t^m(t)\, dx \le \sqrt{2}\|h\|(E^m(t))^{1/2},$$

which, combined with (3.3) and (3.5), implies

$$\frac{d}{dt}E^m(t) \le \left(|\mu_1| + \frac{|\mu_2|}{2}\right)\|\nabla u_t^m\|^2 + \frac{|\mu_2|}{2}\|\nabla u_t^m(t-\tau(t))\|^2 + \sqrt{2}\|h\|(E^m(t))^{1/2}.$$

This shows that

$$\begin{aligned} E^m(t) \le E^m(0) &+ \left(|\mu_1| + \frac{|\mu_2|}{2}\right)\int_0^t \|\nabla u_t^m(s)\|^2\, ds \\ &+ \frac{|\mu_2|}{2}\int_0^t \|\nabla u_t^m(s-\tau(t))\|^2\, ds + \sqrt{2}\|h\|\int_0^t (E^m(s))^{1/2}\, ds. \end{aligned} \tag{3.6}$$

On the other hand, we shall see that

$$\begin{aligned} \int_0^t \|\nabla u_t^m(s-\tau(t))\|^2\, ds &= \int_{-\tau(t)}^0 \|\nabla u_t^m(s)\|^2\, ds + \int_0^{t-\tau(t)} \|\nabla u_t^m(s)\|^2\, ds \\ &= \int_{-\tau(t)}^0 \|\nabla f_0^m(s)\|^2\, ds + \int_0^{t-\tau(t)} \|\nabla u_t^m(s)\|^2\, ds \\ &\le \int_{-\tau(t)}^0 \|\nabla f_0^m(s)\|^2\, ds + \int_0^t \|\nabla u_t^m(s)\|^2\, ds, \end{aligned} \tag{3.7}$$

which, together with (3.6), yields for any $t > 0$

$$\begin{aligned} E^m(t) \le E^m(0) &+ 2(|\mu_1| + |\mu_2|)\int_0^t E^m(s)\, ds + \sqrt{2}\|h\|\int_0^t (E^m(s))^{1/2}\, ds \\ &+ \frac{|\mu_2|}{2}\int_{-\tau(t)}^0 \|\nabla f_0^m(s)\|^2\, ds. \end{aligned} \tag{3.8}$$

By using the Gronwall inequality to (3.8), we can get for any $t \in [0, T]$,

$$E^m(t) \le \left[\left(E^m(0) + \frac{|\mu_2|}{2}\int_{-\tau(t)}^0 \|\nabla f_0^m(s)\|^2\, ds\right)^{1/2} + \frac{\sqrt{2}}{2}\|h\|\right]^2 e^{2(|\mu_1|+|\mu_2|)T}.$$

From the choice of initial data, we can conclude that for all $t \in [0, T]$ and for every $m \in \mathbb{N}$,

$$E^m(t) \le C, \tag{3.9}$$

where $C = C(\|u_1\|, \|\Delta u_0\|, \|\eta_0\|_{\mathcal{M}_2}, \|f_0\|_{H^1(\Omega\times(-\tau(0),0))}, \|h\|) > 0$.

*Step 3: A prior estimate II.* We multiply the equation (3.1) by $-\Delta u_t^m$ and (3.2) by $-\Delta\eta^{t,m}$, respectively, and integrate the results over $\Omega$ to derive that

$$\begin{aligned} \frac{d}{dt}F^m(t) &= -\mu_1\|\Delta u_t^m\|^2 + \mu_2(\Delta u_t^m(t-\tau(t)), \Delta u_t^m) \\ &\quad + \frac{1}{2}\int_0^\infty g'(s)\|\nabla\Delta\eta^{t,m}(s)\|^2\,ds + (f(u^m) - h, \Delta u_t^m), \end{aligned} \tag{3.10}$$

where

$$F^m(t) = \frac{1}{2}\|\nabla u_t^m\|^2 + \frac{1}{2}\|\Delta u_t^m\|^2 + \|\nabla\Delta u^m\|^2 + \|\eta^{t,m}\|^2_{\mathcal{M}_3}.$$

Using Hölder's inequality, we can obtain

$$\mu_2\int_\Omega \Delta u_t^m(t-\tau(t))\Delta u_t^m\,dx \le \frac{|\mu_2|}{2}\|\Delta u_t^m\|^2 + \frac{|\mu_2|}{2}\|\Delta u_t^m(t-\tau(t))\|^2. \tag{3.11}$$

It follows from (2.1)–(2.2), Hölder's inequality and Young's inequality that

$$\begin{aligned} \int_\Omega (f(u^m) - h)\Delta u_t^m\,dx &\le \|f(u^m) - h\|\|\Delta u_t^m\| \\ &\le C + c_f\|\Delta u_t^m\|^2, \end{aligned} \tag{3.12}$$

where $C = C(\|h\|)$. Inserting (3.11)–(3.12) into (3.10) and using the same arguments as (3.8), we know that

$$F^m(t) \le F^m(0) + 2(|\mu_1| + |\mu_2| + c_f)\int_0^t F^m(s)\,ds + \frac{|\mu_2|}{2}\int_{-\tau(t)}^0 \|\Delta f_0^m(s)\|^2\,ds + CT.$$

Hence, noting the choice of initial data, and applying Gronwall's inequality, we finally conclude

$$F^m(t) \le C', \quad \forall\, t \in [0, T],\ \forall\, m \in \mathbb{N}, \tag{3.13}$$

where $C' = C'(\|\nabla u_1\|, \|\nabla\Delta u_0\|, \|\eta_0\|_{\mathcal{M}_3}, \|f_0\|_{H^2(\Omega\times(-\tau(0),0))}, \|h\|, T) > 0$.

The above estimates (3.9) and (3.13) are sufficient to pass limit in the approximate problem (3.1)–(3.2) and hence we can get a stronger weak solution.

*Step 4: Continuous dependence.* Let $U_1(t) = (u, u_t, \eta)$ and $U_2(t) = (\widetilde{u}, \widetilde{u}_t, \widetilde{\eta})$ be two stronger weak solutions of problem (1.6)–(1.11) with initial data $U_1(0) = (u_0, u_1, \eta_0)$,

$u_t(x,t) = f_0(x,t)$ and $U_2(0) = (\widetilde{u}_0, \widetilde{u}_1, \widetilde{\eta}_0)$, $\widetilde{u}_t(x,t) = \widetilde{f}_0(x,t)$, respectively. By setting $\omega = u - \widetilde{u}$ and $\xi = \eta - \widetilde{\eta}$, we know that the function $(\omega, \omega_t, \xi)$ solves the following problem

$$\begin{aligned}\omega_{tt}(t) + \Delta^2\omega(t) + \int_0^\infty g(s)\Delta^2\xi(s)\,ds - \Delta\omega_{tt} - \mu_1\Delta\omega_t(t)\\ - \mu_2\Delta\omega(t-\tau(t)) + f(u) - f(\widetilde{u}) = 0,\end{aligned} \tag{3.14}$$

$$\xi_t + \xi_s = \omega_t, \tag{3.15}$$

with initial data

$$(\omega(0), \omega_t(0), \xi^0) = (u_0 - \widetilde{u}_0, u_1 - \widetilde{u}_1, \eta_0 - \widetilde{\eta}_0) = U_1(0) - U_2(0),$$

and

$$\omega_t(x,t) = g_0(x,t) = f_0(x,t) - \widetilde{f}_0(x,t), \quad t < 0.$$

Multiplying (3.14) by $\omega_t$ in $V_0$, (3.15) by $\xi$ in $\mathcal{M}_2$ respectively, and using integration by parts, we can deduce

$$\begin{aligned}\frac{d}{dt}W(t) = -(\partial_s\xi^t, \xi^t)_{\mathcal{M}_2} - \mu_1\|\nabla\omega_t\|^2 - \mu_2\int_\Omega \nabla\omega_t(t)\nabla\omega_t(t-\tau(t))\,dx\\ - \int_\Omega (f(u) - f(\widetilde{u}))\omega_t\,dx,\end{aligned}$$

where

$$W(t) = \frac{1}{2}\|\omega_t\|^2 + \frac{1}{2}\|\nabla\omega_t\|^2 + \frac{1}{2}\|\Delta\omega\|^2 + \frac{1}{2}\|\xi\|^2_{\mathcal{M}_2}.$$

By using Hölder's and Young's inequalities, we have

$$\begin{aligned}\int_\Omega (f(u) - f(\widetilde{u}))\omega_t\,dx &\le C_R(T)(1 + \|u\|^p_{2(p+1)} + \|\widetilde{u}\|^p_{2(p+1)})\|\omega\|_{2(p+1)}\|\omega_t\|\\ &\le C_1\|\Delta\omega\|^2 + c_f\|\omega_t\|^2,\end{aligned}$$

where we used the embedding $H^2 \hookrightarrow L^{2(p+1)}$.

Following the same arguments as (3.5) and (3.7), we can get that there exists a constant $C_R > 0$ such that

$$\frac{d}{dt}W(t) \le C_R W(t) + \frac{|\mu_2|}{2}\int_\Omega (\nabla f_0 - \nabla\widetilde{f}_0)^2\,dx.$$

Then we have

$$W(t) \le W(0) + C_R\int_0^t W(s)\,ds + \frac{|\mu_2|}{2}\int_\Omega\int_{-\tau(t)}^0 (\nabla f_0 - \nabla\widetilde{f}_0)^2\,dsdx,$$

which, together with Gronwall's inequality, implies

$$W(t) \le \left(W(0) + \frac{|\mu_2|}{2}\int_\Omega\int_{-\tau(t)}^0 (\nabla f_0 - \nabla\widetilde{f}_0)^2\,dsdx\right)e^{C_R t}.$$

From the definition of $W(t)$, we infer that for all $t \in [0,T]$,

$$\begin{aligned}
&\|u_t - \widetilde{u}_t\|^2 + \|\nabla u_t - \nabla \widetilde{u}_t\|^2 + \|\Delta u - \Delta \widetilde{u}\|^2 + \|\eta - \widetilde{\eta}\|^2_{\mathcal{M}_2} \\
&\le C_T\Big( \|u_1 - \widetilde{u}_1\|^2 + \|\nabla u_1 - \nabla \widetilde{u}_1\|^2 + \|\Delta u_0 - \Delta \widetilde{u}_0\|^2 \\
&\qquad + \|\eta - \widetilde{\eta}\|^2_{\mathcal{M}_2} + \|f_0 - \widetilde{f}_0\|^2_{H^1(\Omega\times(-\tau(0),0))}\Big),
\end{aligned}$$

which gives us that stronger weak solutions of problem (1.6)–(1.11) depend continuously on the initial data. Then we know that the stronger weak solutions of problem (1.6)–(1.11) is unique.

By using density arguments, we can get the existence of weak solution and continuous dependence to problem (1.6)–(1.11).

Therefore the proof of Theorem 3.2 is complete. □

## 3.2. The system without rotational inertia

Let us consider (1.6) with $\nu = 0$. The definition of weak solutions is as follows.

**Definition 3.3.** For given $U(0) = (u_0, u_1, \eta_0) \in \mathcal{H}_2$, we say that $U(t) = (u, u_t, \eta^t) \in C([0,T], \mathcal{H}_2)$ is a weak solution of the problem (1.6)–(1.11) for given $T > 0$ if $U(0) = (u_0, u_1, \eta_0)$, and

$$\begin{aligned}
&\frac{d}{dt}(u_t(t), \omega) + (\Delta u(t), \Delta\omega) + (\eta^t, \omega)_{\mathcal{M}_2} \\
&+ \mu_1(\nabla u_t(t), \nabla\omega) + \mu_2(\nabla u_t(t-\tau(t)), \nabla\omega) + (f(u) - h, \omega) = 0, \\
&(\partial_t \eta^t, \xi)_{\mathcal{M}_2} = -(\partial_s \eta^t, \xi)_{\mathcal{M}_2} + (u_t(t), \xi)_{\mathcal{M}_2}
\end{aligned}$$

a.e. in $[0,T]$, for all $\omega \in V_2$, $\xi \in \mathcal{M}_2$.

And the global well-posedness of (1.6)–(1.11) with $\nu = 0$ will be given in the following theorem.

**Theorem 3.4.** *Under the assumptions* (2.1)–(2.7) *and* $|\mu_2| \le \mu_1$*, we have*

(i) *If initial data* $U(0) = (u_0, u_1, \eta_0) \in \mathcal{H}_2$*,* $f_0(x,t) \in H^1(\Omega \times (-\tau(0),0))$ *and* $h(x) \in L^2(\Omega)$*, then problem* (1.6)–(1.11) *has a weak solution* $(u, u_t, \eta^t) \in C(0,T; \mathcal{H}_2)$*,* $\forall T > 0$.

(ii) *If initial data* $U(0) = (u_0, u_1, \eta_0) \in \mathcal{H}_3$*,* $f_0(x,t) \in H^2(\Omega \times (-\tau(0),0))$ *and* $h(x) \in L^2(\Omega)$*, then problem* (1.6)–(1.11) *has a stronger weak solution* $(u, u_t, \eta^t) \in C(0,T; \mathcal{H}_3)$*,* $\forall T > 0$.

(iii) *In both cases the weak solution depends continuously on the initial data in* $\mathcal{H}_2 \times H^1(\Omega \times (-\tau(0), 0))$. *In particular, the weak solution of problem* (1.6)–(1.11) *is unique.*

*Remark* 3.5. We can prove Theorem 3.4 by the same argument as the proof of Theorem 3.2. Here we only give a priori estimate under the assumption $|\mu_2| \le \mu_1$.

We multiply the equation (1.6) by $u_t^m$ with $\nu = 0$, and use the equation (1.7) to obtain for any $t > 0$

$$\begin{aligned}\frac{d}{dt}E^m(t) + \mu_1\|\nabla u_t^m\|^2 &= -\mu_2 \int_\Omega \nabla u_t^m(t)\nabla u_t^m(t-\tau(t))\,dx \\ &\quad + \frac{1}{2}\int_0^\infty g'(s)\|\Delta\eta^{t,m}(s)\|^2\,ds + \int_\Omega h u_t^m(t)\,dx,\end{aligned}$$

where

$$E^m(t) = \frac{1}{2}\|u_t^m(t)\|^2 + \frac{1}{2}\|\Delta u^m(t)\|^2 + \frac{1}{2}\|\eta^{t,m}\|^2_{\mathcal{M}_2} + \int_\Omega \widehat{f}(u^m(t))\,dx.$$

By using (3.5) and (3.7), we can get

$$E^m(t) + (\mu_1 - |\mu_2|)\int_0^t \|\nabla u_t^m\|^2 \le \frac{|\mu_2|}{2}\int_\Omega\int_{-\tau(t)}^0 |\nabla f_0^m|^2\,dsdx + \sqrt{2}\|h\|\int_0^t (E^m(s))^{1/2}\,ds,$$

which, noting that $|\mu_2| \le \mu_1$ and applying Gronwall's inequality, yields that for all $t \in [0, T]$ and for every $m \in \mathbb{N}$,

$$E^m(t) \le C.$$

## 4. Exponential stability

In this section, we shall study the stability of problem (1.6)–(1.11).

### 4.1. The system with rotational inertia

Let us consider the following equation

$$u_{tt} + \Delta^2 u + \int_0^\infty g(s)\Delta^2\eta^t(s)\,ds - \Delta u_{tt} - \mu_2\Delta u_t(t-\tau(t)) = 0, \tag{4.1}$$

together with (1.7) and initial data and boundary conditions (1.8)–(1.11).

The energy functional of problem (4.1) and (1.7)–(1.11) is defined as

$$\begin{aligned}F(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}\|\eta\|^2_{\mathcal{M}_2} \\ &\quad + \frac{\theta|\mu_2|e^{\tau_1}}{2}\int_{t-\tau(t)}^t e^{-(t-s)}\|\nabla u_t(s)\|^2\,ds,\end{aligned} \tag{4.2}$$

where $\theta$ is any real constant satisfying

$$\theta > \frac{1}{\sqrt{1-d}}.$$

Then we can get the following result concerning the stability of problem (4.1) and (1.7)–(1.11).

**Theorem 4.1.** *Let the assumptions* (2.4)–(2.7) *hold. Let the initial data* $U(0) = (u_0, u_1, \eta_0) \in \mathcal{H}$ *and* $f_0(x,t) \in H^1(\Omega \times (-\tau(0), 0))$. *For any* $\theta > 1/\sqrt{1-d}$, *then there exists a constant* $\beta > 0$ *such that the energy, defined by* (4.2), *to problem* (4.1) *and* (1.8)–(1.11) *satisfy for any* $t \geq 0$,

$$F(t) \leq F(0)e^{1-\beta t}$$

*holds for* $|\mu_2| < \mu_0$, *where* $\mu_0$ *is a positive constant defined in* (4.35).

*Remark* 4.2. Since the rotational inertia and memory term, our result is not contained in Nicaise and Pignotti [23]. But if $\nu = 0$ and the memory is absence, our result is a special case of result in Nicaise and Pignotti [23].

Motivated by [1], and noting that the energy $F(t)$ of problem (4.1) and (1.7)–(1.11) is not decreasing, we consider an auxiliary problem. We consider the following problem

$$u_{tt} + \Delta^2 u + \int_0^\infty g(s)\Delta^2\eta^t(s)\,ds - \Delta u_{tt} - \theta|\mu_2|e^{\tau_1}\Delta u_t - \mu_2\Delta u_t(t-\tau(t)) = 0, \tag{4.3}$$

together with (1.7) and initial data and boundary conditions (1.8)–(1.11).

We adopt the method developed by Pignotti [28] to prove Theorem 4.1. For this purpose, we need the following technical lemmas.

**Lemma 4.3.** *For every solution of problem* (4.3) *and* (1.7)–(1.11), *the energy* $F(t)$ *defined in* (4.2) *is decreasing and satisfies for any* $t > 0$,

$$\begin{aligned} F'(t) \leq{} & \frac{1}{2}\int_0^\infty g'(s)\|\Delta\eta(s)\|^2\,ds - \frac{|\mu_2|(\theta e^{\tau_1}\sqrt{1-d}-1)}{2\sqrt{1-d}}\|\nabla u_t\|^2 \\ & - \frac{|\mu_2|\sqrt{1-d}(\theta\sqrt{1-d}-1)}{2}\|\nabla u_t(t-\tau(t))\|^2 \\ & - \frac{|\mu_2|\theta e^{\tau_1}}{2}\int_{t-\tau(t)}^t e^{-(t-s)}\|\nabla u_t\|^2\,ds, \end{aligned} \tag{4.4}$$

*and*

$$\begin{aligned} -\frac{1}{2}\int_S^T\int_0^\infty g'(s)\|\Delta\eta(s)\|^2\,dsdt &\leq F(S), \\ \frac{1}{2}\int_S^T\int_0^\infty g(s)\|\Delta\eta(s)\|^2\,dsdt &\leq \frac{1}{k}F(S). \end{aligned} \tag{4.5}$$

*Proof.* Differentiating (4.2) and using (4.3), we obtain

$$\begin{aligned}F'(t) = &\int_\Omega u_t \left(-\Delta^2 u - \int_0^\infty g(s)\Delta^2\eta(s)\,ds + \Delta u_{tt} + \theta|\mu_2|e^{\tau_1}\Delta u_t + \mu_2\Delta u_t(t-\tau(t))\right) dx\\ &+ \int_\Omega \nabla u_t \cdot \nabla u_{tt}\,dx + \int_\Omega \Delta u \cdot \Delta u_t\,dx + \frac{1}{2}\frac{d}{dt}\|\eta\|^2_{\mathcal{M}_2} + \frac{|\mu_2|\theta e_1^\tau}{2}\|\nabla u_t\|^2\\ &- \frac{|\mu_2|\theta e^{\tau_1}}{2}e^{-\tau(t)}(1-\tau'(t))\|\nabla u_t(t-\tau(t))\|^2 - \frac{|\mu_2|\theta e^{\tau_1}}{2}\int_{t-\tau(t)}^t e^{-(t-s)}\|\nabla u_t(s)\|^2\,ds.\end{aligned}$$

By using integration by parts, (2.6)–(2.7) and noting that

$$(\partial_s\eta,\eta)_{\mathcal{M}_2} = -\frac{1}{2}\int_0^\infty g'(s)\|\Delta\eta(s)\|^2\,ds,$$

we have

$$\begin{aligned}F'(t) \le &\frac{1}{2}\int_0^\infty g'(s)\|\Delta\eta(s)\|^2\,ds - \frac{|\mu_2|\theta e^{\tau_1}}{2}\|\nabla u_t\|^2 - \mu_2\int_\Omega \nabla u_t\cdot\nabla u_t(t-\tau(t))\,dx\\ &- \frac{\theta|\mu_2|}{2}(1-d)\|\nabla u_t(t-\tau(t))\|^2 - \frac{|\mu_2|\theta e^{\tau_1}}{2}\int_{t-\tau(t)}^t e^{-(t-s)}\|\nabla u_t(s)\|^2\,ds.\end{aligned} \tag{4.6}$$

Young's inequality implies

$$-\mu_2\int_\Omega \nabla u_t\cdot\nabla u_t(t-\tau(t))\,dx \le \frac{|\mu_2|}{2\sqrt{1-d}}\|\nabla u_t\|^2 + \frac{|\mu_2|\sqrt{1-d}}{2}\|\nabla u_t(t-\tau(t))\|^2,$$

which, together with (4.6), gives us (4.4).

It follows from (4.4) that

$$-\frac{1}{2}\int_S^T\int_0^\infty g'(s)\|\Delta\eta(s)\|^2\,dsdt \le \int_S^T(-F'(t))\,dt = F(S)-F(T) \le F(S),$$

and using the assumption (2.5), we can directly obtain (4.5). The proof of this lemma is complete. □

**Lemma 4.4.** *Assume*

$$|\mu_2| < \frac{\sqrt{1-d}\lambda_2}{2(\theta e^{\tau_1}\sqrt{1-d}+1)}, \tag{4.7}$$

*then the following estimate holds for any $T \ge S \ge 0$,*

$$\int_S^T \|\Delta u(t)\|^2\,dt \le C_0\int_S^T \|u_t(t)\|^2\,dt + C_1\int_S^T \|\nabla u_t(t)\|^2\,dt + C_2F(S) \tag{4.8}$$

*with*

$$C_0 = 2,\quad C_1 = 2\left(1+\frac{\theta|\mu_2|e^{\tau_1}}{2}\right),\quad C_2 = \frac{4l_0}{k} + 4\left(1+\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right) + \frac{2}{\theta\sqrt{1-d}-1}. \tag{4.9}$$

*Proof.* We begin by multiplying the equation (4.3) by $u$, and integrate the result over $\Omega\times[S,T]$, we shall see that

$$\begin{aligned}\int_S^T\|\Delta u(t)\|^2\,dt=&\int_S^T(\|u_t\|^2+\|\nabla u_t\|^2)(t)\,dt-\int_\Omega(u\cdot u_t+u\cdot\Delta u_t)\,dx\Big|_S^T\\&-\theta|\mu_2|e^{\tau_1}\int_S^T\int_\Omega\nabla u_t\cdot\nabla u\,dxdt\\&-\mu_2\int_S^T\int_\Omega\nabla u_t(t-\tau(t))\cdot\nabla u\,dxdt\\&-\int_S^T\int_\Omega\int_0^\infty g(s)\Delta\eta(s)\cdot\Delta u(t)\,dsdxdt.\end{aligned}\tag{4.10}$$

It follows from Hölder's and Young's inequalities and (4.5) that for any $\varepsilon>0$,

$$\begin{aligned}&-\int_S^T\int_\Omega\int_0^\infty g(s)\Delta\eta(s)\cdot\Delta u(t)\,dsdxdt\\&\le\int_S^T\left(\int_\Omega|\Delta u(t)|^2\,dx\right)^{1/2}\cdot\int_0^\infty g(s)\left(\int_\Omega|\Delta\eta(s)|^2\,dx\right)^{1/2}dsdt\\&\le\frac{\varepsilon}{2}\int_S^T\|\Delta u(t)\|^2\,dt+\frac{1}{2\varepsilon}\int_S^T\left[\int_0^\infty g(s)\left(\int_\Omega|\Delta\eta(s)|^2\,dx\right)^{1/2}ds\right]^2dt\\&\le\frac{\varepsilon}{2}\int_S^T\|\Delta u(t)\|^2\,dt+\frac{1}{2\varepsilon}\int_S^T\left(\int_0^\infty g(s)\,ds\right)\left(\int_0^\infty g(s)\|\Delta\eta(s)\|^2\,ds\right)dt\\&\le\frac{\varepsilon}{2}\int_S^T\|\Delta u(t)\|^2\,dt+\frac{l_0}{2\varepsilon}\int_S^T\int_0^\infty g(s)\|\Delta\eta(s)\|^2\,dsdt\\&\le\frac{\varepsilon}{2}\int_S^T\|\Delta u(t)\|^2\,dt+\frac{l_0}{k\varepsilon}F(S).\end{aligned}\tag{4.11}$$

In addition, we can obtain

$$\begin{aligned}\int_\Omega(u\cdot u_t+u\cdot\Delta u_t)\,dx&\le\frac12\|u_t\|^2+\frac12\|\nabla u_t\|^2+\frac12\|u\|^2+\frac12\|\nabla u\|^2\\&\le F(t)+\left(\frac{1}{2\lambda_1}+\frac{1}{2\lambda_2}\right)\|\Delta u\|^2\\&\le\left(1+\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)F(t),\end{aligned}$$

where we used the fact

$$\frac12\|u_t\|^2+\frac12\|\nabla u_t\|^2+\frac12\|\Delta u\|^2\le F(t).$$

Then we have

$$-\int_\Omega(u\cdot u_t+u\cdot\Delta u_t)\,dx\Big|_S^T\le2\left(1+\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)F(S).\tag{4.12}$$

Inserting (4.11)–(4.12) into (4.10) and using Young's inequality, we shall see below, for any $\varepsilon > 0$,

$$\begin{aligned}\int_S^T \|\Delta u(t)\|^2\,dt \le& \int_S^T (\|u_t\|^2 + \|\nabla u_t\|^2)(t)\,dt + \frac{\varepsilon}{2}\int_S^T \|\Delta u(t)\|^2\,dt + \frac{l_0}{k\varepsilon}F(S)\\ &+ 2\left(1+\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)F(S) + \frac{\theta|\mu_2|e^{\tau_1}}{2}\int_S^T \|\nabla u_t(t)\|^2\,dt\\ &+ \frac{\theta|\mu_2|e^{\tau_1}}{2}\int_S^T \|\nabla u(t)\|^2\,dt + \frac{|\mu_2|}{2\sqrt{1-d}}\int_S^T \|\nabla u(t)\|^2\,dt\\ &+ \frac{|\mu_2|}{2}\sqrt{1-d}\int_S^T \|\nabla u_t(t-\tau(t))\|^2\,dt,\end{aligned}$$

which, by using Poincaré's inequality, yields

$$\begin{aligned}\int_S^T \|\Delta u(t)\|^2\,dt \le& \int_S^T \|u_t(t)\|^2\,dt + \left(\frac{\varepsilon}{2}+\frac{\theta|\mu_2|e^{\tau_1}}{2\lambda_2}+\frac{|\mu_2|}{2\sqrt{1-d}\lambda_2}\right)\int_S^T \|\Delta u(t)\|^2\,dt\\ &+ \left(1+\frac{\theta|\mu_2|e^{\tau_1}}{2}\right)\int_S^T \|\nabla u_t(t)\|^2\,dt + 2\left(1+\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)F(S)\\ &+ \frac{l_0}{k\varepsilon}F(S) + \frac{|\mu_2|}{2}\sqrt{1-d}\int_S^T \|\nabla u_t(t-\tau(t))\|^2\,dt.\end{aligned} \tag{4.13}$$

Taking into account the following estimate

$$\begin{aligned}&\frac{|\mu_2|}{2}\sqrt{1-d}\int_S^T \|\nabla u_t(t-\tau(t))\|^2\,dt\\ =& \frac{1}{\theta\sqrt{1-d}-1}\frac{|\mu_2|(\theta\sqrt{1-d}-1)}{2}\sqrt{1-d}\int_S^T \|\nabla u_t(t-\tau(t))\|^2\,dt\\ \le& \frac{1}{\theta\sqrt{1-d}-1}\int_S^T (-F'(t))\,dt \le \frac{1}{\theta\sqrt{1-d}-1}F(S),\end{aligned}$$

and taking $\varepsilon = 1/2$, we infer from (4.13) that

$$\begin{aligned}\int_S^T \|\Delta u(t)\|^2\,dt \le& \int_S^T \|u_t(t)\|^2\,dt + \left(\frac{1}{4}+\frac{\theta|\mu_2|e^{\tau_1}}{2\lambda_2}+\frac{|\mu_2|}{2\sqrt{1-d}\lambda_2}\right)\int_S^T \|\Delta u(t)\|^2\,dt\\ &+ \left(1+\frac{\theta|\mu_2|e^{\tau_1}}{2}\right)\int_S^T \|\nabla u_t(t)\|^2\,dt + 2\left(1+\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)F(S)\\ &+ \frac{2l_0}{k}F(S) + \frac{1}{\theta\sqrt{1-d}-1}F(S).\end{aligned} \tag{4.14}$$

By using (4.7), we know that

$$1 - \left(\frac{1}{4}+\frac{\theta|\mu_2|e^{\tau_1}}{2\lambda_2}+\frac{|\mu_2|}{2\sqrt{1-d}\lambda_2}\right) > \frac{1}{2}.$$

Then (4.8) follows from (4.14) with constants given by (4.9). Therefore, the proof is complete. □

**Lemma 4.5.** *Assume*

$$|\mu_2| < \frac{l_0}{\theta} e^{-\tau_1}, \tag{4.15}$$

*then the following estimate holds for any $T \geq S \geq 0$ and for any $\varepsilon > 0$,*

$$\int_S^T (\|u_t\|^2 + \|\nabla u_t\|^2)(t)\,dt \leq \varepsilon \int_S^T \|\Delta u_t(t)\|^2\,dt + C_3 F(S) \tag{4.16}$$

*with*

$$\begin{aligned} C_3 = \frac{2}{l_0}\bigg[ & 2\left(1+\frac{2l_0}{\lambda_1}\right) + \frac{4g(0)}{l_0\lambda_1} + \frac{2l_0}{k} + \frac{l_0}{k\varepsilon} \\ & + \frac{\theta|\mu_2|e^{\tau_1}l_0}{2k}\left(1+\frac{1}{\lambda_2}\right) + \frac{1}{\theta\sqrt{1-d}-1} + \frac{|\mu_2|l_0}{k\lambda_2\sqrt{1-d}}\bigg]. \end{aligned} \tag{4.17}$$

*Proof.* Multiplying (4.3) by $\int_0^\infty g(s)\eta(s)\,ds$ and integrating the result over $\Omega \times [S,T]$, we get

$$\begin{aligned} \int_S^T \int_\Omega & \left(u_{tt} + \Delta^2 u + \int_0^\infty g(s)\Delta^2\eta^t(s)\,ds - \Delta u_{tt} - \theta|\mu_2|e^{\tau_1}\Delta u_t - \Delta u_t(t-\tau(t))\right) \\ & \times \left(\int_0^\infty g(s)\eta(s)\,ds\right) dxdt = 0. \end{aligned} \tag{4.18}$$

Using (1.7) and integration by parts, we conclude that

$$\begin{aligned} & \int_S^T \int_\Omega u_{tt} \cdot \int_0^\infty g(s)\eta(s)\,dsdxdt \\ & = \int_\Omega u_t \cdot \int_0^\infty g(s)\eta(s)\,dsdx\bigg|_S^T - \int_S^T\int_\Omega u_t(t)\cdot\int_0^\infty g(s)(u_t(t)-\eta(s))\,dsdxdt \\ & = \int_\Omega u_t \cdot \int_0^\infty g(s)\eta(s)\,dsdx\bigg|_S^T - l_0\int_S^T \|u_t(t)\|^2\,dt \\ & \quad - \int_S^T\int_\Omega u_t(t)\cdot\int_0^\infty g'(s)\eta(s)\,dsdxdt, \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} & -\int_S^T \int_\Omega \Delta u_{tt} \cdot \int_0^\infty g(s)\eta(s)\,dsdxdt \\ & = \int_S^T \int_\Omega \nabla u_{tt} \cdot \int_0^\infty g(s)\nabla\eta(s)\,dsdxdt \\ & = \int_\Omega \nabla u_t \cdot \int_0^\infty g(s)\eta(s)\,dsdx\bigg|_S^T - l_0\int_S^T \|\nabla u_t(t)\|^2\,dt \\ & \quad - \int_S^T\int_\Omega \nabla u_t(t)\cdot\int_0^\infty g'(s)\eta(s)\,dsdxdt. \end{aligned} \tag{4.20}$$

Again using integration by part to the remain terms in (4.18), we derive from (4.19)–(4.20) that

$$
\begin{aligned}
& l_0 \int_S^T (\|u_t\|^2 + \|\nabla u_t\|^2)(t)\, dt \\
&= \underbrace{\int_\Omega (u_t + \nabla u_t)(t) \cdot \int_0^\infty g(s)\eta(s)\, ds dx \Big|_S^T}_{:=I_1} \underbrace{- \int_\Omega (u_t + \nabla u_t)(t) \cdot \int_0^\infty g'(s)\eta(s)\, ds dx dt}_{:=I_2} \\
&\quad + \underbrace{\int_S^T \int_\Omega \Delta u(t) \cdot \int_0^\infty g(s)\Delta\eta(s)\, ds dx dt}_{:=I_3} + \underbrace{\int_S^T \int_\Omega \left( \int_0^\infty g(s)\Delta\eta(s) ds \right)^2 dx dt}_{:=I_4} \\
&\quad + \underbrace{\theta|\mu_2| e^{\tau_1} \int_S^T \int_\Omega \nabla u_t(t) \cdot \int_0^\infty g(s)\nabla\eta(s)\, ds dx dt}_{:=I_5} \\
&\quad + \underbrace{\mu_2 \int_S^T \int_\Omega \nabla u_t(t - \tau(t)) \cdot \int_0^\infty g(s)\nabla\eta(s)\, ds dx dt}_{:=I_6} .
\end{aligned} \tag{4.21}
$$

In the sequel we shall estimate the terms $I_i$ $(i = 1, \ldots, 6)$ on the right-hand side of (4.21). It follows from Hölder's inequality and Young's inequality that

$$
\begin{aligned}
\int_\Omega (u_t + \nabla u_t)(t) \cdot \int_0^\infty g(s)\eta(s)\, ds dx &\le \frac{1}{2}(\|u_t\|^2 + \|\nabla u_t\|^2) + \int_\Omega \left( \int_0^\infty g(s)\eta(s)\, ds \right)^2 ds \\
&\le \frac{1}{2}(\|u_t\|^2 + \|\nabla u_t\|^2) + \frac{l_0}{\lambda_1} \int_0^\infty g(s)\|\Delta\eta(s)\|^2\, ds \\
&\le F(t) + \frac{2l_0}{\lambda_1} F(t),
\end{aligned}
$$

which gives us

$$
I_1 \le 2\left(1 + \frac{2l_0}{\lambda_1}\right) F(S). \tag{4.22}
$$

Moreover, we have for any $\delta > 0$,

$$
\begin{aligned}
I_2 &\le \frac{\delta}{2} \int_S^T (\|u_t\|^2 + \|\nabla u_t\|^2)(t)\, dt + \frac{1}{\delta} \int_S^T \left\| \int_0^\infty g'(s)\eta(s)\, ds \right\|^2 dt \\
&\le \frac{\delta}{2} \int_S^T (\|u_t\|^2 + \|\nabla u_t\|^2)(t)\, dt \\
&\quad + \frac{1}{\delta} \int_S^T \int_\Omega \left( - \int_0^\infty g'(s)\, ds \right) \left( \int_0^\infty (-g'(s))|\eta(s)|^2\, ds \right) dx dt \\
&\le \frac{\delta}{2} \int_S^T (\|u_t\|^2 + \|\nabla u_t\|^2)(t)\, dt - \frac{g(0)}{\delta\lambda_1} \int_S^T \int_0^\infty g'(s)\|\Delta\eta(s)\|^2\, ds dt \\
&\le \frac{\delta}{2} \int_S^T (\|u_t\|^2 + \|\nabla u_t\|^2)(t)\, dt + \frac{2g(0)}{\delta\lambda_1} F(S),
\end{aligned} \tag{4.23}
$$

$$I_4 \le l_0 \int_S^T \int_0^\infty g(s)\|\Delta\eta(s)\|^2\, dsdt \le \frac{2l_0}{k} F(S). \tag{4.24}$$

By virtue of (4.24), we get for any $\varepsilon > 0$,

$$\begin{aligned} I_3 &\le \frac{\varepsilon}{2} \int_S^T \|\Delta u(t)\|^2\, dt + \frac{1}{2\varepsilon} \int_S^T \left\| \int_0^\infty g(s)\Delta\eta(s)\, ds \right\|^2 dt \\ &\le \frac{\varepsilon}{2} \int_S^T \|\Delta u(t)\|^2\, dt + \frac{l_0}{k\varepsilon} F(S). \end{aligned} \tag{4.25}$$

By using integration by parts, Hölder's inequality and Young's inequality, we derive

$$\begin{aligned} I_5 &= \frac{1}{2}\theta|\mu_2|e^{\tau_1} \int_S^T \int_\Omega (-\Delta u_t(t)) \int_0^\infty g(s)\eta(s)\, dsdxdt \\ &\quad + \frac{1}{2}\theta|\mu_2|e^{\tau_1} \int_S^T \int_\Omega (-\Delta u_t(t)) \int_0^\infty g(s)\eta(s)\, dsdxdt \\ &= -\frac{1}{2}\theta|\mu_2|e^{\tau_1} \int_S^T \int_\Omega u_t(t) \int_0^\infty g(s)\Delta\eta(s)\, dsdxdt \\ &\quad + \frac{1}{2}\theta|\mu_2|e^{\tau_1} \int_S^T \int_\Omega \nabla u_t(t) \int_0^\infty g(s)\nabla\eta(s)\, dsdxdt \\ &\le \frac{\theta|\mu_2|e^{\tau_1}}{4} \int_S^T (\|u_t\|^2 + \|\nabla u_t\|^2)(t)\, dt + \frac{\theta|\mu_2|e^{\tau_1}}{4} \int_S^T \left\| \int_0^\infty g(s)\Delta\eta(s)\, ds \right\|^2 dt \\ &\quad + \frac{\theta|\mu_2|e^{\tau_1}}{4} \int_S^T \left\| \int_0^\infty g(s)\nabla\eta(s)\, ds \right\|^2 dt \\ &\le \frac{\theta|\mu_2|e^{\tau_1}}{4} \int_S^T (\|u_t\|^2 + \|\nabla u_t\|^2)(t)\, dt + \frac{\theta|\mu_2|e^{\tau_1} l_0}{2k} \left(1 + \frac{1}{\lambda_2}\right) F(S) \end{aligned} \tag{4.26}$$

and

$$\begin{aligned} I_6 &\le \frac{|\mu_2|}{2}\sqrt{1-d} \int_S^T \|\nabla u_t(t-\tau(t))\|^2\, dt \\ &\quad + \frac{|\mu_2|}{2\sqrt{1-d}} \int_S^T \left\| \int_0^\infty g(s)\nabla\eta(s)\, ds \right\|^2 dt \\ &\le \frac{|\mu_2|}{2}\sqrt{1-d} \int_S^T \|\nabla u_t(t-\tau(t))\|^2\, dt + \frac{|\mu_2| l_0}{k\lambda_2\sqrt{1-d}} F(S) \\ &\le \left( \frac{1}{\theta\sqrt{1-d}-1} + \frac{|\mu_2| l_0}{k\lambda_2\sqrt{1-d}} \right) F(S). \end{aligned} \tag{4.27}$$

Inserting (4.22)–(4.27) into (4.21), we have for any $\varepsilon > 0$ and $\delta > 0$,

$$\begin{aligned} &\left( l_0 - \frac{\theta|\mu_2|e^{\tau_1}}{4} - \frac{\delta}{2} \right) \int_S^T (\|u_t\|^2 + \|\nabla u_t\|^2)(t)\, dt \\ &\le \frac{\varepsilon}{2} \int_S^T \|\Delta u(t)\|^2\, dt + \left[ 2\left(1 + \frac{2l_0}{\lambda_1}\right) + \frac{2g(0)}{\delta\lambda_1} + \frac{2l_0}{k} + \frac{2l_0}{k\varepsilon} \right. \\ &\quad \left. + \frac{\theta|\mu_2|e^{\tau_1} l_0}{2k}\left(1 + \frac{1}{\lambda_2}\right) + \frac{1}{\theta\sqrt{1-d}-1} + \frac{|\mu_2| l_0}{k\lambda_2\sqrt{1-d}} \right] F(S). \end{aligned} \tag{4.28}$$

Taking $\delta = l_0/2$ and using (4.15), we know that

$$l_0 - \frac{\theta|\mu_2|e^{\tau_1}}{4} - \frac{\delta}{2} > \frac{l_0}{2}.$$

Then (4.16) follows from (4.28) with a constant $C_3$ defined by (4.17). The proof is complete. □

**Lemma 4.6.** *Assume*

$$|\mu_2| < \min\left\{\frac{\sqrt{1-d}\lambda_2}{2(\theta e^{\tau_1}\sqrt{1-d}+1)}, \frac{l_0}{\theta}e^{-\tau_1}\right\}$$

*then the following estimate holds for any* $T \geq S \geq 0$,

$$\frac{1}{2}\int_S^T \|\Delta u(t)\|^2\,dt + \frac{1}{2}\int_S^T (\|u_t\|^2 + \|\nabla u_t\|^2)(t)\,dt \leq C_4 F(S) \tag{4.29}$$

*with*

$$C_4 = (C_0C_3 + C_1C_3 + C_2) + \frac{C_3}{2},$$

*where the constants* $C_0$, $C_1$, $C_2$ *is defined in* (4.9), *and*

$$\begin{aligned} C_3 = \frac{2}{l_0}\Bigg[&2\left(1+\frac{2l_0}{\lambda_1}\right) + \frac{4g(0)}{l_0\lambda_1} + \frac{2l_0}{k} + \frac{l_0(2C_0+2C_1+1)}{k} \\ &+ \frac{\theta|\mu_2|e^{\tau_1}l_0}{2k}\left(1+\frac{1}{\lambda_2}\right) + \frac{1}{\theta\sqrt{1-d}-1} + \frac{|\mu_2|l_0}{k\lambda_2\sqrt{1-d}}\Bigg]. \end{aligned}$$

*Proof.* It follows from (4.8) and (4.16) that for any $\varepsilon > 0$,

$$\int_S^T \|\Delta u(t)\|^2\,dt \leq (C_0+C_1)\varepsilon\int_S^T \|\Delta u(t)\|^2\,dt + (C_0C_3 + C_1C_3 + C_2)F(S). \tag{4.30}$$

We infer from (4.16) and (4.30) that for any $\varepsilon > 0$,

$$\begin{aligned} &\int_S^T \|\Delta u(t)\|^2\,dt + \frac{1}{2}\int_S^T (\|u_t\|^2 + \|\nabla u_t\|^2)(t)\,dt \\ &\leq \left[(C_0+C_1)+\frac{1}{2}\right]\varepsilon\int_S^T \|\Delta u(t)\|^2\,dt + \left[(C_0C_3 + C_1C_3 + C_2) + \frac{C_3}{2}\right]F(S), \end{aligned}$$

which, choosing $\varepsilon > 0$ satisfying

$$\varepsilon = \frac{1}{2C_0 + 2C_1 + 1},$$

implies

$$\frac{1}{2}\int_S^T \|\Delta u(t)\|^2\,dt + \frac{1}{2}\int_S^T (\|u_t\|^2 + \|\nabla u_t\|^2)(t)\,dt \leq \left[(C_0C_3 + C_1C_3 + C_2) + \frac{C_3}{2}\right]F(S).$$

Hence the proof is complete. □

For problem (4.3) and (1.7)–(1.11), we can get the following stability result.

**Theorem 4.7.** *Assume*

$$|\mu_2| < \overline{\mu_2} := \min\left\{\frac{\sqrt{1-d}\lambda_2}{2(\theta e^{\tau_1}\sqrt{1-d}+1)}, \frac{l_0}{\theta}e^{-\tau_1}\right\}. \tag{4.31}$$

*Let the assumptions* (2.4)*–*(2.7) *hold. Let the initial data* $U(0) = (u_0, u_1, \eta_0) \in \mathcal{H}$ *and* $f_0(x,t) \in H^1(\Omega \times (-\tau(0), 0))$*. For any* $\theta > 1/\sqrt{1-d}$*, then there exists a constant* $\widetilde{\beta} > 0$ *such that the energy* $F(t)$ *of the auxiliary problem* (4.3) *and* (1.7)*–*(1.11) *defined in* (4.2) *satisfies for any* $t \geq 0$*,*

$$F(t) \leq F(0)e^{1-\widetilde{\beta}t}. \tag{4.32}$$

*Proof.* It follows from (4.4) that

$$\frac{|\mu_2|\theta e^{\tau_1}}{2}\int_{t-\tau(t)}^{t} e^{-(t-s)}\|\nabla u_t\|^2\,ds \leq -\int_S^T F'(t)\,dt \leq F(S),$$

which, together with (4.5) and (4.29) and noting that $|\mu_2| < \overline{\mu_2}$, gives us

$$\int_S^T F(t)\,dt \leq \left(C_4 + \frac{1}{k} + 1\right)F(S).$$

Then using Lemma 2.1, we can get the desired estimate (4.32) with

$$C = C_4 + \frac{1}{k} + 1, \quad \widetilde{\beta} = \frac{1}{C}. \tag{4.33}$$

The proof is complete. □

*Proof of Theorem* 4.1. By using Theorems 2.2 and 4.7, we can obtain that Theorem 4.1 holds with $\beta = \widetilde{\beta} - e\theta|\mu_2|e^{\tau_1}$ if

$$-\widetilde{\beta} + e\theta|\mu_2|e^{\tau_1} < 0,$$

i.e., if the coefficient of delay $\mu_2$ satisfies

$$|\mu_2| < \pi(|\mu_2|) := \frac{1}{Ce\theta e^{\tau_1}}, \tag{4.34}$$

and $C > 0$ is the constant defined in (4.33). Noting that $\pi(0) > 0$, we first know that (4.34) holds for $\mu_2 = 0$. In addition, it follows from the definition of the constants $C_0$, $C_1$, $C_2$ and $C_3$ that the function $\pi\colon [0,\infty) \to [0,\infty)$ is a continuous decreasing function satisfying

$$\pi(|\mu_2|) \to 0 \quad \text{for } |\mu_2| \to \infty.$$

Thus there exists a unique positive constant $\widehat{\mu_2}$ such that $\widehat{\mu_2} = \pi(\widehat{\mu_2})$. Therefore for any $\theta$ in (4.2), inequality (4.34) is satisfied for every $\mu_2$ with

$$|\mu_2| < \mu_0 := \min\{\widehat{\mu_2}, \overline{\mu_2}\}, \tag{4.35}$$

where $\overline{\mu_2}$ is defined in (4.31). The proof of Theorem 4.1 is complete. □

*Remark* 4.8. Following the same arguments as in [1], we can also compute an explicit lower bound for $\mu_0$. Here we omit the proof.

*Remark* 4.9. The stability result also holds for the plate equation with strong anti-damping, that is, the case $\tau(t) = 0$ and $\mu_2 < 0$. In fact we can take $\theta = 1$ to get the stability result under the condition

$$|\mu_2| < \left(\frac{9}{2}C_3 + C_2 + \frac{1}{k} + 1\right)^{-1} e^{-1},$$

where

$$C_2 = \frac{4l_0}{k} + 4\left(1 + \frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right) \quad \text{and} \quad C_3 = \frac{2}{l_0}\left[2\left(1 + \frac{2l_0}{\lambda_1}\right) + \frac{4g(0)}{l_0\lambda_1} + \frac{11l_0}{k}\right].$$

4.2. The system without rotational inertia

In this subsection, let us consider (1.6) with $\nu = 0$, and study the following system

$$u_{tt} + \Delta^2 u + \int_0^\infty g(s)\Delta^2\eta^t(s)\,ds - \mu_1\Delta u_t - \mu_2\Delta u_t(t-\tau(t)) + f(u) = 0, \tag{4.36}$$

together with (1.7) and initial data and boundary conditions (1.8)–(1.11).

We define energy functional $G(t)$ of problem (4.36) and (1.7)–(1.11) as

$$\begin{aligned} G(t) &= \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 + \frac{1}{2}\|\eta^t\|^2_{\mathcal{M}_2} + \int_\Omega \widehat{f}(u(t))\,dx \\ &\quad + \frac{\xi}{2}\int_{t-\tau(t)}^t \int_\Omega e^{\lambda(s-t)}|\nabla u_t(x,s)|^2\,dxds, \end{aligned} \tag{4.37}$$

where $\xi > 0$ is a constant will be determined later and the constant $\lambda > 0$, as below, has been introduced in [22],

$$\lambda < \frac{1}{\tau_1}\left|\log\frac{|\mu_2|}{\sqrt{1-d}}\right|.$$

Then we can get the following stability result for problem (4.36) and (1.7)–(1.11).

**Theorem 4.10.** *Let the assumptions* (2.4)–(2.7) *hold. Assume* $|\mu_2| < \sqrt{1-d}\mu_1$. *Let the initial data* $U(0) = (u_0, u_1, \eta_0) \in \mathcal{H}_2$ *and* $f_0(x,t) \in H^1(\Omega \times (-\tau(0), 0))$. *Then there exist two constants* $\gamma > 0$ *and* $\alpha > 0$ *such that the energy* $G(t)$, *defined by* (4.37), *to problem* (4.36) *and* (1.8)–(1.11) *satisfies*

$$G(t) \le \gamma e^{-\alpha t} \quad \textit{for all } t \ge 0. \tag{4.38}$$

In the sequel we shall prove Theorem 4.10, which will be divided into the following lemmas.

**Lemma 4.11.** *Under the assumptions of Theorem* 4.10, *the energy functional defined by* (4.37) *satisfies for any* $t \geq 0$,

$$
\begin{aligned}
G'(t) \leq & \left(\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2}\right)\|\nabla u_t(t)\|^2 \\
& + \left(\frac{|\mu_2|}{2}\sqrt{1-d} - \frac{\xi}{2}e^{-\lambda\tau_1}(1-d)\right)\|\nabla u_t(t-\tau(t))\|^2 \\
& + \frac{1}{2}\int_0^\infty g'(s)\|\Delta\eta^t\|^2\,ds - \frac{\xi\lambda}{2}\int_{t-\tau(t)}^t e^{-\lambda(t-s)}\|\nabla u_t(s)\|^2\,ds.
\end{aligned} \tag{4.39}
$$

*Proof.* Differentiating (4.37) with respect to $t$, we have

$$
\begin{aligned}
G'(t) = & \int_\Omega u_{tt}u_t\,dx + \int_\Omega \Delta u\cdot\Delta u_t\,dx + \int_\Omega\int_0^\infty g(s)\Delta\eta^t\cdot\Delta\eta_t^t\,dsdx \\
& + \int_\Omega f(u)u_t\,dx + \frac{\xi}{2}\|\nabla u_t\|^2 - \frac{\xi}{2}e^{-\lambda\tau(t)}(1-\tau'(t))\|\nabla u_t(t-\tau(t))\|^2 \\
& - \frac{\lambda\xi}{2}\int_{t-\tau(t)}^t e^{-\lambda(t-s)}\|\nabla u_t(s)\|^2\,ds.
\end{aligned}
$$

By using (4.36), (1.7) and (2.6)–(2.7), we can obtain for any $t > 0$,

$$
\begin{aligned}
G'(t) \leq & \frac{1}{2}\int_0^\infty g'(s)\|\Delta\eta^t(s)\|^2\,ds - \mu_1\|\nabla u_t\|^2 \\
& - \mu_2\int_\Omega \nabla u_t(t)\cdot\nabla u_t(t-\tau(t))\,dx + \frac{\xi}{2}\|\nabla u_t\|^2 \\
& - \frac{\xi}{2}(1-d)e^{-\lambda\tau_1}\|\nabla u_t(t-\tau(t))\|^2 - \frac{\lambda\xi}{2}\int_{t-\tau(t)}^t e^{-\lambda(t-s)}\|\nabla u_t(s)\|^2\,ds.
\end{aligned} \tag{4.40}
$$

Using Young's inequality, we get

$$
-\mu_2\int_\Omega \nabla u_t(t)\cdot\nabla u_t(t-\tau(t))\,dx \leq \frac{|\mu_2|}{2\sqrt{1-d}}\|\nabla u_t\|^2 + \frac{|\mu_2|}{2}\sqrt{1-d}\|\nabla u_t(t-\tau(t))\|^2,
$$

which, together with (4.40), implies (4.39). The proof is complete. □

**Lemma 4.12.** *We define the functional* $\phi(t)$ *by*

$$
\phi(t) = \int_\Omega u(t)u_t(t)\,dx.
$$

*Then under the assumptions of Theorem* 4.10, *there exist two positive constants* $c_1$ *and* $c_2$ *such that*

$$
\begin{aligned}
\phi'(t) \leq & -G(t) - \frac{1}{4}\|\Delta u\|^2 + c_1\|\nabla u_t\|^2 + c_1\|\nabla u_t(t-\tau(t))\|^2 - \int_\Omega f(u)u\,dx \\
& - c_2\int_0^\infty g'(s)\|\Delta\eta^t(s)\|^2\,ds + \frac{\xi}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\|\nabla u_t(s)\|^2\,ds, \quad \forall\, t > 0.
\end{aligned} \tag{4.41}
$$

*Proof.* It follows from (4.36) and integration by parts that

$$\tag{4.42} \begin{aligned} \phi'(t) = \|u_t\|^2 - \|\Delta u\|^2 - \int_\Omega \Delta u(t) \int_0^\infty g(s)\Delta\eta^t(s)\, dsdx \\ - \mu_1 \int_\Omega \nabla u \cdot \nabla u_t \, dx - \mu_2 \int_\Omega \nabla u \cdot \nabla u_t(t-\tau(t))\, dx - \int_\Omega f(u)u\, dx. \end{aligned}$$

By using Hölder's inequality and Young's inequality, we derive that for any $\varepsilon > 0$,

$$\tag{4.43} -\int_\Omega \Delta u(t) \cdot \int_0^\infty g(s)\Delta\eta^t(s)\, dsdx \le \frac{1}{8}\|\Delta u\|^2 + 2l_0\|\eta^t\|^2_{\mathcal{M}_2},$$

$$\tag{4.44} -\mu_1 \int_\Omega \nabla u \cdot \nabla u_t \, dx \le \frac{|\mu_1|\varepsilon}{\lambda_2}\|\Delta u\|^2 + \frac{|\mu_1|}{4\varepsilon}\|u_t\|^2$$

and

$$-\mu_2 \int_\Omega \nabla u \cdot \nabla u_t(t-\tau(t))\, dx \le \frac{|\mu_2|\varepsilon}{\lambda_2}\|\Delta u\|^2 + \frac{|\mu_2|}{4\varepsilon}\|\nabla u_t(t-\tau(t))\|^2,$$

which, along with (4.42)–(4.44), gives us

$$\begin{aligned} \phi'(t) \le -\left(\frac{7}{8} - \frac{|\mu_1|\varepsilon}{\lambda_2} - \frac{|\mu_2|\varepsilon}{\lambda_2}\right)\|\Delta u\|^2 + 2l_0\|\eta^t\|^2_{\mathcal{M}_2} + \left(\frac{1}{\lambda_1} + \frac{|\mu_1|}{4\varepsilon}\right)\|\nabla u_t\|^2 \\ + \frac{|\mu_2|}{4\varepsilon}\|\nabla u_t(t-\tau(t))\|^2 - \int_\Omega f(u)u\, dx. \end{aligned}$$

Noting (4.37), we have for any $\varepsilon > 0$,

$$\begin{aligned} \phi'(t) \le -G(t) - \left(\frac{3}{8} - \frac{|\mu_1|\varepsilon}{\lambda_2} - \frac{|\mu_2|\varepsilon}{\lambda_2}\right)\|\Delta u\|^2 + \left(\frac{1}{2} + 2l_0\right)\|\eta^t\|^2_{\mathcal{M}_2} \\ + \left(\frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{|\mu_1|}{4\varepsilon}\right)\|\nabla u_t\|^2 + \frac{|\mu_2|}{4\varepsilon}\|\nabla u_t(t-\tau(t))\|^2 \\ + \frac{\xi}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\|\nabla u_t(s)\|^2\, ds - \int_\Omega f(u)u\, dx. \end{aligned}$$

At this point, choosing $\varepsilon > 0$ sufficiently small that

$$\frac{|\mu_1|\varepsilon}{\lambda_2} + \frac{|\mu_2|\varepsilon}{\lambda_2} \le \frac{1}{8},$$

and since

$$\|\eta^t\|^2_{\mathcal{M}_2} \le -\frac{1}{k}\int_0^\infty g'(s)\|\Delta\eta^t(s)\|^2\, ds,$$

thus we can obtain (4.41) with

$$c_1 = \max\left\{\frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{|\mu_1|}{4\varepsilon}, \frac{|\mu_2|}{4\varepsilon}\right\}, \quad c_2 = \frac{1}{2k} + \frac{2l_0}{k}.$$

The proof is hence complete. ∎

**Lemma 4.13.** *We define the functional* $\psi(t)$ *as*

$$\psi(t) = -\int_\Omega u_t(t)\cdot\left(\int_0^\infty g(s)\eta^t(s)\,ds\right)\,dx.$$

*Then under the assumptions of Theorem* 4.10*, the functional* $\psi(t)$ *satisfies that for any* $\delta_1 > 0$ *and* $\delta_2 > 0$*,*

$$\begin{aligned}\psi'(t) \le& -\frac{3}{4}l_0\|u_t\|^2 + \delta_2\|\Delta u\|^2 + \delta_1\|\nabla u_t(t-\tau(t))\|^2 + \delta_1\|\nabla u_t\|^2\\ &- c_3\int_0^\infty g'(s)\|\Delta\eta^t(s)\|^2\,ds,\end{aligned} \tag{4.45}$$

*where* $c_3 > 0$ *is a constant.*

*Proof.* First we can easily get

$$\begin{aligned}\psi'(t) =& -\int_\Omega u_{tt}\cdot\left(\int_0^\infty g(s)\eta^t(s)\,ds\right)\,dx - \int_\Omega u_t\cdot\left(\int_0^\infty g(s)\eta^t_t(s)\,ds\right)\,dx\\ =& \int_\Omega\left(\Delta^2 u + \int_0^\infty g(s)\Delta^2\eta^t(s)\,ds - \mu_1\nabla u_t - \mu_2\nabla u_t(t-\tau(t)) + f(u)\right)\\ &\cdot\left(\int_0^\infty g(s)\eta^t(s)\,ds\right)\,dx - \int_\Omega u_t\cdot\left(\int_0^\infty g(s)\eta^t_t(s)\,ds\right)\,dx.\end{aligned} \tag{4.46}$$

Therefore, from integration by parts, Young's inequality and Hölder's inequality, we infer that for any $\delta_1 > 0$,

$$\int_\Omega \Delta^2 u(t)\cdot\left(\int_0^\infty g(s)\eta^t(s)\,ds\right)\,dx \le \delta_1\|\Delta u\|^2 + \frac{l_0}{4\delta_1}\|\eta^t\|^2_{\mathcal{M}_2}, \tag{4.47}$$

$$\begin{aligned}&\int_\Omega\left(\int_0^\infty g(s)\Delta^2\eta^t(s)\,ds\right)\cdot\left(\int_0^\infty g(s)\eta^t(s)\,ds\right)\,dx\\ &= \int_\Omega\sum_{j=1}^n\left(\int_0^\infty g(s)\frac{\partial^2\eta^t}{\partial x_j^2}\right)^2\,dx \le l_0\|\eta^t\|^2_{\mathcal{M}_2},\end{aligned} \tag{4.48}$$

$$\begin{aligned}\mu_1\int_\Omega\nabla u_t(t)\cdot\left(\int_0^\infty g(s)\eta^t(s)\,ds\right)\,dx &\le \delta_1\|\nabla u_t\|^2 + \frac{\mu_1^2 l_0}{4\delta_1}\int_0^\infty g(s)\|\eta^t(s)\|^2\,ds\\ &\le \delta_1\|\nabla u_t\|^2 + \frac{\mu_1^2 l_0}{4\delta_1\lambda_1}\int_0^\infty g(s)\|\Delta\eta^t(s)\|^2\,ds,\end{aligned} \tag{4.49}$$

$$\begin{aligned}&\mu_2\int_\Omega\nabla u_t(t-\tau(t))\cdot\left(\int_0^\infty g(s)\eta^t(s)\,ds\right)\,dx\\ &\le \delta_1\|\nabla u_t(t-\tau(t))\|^2 + \frac{\mu_2^2 l_0}{4\delta_1\lambda_1}\int_0^\infty g(s)\|\Delta\eta^t\|^2\,ds.\end{aligned} \tag{4.50}$$

Noting that

$$\int_0^\infty g(s)\eta^t_t(s)\,ds = -\int_0^\infty g(s)\eta^t_s(s) + \int_0^\infty u_t(t)g(s)\,ds = \int_0^\infty g'(s)\eta^t(s)\,ds + l_0 u_t,$$

we shall deduce that

$$
\begin{aligned}
& -\int_\Omega u_t \cdot \left( \int_0^\infty g(s)\eta_t^t(s)\,ds \right) dx \\
& \le -l_0\|u_t\|^2 + \frac{l_0}{4}\|u_t\|^2 + \frac{1}{l_0}\int_\Omega \left( \int_0^\infty g'(s)\eta^t(s)\,ds \right)^2 dx \\
& \le -\frac{3}{4}l_0\|u_t\|^2 + \frac{1}{l_0}\int_\Omega \left( \int_0^\infty -g'(s)\,ds \right) \cdot \left( \int_0^\infty -g'(s)(\eta^t(s))^2\,ds \right) dx \\
& \le -\frac{3}{4}l_0\|u_t\|^2 - \frac{g(0)}{l_0\lambda_1}\int_0^\infty g'(s)\|\Delta\eta^t(s)\|^2\,ds.
\end{aligned} \tag{4.51}
$$

Clearly, $e^{\lambda\tau_1}$ goes to 1 as $\lambda \to 0$. By virtue of the continuity of the set of real numbers, we take $\lambda > 0$ sufficiently small that there exists a constant $\xi > 0$ such that

$$
\frac{e^{\lambda\tau_1}|\mu_2|}{\sqrt{1-d}} < \xi < \mu_1, \tag{4.52}
$$

which implies

$$
\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2} < 0 \quad \text{and} \quad \frac{|\mu_2|}{2}\sqrt{1-d} - \frac{\xi}{2e^{\lambda\tau_1}}(1-d) < 0. \tag{4.53}
$$

Thus we know from (4.39) and (4.53) that

$$
G'(t) \le 0. \tag{4.54}
$$

It follows from (2.1) and (4.54) that for any $\delta_3 > 0$,

$$
\int_\Omega f(u) \cdot \left( \int_0^\infty g(s)\eta^t(s)\,ds \right) dx \le \frac{\delta_3}{\lambda_1}G^p(0)\|\Delta u\|^2 + \frac{l_0}{4\delta_3\lambda_1}\|\eta^t\|^2_{\mathcal{M}_2},
$$

which, combined with (4.46)–(4.51), yields (4.45) with

$$
\delta_2 = \delta_1 + \frac{\delta_3}{\lambda_1}G^p(0) \quad \text{and} \quad c_3 = \frac{l_0^2}{4\delta_1 k} + \frac{l_0}{k} + \frac{\mu_1^2 l_0}{4\delta_1\lambda_1 k} + \frac{\mu_2^2 l_0}{4\delta_1\lambda_1 k} + \frac{g(0)}{l_0\lambda_1} + \frac{l_0^2}{4\delta_3\lambda_1 k}.
$$

The proof is hence complete. □

Now we define the Lyapunov functional $\mathcal{L}(t)$ as

$$
\mathcal{L}(t) := G(t) + \varepsilon_1\phi(t) + \varepsilon_2\psi(t), \tag{4.55}
$$

where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are constants will be taken later. First we can easily deduce, for $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ sufficiently small, that for any $t > 0$,

$$
\frac{1}{2}G(t) \le \mathcal{L}(t) \le \frac{3}{2}G(t). \tag{4.56}
$$

*Proof of Theorem* 4.10. Combining (4.39), (4.41) and (4.45) with (4.55), we can derive for any $t > 0$,

$$\begin{aligned}\mathcal{L}'(t) &= G'(t) + \varepsilon_1\phi'(t) + \varepsilon_2\psi'(t)\\ &\leq -\varepsilon_1 G(t) - \frac{3}{4}l_0\|u_t\|^2 + \left[\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2} + c_1\varepsilon_1 + \varepsilon_2\delta_1\right]\|\nabla u_t\|^2\\ &\quad + \left(\frac{|\mu_2|}{2}\sqrt{1-d} - \frac{\xi}{2}e^{-\lambda\tau_1}(1-d) + c_1\varepsilon_1 + \varepsilon_2\delta_1\right)\|\nabla u_t(t-\tau(t))\|^2\\ &\quad + \left(\frac{1}{2} - c_2\varepsilon_1 - c_3\varepsilon_2\right)\int_0^\infty g'(s)\|\Delta\eta^t(s)\|^2\,ds + \left(\varepsilon_2\delta_2 - \frac{\varepsilon_1}{4}\right)\|\Delta u\|^2\\ &\quad + \left(\frac{\xi\varepsilon_1}{2} - \frac{\lambda\xi}{2}\right)\int_{t-\tau(t)}^t e^{-\lambda(t-s)}\|\nabla u_t(s)\|^2\,ds - \varepsilon_1\int_\Omega f(u)u\,dx.\end{aligned}$$

Since (4.52)–(4.53) hold, at this point we first take $\varepsilon_1 > 0$ sufficiently small such that (4.56) holds, and further,

$$\varepsilon_1 < \min\left\{\frac{\lambda}{2}, \frac{1}{4c_2}, \frac{\xi}{4c_1}e^{-\lambda\tau_1}(1-d) - \frac{|\mu_2|}{4c_1}\sqrt{1-d}, \frac{\mu_1}{2c_1} - \frac{|\mu_2|}{4c_1\sqrt{1-d}} - \frac{\xi}{4c_1}\right\},$$

which gives us

$$\frac{\xi\varepsilon_1}{2} - \frac{\lambda\xi}{2} < -\frac{\lambda\xi}{4}, \quad \frac{1}{2} - c_2\varepsilon_1 > \frac{1}{4},$$

and

$$\begin{gathered}\frac{|\mu_2|}{2}\sqrt{1-d} - \frac{\xi}{2}e^{-\lambda\tau_1}(1-d) + c_1\varepsilon_1 < \frac{|\mu_2|}{4}\sqrt{1-d} - \frac{\xi}{4}e^{-\lambda\tau_1}(1-d),\\ \frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2} + c_1\varepsilon_1 < \frac{|\mu_2|}{4\sqrt{1-d}} - \frac{\mu_1}{2} + \frac{\xi}{4}.\end{gathered}$$

For any fixed $\delta_1, \delta_2 > 0$ and $\varepsilon_1 > 0$, we choose $\varepsilon_2 > 0$ sufficiently small such that (4.56) holds, and further,

$$\varepsilon_2 < \min\left\{\frac{\varepsilon_1}{8\delta_2}, \frac{1}{8c_3}, \frac{\xi}{8\delta_1}e^{-\lambda\tau_1}(1-d) - \frac{|\mu_2|}{8\delta_1}\sqrt{1-d}, \frac{\mu_1}{4\delta_1} - \frac{|\mu_2|}{8\delta_1\sqrt{1-d}} - \frac{\xi}{8\delta_1}\right\},$$

which implies

$$\varepsilon_2\delta_2 - \frac{\varepsilon_1}{4} < -\frac{\varepsilon_1}{8}, \quad \frac{1}{4} - c_3\varepsilon_2 > \frac{1}{8},$$

and

$$\begin{gathered}\frac{|\mu_2|}{4}\sqrt{1-d} - \frac{\xi}{4}e^{-\lambda\tau_1}(1-d) + \delta_1\varepsilon_2 < \frac{|\mu_2|}{8}\sqrt{1-d} - \frac{\xi}{8}e^{-\lambda\tau_1}(1-d),\\ \frac{|\mu_2|}{4\sqrt{1-d}} - \frac{\mu_1}{2} + \frac{\xi}{4} + \delta_1\varepsilon_2 < \frac{|\mu_2|}{8\sqrt{1-d}} - \frac{\mu_1}{4} + \frac{\xi}{8}.\end{gathered}$$

From above and (4.37) and (4.56), we can conclude that there exists a positive constant $\rho$ such that for any $t > 0$,

$$\mathcal{L}'(t) \leq -\rho G(t) \leq -\frac{\rho}{2}\mathcal{L}(t),$$

then we have

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\frac{\varrho}{2}t}.$$

By using (4.56) again, we see that

$$G(t) \leq 3G(0)e^{-\frac{\varrho}{2}t}. \tag{4.57}$$

Therefore by renaming the constants in (4.57), we can get the desired estimate (4.38). The proof of Theorem 4.10 is complete. □

*Remark* 4.14. For plate equation without rotational inertia, our result concerning global well-posedness only holds for $|\mu_2| \leq \mu_1$, concerning exponential stability only holds for $|\mu_2| < \sqrt{1-d}\mu_1$ and $\mu_1 \neq 0$. Whether the two results hold for $\mu_1 = 0$ is still open.

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## References

[1] F. Alabau-Boussouira, S. Nicaise and C. Pignotti, *Exponential stability of the wave equation with memory and time delay*, in: *New Prospects in Direct, Inverse and Control Problems for Evolution Equations*, 1–22, Springer INdAM Ser. **10**, Springer, Cham, 2014.

[2] C. M. Dafermos, *Asymptotic stability in viscoelasticity*, Arch. Rational Mech. Anal. **37** (1970), 297–308.

[3] Q. Dai and Z. Yang, *Global existence and exponential decay of the solution for a viscoelastic wave equation with a delay*, Z. Angew. Math. Phys. **65** (2014), no. 5, 885–903.

[4] R. Datko, J. Lagnese and M. P. Polis, *An example on the effect of time delays in boundary feedback stabilization of wave equations*, SIAM J. Control Optim. **24** (1986), no. 1, 152–156.

[5] G. Duvaut and J.-L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.

[6] B. Feng, *Global well-posedness and stability for a viscoelastic plate equation with a time delay*, Math. Probl. Eng. **2015** (2015), Art. ID 585021, 10 pp.

[7] ———, *Well-posedness and exponential stability for a plate equation with time-varying delay and past history*, Z. Angew. Math. Phys. **68** (2017), no. 1, Art. 6, 24 pp.

[8] ———, *General decay for a viscoelastic wave equation with strong time-dependent delay*, Bound. Value Probl. **2017** (2017), no. 57, 11 pp.

[9] ———, *General decay for a viscoelastic wave equation with density and time delay term in* $\mathbb{R}^n$, Taiwanese J. Math. **22** (2018), no. 1, 205–223.

[10] ———, *Long-time dynamics of a plate equation with memory and time delay*, Bull. Braz. Math. Soc. (N.S.) **49** (2018), no. 2, 395–418.

[11] B. Feng, K. Zennir and L. K. Laouar, *Decay of an extensible viscoelastic plate equation with a nonlinear time delay*, accepted in Bull. Malays. Math. Sci. Soc., 2018, 20 pp.

[12] P. Freitas and E. Zuazua, *Stability results for the wave equation with indefinite damping*, J. Differential Equations **132** (1996), no. 2, 338–352.

[13] M. Kafini, S. A. Messaoudi and S. Nicaise, *A blow-up result in a nonlinear abstract evolution system with delay*, NoDEA Nonlinear Differential Equations Appl. **23** (2016), no. 2, Art. 13, 14 pp.

[14] M. Kirane and B. Said-Houari, *Existence and asymptotic stability of a viscoelastic wave equation with a delay*, Z. Angew. Math. Phys. **62** (2011), no. 6, 1065–1082.

[15] V. Komornik, *Exact Controllability and Stabilization: The multiplier method*, RMA: Research in Applied Mathematics, Masson, Paris, John Wiley & Sons, Chichester, 1994.

[16] W. Liu, *General decay of the solution for a viscoelastic wave equation with a time-varying delay term in the internal feedback*, J. Math. Phys. **54** (2013), no. 4, 043504, 9 pp.

[17] ———, *General decay rate estimate for the energy of a weak viscoelastic equation with an internal time-varying delay term*, Taiwanese J. Math. **17** (2013), no. 6, 2101–2115.

[18] G. Liu and H. Zhang, *Well-posedness for a class of wave equation with past history and a delay*, Z. Angew. Math. Phys. **67** (2016), no. 1, Art. 6, 14 pp.

[19] S. A. Messaoudi, A. Fareh and N. Doudi, *Well posedness and exponential stability in a wave equation with a strong damping and a strong delay*, J. Math. Phys. **57** (2016), no. 11, 111501, 13 pp.

[20] S. Nicaise and C. Pignotti, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks*, SIAM J. Control Optim. **45** (2006), no. 5, 1561–1585.

[21] ______, *Stabilization of the wave equation with boundary or internal distributed delay*, Differential Integral Equations **21** (2008), no. 9-10, 935–958.

[22] ______, *Interior feedback stabilization of wave equations with time dependent delay*, Electron. J. Differential Equations **2011** (2011), no. 41, 20 pp.

[23] ______, *Exponential stability of abstract evolution equations with time delay*, J. Evol. Equ. **15** (2015), no. 1, 107–129.

[24] S. Nicaise and J. Valein, *Stabilization of second order evolution equations with unbounded feedback with delay*, ESAIM Control Optim. Calc. Var. **16** (2010), no. 2, 420–456.

[25] S. Nicaise, J. Valein and E. Fridman, *Stability of the heat and of the wave equations with boundary time-varying delays*, Discrete Contin. Dyn. Syst. Ser. S **2** (2009), no. 3, 559–581.

[26] S. H. Park, *Decay rate estimates for a weak viscoelastic beam equation with time-varying delay*, Appl. Math. Lett. **31** (2014), 46–51.

[27] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences **44**, Springer-Verlag, New York, 1983.

[28] C. Pignotti, *A note on stabilization of locally damped wave equations with time delay*, Systems Control Lett. **61** (2012), no. 1, 92–97.

[29] ______, *Stability results for second-order evolution equations with memory and switching time-delay*, J. Dynam. Differential Equations **29** (2017), no. 4, 1309–1324.

[30] G. Xu, S. P. Yung and L. Li, *Stabilization of wave systems with input delay in the boundary control*, ESAIM Control Optim. Calc. Var. **12** (2006), no. 4, 770–785.

[31] Z. Yang, *Existence and energy decay of solutions for the Euler-Bernoulli viscoelastic equation with a delay*, Z. Angew. Math. Phys. **66** (2015), no. 3, 727–745.

Baowei Feng
Department of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, P. R. China
*E-mail address*: `bwfeng@swufe.edu.cn`

Gongwei Liu
College of Science, Henan University of Technology, Zhengzhou 450001, P. R. China
*E-mail address*: gongweiliu@126.com